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Abstract

We study pure exchange economies with symmetries on preferences up to taste intensity transformations. In a 2-person, 2-good endowment economy, we show that bilateral symmetry on each utility functional precludes a rectangle in the Edgeworth box as the location of Pareto optimal allocations. Under strictly quasi-concave preferences, a larger set can be ruled out. The inadmissible region is still larger when preferences are homothetic and identical up to taste intensity parameters. Symmetry also places bounds on the relations between terms of trade and efficient allocation. The inferences can be extended to an n-person, m-good endowment economy under generalized permutation group symmetries on preferences.

JEL classification: D51, D61, C60

KEYWORDS: bilateral symmetry, general equilibrium, group majorize, homothetic preferences, permutation groups, taste differences, terms of trade.

1. INTRODUCTION

SYMMETRY AND MARKET EXCHANGE ARE INTIMATELY related phenomena. Market transactions are motivated by asymmetries in tastes or endowments. Absent transactions costs, these asymmetries are exploited through exchange where the terms of trade are invariant to the parties involved. Consequently, in a pure exchange economy there should be fundamental structural relationships between the nature of heterogeneities among consumers, equilibrium decisions by consumers, and the equilibrium prices that guide these decisions. Our interest is in identifying and characterizing how heterogeneities in consumer tastes over the set of available goods affect general equilibrium. We show that particular forms of heterogeneities among consumers imply that certain consumption bundles cannot be supported in equilibrium regardless of consumer income levels. The reason is because the terms of trade to support these equilibria are inconsistent with consumer preferences, when considered collectively.

To identify cardinal aspects of general equilibrium relationships in a pure exchange economy it is necessary to employ tools that model the structure of similarities and dissimilarities in preferences and endowments. Mathematical science provides a number of related tools, such as the theories of group structures and vector majorization, to model structural symmetries. These tools have been used elsewhere in economics. Work by Koopmans, Diamond, and Williamson (1964) invoked an analogy with group structures on invariant measures to understand consumption preferences over time. Sato (1977, 1981) and others have found a number of applications in the theory of dual structures. The role of symmetries in index number theory was apparent to Samuelson and Swamy (1974) some 30 years ago. In related research, the group structures underlying aggregators have received some attention since then by Vogt and Barta (1997), among others.[1] In trade theory, Samuelson (2001) has applied symmetry to facilitate the numerical accounting of gains from trade when comparing the Ricardo and Sraffa models.

Majorization, a pre-ordering on vectors that relies on complete symmetry, has been applied by Atkinson (1970) in the study of income dispersion and social equity. Since risk may be viewed as a dispersion attribute on a random variable, it is not surprising that majorization relations should also arise in considerations of decision making under uncertainty. Rothschild and Stiglitz (1970) and, in a more general framework, Chambers and Quiggin (2000) have used versions of the concept to analyze comparative statics under risk.

General equilibrium has also been subjected to group-oriented analyses. Balasko (1990) has observed that sunspot equilibria may be interpreted as broken symmetries in general equilibrium that arise due to market incompleteness. This observation allows for a more comprehensive characterization of admissible equilibria. While we also seek to understand the nature of general equilibrium, our purpose is more microeconomic in flavor. From a methodological perspective, Hennessy and Lapan (2003a, 2003b), who studied firm-level production decisions (2003a) and investor-level portfolio allocation decisions (2003b), provide the most direct links to the approach that we will take. Their analyses exploit the symmetries of a functional when a group acts on the functional's arguments. Contradictions then generate bounds on optimal decision vectors.

The present analysis is also built upon the contradictions that symmetries can generate. Section 2 focuses on a 2-agent, 2-good pure exchange economy. There, we use group invariances to show that when monotone utility functions are bilaterally symmetric up to symmetry-breaking scale parameters then conditions exist such that one rectangle in the Edgeworth box must be eliminated as candidate solutions for an efficient equilibrium. These conditions pertain to the relative strength of

[1] Fixed point theorems have an algebraic structure. While algebraic topology has been applied to better understand Nash equilibria, as in Herings and Peeters (2001), the research does not appear to

preferences and the relative scarcity of endowments. If, in addition to bilateral symmetry, the utility functions are quasi-concave then two further results may be obtained. First, the inadmissible region in the allocation box may be expanded to include two additional contiguous triangles. Second, use of majorization theory demonstrates that the equilibrium terms of trade must bear a particular relationship with the equilibrium consumption point. If, aswell, it is assumed that preferences are homothetic, are identical apart from taste intensity parameters, and possess an elasticity of substitution less than unity, then the inadmissible region can be shown to be larger still. When considered separately, neither identical preferences nor homothetic preferences support a larger precluded region.

In the remainder of the paper the inferences arising from the bilateral symmetry context of Section 2 are extended to a multi-good, multi-consumer pure exchange economy under generalized symmetry structures. The most obvious extension, and the one which we make, is to finite permutation groups, i.e., utility functional invariances under permutations of a finite number of arguments.

2. MOTIVATION

2.1. *2×2 Model*

In a 2-person pure exchange economy, goods A and B are available in the amounts $\bar{q}_a$ and $\bar{q}_b$. Person 1 has the composite utility function $U^1[T^{1,a}(q_{1,a}),T^{1,b}(q_{1,b})]$ while Person 2 has utility $U^2[T^{2,a}(q_{2,a}),T^{2,b}(q_{2,b})]$. The functions $T^{1,a}(q_{1,a})$, $T^{1,b}(q_{1,b})$, $T^{2,a}(q_{2,a})$, and $T^{2,b}(q_{2,b})$ are $\mathbb{R}\to\mathbb{R}$ continuously differentiable, monotone increasing maps, while the functionals $U^1[\cdot,\cdot]$ and $U^2[\cdot,\cdot]$ are

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have appealed to any underlying fundamental symmetries.

$\mathbb{R}^2 \to \mathbb{R}$. All are strictly increasing (i.e., monotone or non-satiated) and once continuously differentiable. There is no waste in allocation and the goods are scarce, so that efficiency requires both $q_{1,a} + q_{2,a} = \bar{q}_a$ and $q_{1,b} + q_{2,b} = \bar{q}_b$. The sorts of question that we are interested in involve what can be inferred about Pareto efficient divisions of $\bar{q}_a$ and $\bar{q}_b$, as well as how prices and quantities relate in general equilibrium.

Further assumptions are clearly necessary. Because Pareto efficiency requires the exhaustion of exchange opportunities that arise from consumer heterogeneities, the assumptions will involve restrictions on how goods substitute within a consumer's basket of purchases. The restrictions we employ involve symmetries that place structure on a consumer's iso-utility trade-offs. Pareto improving re-allocations can then be identified by using symmetries to hold the utility level of one consumer constant while freeing up endowments to make the other consumer better off.

Asymmetries are necessary to model taste differences, while it is necessary to modularize the asymmetries if meaningful insights are to be found. For the moment, we model these asymmetries through the superscripted $T(\cdot)$ functions. The structure on the symmetries are modeled through the assumption that $U^i[T^{i,a}, T^{i,b}] = U^i[T^{i,b}, T^{i,a}], i \in \Omega_2 = \{1,2\}$.

To illustrate, model the superscripted $T(\cdot)$ functions as ray linear; $T^{i,a}(q_{i,a}) = \theta_{i,a} q_{i,a}, i \in \Omega_2$, $T^{i,b}(q_{i,b}) = \theta_{i,b} q_{i,b}, i \in \Omega_2$, where the θ values are strictly positive, symmetry-breaking, taste intensity heterogeneities. The central concept in our analysis is the notion of invariance, and this example will show how to use the invariances that arise from symmetries when developing inferences about efficient allocations. Our present interest is in exploiting the invariances of the two functions $U^1[\theta_{1,a} q_{1,a}, \theta_{1,b} q_{1,b}]$ and $U^2[\theta_{2,a} q_{2,a}, \theta_{2,b} q_{2,b}]$. The ratios $z_1 = \theta_{1,a} / \theta_{1,b}$ and $z_2 =$ $\theta_{2,a} / \theta_{2,b}$ are clearly important in this regard because they gauge a consumer's personal relative intensity of preference for good A. Ratio $z_q = \overline{q}_b / \overline{q}_a$ should also be important because it measures the relative economy-wide scarcity of good A.

2.2. *Symmetric and Monotone Utility*

Denote the set of Pareto efficient allocations as $\tilde{Q}$ with sample elements given by the allocation 2-vector $\{(\tilde{q}_{1,a}, \tilde{q}_{1,b})', (\tilde{q}_{2,a}, \tilde{q}_{2,b})'\}$, where "'" is the vector transpose operation. The endowment constraint determines two of the points in this quadruple. Apart from singularities arising from any of the relations $z_q = z_1$, $z_q = z_2$, or $z_1 = z_2$, there are essentially two contexts to be considered. When $z_1 > (<) z_2$ then person 1 (person 2) has a stronger comparative preference for good A than does person 2 (person 1). Without further loss of generality, and ignoring equality in relative preference intensities for the moment, we may assign the order $z_1 > z_2$.

Upon, again, ignoring singularities that will be studied separately, we may assume that either $z_q \in (z_2, z_1)$ or $z_q \notin [z_2, z_1]$. Defining $\tilde{q}_{1,a} = \delta_a \overline{q}_a$ and $\tilde{q}_{1,b} = \delta_b \overline{q}_b$, the endowment constraints require $\tilde{q}_{2,a} = (1-\delta_a)\overline{q}_a$ and $\tilde{q}_{2,b} = (1-\delta_b)\overline{q}_b$. After applying some symbol manipulations to exploit the invariances, it is clear that bilateral symmetry in the utility functionals imply $U^1[\theta_{1,a}\delta_a\overline{q}_a, \theta_{1,b}\delta_b\overline{q}_b]$ $= U^1[\theta_{1,a}\delta_b\overline{q}_b / z_1, \theta_{1,b}\delta_a\overline{q}_a z_1]$. Denoting the bundle indifference relation by $\sim$ and identifying vectors $\vec{c}_1 = (\delta_a\overline{q}_a, \delta_b\overline{q}_b)'$, $\vec{c}_1^{\,+} = (\delta_b\overline{q}_b / z_1, \delta_a\overline{q}_a z_1)'$, we have $\vec{c}_1 \sim \vec{c}_1^{\,+}$. Similarly, if we define $\vec{c}_2 =$ $((1-\delta_a)\overline{q}_a, (1-\delta_b)\overline{q}_b)'$ and $\vec{c}_2^{\,+} = ((1-\delta_b)\overline{q}_b / z_2, (1-\delta_a)\overline{q}_a z_2)'$, then $\vec{c}_2 \sim \vec{c}_2^{\,+}$. However, and this is the foundation of our analysis, if the pair $\{\vec{c}_1^{\,+}, \vec{c}_2^{\,+}\}$ frees up resources then the pair $\{\vec{c}_1, \vec{c}_2\}$ cannot be Pareto efficient.

Our interest is in ascertaining the feasibility of certain points so that a determination can be made as to whether the point could be Pareto efficient. To this end we define convex combinations of iso-utility bundles. Restrict $\lambda_i \in [0,1], i \in \Omega_2$, and define the semi-norm

$$(2.1) \qquad \begin{aligned} D(\lambda_1, \lambda_2, \vec{c}_1, \vec{c}_2, \vec{c}_1^{\,+}, \vec{c}_2^{\,+}) &= 1 \quad \text{whenever} \quad \vec{v} \geq \vec{0},\ \vec{v} \neq \vec{0}, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

where $\vec{v} = (\overline{q}_a, \overline{q}_b)' - \lambda_1 \vec{c}_1 - (1-\lambda_1)\vec{c}_1^{\,+} - \lambda_2 \vec{c}_2 - (1-\lambda_2)\vec{c}_2^{\,+}$. If $D(\lambda_1, \lambda_2, \vec{c}_1, \vec{c}_2, \vec{c}_1^{\,+}, \vec{c}_2^{\,+}) > 0$ for some reallocation of endowments that weakly increases all utility levels, then the candidate allocation $\{(\tilde{q}_{1,a}, \tilde{q}_{1,b})', (\tilde{q}_{2,a}, \tilde{q}_{2,b})'\}$ cannot be Pareto efficient for utilities that are strictly monotone.

When all we know of the utility functions are that they are symmetric and monotone, then invariance only allows us to make deductions for the lattice points of the unit square, $(\lambda_1, \lambda_2) \in \Lambda^{lp}$ $= \{(0,0), (0,1), (1,0), (1,1)\}$. When, in addition, strict quasi-concavity is known to hold then we may seek violations on any $(\lambda_1, \lambda_2) \in ch(\Lambda^{lp}) = [0,1] \times [0,1]$ where $ch(\cdot)$ is the convex hull set operation.

The comparisons in (2.1) reduce to the assertion that $D(\cdot) = 1$ whenever

$$(2.2) \qquad \begin{aligned} &\lambda_1 \delta_a + (1-\lambda_1)\delta_b \frac{z_q}{z_1} + \lambda_2(1-\delta_a) + (1-\lambda_2)(1-\delta_b)\frac{z_q}{z_2} \leq 1, \\ &\lambda_1 \delta_b + (1-\lambda_1)\delta_a \frac{z_1}{z_q} + \lambda_2(1-\delta_b) + (1-\lambda_2)(1-\delta_a)\frac{z_2}{z_q} \leq 1, \end{aligned}$$

and one does not bind. Clearly the critical parameters are $r_1 = z_1 / z_q$ and $r_2 = z_2 / z_q$. On Λ^{lp}, i.e., for monotone, symmetric utilities only, then $D(\cdot) = 1$ if

$$(2.3) \qquad \frac{\delta_b}{r_1} + \frac{(1-\delta_b)}{r_2} \leq 1, \qquad \delta_a r_1 + (1-\delta_a) r_2 \leq 1,$$

where one inequality is strict. The solution interval is non-degenerate only if $r_i > 1, r_j < 1, i \neq j$; $i, j \in \Omega_2$. Consequently, when all we know about preferences is that they are symmetric and monotone then the only allocations that may be excluded are (for $r_1 > 1 > r_2$) the Edgeworth box points defined by

$$(2.4) \qquad \delta_a \le \frac{1-r_2}{r_1-r_2}, \qquad \delta_b \ge \frac{(1-r_2)r_1}{r_1-r_2},$$

with one inequality strict.

The situation is depicted in Figure 1, where $Q_a = (\overline{q}_a, 0)$ and $Q_b = (0, \overline{q}_b)$. The line L1: $q_{1,b} = \theta_{1,a} q_{1,a} / \theta_{1,b} = z_1 q_{1,a}$ from O1, the consumer 1 origin, gives the locus of consumption bundles that are invariant under the iso-utility symmetry for that consumer. These are the bundles that sit on the consumer 1 axis of symmetry (AS_1). The line L2: $q_{2,b} = \theta_{2,a} q_{2,a} / \theta_{2,b} = z_2 q_{2,a}$ performs the same role for consumer 2. The lines L*i* are rays from the origin O*i* because $U^1[0,0]$ is invariant to permutation on the arguments. The conditions $r_1 > 1 > r_2$ require that these two AS lines intersect in the interior of the box and to the northwest of the diagonal (denoted by L3) between the two origins. The point of intersection is given by

$$(2.5) \qquad Y = (\hat{\delta}_a \overline{q}_a, \hat{\delta}_b \overline{q}_b), \qquad \hat{\delta}_a = \frac{1-r_2}{r_1-r_2}, \qquad \hat{\delta}_b = \frac{(1-r_2)r_1}{r_1-r_2},$$

so that (2.4) may be interpreted as the pair of requirements $\delta_a \le \hat{\delta}_a, \delta_b \ge \hat{\delta}_b$.

Under these conditions we may rule out all points except the southeastern vertex, *Y*, of the northwestern rectangle in the Edgeworth box. The rectangle is depicted in Figure 1. The intuition is that for any point inside this inadmissible region there is a point somewhere else in the Edgeworth box such that both consumers are as well off while at least one of the resource constraints is slack. The vertex *Y* is special because it is the unique fixed point where the known invariances of both consumer utilities do not even alter the values of either bundle. Due to these invariances on the bundles, no opportunity can exist to exploit opportunities that arise from invariances on the iso-utility curves.

For a more detailed version of the argument, pick a candidate equilibrium point $\{(\tilde{q}_{1,a},\tilde{q}_{1,b})',(\tilde{q}_{2,a},\tilde{q}_{2,b})'\}$ that happens to be in the inadmissible region of the Edgeworth box. There $(\tilde{q}_{1,a},\tilde{q}_{1,b})$, as measured from O1, coincides with $(\tilde{q}_{2,a},\tilde{q}_{2,b})$, as measured from O2. The points must coincide for a Pareto efficient equilibrium under strict monotonicity. The map $(\tilde{q}_{1,a},\tilde{q}_{1,b}) \rightarrow$ $(\hat{q}_{1,a},\hat{q}_{1,b})$, with $\hat{q}_{1,a} = \tilde{q}_{1,b} / z_1$ and $\hat{q}_{1,b} = \tilde{q}_{1,a} z_1$, is also provided in the diagram. The linear map $(\tilde{q}_{2,a},\tilde{q}_{2,b}) \rightarrow (\hat{q}_{2,a},\hat{q}_{2,b})$, with $\hat{q}_{2,a} = \tilde{q}_{2,b} / z_2$ and $\hat{q}_{2,b} = \tilde{q}_{2,a} z_2$, is distinct, and so one must be careful that the endowment budgets are not broken. In matrix form, these maps are given as

$$\text{(2.6)} \qquad \begin{pmatrix} 0 & z_i^{-1} \\ z_i & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_{i,a} \\ \tilde{q}_{i,b} \end{pmatrix} = \begin{pmatrix} \hat{q}_{i,a} \\ \hat{q}_{i,b} \end{pmatrix}, \quad i \in \Omega_2.$$

The endowment constraints are not broken because the slope (really an arc marginal rate of substitution) for map $(\tilde{q}_{1,a},\tilde{q}_{1,b}) \rightarrow (\hat{q}_{1,a},\hat{q}_{1,b})$, being $-z_1$, differs from the slope for map $(\tilde{q}_{2,a},\tilde{q}_{2,b})$ $\rightarrow (\hat{q}_{2,a},\hat{q}_{2,b})$, $-z_2$. The supermodular nature of the transformations, θq, ensures that resources are freed up. The released resources are represented by the vector between the map image points in Figure 1.

2.3. *Allocation Under Strictly Quasi-concave Utility*

At this point we make the additional assumption that both utility functions are strictly quasi-concave so that the level sets are strictly convex and any equilibrium is unique. Symmetry, together with continuous differentiability and quasi-concavity imply that the Schur condition holds,[2]

$$(2.7) \qquad (U_1^i[\cdot]-U_2^i[\cdot])(T^{i,a}-T^{i,b}) \le 0, \quad i \in \Omega_2,$$

where $U_j^i[\cdot]$ represents the derivative with respect to the functional's j^{th} argument. With scaling symmetry breakers we have that efficient equilibria must satisfy

$$(2.8) \qquad (U_1^i[\cdot]-U_2^i[\cdot])(\theta_{i,a}\tilde{q}_{i,a}-\theta_{i,b}\tilde{q}_{i,b}) \le 0, \quad i \in \Omega_2.$$

Notice that, due to continuity, $U_1^i[\cdot]=U_2^i[\cdot]$ on the respective AS lines under quasi-concavity so that the marginal rates of substitution along the AS lines are given by

$$(2.9) \qquad \frac{U_2^1[\cdot]\theta_{1,b}}{U_1^1[\cdot]\theta_{1,a}}\Big|_{\substack{(q_{1,a},q_{1,b})\in \mathrm{L1} \\ U^1[\cdot]\text{ fixed}}} = \frac{\theta_{1,b}}{\theta_{1,a}} = -\frac{\partial q_{1,a}}{\partial q_{1,b}}, \qquad \frac{U_2^2[\cdot]\theta_{2,b}}{U_1^2[\cdot]\theta_{2,a}}\Big|_{\substack{(q_{2,a},q_{2,b})\in \mathrm{L2} \\ U^2[\cdot]\text{ fixed}}} = \frac{\theta_{2,b}}{\theta_{2,a}} = -\frac{\partial q_{2,a}}{\partial q_{2,b}}.$$

We see that the marginal rates of substitution are constant along an axis of symmetry. The symmetry assumption, together with the scalar structure of the transformation functions, impose a local form of ray homotheticity on preference structures. The importance of strict quasi-concavity lies in the fact that any interior convex combinations of iso-utility points are Pareto improving, if feasible.

Returning to the program provided in (2.2) and now choosing over any $(\lambda_1,\lambda_2) \in ch(\Lambda^{lp}) =$ $[0,1]\times[0,1]$, some manipulation of (2.2) shows that allocations adhering to

$$(2.10) \qquad \begin{aligned} &(1-\lambda_2)r_1M_1 \le (1-\lambda_1)r_2M_2 \le (1-\lambda_2)r_2M_1, \\ &M_1 \equiv (1-\delta_b)-(1-\delta_a)r_2, \quad M_2 \equiv \delta_a r_1 - \delta_b, \end{aligned}$$

with one strict, constitute a violation of Pareto efficiency on the part of candidate optimum $\{(\tilde{q}_{1,a},\tilde{q}_{1,b})',(\tilde{q}_{2,a},\tilde{q}_{2,b})'\}$. Obviously the pair of inequalities is always satisfied when $\lambda_1=\lambda_2=1$, i.e.,

[2] See Marshall and Olkin (1979, p. 57 and p. 69). See Chambers and Quiggin (2000) for economic applications of the condition.

when there is zero displacement along either arc between an initial consumption bundle and an iso-utility bundle.

If we assume that $r_1 \geq r_2$, then we need only consider two cases: $r_1 > r_2$ and $r_1 = r_2$ where we identify the common ratio as r. For the latter case the interval in (2.10) that r_2 must satisfy is degenerate. In that case, it is clear that the set of points (δ_a, δ_b) satisfying $D(\cdot) = 1$ in (2.1) has positive measure if and only if M_1 and M_2 have the same sign, i.e., the only possible Pareto efficient solutions are such that $M_1 M_2 < 0$ or $M_1 = M_2 = 0$. Thus, with $r_1 = r_2 = r$, the following allocations are the only potential Pareto efficient allocations:[3]

$$\text{(2.11)} \quad \begin{array}{llll} (a) & r < 1: & \text{all allocations such that} & (\delta_b/r) > \delta_a > (\delta_b/r) + ((r-1)/r), \\ (b) & r = 1: & \text{all allocations such that} & \delta_a = (\delta_b/r), \\ (c) & r > 1: & \text{all allocations such that} & (\delta_b/r) < \delta_a < (\delta_b/r) + ((r-1)/r). \end{array}$$

All allocations not satisfying these bounds may be excluded. Observe that the most informative situation arises in case (*b*) when both taste intensity indices equal the index of relative scarcity, i.e., $r_1 = r_2 = 1$. Then the Pareto efficient locus must be the main diagonal. In both cases (*a*) and (*c*) the remaining admissible region is a parallelogram between the axes of symmetry.

Turning to the situation where $r_1 > r_2$, the set of solutions ruled out by (2.10) under strict quasi-concavity is empty if $M_1 > 0$ or if $M_2 > 0 > M_1$. We may, however exclude all points such that both $M_1 \leq 0$ and $M_2 \leq 0$. In summary, if $r_1 > r_2$ then all points $(\delta_a, \delta_b) \in [0,1] \times [0,1]$ such that

$$\text{(2.12)} \qquad \delta_b \geq \max\left[\delta_a r_1, \delta_a r_2 + 1 - r_2\right]$$

may be excluded.[4]

[3] To conserve on space we have not drawn the associated regions. However, we encourage the interested reader to do so.
[4] Notice that $\delta_b / r_1 = (\delta_b + r_2 - 1) / r_2 = \delta_a$ defines point Y as given in Figure 1.

Another approach to establishing (2.12) is to note that all points such that the marginal rates of substitution across consumers are not equal can be precluded. From (2.8) and (2.9) we have

$$(2.13)\qquad \begin{aligned} -\frac{\partial q_{1,b}}{\partial q_{1,a}}\Big|_{U^1[\cdot]\text{ fixed}} &= \frac{U_1^1[\cdot]\theta_{1,a}}{U_2^1[\cdot]\theta_{1,b}} \geq (\leq) z_1 \quad \text{whenever} \quad \frac{\theta_{1,a}q_{1,a}}{\theta_{1,b}q_{1,b}} \equiv \frac{z_1}{z_q}\frac{\delta_a}{\delta_b} \leq (\geq) 1, \\ -\frac{\partial q_{2,b}}{\partial q_{2,a}}\Big|_{U^2[\cdot]\text{ fixed}} &= \frac{U_2^2[\cdot]\theta_{2,a}}{U_1^2[\cdot]\theta_{2,b}} \geq (\leq) z_2 \quad \text{whenever} \quad \frac{\theta_{2,a}q_{2,a}}{\theta_{2,b}q_{2,b}} \equiv \frac{z_2}{z_q}\frac{(1-\delta_a)}{(1-\delta_b)} \leq (\geq) 1. \end{aligned}$$

Upon requiring $z_1 > z_2$, the bounds in (2.13) also deliver (2.12).

If, instead, we knew that $r_1 = r_2$, i.e., $z_1 = z_2$ as in (2.11), then a larger region in the Edgeworth box could be precluded. Then we may rule out points such that

$$(2.14)\qquad \frac{z_1\delta_a}{z_q\delta_b} \geq 1 > \frac{z_2}{z_q}\frac{(1-\delta_a)}{(1-\delta_b)},$$

as well as those such that

$$(2.15)\qquad \frac{z_1\delta_a}{z_q\delta_b} < 1 \leq \frac{z_2}{z_q}\frac{(1-\delta_a)}{(1-\delta_b)}.$$

The geometry of the excluded region depends upon the magnitudes of the r_i relative to unity. The situation for $r_1 > 1 > r_2$ is depicted in Figure 2. We leave it to the interested reader to study the other cases. The dashed parallel lines are tangents to some isoquant along that utility function's AS line. Because the AS lines intersect inside the box, at *Y*, it is clear from (2.9) above that the tangents on the AS lines must intersect to the north of L1 and to the west of L2. But then any point to the north of L1 and west of L2 cannot be efficient because the utility indifference curves cannot be tangent there, i.e., $-\theta_{1,a}/\theta_{1,b} < -\theta_{2,a}/\theta_{2,b}$. For any given pair of AS, this trapezoid is larger than the area that was precluded in Figure 1. The quasi-concavity assumption buys us the difference, namely two right-angled triangles each with a vertex at point *Y*.

A geometric interpretation of quasi-concavity is that, to exclude a point, we only need to know that some point on the line segment connecting the original consumption point for person 2 with its reflection lies to the northeast of some point on the comparable line segment for person 1. Suppose that point K in Figure 2 is posited as being Pareto efficient. It lies outside the excluded rectangle given in Figure 1, but it satisfies condition (2.12). To see why it can be excluded, observe the point's reflections through the two axes. For person 1 the reflected point is K', while for person 2 it is K''. While K' and K'' are not comparable, a point on the segment KK'' does lie to the northeast of a point on the line segment KK'. This means that by giving each person some convex combination of his original point and its reflection, a surplus of goods can be created. But at these same convex combinations the respective consumers are at least weakly better off, and so the original point can be precluded as a Pareto efficient equilibrium under strictly monotone utilities.

2.4. *Generalized Transformations*

While scale transformations are convenient for describing the symmetry structures, all the observations thus far generalize to a larger set of transformations. Define $\mu_{i,j} = T^{i,j}(q_{i,j})$ and label the inverse relation as $S^{i,j}(\mu_{i,j}) = (T^{i,j})^{-1}(\mu_{i,j}) = q_{i,j}$. Then bilateral symmetry in an utility functional can be described as $U^i[T^{i,a}(q_{i,a}), T^{i,b}(q_{i,b})] = U^i[T^{i,b}(q_{i,b}), T^{i,a}(q_{i,a})]$, $i \in \Omega_2$, or

$$U^i[T^{i,a}(q_{i,a}), T^{i,b}(q_{i,b})] = U^i[T^{i,a}(\hat{q}_{i,a}), T^{i,b}(\hat{q}_{i,b})], \quad i \in \Omega_2, \tag{2.16}$$

where $\hat{q}_{i,a} = S^{i,a}[T^{i,b}(q_{i,b})]$ and $\hat{q}_{i,b} = S^{i,b}[T^{i,a}(q_{i,a})]$. In this way we may conclude, for points not on both $\mu_{1,a}(q_{1,a}) = \mu_{1,b}(q_{1,b})$ and $\mu_{2,a}(q_{2,a}) = \mu_{2,b}(q_{2,b})$, that at least one of the weak inequalities

$$\begin{aligned} \overline{q}_a &\geq S^{1,a}[T^{1,b}(q_{1,b})] + S^{2,a}[T^{2,b}(\overline{q}_b - q_{1,b})], \\ \overline{q}_b &\geq S^{1,b}[T^{1,a}(q_{1,a})] + S^{2,b}[T^{2,a}(\overline{q}_a - q_{1,a})], \end{aligned} \tag{2.17}$$

fails because otherwise an utility-preserving endowment surplus would exist.

To identify Pareto inefficient points under symmetry and monotonicity only, let equations

$$\begin{aligned} K^a(q'_{1,b}) &= S^{1,a}[T^{1,b}(q'_{1,b})] + S^{2,a}[T^{2,b}(\overline{q}_b - q'_{1,b})], \\ K^b(q'_{1,a}) &= S^{1,b}[T^{1,a}(q'_{1,a})] + S^{2,b}[T^{2,a}(\overline{q}_a - q'_{1,a})], \end{aligned} \tag{2.18}$$

implicitly define a vector-valued function of $(q'_{1,a}, q'_{1,b})$ on the non-empty, compact, convex set $[\overline{q}_a, 0] \times [0, \overline{q}_b]$. If the function, and for both arguments, is continuous and into then Brouwer's fixed point theorem is satisfied (MasColell, Whinston, and Green (1995, p. 952)) and a fixed point exists in the Edgeworth box. If, in addition, one function is strictly increasing and the other is strictly decreasing then there exists an inadmissible rectangle interior to the Edgeworth box. These monotonicity conditions are satisfied whenever one of

$$\frac{t^{1,a}(q)}{t^{1,b}(q)} > \frac{t^{2,a}(\overline{q}_a - q)}{t^{2,b}(\overline{q}_b - q)} \;\; \forall q \in [0, \min[\overline{q}_a, \overline{q}_b]]; \quad \frac{t^{1,a}(q)}{t^{1,b}(q)} < \frac{t^{2,a}(\overline{q}_a - q)}{t^{2,b}(\overline{q}_b - q)} \;\; \forall q \in [0, \min[\overline{q}_a, \overline{q}_b]]; \tag{2.19}$$

holds, where $t^{i,j}(q_{i,j}) = dT^{i,j}(q_{i,j}) / dq_{i,j}$. The rectangle has $(q'_{1,a}, q'_{1,b})$ as one vertex, is bounded by the axes, and cannot contain a consumer origin.

The analysis in Section 2.3 can also be extended to the more general context. Since $U_1^i[\cdot] =$ $U_2^i[\cdot]$ on the AS lines, the marginal rates of substitution along the AS lines are given by

$$\begin{aligned} \frac{U_2^1[\cdot] t^{1,b}(q_{1,b})}{U_1^1[\cdot] t^{1,a}(q_{1,a})}\Big|_{\substack{(q_{1,a}, q_{1,b}) \in \mathrm{L1} \\ U^1[\cdot]\text{ fixed}}} &= \frac{t^{1,b}(q_{1,b})}{t^{1,a}(q_{1,a})} = -\frac{\partial q_{1,a}}{\partial q_{1,b}}, \\ \frac{U_2^2[\cdot] t^{2,b}(q_{2,b})}{U_1^2[\cdot] t^{2,a}(q_{2,a})}\Big|_{\substack{(q_{2,a}, q_{2,b}) \in \mathrm{L2} \\ U^2[\cdot]\text{ fixed}}} &= \frac{t^{2,b}(q_{2,b})}{t^{2,a}(q_{2,a})} = -\frac{\partial q_{2,a}}{\partial q_{2,b}}, \end{aligned} \tag{2.20}$$

where L*i* refers to the line identified by the equation $T^{i,a}(q_{i,a}) = T^{i,b}(q_{i,b})$. Section 2.3 may now be adapted, except that $t^{i,a}(q_{i,a}) / t^{i,b}(q_{i,b})$ replaces z_i.

The properties of strict quasi-concavity, strict monotonicity, differentiability, and bilateral symmetry (on the functional) also allow us to make deductions about equilibrium prices. In general equilibrium, (2.20) implies that

$$\text{(2.21)} \qquad \frac{U_2^1[\cdot]t^{1,b}(\tilde{q}_{1,b})}{U_1^1[\cdot]t^{1,a}(\tilde{q}_{1,a})} = \frac{U_2^2[\cdot]t^{2,b}(\tilde{q}_{2,b})}{U_1^2[\cdot]t^{2,a}(\tilde{q}_{2,a})} = \frac{P_b}{P_a},$$

for Pareto efficient points so that (2.8) modifies to

$$\text{(2.22)} \qquad \begin{aligned} &[P_a t^{1,b}(\tilde{q}_{1,b}) - P_b t^{1,a}(\tilde{q}_{1,a})][T^{1,a}(\tilde{q}_{1,a}) - T^{1,b}(\tilde{q}_{1,b})] \le 0, \\ &[P_a t^{2,b}(\overline{q}_b - \tilde{q}_{1,b}) - P_b t^{2,a}(\overline{q}_a - \tilde{q}_{2,a})][T^{2,a}(\overline{q}_a - \tilde{q}_{1,a}) - T^{2,b}(\overline{q}_b - \tilde{q}_{1,b})] \le 0, \end{aligned}$$

upon imposing general equilibrium efficiency conditions. Thus, attending any solution $(\tilde{q}_{i,a}, \tilde{q}_{i,b})$ are constraints on the equilibrium price ratio P_b / P_a.

To this point we have not imposed the assumption of homotheticity, a property that has long been exploited in studies of efficiency. It is, however, true that our assumptions require local homotheticity along a ray. Homotheticity carries with it strong structure on symmetries among consumption bundles. The subsection to follow addresses the question of what, in addition to that already established about Pareto efficient bundles, homotheticity allows us to assert.

2.5. *Allocation under Strictly Quasi-concave and Homothetic Utility*

As one might expect, the imposition of homotheticity can further winnow down the set on which the efficient solution might live. The argument concerns a comparison of slopes away from the axes of symmetry. We have

$$\text{(2.23)} \qquad -\frac{dq_{1,b}}{dq_{1,a}}\Big|_{U^1[\cdot]\text{ fixed}} = \frac{\theta_{1,a}U_1^1[\theta_{1,a}q_{1,a}, \theta_{1,b}q_{1,b}]}{\theta_{1,b}U_2^1[\theta_{1,a}q_{1,a}, \theta_{1,b}q_{1,b}]} = z_1\phi^1(x_1), \quad x_1 \equiv \frac{\theta_{1,a}q_{1,a}}{\theta_{1,b}q_{1,b}} = \frac{\delta_a r_1}{\delta_b},$$

where $\phi^1(\cdot)$ is the marginal rate of substitution function with respect to the transformed 'goods' $\theta_{1,a}q_{1,a}$ and $\theta_{1,b}q_{1,b}$, and where homotheticity has been used to express the ratio in terms of relative consumption. If we also assume that Person 2 has a homothetic utility function, then we have

$$(2.24) \qquad -\frac{dq_{2,b}}{dq_{2,a}}\Big|_{U^2[\cdot]\text{ fixed}} = \frac{\theta_{2,a}U_1^2[\theta_{2,a}q_{2,a},\theta_{2,b}q_{2,b}]}{\theta_{2,b}U_2^2[\theta_{2,a}q_{2,a},\theta_{2,b}q_{2,b}]} = z_2\phi^2(x_2), \quad x_2 \equiv \frac{\theta_{2,a}q_{2,a}}{\theta_{2,b}q_{2,b}} = \frac{(1-\delta_a)r_2}{(1-\delta_b)},$$

with $\phi^2(\cdot)$ as the marginal rate of substitution function with respect to transformed 'goods' $\theta_{2,a}q_{2,a}$ and $\theta_{2,b}q_{2,b}$.

Following the earlier analysis we would like to identify a domain on which the marginal rates of substitution cannot be common across the consumers. In this regard it appears that identical preferences, where we mean that $U^1(T',T'') \equiv U^2(T',T'')$, by itself does not help. Similarly, homotheticity by itself does not help. However, with the assumption of identical and homothetic preferences, so that bilateral symmetry then implies $\phi^1(1) = \phi^2(1) = 1$, we can conclude:

$$(2.25) \qquad z_1 > z_2 \quad \text{implies} \quad -\frac{dq_{1,b}}{dq_{1,a}}\Big|_{U^1[\cdot]\text{ fixed}} > -\frac{dq_{2,b}}{dq_{2,a}}\Big|_{U^2[\cdot]\text{ fixed}} \quad \text{whenever} \quad x_1 \le x_2.$$

We cannot extend the deduction to the half-space $x_1 > x_2$ because we do not know how rapidly the marginal rate of substitution declines.

To summarize the consequences of (2.25), if $z_1 = z_2$ then we have the well-known conclusion: under identical and homothetic preferences all allocations other than the main diagonal may be excluded as Pareto inefficient points. This is because (2.25) then provides $x_1 \le x_2$ and $x_1 \ge x_2$ where $x_1 = x_2$ defines the main diagonal. Instead, if $z_1 > z_2$ then we can exclude the set defined by

$$(2.26) \qquad \delta_b \ge \delta_b^* \equiv \frac{\delta_a r_1}{\delta_a r_1 + (1-\delta_a)r_2}.$$

An inspection of (2.5) reveals that $\delta_b^* \geq (\leq) \delta_a r_1$ according as $\delta_a \leq (\geq) \hat{\delta}_a$, while we also have that $\delta_b^* \geq (\leq) \delta_a r_2 + 1 - r_2$ according as $\delta_a \geq (\leq) \hat{\delta}_a$. Put another way, we can write $\delta_b^* = \delta_b^*(\delta_a)$ and make the following observations. The function passes through the point $(\hat{\delta}_a, \hat{\delta}_b)$. It crosses L1 just once on the interior, and from above as δ_a increases. The function also crosses L2 just once (again at $(\hat{\delta}_a, \hat{\delta}_b)$) on the interior, but from below as δ_a increases. Partitioning the decision space, these observations require

$$
\text{(2.27)} \qquad \begin{aligned} &(\delta_a \leq \hat{\delta}_a) \Rightarrow (\delta_a r_2 + 1 - r_2 \geq \delta_b^* \geq \delta_a r_1), \\ &(\delta_a > \hat{\delta}_a) \Rightarrow (\delta_a r_2 + 1 - r_2 \leq \delta_b^* \leq \delta_a r_1). \end{aligned}
$$

Upon imposing the weaker of the two inequalities in either direction we have

$$
\text{(2.28)} \qquad \max[\delta_a r_1, \delta_a r_2 + 1 - r_2] \geq \delta_b^* \geq \min[\delta_a r_1, \delta_a r_2 + 1 - r_2]
$$

regardless of the evaluation of δ_a. Comparing with the bound in (2.12), $\delta_b \geq \max[\delta_a r_1, \delta_a r_2 + 1 - r_2]$, the joint impositions of identical homothetic $U^i(\cdot)$ does (weakly at any rate) extend the set of excludable points on $(\delta_a, \delta_b) \in [0,1] \times [0,1]$.

Whether the additional assumptions do lead to a ruling out of a strictly larger area depends upon where the efficiency locus occurs relative to the principal diagonal. This is because with homothetic preferences it is well-known that the efficiency locus cannot cut the diagonal, i.e., it either coincides with the diagonal or only the end points are common. If $z_1 = z_2$ then the efficiency locus is the main diagonal. Relative to this benchmark and for a given pair of allocation vectors, suppose we then increase the value of z_1. The effect is to increase the marginal rate of substitution for person 1 whenever the elasticity of substitution exceeds unity, and to decrease the marginal rate whenever the elasticity is less than unity. For $z_1 > z_2$ we see then that the efficiency locus must be above the principal diagonal whenever the elasticity of substitution is less that unity. In that case, (2.26) combines with the diagonal to provide tight bounds on the set of Pareto efficient points. However, when the elasticity of substitution is greater than unity then the efficiency locus will be below the main diagonal and (2.26) bears no information.

At this juncture we turn back to the issue of generalizing our analysis. A generalization of the transformations, as in subsection 2.4, is not the only way in which our model of pure exchange equilibrium can be extended. While we could further extend the set of transformations that can be studied, the emphasis in the remains of this paper will be to extend our insights thus far to an arbitrary finite good, finite person exchange economy. Before doing so, however, we will provide a brief overview of the permutation groups we will use to accommodate the wider variety of symmetries that can arise in larger dimensioned economies.

3. GROUP ALGEBRA

The symmetries assumed in Section 2 may seem natural in a 2×2 pure exchange economy because there are only two ways of internally exchanging two goods. This is not true, however, when three or more goods are available for consumption. When there are three goods, A, B, and C, then there are several ways in which utility functional $U^i[T^{i,a}(q_{i,a}),T^{i,b}(q_{i,b}),T^{i,c}(q_{i,c})]$ can be left invariant upon interchanging arguments.[5] These include the interchanges $T^{i,a} \leftrightarrow T^{i,b}$, $T^{i,a} \leftrightarrow T^{i,b} \leftrightarrow T^{i,c}$, and $T^{i,a} \to T^{i,b} \to T^{i,c} \to T^{i,a}$. It is clear that as the number of goods entering the utility function increases, the set of invariances that may arise increases at a much greater rate. Further,

[5] We consider only permutations on the argument set $\{T^{i,a},T^{i,b},T^{i,c}\}$ where arguments cannot be combined.

these symmetries may differ across consumer utility functions. Group theory allows for a general treatment of these invariances and their implications for Pareto efficient equilibria.[6]

DEFINITION 3.1: A *group*, $\ddot{G}$, is a set of elements, G with cardinality $|G|$, together with a single-valued binary operation, *, such that the structure satisfies all of: I) closure; G is closed with respect to *, II) identity element; $\exists\, e \in G$ such that $g * e = e * g = g \;\forall g \in G$, III) inverse elements; $\forall g \in G$ there exists an unique element, labeled $g^{-1} \in G$, such that $g * g^{-1} = g^{-1} * g = e$, IV) associativity; $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3) \;\forall g_1, g_2, g_3 \in G$ where the operations in parentheses occur first.

It should by now be clear as to how we will employ group algebra; the arguments of an utility functional comprise a set on which the group acts. The elements of a group are not the utility functional arguments, but rather the operations on the functional arguments that leave the utility level invariant. Define $k \in \Omega_m = \{1, 2, ..., m\}$ where $k \sim T^{i,k}(q_{i,k})$. In our notation we write that $g(k) = l$ whenever group element (i.e., operation) g replaces a function's k^{th} argument with its l^{th} argument. In this way the group can be viewed as a set bijection $g(\Omega_m) = \Omega_m$, i.e., $g(k) = l \in \Omega_m$ $\forall k \in \Omega_m$.

EXAMPLE 3.1 (Dihedral 4 group): Suppose person 1 has a scale-transformed utility function on four goods, A, B, C, and D, where utility is known to have two invariances. These are the group operation g_1, which represents the permutation $U^1[\theta_{1,a}q_{1,a}, \theta_{1,b}q_{1,b}, \theta_{1,c}q_{1,c}, \theta_{1,d}q_{1,d}] \equiv$

---

[6] Useful treatments of the groups applied in this paper can be found in Hungerford (1974) and

$U^1[\theta_{1,b}q_{1,b}, \theta_{1,c}q_{1,c}, \theta_{1,d}q_{1,d}, \theta_{1,a}q_{1,a}]$, and the group operation g_4, which represents the permutation $U^1[\theta_{1,a}q_{1,a}, \theta_{1,b}q_{1,b}, \theta_{1,c}q_{1,c}, \theta_{1,d}q_{1,d}] \equiv U^1[\theta_{1,b}q_{1,b}, \theta_{1,a}q_{1,a}, \theta_{1,c}q_{1,c}, \theta_{1,d}q_{1,d}]$.[7] Clearly the invariances cannot end at this point because $g_2 = g_1 * g_1$, i.e., $U^1[\theta_{1,a}q_{1,a}, \theta_{1,b}q_{1,b}, \theta_{1,c}q_{1,c}, \theta_{1,d}q_{1,d}] \equiv U^1[\theta_{1,c}q_{1,c}, \theta_{1,d}q_{1,d}, \theta_{1,a}q_{1,a}, \theta_{1,b}q_{1,b}]$, must also be true. To ensure closure, as required by the definition, we must also have an element $g_5 = g_1 * g_4$, an element $g_7 = g_4 * g_1$, and so on. In this way we see that the elements g_1 and g_4 generate a group upon iteration until closure occurs. This group is the dihedral 4 group, best known as the group of symmetries on the square. In addition to the above defined elements and to the identity *e*, there are $g_3 = g_1 * g_1 * g_1$ and $g_6 = g_4 * g_2$. The complete table of group element compositions, often called the Cayley Table, is given in Table I.

When reading Table I one may note that set of group elements $\{e, g_1, g_2, g_3\}$ forms the element set of a group in its own right. So do several other sets of group elements, including $\{e, g_2, g_4, g_6\}$, $\{e, g_2\}$, $\{e, g_4\}$, $\{e, g_5\}$, $\{e, g_6\}$, and $\{e, g_7\}$. These sets of elements each give rise to a subgroup of group $\ddot{G}$.

DEFINITION 3.2: A *subgroup* $\ddot{H}$ of group $\ddot{G}$ is a subset, *H*, of set *G* that is closed under the same operation *. It is written as $\ddot{H} \le \ddot{G}$ where it is understood that $\ddot{G} \le \ddot{G}$, $\langle e \rangle \le \ddot{G}$, and $\langle e \rangle$ is the trivial subgroup represented by the identity element.

Dixon and Mortimer (1996).

[7] For other examples, see Hennessy and Lapan (2003a, 2003b).

The sorts of groups that we will work with in Section 4 are permutation groups, or subgroups of the symmetric group on *m* arguments where there are *m* available consumption goods in the exchange economy.[8]

DEFINITION 3.3: Let Ω_m be a finite non-empty set of objects with cardinality *m*. A bijection of Ω_m onto itself is called a permutation of Ω_m. The set of all such permutations forms a group under the composition of bijections. This is the *symmetric group* of Ω_m, and is denoted by $\ddot{S}_m$. Group $\ddot{S}_m$ is said to *act on* set Ω_m. Any subgroup of $\ddot{S}_m$ is called a *permutation group*.

All the groups and subgroups that arose in Example 3.1 may be viewed as permutation subgroups of $\ddot{S}_4$.

4. M-GOOD, N-PERSON MODEL

Now there are *n* people, labeled $i \in \Omega_n = \{1,...,n\}$, in a pure exchange economy where there are *m* exchangeable goods, labeled $j \in \Omega_m$. The endowments are $\overline{q}_j$, $j \in \Omega_m$, where the i^{th} consumer uses $q_{i,j} \geq 0$. Under non-satiation, the endowment constraints are then

(4.1) $$\textstyle\sum_{i=1}^{n} q_{i,j} = \overline{q}_j \;\; \forall j \in \Omega_m.$$

[8] The insights developed in Section 2 can be generalized much further. For instance, permutation groups are a restricted subclass of the class of general linear groups. This latter class of groups would allow symmetries in which linear combinations of consumption goods are permuted. We confine our attention to permutations on uncombined consumption goods only because this focus allows for a more transparent exposition.

The i^{th} person's utility functional is given by $U^i(\theta_{i,1}q_{i,1}, \theta_{i,2}q_{i,2}, \ldots, \theta_{i,m}q_{i,m})$, and the group of invariances on this functional's arguments is assumed to be $\ddot{G}^i$ with the element set G^i = $\{e^i, g_1^i, g_2^i, \ldots, g_{|G^i|-1}^i\}$ where e^i is the i^{th} group identity element. We write the symmetry structure on the n utility functionals as $\overline{\overline{G}}^n = \{\ddot{G}^1, \ddot{G}^2, \ldots, \ddot{G}^n\}$. This section will generalize the observations in Section 2 to the larger permutation group setting.

4.1. *Symmetric and Monotone Utility*

Just as frictionless markets allow barter trades across all goods and trading consumers, the natural extension of the observation in Figure 1 to the m-good, n-person context is to exploit all the structured symmetries. We find

THEOREM 1: *Let the n non-satiated consumers have scale transformation functions on the arguments entering their respective utility functionals, and let the symmetry structure on utility functionals be given by $\overline{\overline{G}}^n$. Define the set of Pareto efficient equilibria as*

(4.2) $$\tilde{Q} = \{\vec{q}^1, \vec{q}^2, \ldots, \vec{q}^n\}, \quad \vec{q}^i = (\tilde{q}_{i,1}, \tilde{q}_{i,2}, \ldots, \tilde{q}_{i,m})', \quad i \in \Omega_n.$$

With one qualification any $\tilde{Q}$ must, in addition to satisfying the constraint set (4.1) and the non-negativity constraints, violate at least one of

(4.3) $$\textstyle\sum_{i=1}^{n} \hat{q}_{i,j} \le \overline{q}_j \ \forall j \in \Omega_m, \forall \hat{g}^i \in G^i, \forall i \in \Omega_n,$$

where (fixing $\hat{g}^i$) $\hat{q}_{i,j} = \theta_{i,\hat{g}^i(j)}\tilde{q}_{i,\hat{g}^i(j)} / \theta_{i,j}$ and $\hat{g}^i$ solves $\hat{g}^i(j) = k$ for some $k \in \Omega_m$. The qualification is that no constraint in (4.3) will be violated whenever $\tilde{Q}$ is invariant under $\forall \hat{g}^i \in$ $G^i, \forall i \in \Omega_n$.

Were the assertion not true one could then, e.g., distribute the spare endowments in equal amounts among all consumers. A further observation is that system (4.3), together with the non-negativity constraints, forms a convex set. If we ignore all points that satisfy all (4.3) with equality, then the theorem rules out a convex set of candidate efficient allocations.

EXAMPLE 4.1 (System of cyclic 3 groups): Table II represents the Cayley table for the cyclic 3 group, $\ddot{C}_3$. This group represents the invariances of the utility functional $U^i(\theta_{i,1}q_{i,1},\theta_{i,2}q_{i,2},\theta_{i,3}q_{i,3})$ when $U^i(\theta_{i,1}q_{i,1},\theta_{i,2}q_{i,2},\theta_{i,3}q_{i,3}) \equiv U^i(\theta_{i,2}q_{i,2},\theta_{i,3}q_{i,3},\theta_{i,1}q_{i,1}) \equiv U^i(\theta_{i,3}q_{i,3},\theta_{i,1}q_{i,1},\theta_{i,2}q_{i,2})\ \forall \Omega_n$. In an 3-good, 3-person economy, if each utility functional $U^i(\cdot)$ is invariant under the $\ddot{C}_3$ group, then relation (4.3) provides 3^3=27 inequalities. Some of these, such as when $\hat{g}^i = e^i \in G^i$, $\forall i \in \Omega_3$, may be trivial.

4.2. *Allocations Under Strictly Quasi-concave Utility*

In this sub-section we will extend the observation in Figure 2, but we will need one further concept to do so. The majorization pre-ordering is intimately related with the symmetric group, $\ddot{S}_m$. It generates partial order on a convex hull of the set of points that are generated when $\ddot{S}_m$, for some *m*, acts on a single point in $\mathbb{R}^m$. Our interest lies, however, in an arbitrary subgroup of some symmetric group. The class of *G*-majorization pre-orderings, of which the usual majorization pre-ordering is one, will prove useful when utility functions are symmetric under some group.

DEFINITION 4.1: (Marshall and Olkin, 1979, p. 422) Let $\ddot{G}$ be a group of linear transformations mapping $\mathbb{R}^m$ to $\mathbb{R}^m$. Then vector $\vec{q}'$ is *group majorized* by $\vec{q}''$ with respect to group $\ddot{G}$, written as $\vec{q}' \leq_{\ddot{G}} \vec{q}''$, if $\vec{q}'$ lies in the convex hull of the orbit of $\vec{q}''$ under the action of group $\ddot{G}$. Here, the orbit of $\vec{q}''$ is the set of points $g(\vec{q}''), g \in G$.

This pre-ordering is of interest when we assume that each of the *n* utility functionals, $U^i(\theta_{i,1}q_{i,1}, \theta_{i,2}q_{i,2}, \ldots, \theta_{i,m}q_{i,m})$, is strictly quasi-concave in the underlying consumption goods and is symmetric under some permutation group. Quasi-concavity implies convexity of the level sets. Given one point on a level set, the group symmetries generate $\left|G^i\right| - 1$ additional points on the level set. Any (strictly) convex combination of these points will generate a level of utility that is (strictly) larger. Of course, the convex combination must be feasible.

THEOREM 2: *Consider some Pareto efficient equilibrium as specified by (4.2) above. For each* $\vec{q}^i \in \tilde{Q}, \hat{g}^i \in G^i$, *and* $i \in \Omega_n$, *define* $\vec{q}^i_{\hat{g}^i} = (\theta_{i,\hat{g}^i(1)}\tilde{q}_{i,\hat{g}^i(1)} / \theta_{i,1}, \theta_{i,\hat{g}^i(2)}\tilde{q}_{i,\hat{g}^i(2)} / \theta_{i,2}, \ldots, \theta_{i,\hat{g}^i(m)}\tilde{q}_{i,\hat{g}^i(m)} / \theta_{i,m})$. *Specify the convex hull of the resulting* $|G^i|$ *vectors as* $ch(\vec{q}^i, \vec{\theta}^i; \ddot{G}^i)$, $\vec{\theta}^i = (\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,m})'$, *where the interior of the set is given by* $ch^{\text{int}}(\vec{q}^i, \vec{\theta}^i; \ddot{G}^i)$. *If,* $\forall i \in \Omega_n$, *the* i^{th} *utility functional is* $\ddot{G}^i$-*symmetric, strictly monotone, and strictly quasi-concave then any point in*

$$\text{(4.4)} \qquad \tilde{Q}_* = \{\vec{q}^1_*, \vec{q}^2_*, \ldots, \vec{q}^n_*\}, \quad \vec{q}^i_* \in ch^{\text{int}}(\vec{q}^i, \vec{\theta}^i; \ddot{G}^i), \quad i \in \Omega_n,$$

must violate at least one of the inequalities in (4.3).

The theorem is constructive in the sense that, just as in Figure 2, a linear program can be constructed to discard candidate equilibria. If convex weights can be found such that (4.4) is satisfied, and if the resulting allocation $\tilde{Q}_*$ is feasible, then the initial allocation $\tilde{Q}$ cannot be Pareto efficient.

EXAMPLE 4.2: Returning to the cyclic 3 groups in Example 4.1, the invariances of each utility functional define a convex hull as the convex combinations of the respective sets of three iso-utility points generated by the $\ddot{G}^i$. If the $\tilde{Q}_*$ of (4.4) is adapted to this problem and the three convex hulls have non-empty interiors, then all consumers can be made strictly better off; a clear violation of Pareto efficiency.

4.3. *Price Bounds Under Strictly Quasi-concave Utility*

An utility functional that is quasi-concave and invariant under some group is said to be decreasing with respect to that group, or $\ddot{G}$-decreasing (Eaton and Perlman, 1977). More formally, if $\vec{x}' \leq_{\ddot{G}} \vec{x}''$ and $f(\vec{x}) : \mathbb{R}^N \to \mathbb{R}$ is both quasi-concave and symmetric under $\ddot{G}$ then $f(\vec{x}') \geq f(\vec{x}'')$. Suppose further that $f(\vec{x})$ is differentiable on its domain. Then Eaton and Perlman (1977) have shown that

(4.5) $$\vec{x} \bullet [g(\partial f(\vec{x})/\partial \vec{x}) - \partial f(\vec{x})/\partial \vec{x}] \geq 0 \quad \forall g \in G, \forall \vec{x} \in \mathbb{R}^N,$$

where $\bullet$ is the inner product operator.

EXAMPLE 4.3: In Example 4.1, functional $U^1(\theta_{1,1}q_{1,1},\theta_{1,2}q_{1,2},\theta_{1,3}q_{1,3})$ is held to be invariant under the $\ddot{C}_3$ group. If it is also strictly quasi-concave and once continuously differentiable then candidate equilibria must satisfy

$$
\begin{aligned}
&\theta_{1,1}\tilde{q}_{1,1}(U_2^1[\cdot]-U_1^1[\cdot])+\theta_{1,2}\tilde{q}_{1,2}(U_3^1[\cdot]-U_2^1[\cdot])+\theta_{1,3}\tilde{q}_{1,3}(U_1^1[\cdot]-U_3^1[\cdot])\geq 0,\\
&\theta_{1,1}\tilde{q}_{1,1}(U_3^1[\cdot]-U_1^1[\cdot])+\theta_{1,2}\tilde{q}_{1,2}(U_1^1[\cdot]-U_2^1[\cdot])+\theta_{1,3}\tilde{q}_{1,3}(U_2^1[\cdot]-U_3^1[\cdot])\geq 0,\quad \forall\vec{q}\in\mathbb{R}^3.
\end{aligned}
\tag{4.6}
$$

Together with general equilibrium efficiency conditions, we then have

$$
\begin{aligned}
&\theta_{1,1}\tilde{q}_{1,1}\left(\frac{P_2}{\theta_{1,2}}-\frac{P_1}{\theta_{1,1}}\right)+\theta_{1,2}\tilde{q}_{1,2}\left(\frac{P_3}{\theta_{1,3}}-\frac{P_2}{\theta_{1,2}}\right)+\theta_{1,3}\tilde{q}_{1,3}\left(\frac{P_1}{\theta_{1,1}}-\frac{P_3}{\theta_{1,3}}\right)\geq 0,\\
&\theta_{1,1}\tilde{q}_{1,1}\left(\frac{P_3}{\theta_{1,3}}-\frac{P_1}{\theta_{1,1}}\right)+\theta_{1,2}\tilde{q}_{1,2}\left(\frac{P_1}{\theta_{1,1}}-\frac{P_2}{\theta_{1,2}}\right)+\theta_{1,3}\tilde{q}_{1,3}\left(\frac{P_2}{\theta_{1,2}}-\frac{P_3}{\theta_{1,3}}\right)\geq 0,\quad \forall(\tilde{q}_{1,1},\tilde{q}_{1,2},\tilde{q}_{1,3})\in\mathbb{R}_+^3.
\end{aligned}
\tag{4.7}
$$

In general, for permutation groups and scaling functionals the $\ddot{G}$-decreasing property implies[9]

THEOREM 3: *If the i^{th} utility functional has scale transformations, is $\ddot{G}^i$-symmetric, strictly monotone, and strictly quasi-concave then the relation between general equilibrium prices and the i^{th} person's decision vector must satisfy*

$$
\sum_{j=1}^{m}\theta_{i,j}\tilde{q}_{i,j}\left(\frac{P_{\hat{g}^i(j)}}{\theta_{i,\hat{g}^i(j)}}-\frac{P_j}{\theta_{i,j}}\right)\geq 0,\quad \forall\hat{g}^i\in G^i.
\tag{4.8}
$$

5. CONCLUSION

By way of the notion of exchange, we have developed a number of sets of relationships that symmetries and controlled heterogeneities in the primitives underlying a pure endowment economy imply for an efficient equilibrium. Some of these sets of relationships are quite straightforward, but others could hardly be developed without the formal frameworks that are provided by group and majorization algebras. While we have confined the analysis to permutation groups, the framework naturally extends to more general sets of invariances. Because symmetry structures have such strong implications for the nature of an efficient equilibrium, they should also have some implications for how a failure in the conditions underlying efficiency affect equilibrium. A number of extensions to the present work then arise naturally. These include a study of the implications of symmetries in technologies and preferences when market power leads to strategic interactions. It would seem too that symmetry structures on consumer preferences should provide further insights on tying, bundling, and price discrimination strategies by imperfectly competitive producers of differentiated goods. The present framework could also be expanded to accommodate an Arrow-Debreu state-contingent equilibrium, and perhaps even when markets are incomplete. Hopefully, such efforts would also point to ways through which the insights provided in this paper can be sharpened.

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[9] A revealed preference argument can readily show that the quasiconcavity condition in Theorem 3 may be discarded.

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TABLE I

DIHEDRAL 4 GROUP

*=after	e	g_1	g_2	g_3	g_4	g_5	g_6	g_7
e	e	g_1	g_2	g_3	g_4	g_5	g_6	g_7
g_1	g_1	g_2	g_3	e	g_5	g_6	g_7	g_4
g_2	g_2	g_3	e	g_1	g_6	g_7	g_4	g_5
g_3	g_3	e	g_1	g_2	g_7	g_4	g_5	g_6
g_4	g_4	g_7	g_6	g_5	e	g_3	g_2	g_1
g_5	g_5	g_4	g_7	g_6	g_1	e	g_3	g_2
g_6	g_6	g_5	g_4	g_7	g_2	g_1	e	g_3
g_7	g_7	g_6	g_5	g_4	g_3	g_2	g_1	e

TABLE II

CYCLIC 3 GROUP, $\ddot{C}_3$

*=after	e	h_1	h_2
e	e	h_1	h_2
h_1	h_1	h_2	e
h_2	h_2	e	h_1

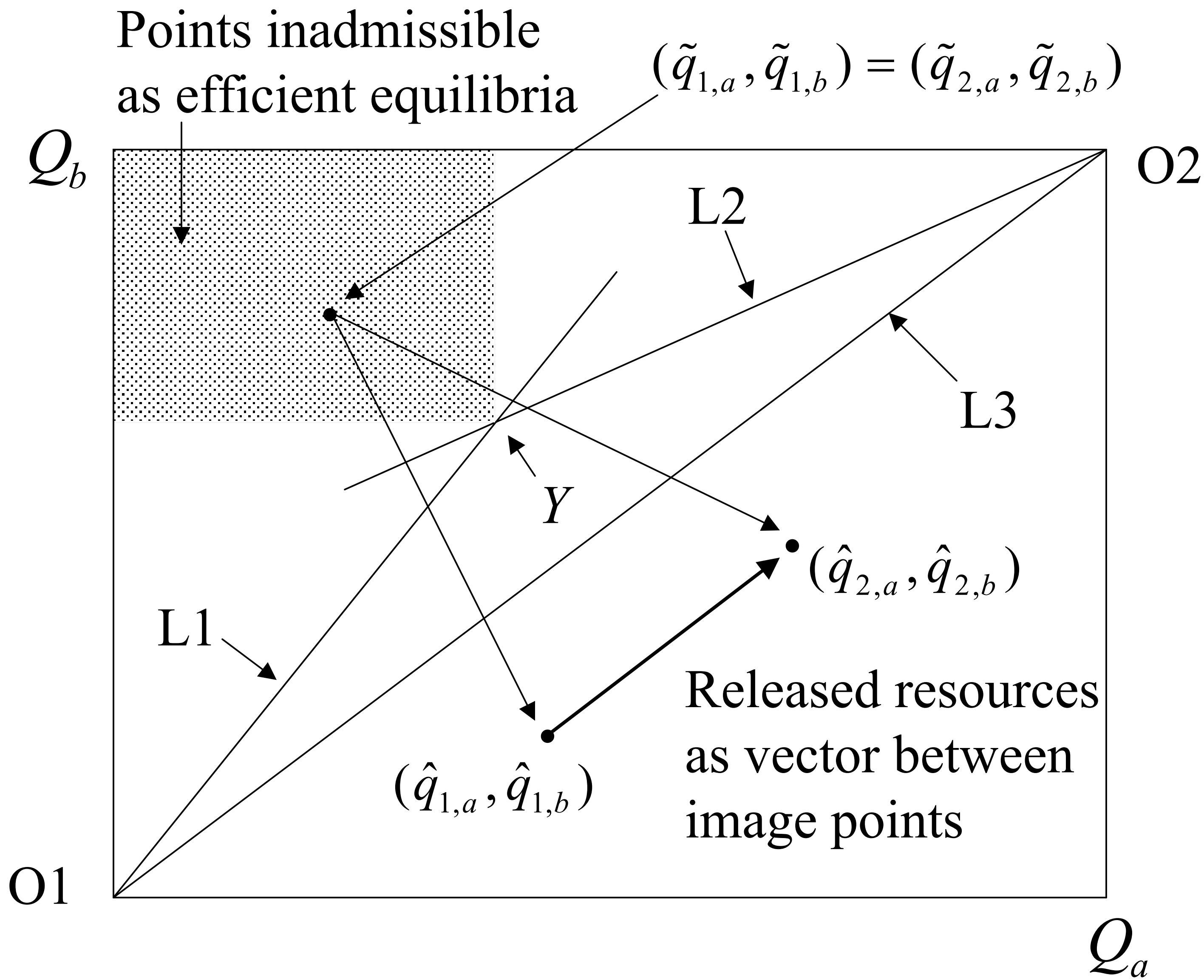

FIGURE 1.–Inadmissible equilibria under symmetry and strict monotonicity, with $r_1 > 1 > r_2$

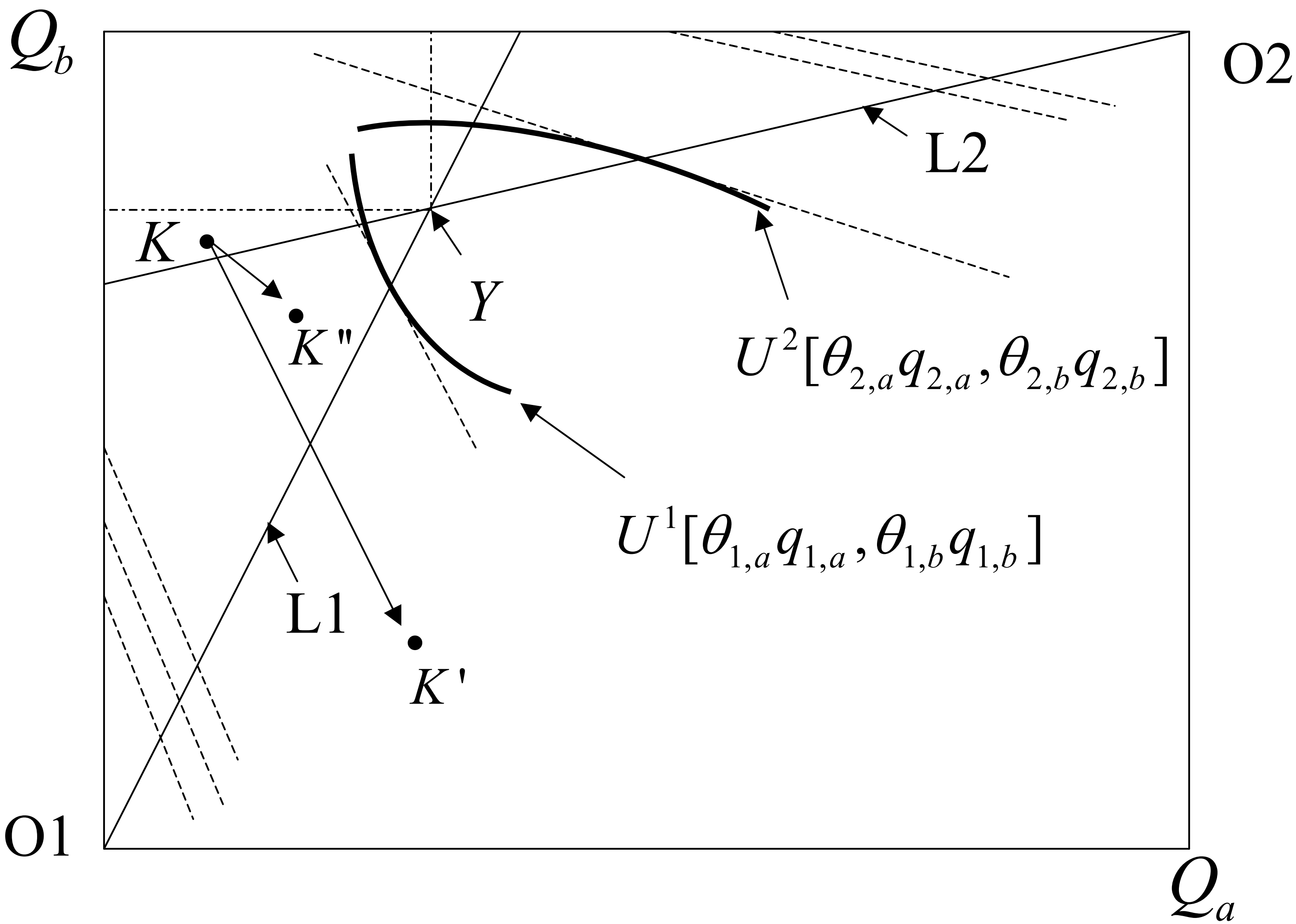

FIGURE 2.–Implications of bilateral symmetry for marginal rates of substitution under quasi-concave preferences.