Dynamic Price Competition with Network Effects

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Abstract

I consider a dynamic model of competition between two proprietary networks. Consumers die and are replaced with a constant hazard rate. Firms compete for new consumers to join their network by offering network entry prices (which may be below cost). New consumers have a privately known preference for each network. Upon joining a network, in each period consumers enjoy a benefit which is increasing in network size during that period. Firms receive revenues from new consumers as well as from consumers already belonging to their network.

Using a combination of analytical and numerical methods, I discuss various properties of the equilibrium. I show that very small or very large networks tend to price higher than networks of intermediate size. I also show that, around symmetric states, the gap between the large and the small network tends to widen (increasing dominance) whereas the opposite is true (reversion to the mean) around very asymmetric states.

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1 Introduction

Many industries exhibit some form of network effects, the situation whereby a consumer's valuation is increasing in the number of other consumers buying the same product (that is, the number of consumers in the same "network"). The most obvious source of network effects is direct network effects. Take the example of operating systems. If I use the Windows OS then, when I travel, it is more likely I will find a computer that I can use (both in terms of knowing how to use it and in terms of being able to run files and programs I carry with me).[1]

A second source of network effects is the availability of complementary products. For example, it seems reasonable that the variety and quality of software available for the Palm system is greater the more users buy PDAs that run Palm OS. A similar argument applies for complementary services. For example, the greater the number of Canon photocopiers are sold, the more likely it is I will be able to find good post-sale service providers.

Finally, a third source of network effects is the pricing of network services.[2] Take the example of wireless telecommunications. To the extent that operators set different on-net and off-net prices, the utility from being connected to a given network is increasing in the number of other users on the same network.

In this paper, I consider a dynamic model of competition between two proprietary networks. Consumers die and are replaced with a constant hazard rate. Firms compete for new consumers to join their network by offering network entry prices (which may be below cost). New consumers have a privately known preference for each network. Upon joining a network, in each period consumers enjoy a benefit which is increasing in network size during that period. Firms receive revenues from new consumers as well as from consumers already belonging to their network.

I develop a general model with the above features. I derive the firms' and the consumers' value functions, both of which are a function of current network sizes. I provide conditions such that there exists a unique Markov equilibrium. The key is that, differently from static models, in an overlap-

1. Another source of direct network effects would be file sharing. While this is frequently proposed as the main source of direct network effects, in the example at hand I think it is relatively less important.
2. Laffont, Rey and Tirole (1998a) refer to this case as "tariff-mediated network externalities."

ping generations framework consumers effectively make their network choices sequentially.

I then characterize the equilibrium, using a combination of analytical and numerical methods. One set of results pertains to the pricing function. As is frequently the case with dynamic games, there are two effects to consider. Larger networks are more attractive to (forward looking) consumers. This implies that, ignoring the firms' future payoffs, larger networks should set higher prices. However, in terms of future payoffs, larger firms have more to gain from increasing their network size then small firms. This dynamic version of the "efficiency effect" (duopoly joint profits are greater the greater the asymmetry between firms) leads larger firms to price lower. I provide conditions such that each of these effects dominates. In general, they result in a U-shaped pricing function, with small and large networks charging the highest prices.

Although the equilibrium is symmetric, both the birth and death processes are stochastic. Consequently, the actual state of the system (each firm's network size) is generally asymmetric. I show that a larger network is always more likely to attract a new consumer. Moreover, if network effects are sufficiently strong, then the larger network tends to increase in size, unless it holds close to 100% of the market, in which case it tends to decrease in size. In other words, I show that market share dynamics feature increasing dominance around the symmetric state and reversion to the mean around the extreme states. As a result, when network effects are sufficiently strong the stationary distribution of market shares is typically bimodal — the system spends most of the time at states where the large network has a market share between 50 and 100 percent.

Finally, I consider two applications of my general framework: I estimate the barrier to entry implied by network effects; and I characterize the dynamic implications of asymmetric product improvement.

■ **Related literature.** Following seminal work by Katz and Shapiro (1985), the early literature on oligopoly with network effects focused on relatively simple, static models.[3] Since then, the industrial organization literature developed in two directions. One strand attempts to empirically measure

3. Other important early work includes Farrell and Saloner (1985), who focus on consumer behavior, and Arthur (1989), who presents an infinite period model but assumes non-proprietary networks (and thus excludes strategic behavior).

the size of network effects.[4] Another strand investigates further implications of network effects in an oligopoly context.[5] Despite important developments, most of this literature has followed a static, or finite period, approach.[6]

More recent work attempts to explicitly address the issue of dynamic competition between proprietary networks. Doganoglu (2003), Mitchell and Skrzypacz (2006), derive the Markov Perfect Equilibrium of an infinite period game where consumer's utility is an increasing function of past market shares. Markovich (2004), Markovich and Moenius (2004) develop computational models of industries with "hardware" and "software" components (very much like my paper). They assume consumers live for two periods and benefit from indirect network effects through the quality of products available. Doraszelski, Chen, and Harrington (2007) also develop a computational dynamic model. In many respects, their analysis goes beyond my paper: they consider R&D investments and compatibility decisions. However, like Doganoglu (2003), Mitchell and Skrzypacz (2006), they assume consumer benefits are an increasing function of network size at the time of purchase (that is, consumers are not forward looking). In sum, all of these papers assume relatively simple behavior on the part of consumers: either consumers are short-lived, or they are myopic, or they are backward looking.[7] By contrast, I assume that consumers live for potentially many periods (that is, die with a constant hazard rate), and make their decisions in a rational, forward looking way. My paper also differs from theirs in that I look at a different set of issues.

In this sense, the the papers that come closest to mine are Fudenberg and Tirole (2000), Driskill (2007), Laussel and Resende (2007), and Lee (2007), all of which consider forward looking consumers. The framework considered by Fudenberg and Tirole (2000) is very specific: two consumers, two consumer types, etc. The number and type of questions that can be addressed with such a simple model is limited. Driskill (2007) considers a deterministic, continuous-time model where consumers are forward looking. My paper differs from his in that I consider idiosyncratic consumer preferences, which generate stochastic dynamics; and in the fact I focus on different issues of

4. See, for example, Gandal (1994), Goolsbee and Klenow (2002), Rysman (2004).

5. See, for example, Laffont, Rey and Tirole (1998a,b).

6. Farrell and Klemperer (2006) present an excellent survey of this literature. See also Economides (1996).

7. See also Kandori and Rob (1998), Auriol and Benaim (2000), who approach the problem from a stochastic evolutionary perspective.

equilibrium and comparative dynamics. Laussel and Resende (2007) also consider forward looking consumers. However, unlike my paper they restrict to linear Markov strategies. (As I show below, equilibrium strategies in my model are highly nonlinear.) Finally, Lee (2007) addresses different issues and uses a fairly different framework.

Following the seminal contributions by Gilbert and Newbery (1982) and Reinganum (1983), a series of papers have addressed the issue of persistence of firm dominance. Contributions to this literature include Budd, Harris and Vickers (1993), Cabral and Riordan (1994), Athey and Schmutzler (2001), Cabral (2002). These papers provide conditions under which larger firms tend to become larger (increasing dominance). Intuitively, the reason for such dynamics corresponds to some form of the "efficiency effect" characterized by Gilbert and Newbery (1982), the fact that joint profits are greater the closer to monopoly market structure is. My framework and results are consistent with the idea of increasing dominance. Specifically, I show that, if network effects are sufficiently strong and the large firm is not too large, then increasing dominance holds.

Another related literature is that of switching costs, in particular the paper by Beggs and Klemperer (1992). They consider an infinite period model with two sellers and a stationary mass of consumers. In each period a fraction of consumers dies and a corresponding fraction enters the market. Each new consumer chooses one of the sellers and sticks with it for the rest of the consumer's life. Beggs and Klemperer derive Markov equilibria such that the firms' strategies and the consumers' value functions are linear functions of the state. They show that prices are higher the greater switching costs are. My paper differs in two important ways. First, I consider both a primary and a secondary market. As a result, I allow sellers to (partially) discriminate between old consumers and new consumers. Second, I consider the possibility of network events. In terms of results, there is also an important difference, namely price strategies are quite different from affine functions.

2 Model

I consider an infinite period model of price competition between two proprietary networks, owned by firms A and B. Since I analyze symmetric Markov equilibria, with some abuse of notation I denote each firm by the size of its network, i or j. Network size evolves over time due to consumer birth and death. In each period, a consumer dies and a new consumer is born. The new consumer chooses between one of the existing networks and stays with it until death.[8,9]

Specifically, the timing of moves in each period is summarized in Table 1. Initially, a total of $\eta - 1$ consumers are distributed between the two firms, so $i + j = \eta - 1$. A new consumer is born and firms simultaneously set prices $p(i)$, $p(j)$ for the consumer to join their network. If the new consumer opts for network i, then firm i receives a profit $p(i)$, whereas the consumer receives a onetime benefit from joining network i, ζ_i.[10]

After the new consumer makes his choice, there are a total of η consumers divided between the two networks. During the remainder of the period, firm i receives a payoff $\theta(i)$, whereas a consumer attached to network i enjoys a benefit $\lambda(i)$. In others words, I treat network choice as a durable good, and assume there is some non-durable good attached to the durable good "network membership." I denote the market for the non-durable good as the aftermarket. Finally, at the end of the period one consumer dies (each with equal probability). In other words, a consumer from firm i's network dies with probability i/η.

■ **After market.** Since my main goal is to understand the evolution of network size over time, I take the values $\theta(i)$ and $\lambda(i)$ as given, that is, I treat them as the reduced form of the stage game played in the aftermarket. I make very minimal assumptions regarding the values of $\theta(i)$, $\lambda(i)$:

Assumption 1 *(i) $\theta(i+1) - \theta(i)$ is (weakly) increasing in i; (ii) $\lambda(i)$ is (weakly) increasing in i.*

Note that Part (i) is equivalent to $\frac{\theta(i+1)-\theta(0)}{i+1} \geq \frac{\theta(i)-\theta(0)}{i}$. It thus states that

8. In this sense, my framework is similar to that of Beggs and Klemperer (1992). They too consider a stationary number of consumers and assume that a newborn consumer, having chosen one of the sellers, sticks with it until death.

9. Although I work with a discrete time model, the underlying reality I have in mind is one of continuous time. Suppose that consumers die according to a Poisson process with arrival rate λ. Essentially, I consider the time between two consecutive deaths as a period in my discrete time model. By assuming risk-neutral agents, I can summarize the Poisson arrival process in a discount factor δ that reflects the average length of a discrete period: $\delta = \exp(-r/\lambda)$, where r is the continuous time discount rate.

10. For simplicity, I assume zero cost. Alternatively, we can think of $p(i)$ as markup over marginal cost.

Table 1: Timing of model: events occurring in each period t.

Event	Value functions	State of the game
Firms set network entry prices $p(i)$	$v(i)$	$i \in \{0, \ldots, \eta - 1\}$
Nature chooses x_i, new consumer's preference for network i		
New consumer chooses network	$u(i)$	$i \in \{0, \ldots, \eta\}$
Stage competition takes place: period profits $\theta(i)$, consumer surplus $\lambda(i)$		
One consumer dies (probability $\frac{1}{\eta}$)		$i \in \{0, \ldots, \eta - 1\}$

firms (weakly) enjoy network benefits, in the sense that aftermarket profit per consumer is nondecreasing in network size. Part (ii) states that consumers (weakly) enjoy network benefits, in the sense that aftermarket benefit is non decreasing in the number of consumers in the same network.

Different industries are naturally be associated to different values of $\theta(i)$ and $\lambda(i)$. Here are two possible examples:

- After sales service (e.g., photocopiers, printers, cameras). In these examples, purchase of equipment leads to a stream of future benefits. Suppose that the value of after sales service depends on seller investment. By an appropriate change of units, suppose value equals investment. Let the cost of investment be given by $\alpha\, v^2$, where v is value. Suppose moreover that the seller captures all of the surplus it creates in the aftermarket. Then the seller's optimal v maximizes $i\, v - \alpha\, v^2$. Substituting the optimum v in the payoff function yields $\theta(i) = \psi\, i^2$, where $\psi = \frac{1}{4\,\alpha}$, whereas $\lambda(i) = 0$.

- Handheld operating systems. Buying a Palm OS handheld gives the consumer access to a wealth of applications written for that OS. Most of these are written by third party suppliers. Suppose there is free entry into the aftermarket. Firms compete a la Cournot. Each firm's cost function is $C = F + c\,q$, where q is output. Each consumer's demand is given by $q = a - p$. Ignoring the integer constraint, free entry leads to n active firms in the aftermarket, where $n = (a-c)\sqrt{i/F} - 1$. This in turn leads to a net consumer surplus of $i\left((a-c)\sqrt{i/F}-1\right)^2 F$. Finally, on a per-consumer basis we get $\lambda(i) = \left((a-c)\sqrt{i/F}-1\right)^2 F$, whereas $\theta(i)=0$.

My focus is on the firms' pricing decision and the consumer's network choice. Specifically, I consider equilibria in Markov pricing and network choice decisions. The state is defined by i, the size of firm i's network at the beginning of the period, that is, when firms set prices and the newborn consumer chooses one of the networks.[11] I next derive the consumer's and the firm's decisions in a Markov equilibrium.

■ **Consumer choice.** Each consumer's utility is given by two components: ζ_i and $\lambda(i)$. The first component is the consumer's idiosyncratic preference for firm i, which I assume depends on the identity of firm i but not on the size of its network (thus the use of a subscript rather than an argument). The value of ζ_i is also the consumer's private information. The second component is network benefit from a network with size i (*including* the consumer in question), which I assume is independent of the firm's identity. I assume that consumers receive the ζ_i component the moment they join a network, whereas $\lambda(i)$ is received each period that a consumer is still alive (and thus varies according to the size of the network during each future period).[12]

I assume that the values of ζ_i are sufficiently high so that a newborn consumer always chooses one of the available networks (that is, the outside option is always dominated). This assumption allows me to concentrate on the value of $\xi_i \equiv \zeta_i - \zeta_j$, the consumer's idiosyncratic *relative* preference for firm i's network. Notice that $\xi_j = -\xi_i$. I assume that ξ_i is distributed according to $\Phi(\xi)$ (density $\phi(\xi)$), which satisfies the following properties:

Assumption 2 *(i) $\Phi(\xi)$ is continuously differentiable; (ii) $\phi(\xi)=\phi(-\xi)$; (iii) $\phi(\xi)>0$, $\forall \xi$; (iv) $\Phi(\xi)/\phi(\xi)$ is strictly increasing.*

Many common distributions, including the Normal, satisfy Assumption 2.

11. Note that, when consumers enjoy network benefits, there are a total of η consumers, divided between the two networks. However, at the time that prices are set there are only $\eta-1$ consumers, so $i \in \{0,\ldots,\eta-1\}$ at that moment. The state and the firm value functions are defined at this moment (beginning of period).

12. The assumption that ζ_i is received at birth is not important. I could have the consumer receive ζ_i each period of his or her lifetime. However, this way of accounting for consumer utility simplifies the calculations.

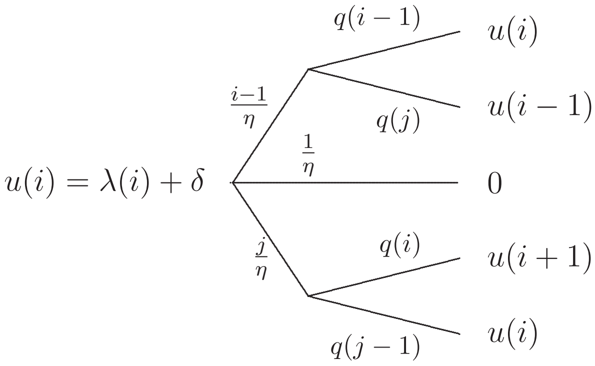

Figure 1: Consumer's value function.

Let $u(i)$ be a consumer's aftermarket value function, that is, the discounted value of payoff streams $\lambda(i)$ received while the consumer is alive (thus excluding both ζ_i and the price paid to join the network).

Consider a new consumer's decision. At state i, the indifferent consumer will have $\xi_i = x(i)$, where the latter is given by

$$x(i) - p(i) + u(i+1) = -p(j) + u(j+1), \tag{1}$$

or simply

$$x(i) = p(i) - p(j) - u(i+1) + u(j+1). \tag{2}$$

where $p(i)$ is firm i's price. This looks very much like a Hotelling consumer decision, except for the fact that $u(i+1)$ and $u(j+1)$ and endogenous values.

Firm i's demand is the probability of attracting the new consumer to its network. It is given by

$$\begin{aligned} q(i) &= 1 - \Phi\Big(x(i)\Big) \\ &= 1 - \Phi\Big(p(i) - p(j) - u(i+1) + u(j+1)\Big). \end{aligned} \tag{3}$$

The consumer value functions, introduced above, are illustrated in Figure 1. The corresponding formula is given by

$$u(i) = \lambda(i) + \delta \left(\frac{j}{\eta}\, q(i)\, u(i+1) + \left(\frac{j}{\eta}\, q(j-1) + \frac{i-1}{\eta}\, q(i-1) \right) u(i) + \right.$$

$$+ \frac{i-1}{\eta}\, q(j)\, u(i-1) \Bigg), \tag{4}$$

where $q(i)$ is given by (3), $i = 1, \ldots, \eta$, and $j = \eta - i$.[13] In words, a consumer's value is given, to begin with, by the current aftermarket benefit $\lambda(i)$. In terms of future value, there are three possibilities: with probability $1/\eta$, the consumer dies, in which case I assume continuation utility is zero.[14] With probability $(i-1)/\eta$, a consumer from the same network dies. This loss is compensated by the newborn consumer joining network i, which happens with probability $q(i-1)$, in which case next period's aftermarket state reverts back to i. With probability $1 - q(i-1)$, the new consumer opts for the rival network, in which case next period's aftermarket state drops to $i-1$. Finally, with probability j/η, a consumer from the rival network dies. This loss is compensated by the newborn consumer joining network j, which happens with probability $q(j-1)$, in which case next period's aftermarket state reverts back to i. With probability $1 - q(j-1) = q(i)$, the new consumer opts for network i, in which case next period's aftermarket state increases to $i+1$.

■ **Firm's pricing decision.** Firm i's value function is given by

$$\begin{aligned} v(i) = q(i) \left(p(i) + \theta(i+1) + \delta\, \frac{j}{\eta}\, v(i+1) + \delta\, \frac{i+1}{\eta}\, v(i) \right) \\ + \Big(1 - q(i)\Big) \left(\theta(i) + \delta\, \frac{j+1}{\eta}\, v(i) + \delta\, \frac{i}{\eta}\, v(i-1) \right), \end{aligned} \tag{5}$$

where $i = 0, \ldots, \eta - 1$ and $j = \eta - 1 - i$.[15] This is illustrated in Figure 2. With probability $q(i)$, firm i attracts the new consumer and receives $p(i)$. This moves the aftermarket state to $i+1$, yielding a period payoff of $\theta(i+1)$; following that, with probability $(i+1)/\eta$ network i loses a consumer, in which case the state reverts back to i, whereas with probability j/η network j loses a consumer, in which case the state stays at $i+1$. With probability $q(j)$,

13. Recall that the argument of u includes the network adopter to whom the value function applies, thus i must be strictly positive in order for the value function to apply. For the extreme values $i = 1$ and $i = \eta$, (4) calls for values of $q(\cdot)$ and $u(\cdot)$ that are not defined. However, these values are multiplied by zero.

14. Alternatively, I can consider a constant continuation utility upon death.

15. Again, notice that, for the extreme case $i = 0$, (7) calls for values of $v(\cdot)$ which are not defined. However, these values are multiplied by zero.

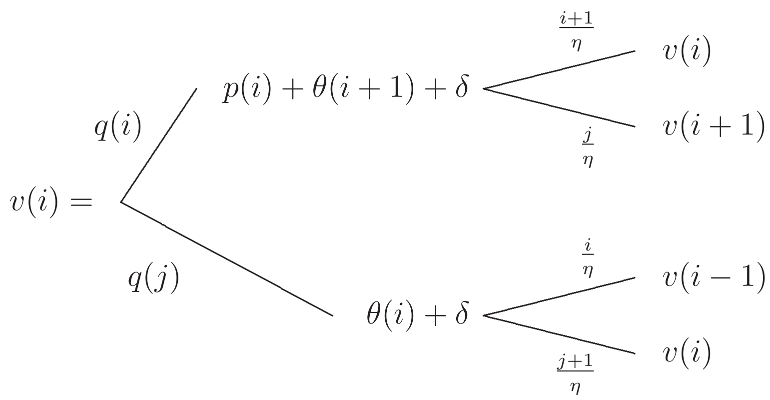

Figure 2: Firm's value function.

the rival firm makes the current sale. Firm i gets no revenues in the primary market. In the aftermarket, it gets $\theta(i)$ in the current period; following that, with probability i/η network i loses a consumer, in which case the state drops to $i-1$, whereas with probability $(j+1)/\eta$ network j loses a consumer, in which case the state reverts back to i.

Equation (5) leads to the following first-order conditions for firm value maximization:

$$q(i) + \frac{\partial\, q(i)}{\partial\, p(i)} \Bigg(p(i) + \theta(i+1) - \theta(i) + \delta\, \frac{j}{\eta}\, v(i+1) + \delta\, \frac{i+1}{\eta}\, v(i) - \delta\, \frac{j+1}{\eta}\, v(i) - \delta\, \frac{i}{\eta}\, v(i-1) \Bigg) = 0,$$

or simply

$$p(i) = h(i) - w(i), \tag{6}$$

where

$$\begin{aligned} h(i) &\equiv \frac{q(i)}{-q'(i)} = \frac{1 - \Phi\left(x(i)\right)}{\phi\left(x(i)\right)} \\ w(i) &\equiv \Big(\theta(i+1) - \theta(i)\Big) + \delta \left(\frac{j}{\eta}\, v(i+1) + \frac{i-j}{\eta}\, v(i) - \frac{i}{\eta}\, v(i-1) \right) \end{aligned}$$

The first-order condition (6) is fairly intuitive and plays an important role in explaining several results below. The first term on the right-hand side corresponds to the standard markup under monopoly pricing. The only difference is that consumer demand includes the endogenous value difference $u(i+1) - u(j+1)$: recall that the indifferent consumer "address" $x(i)$ is given by $x(i) = p(i) - p(j) - u(i+1) + u(j+1)$.

The second term of the right-hand side of (6), $w(i)$, is firm i's incremental future value from winning the current sale. By "future" I mean beginning with the current period's aftermarket. In terms of current period's payoff, the difference comes to $\theta(i+1) - \theta(i)$. In terms of future payoffs, we have the difference in value function between states $i+1$ and i (if consumer death takes place in network j) or between states i and $i-1$ (if consumer death takes place in network i).

Finally, substituting (6) into (5) and simplifying, we get

$$v(i) = r(i) + \theta(i) + \delta \left(\frac{j+1}{\eta} \, v(i) + \frac{i}{\eta} \, v(i-1) \right),$$

where

$$r(i) \equiv \Big(1 - \Phi\big(x(i)\big)\Big)\, h(i) = \frac{\big(1 - \Phi\left(x(i)\right)\big)^2}{\phi\left(x(i)\right)}.$$

This is a recursive system, the solution of which is given by

$$v(i) = \left(1 - \delta \, \frac{\eta - i}{\eta}\right)^{-1} \left(r(i) + \theta(i) + \delta \, \frac{i}{\eta} \, v(i-1) \right), \tag{7}$$

$i = 0, \ldots, \eta - 1$.

■ **Transition matrix and steady state distribution.** Given the equilibrium values of $q(i)$, I can compute a Markov transition matrix $M = m(i,k)$ where $m(i,k)$ is the probability of moving from state i to state k. For $0 < i < \eta - 1$, we have

$$\begin{aligned} m(i, i-1) &= \frac{i}{\eta} \, (1 - q(i)) \\ m(i,i) &= \frac{i+1}{\eta} \, q(i) + \frac{\eta - i}{\eta} \, (1 - q(i)) \\ m(i, i+1) &= \frac{\eta - 1 - i}{\eta} \, q(i) \end{aligned} \tag{8}$$

Moreover, $m(i,k) = 0$ if $k < i-1$ or $k > i+1$. Finally, the boundary values are obtained as follows. For $i = 0$, I apply the general equations and add the value obtained for $m(0,-1)$ to the value of $m(0,0)$. For $i = \eta - 1$, again I apply the general equations and add the value obtained for $m(\eta-1,\eta)$ to $m(\eta-1,\eta-1)$. As a result, I get

$$\begin{aligned} m(0,0) &= 1 - \frac{\eta-1}{\eta}\, q(0) \\ m(0,1) &= \frac{\eta-1}{\eta}\, q(0) \\ m(\eta-1,\eta-2) &= \frac{\eta-1}{\eta}\, \Big(1 - q(\eta-1)\Big) \\ m(\eta-1,\eta-1) &= 1 - \frac{\eta-1}{\eta}\, \Big(1 - q(\eta-1)\Big) \end{aligned}$$

Given the assumption that $\Phi(\cdot)$ has full support (Assumption 2), $q(i) \in (0,1)\ \forall i$, that is, there are no corner solutions in the pricing stage. It follows that the Markov process is ergodic and I can compute the stationary distribution over states. This is given by the (transposed) vector d that solves $d\,M = d$.[16]

■ **Consumer welfare and social welfare.** So far I have only considered the after-market value function for an individual consumer. I now turn to the task of measuring total consumer welfare at state i. Although $u(i)$ is measured at the time the new consumer has already chosen a network, it is now simpler if I simply compute total consumer welfare at the time firms set prices $p(i)$, that is, before network choice takes place. Consumer welfare, $c(i)$, is then recursively given by

$$\begin{aligned} c(i) = \sum_{\substack{k,\ell = i,j \\ \ell \neq k}} \quad & q(k) \bigg(E\Big(\zeta_k \mid \xi_k > x(k)\Big) - p(k) + \\ & + (k+1)\,\lambda(k+1) + \ell\,\lambda(\ell) + \\ & + \delta\,\frac{k+1}{\eta}\,c(k) + \delta\,\frac{\ell}{\eta}\,c(k+1)\bigg), \end{aligned} \tag{9}$$

16. This vector d can also be computed by repeatedly multiplying M by itself. That is, $\lim_{k\to\infty} M^k$ is a matrix with d in every row.

$i = 0, \ldots, \eta - 1$. There are three components to $c(i)$, each corresponding to a row in the right-hand side of (9). Suppose the newborn consumer joins network i. The immediate benefit for that consumer is given by the value of the idiosyncratic component ξ_i minus price $p(i)$. By joining network i, the newborn consumer increases network size to $(i+1)$, leading to a total benefit $(i + 1)\,\lambda(i + 1)$ to all consumers in network i. In the meantime, consumers in the rival network jointly receive a total benefit of $j\,\lambda(j)$. Finally, we need consider continuation payoffs. With probability $\frac{i+1}{\eta}$, a consumer from network i (now of size $i + 1$) dies, bringing network size back to i and total consumer value to $c(i)$. With probability $\frac{j}{\eta}$, a consumer form network j dies, bringing the state to $i + 1$, in which case total consumer surplus is given by $c(i + 1)$.

Given the (equilibrium) values of $q(i)$, $p(i)$ and $u(i)$, (9) is a well-defined linear system, yielding a unique solution $c(i)$. Notice that, by symmetry, $c(i) = c(j)$, where $j = \eta - 1 - i$. Finally, social welfare, $s(i)$, is simply consumer welfare plus firm value, that is,

$$s(i) = c(i) + v(i) + v(j), \tag{10}$$

$i = 0, \ldots, \eta - 1$, $j = \eta - 1 - i$. Once again, by symmetry $s(j) = s(i)$, where $j = \eta - 1 - i$.

A summary of the model's notation is given in Table 2. To help the reader navigate through the extensive set of variables, I follow the rule of using Greek letters to denote exogenous values and Roman letters to denote endogenous values.

■ **Numerical computation.** The above dynamic system has no closed-form solution. Below I derive analytical results for the case when the discount factor is low or the case when $\eta = 2$. An alternative strategy is to numerically compute the equilibrium. Numerical computation serves several purposes. First, it allows me to extend the analytical results to additional parameter values and additional properties. Second, by choosing reasonable parameter values, numerical simulation leads to particular values of the variables of interest. Finally, by performing various exercises of comparative dynamics, numerical simulation allows me to uncover additional qualitative results that are difficult or impossible to derive from analytically solving the model.

I now describe the process of numerically solving the model. The exogenous parameters correspond to the stage game profit function, $\theta(i)$, and

Table 2: Notation.

i	Firm i's network size (also j, k, ℓ).
η	Market size (number of consumers).
δ	Discount factor.
ξ_i	Consumers's idiosyncratic preference for firm i.
$\Phi(\xi_i)$	Distribution of ξ_i.
$\theta(i)$	Firm's aftermarket profit in state i.
$\lambda(i)$	Consumer's aftermarket benefit in state i.
$x(i)$	Indifferent consumer's relative preference for firm i.
$p(i)$	Price in state i (for new consumer).
$q(i)$	Probability of a sale in state i (to new consumer).
$u(i)$	Individual consumer's value in state i.
$v(i)$	Firm's value in state i.
$c(i)$	Total consumer's surplus in state i.
$s(i)$	Social surplus in state i.
$m(i,j)$	Transition probability from state i to state j.
$d(i)$	Stationary probability density of state i.

consumer surplus function, $\lambda(i)$; the discount factor, δ; and the distribution of ξ_i, $\Phi(\cdot)$ (Greek letters). The endogenous variables are the price function $p(i)$, the demand function $q(i)$, the firm value function $v(i)$, and the consumer value function $u(i)$ (and all the other Roman letter derived functions presented above, such as the stationary distribution over states).

In order to obtain an equilibrium solution, I follow a Gaussian method similar to that proposed by Pakes and McGuire (1994):[17]

1. Start with "naïve" consumer and firm value functions: $u_0(i) = \lambda(i)/(1-\delta)$ and $v_0(i) = \theta(i)/(1-\delta)$.[18]

2. At iteration $t = 1, \ldots$, use (6) to compute equilibrium prices and (3) to compute equilibrium demand given the latest estimate of u and v, that is, $u_{t-1}(i)$ and $v_{t-1}(i)$.

17. See also Doraszelski and Pakes (2006), Doraszelski and Satterthwaite (2007) for a discussion of various numerical methods.
18. I tried different starting conditions and always obtained the same equilibrium.

3. Use equilibrium prices and demands to compute the value functions at iteration t. The consumer value functions, $u_t(i)$, are obtained from (4). The firm value functions, $v_t(i)$, are obtained from (7).

4. Compute

$$\Delta \equiv \frac{\sum_{i=1}^{\eta} \left| u_t(i) - u_{t-1}(i) \right| + \sum_{i=0}^{\eta-1} \left| v_t(i) - v_{t-1}(i) \right|}{\sum_{i=1}^{\eta} u_t(i) + \sum_{i=0}^{\eta-1} v_t(i)}$$

Return to Step 2 until $\Delta < \Delta'$. I use $\Delta' = 1\text{E}{-6}$, though convergence is not greatly affected by considering different values.[19]

For the purpose of numerical computation, the following is a useful theoretical result:

Lemma 1 *Given $u(i)$ and $v(i)$, there exist unique $p(i)$ and $q(i)$; given $p(i)$ and $q(i)$, there exist unique $u(i)$ and $v(i)$.*

The proof of this and all other results may be found in the Appendix. The proof shows that (6) yields a unique solution $p(i)$ given the value functions $u(i), v(i)$. As a result, we also get a unique solution $q(i)$ given $u(i)$ and $v(i)$. Moreover, the systems determining $u(i), v(i)$ are linear, and so given $p(i), q(i)$ we get unique values of $u(i)$ and $v(i)$. Lemma 1 implies that the process described in Steps 1–4 above yields a unique solution at each iteration. Therefore, insofar as the sequence of p, q, u, v converges, we obtain a specific Nash equilibrium of the game.

A solution consists of four primary outputs: the mappings $p(i)$ and $q(i)$, which give equilibrium price and probability of a sale at each state $i = 0, \ldots, \eta - 1$; the firm value functions $v(i),\ i = 0, \ldots, \eta - 1$; and the consumer value functions $u(i),\ i = i, \ldots, \eta$. From these I can also derive the Markov transition matrix M, which is given by (8); the stationary distribution over states, $d(i)$, which is given by the solution to $d\,M = d$; consumer welfare at state i, $c(i)$, which is given by (9); and social welfare at state i, $s(i)$, which is given by (10).

In computing $c(i)$, one step still needs to be taken, namely computing the value of $E\left(\zeta_i \mid \xi_i > x(i)\right)$, which appears in (9). I make the Hotelling-like

19. I have also experimented with computing a moving average of values of u and v from previous iterations. This alternative procedure improved the speed of convergence for some parameter values.

assumption that $\zeta_i = \zeta' + \frac{1}{2}\,\xi_i$, where ζ' is a constant, which I normalize to zero. Moreover, I assume that $\xi_i \sim N(0, \sigma)$. This implies that

$$E\left(\zeta_i \mid \xi_i > x(i)\right) = \left(\frac{\sigma}{2}\right)\left(\frac{\phi\left(x(i)\right)}{1 - \Phi\left(x(i)\right)}\right),$$

where $\phi(x)$ and $\Phi(i)$ are the standardized normal density and distribution functions, respectively.

3 Equilibrium existence and uniqueness

One of the limitations of static analysis is that multiple equilibria emerge when the degree of network effects is strong. By contrast, I provide conditions such that a unique equilibrium exists.

Proposition 1 *There exists a δ' such that, if $\delta < \delta'$, then there exists a unique Markov equilibrium, which is symmetric.*

Notice that, in Proposition 1, symmetry is a derived property, not an assumption.[20]

The critical difference of my model with respect to static models is that I assume an overlapping generations framework. As a result, the consumers' decisions are interdependent but not simultaneous. In other words, simultaneity of adoption decisions overstates the coordination problem in the static model.

For $\eta = 2$, I can prove uniqueness of symmetric equilibria for *any* value of δ:

Proposition 2 *If $\eta = 2$, then there exists a unique symmetric Markov equilibrium.*

I conjecture that, if the discount factor is sufficiently close to 1, then there may exist multiple equilibria, supported by asymmetric beliefs. Even then, I conjecture that there exists a unique symmetric equilibrium; and symmetry seems a reasonable equilibrium selection criterion.

20. Caplin and Nalebuff (1991) prove equilibrium uniqueness (Proposition 6) for a game similar to the static game obtained when $\delta = 0$. Instead of my Assumption 2, they assume the density ϕ is log-concave. The relation between the two assumptions is not clear to me. Both are satisfied by most commonly used density functions.

In the next section, I present additional results that characterize the equilibrium of the dynamic game and compare it to the equilibria of a static model that approximates the dynamic model as much as possible.

4 Equilibrium properties

Figure 3 shows equilibrium values for a particular parameterization. I consider the case when $\lambda(i) = \psi\, i/\eta$, that is, consumer benefit is proportional to network i's market share, with the coefficient ψ measuring the degree of network effects. I also consider the case when $\theta(i) = \psi\, i^2/\eta$. This corresponds to the assumption that aftermarket surplus is equally split between consumers and firms (notice that firm i has i consumers and thus the term i becomes squared).[21] Finally, I assume ξ_i is normally distributed with zero mean and unit variance, $\eta = 101$, $\delta = .9$, and $\psi = .1$. Recall that the state space is given by $\{0, \ldots, \eta - 1\}$, in this case $\{0, \ldots, 100\}$. Thus, my particular choice of η has the advantage that i denotes both firm i's network size and firm i's market share.

■ **Pricing.** The top left panel of Figure 3 shows the price function, which is U shaped: decreasing at lower market shares, increasing for high market shares. As I mentioned earlier, there are two forces that influence equilibrium price: short run market power and long run market power. This can be seen from equation (6), which states that $p(i) = h(i) - w(i)$. Here the $h(i)$ term represents market power over the current newborn consumer, whereas the term $w(i)$ represents the quest for market power in the aftermarket and in future periods. The first term is increasing in i, whereas the second term is generally decreasing in i. In what follows, I describe each of the effects in greater detail.

The short run effect is that the larger a firm is the higher its price is. This corresponds to firms harvesting their market power. A firm with a greater network promises consumers a greater utility than its rival, to the tune of $u(i + 1) - u(j + 1)$. Consequently, firm i's demand curve expands as its network size increases. Under fairly general conditions, an expansion of the demand curve leads to higher prices. In particular, Assumption 2 implies

21. In other words, suppose that the value generated in the aftermarket is given by $2\,\psi\, i/\eta$ per consumer. Suppose moreover that this value is equally split between buyer and seller. Then we obtain the expressions above.

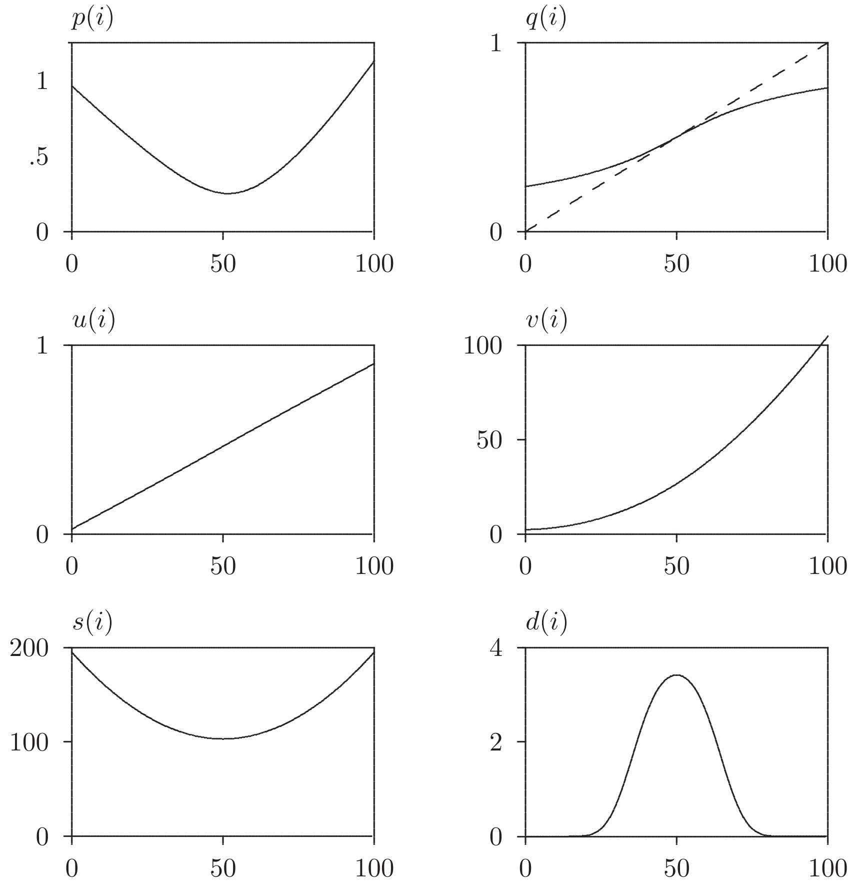

Figure 3: Equilibrium values as a function of state (horizontal axis) when F is normal with standard deviation σ and assuming $\theta(i) = \psi\, i^2/\eta,\ \lambda(i) = \psi\, i/\eta,\ \psi = .1,\ \sigma = 1,\ \delta = .9,\ \eta = 101.$

that, if the game were played in one period only, then the higher i, the higher $u(i)$ is, and the higher optimal price is.

The long-run effect is the investment effect. To the extent that the firm's value function is convex (which can be seen in the middle right panel), the greater the firm's market share the greater the benefit from making a sale, in terms of future profits. In fact, the benefit from increasing market share is measured by the difference between $v(i+1)$ and $v(i)$, that is, the "derivative" of $v(i)$. A convex value function implies that this value is greater the greater i is.

Convexity of the value function is also found in the models of Budd, Harris and Vickers (1993) and Cabral and Riordan (1994). Essentially, it corresponds to the dynamic version of Gilbert and Newbery's (1982) efficiency effect: joint profits are increasing in asymmetry, or equivalently, a large firm has more to lose from decreasing its market share than a small firm has from gaining the same increment in market share.

We thus have two opposite effects on price. Numerical simulation suggests that they combine into a U shaped function, with lowest prices around the relatively symmetric states. Although I conjecture that there may be sufficient conditions leading to this pattern, I have not been able to prove it analytically. I can, however, provide a more complete explanation for the observed pattern. The harvesting effect is typically a convex function of network i's advantage in the eyes of consumers, $u(i+1) - u(j+1)$.[22] This implies that the harvesting effect is very large for very large networks. The investment effect, in turn, is proportional to $v(i+1) - v(i)$. This function is virtually flat at $i = 0$, very "convex" around symmetric states, and close to linear at $i = \eta - 1$.

The result of these two patterns is that, for very small networks, the harvesting effect is small but the investment effect is very small (nearly flat value function). For very large networks, the harvesting effect is very large and the investment effect is large but not too large. Finally, for intermediate size networks, the investment effects is very strong relative to the harvesting effect, leading to lower price levels relative to the extremes of very large or very small networks.

22. This depends on the assumption that there is no outside option, that is, each consumer always chooses one of the networks. The harvesting effect is then given by the hazard rate $h(x) \equiv \frac{1-\Phi(x)}{\phi(x)}$, which is convex in x for most common distribution functions. Without the no-outside-good assumption, a large network won't have as much market power as my model suggests.

Although I cannot provide an analytical proof of the above results, I can derive sufficient conditions such that the pricing function is increasing or decreasing:

Proposition 3 *There exists a δ' such that: (a) if $\delta < \delta'$, $\theta(i+1) - \theta(i)$ is constant, and $\lambda(i)$ is strictly increasing, then $p(i)$ is strictly increasing; (b) if $\delta < \delta'$, $\lambda(i)$ is constant, and $\theta(i+1) - \theta(i)$ is strictly increasing, then $p(i)$ is strictly decreasing.*

Again, Proposition 3 highlights the two main forces impacting on the firms' pricing incentives: market power over the current newborn consumer and the quest for market power in the aftermarket and in future periods. Proposition 3 considers the case when the discount factor is small. In this case, most of the effects are reflected in (current period's) after market payoffs, which in turn allow me to derive conditions under which the first or the second effects dominate.

Specifically, given my assumption that the discount factor is small, $w(i)$ is essentially given by after-market profits, $\theta(i)$. Under case (a), aftermarket profits are an affine function of network size. This implies that the benefit from winning a new customer, in terms of aftermarket profits, does not depend on network size. Differences in pricing are thus exclusively driven by market power considerations related to the newborn consumer. Now, consumers are willing to pay more for a firm with a bigger network. In equilibrium, this is reflected in a higher price by the firm with a larger network. Thus $p(i)$ is increasing in i.

Consider now the case (b), the case when consumers do not care about network size. (Notice this does not mean there are no network externalities, rather that sellers completely capture the added consumer surplus resulting from network externalities.) If it were not for aftermarket and future profits, firms would set the same price, as their products are identical in the eyes of consumers. But to the extent that $\theta(i) - \theta(i-1)$ is increasing, the firm with a bigger network size has more to gain from making the next sale. This implies that it discounts price (with respect to the static price) to a greater extent. Thus $p(i)$ is decreasing in i.

■ **Increasing dominance.** How does the system, that is, the size of each network, evolve over time? The evolution of firm size is the net result of birth rates and death rates. I first characterize birth rates.

Proposition 4 *There exists a δ' such that, if $\delta < \delta'$, then $q(i) \geq \frac{1}{2}$ if $i > i^*$.*

For the case when $\eta = 2$, I can prove a stronger result:

Proposition 5 *If $\eta = 2$, then $q(1) > q(0)$.*

In other words, the leader always has a higher probability of attracting a newborn consumer. This is fairly intuitive. It corresponds to the idea that, in a static duopoly, a firm with a better product sells more (and at a higher price) than a firm with a worse product. The top right panel confirms the results of Proposition 4: the greater market share, the greater the probability of attracting a new consumer.

Notice that Propositoin 4 does not imply that the large firm will tend to become larger. In fact, larger firms also have larger death rates. The important comparison is between birth rates and death rates. My next result concerns precisely the issue of expected motion across states. Before presenting the result, I define by $i^* \equiv \frac{\eta-1}{2}$ the "symmetric" state. If η is even, then there exists no symmetric state, but the result below applies nonetheless.[23]

Proposition 6 *There exist δ', λ', θ' such that, if $\delta < \delta'$, then: (a) if i is close to zero or close to $\eta - 1$, then the state moves toward i^* in expected terms; (b) if i is close to i^*, and either $\lambda(i^*+1) - \lambda(i^*) > \lambda'$ or $\theta(i^*+1) + \theta(i^*-1) - 2\,\theta(i^*) > \theta'$, then the state moves away from i^* in expected terms.*

This result is in the same spirit as Budd, Harris and Vickers (1993). In a one-dimensional R&D race between two players, they show that, in expected terms, the system moves away from the symmetric state. In my model this is true if and only if network effects are sufficiently strong.

The top right panel of Figure 3 is apparently at odds with Proposition 6. In fact, if $i > i^* = 50$ then the birth rate is lower than the death rate (the dashed line), even for i close to $i^* = 50$. However, Proposition 6 requires that network effects be sufficiently strong. I next confirm numerically that, as the degree of network effects increases, increasing dominance emerges.

The particular parametrization I consider, $\theta(i) = \psi\, i^2/\eta$ and $\lambda(i) = \psi\, i/\eta$ allows me to easily consider the effects of increasing network effects. Figure 4 plots equilibrium mappings for three values of ψ: 0, .1, and .2. As expected,

23. Specifically, if η is odd, then i^* is the symmetric state. If η is even, then there is no symmetric state; $[i^*]$ and $[i^*]+1$ are the two states closest to the symmetric state, where $[x]$ denotes the highest integer lower than x.

if $\psi = 0$ then the price and the quantity functions are flat (the latter at $\frac{1}{2}$). The same is true for the consumer and the firm's value functions (the former at zero).

As ψ increases, the price function becomes U shaped (lower prices at intermediate states). The price difference $p(i) - p(j)$ does not change that much as ψ increases; but the consumers' utility difference $u(i+1) - u(j+1)$ does increase considerably. As a result, the demand function becomes steeper, as the top right panels shows. Ultimately, this leads to a more dispersed stationary distribution. In particular, notice that as ψ goes from .1 to .2, the slope of the $q(i)$ function at the symmetric state becomes greater than 1. This implies that, for a market share greater than, but close to, the symmetric state, the birth rate is greater than the death rate. This is consistent with Proposition 6: around the symmetric state, the system tends to move away from the symmetric state, that is, increasing dominance holds.

Since all consumers die with equal probability, the death rates ranges from 0 to 1 as market share varies from 0 to 100. By Assumption 2, birth rates $q(i)$ are always in (0,1). This implies that, at $i = 0$, the birth rate is greater than the death rate, and at $i = 100$ the opposite is true. This property (reversion to the mean), which is illustrated on the top right panel of Figure 4, is consistent with the first part of Proposition 6.

In terms of the stationary distribution, increasing dominance at i^* and reversion to the mean at the extreme states imply that the stationary distribution is bimodal. In fact, the modes of the stationary distribution are given by the points at which the $q(i)$ map crosses the main diagonal from above.

■ **Tipping.** An alternative way to judge the impact of network effects is to generate typical time paths for the state space (market share). Figure 5 does that. On the left panel, two times series are plotted, one corresponding to $\psi = 0$, one corresponding to $\psi = .2$. The two time series are constructed based on the same random series of ζ_{it}, time t's newborn consumer preference for network i. This allows for a better understanding of the relative role of randomness and network effects. (It also explains why the two time series are so highly correlated.)

Three values of market share are marked with dashed lines: 50%, the mode of the stationary distribution when $\psi = 0$; and the two modes of the asymmetric distribution corresponding to $\psi = .2$, approximately 20% and 80%. As expected, if there are no network effects ($\psi = 0$), then market share is typically around 50%. Not so when network effects are significant. As can

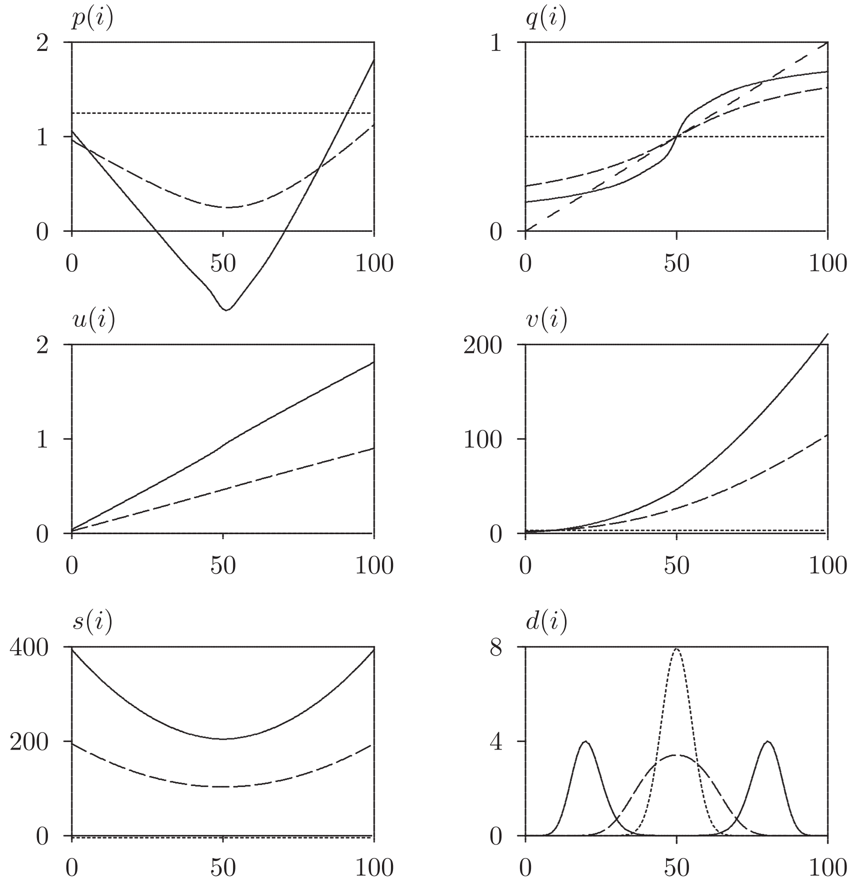

Figure 4: Equilibrium values as a function of state (horizontal axis) when F is normal with standard deviation σ and assuming $\theta(i) = \psi\, i^2/\eta$, $\lambda(i) = \psi\, i/\eta$, $\sigma = 1$, $\delta = .9$, $\eta = 101$. Three values of ψ are considered: 0 (dotted lines), .1 (dashed lines), .2 (solid lines).

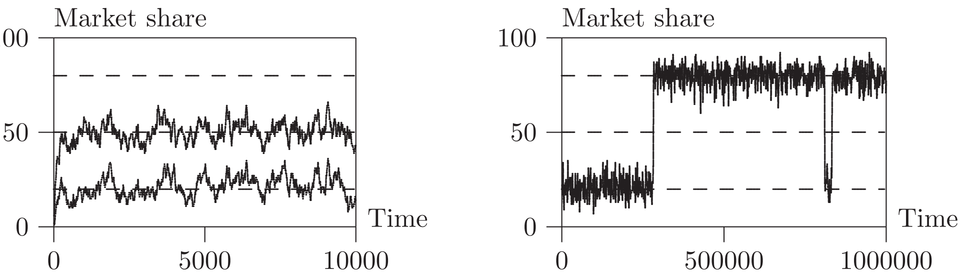

Figure 5: Time series of market share with network effects (lower line) or no network effects (higher line). The dashed lines represent the modes of the stationary distribution of market shares (cf Figure 4).

be seen, if $\psi = .2$, then, starting from $i = 0$, the system rapidly converges to one of the asymmetric states corresponding to the closest mode.

Although there is significant inertia around this asymmetric state, tipping is possible. Tipping does not take place in the particular series and time horizon plotted on the left panel of Figure 5. However, if I extend the series for long enough, then tipping eventually takes place. This is shown on the right panel of Figure 5, where the lower line series of the left panel is extended to 10^6 rather than 10^4 periods.[24]

■ **Static vs dynamic model.** In order to better understand the relation between static and dynamic analysis, I propose a static model that is as close as possible to my dynamic model. Suppose that $\eta - 1$ consumers simultaneously receive preference shocks ξ_i and must decide between one of the networks. As in the dynamic model, I assume that consumers receive a stream of surplus $\lambda(i)$, with the difference that i, the number of consumers in network i, is set fixed by the initial decisions of all $\eta - 1$ consumers. This being a symmetric equilibrium, I also assume that both firms set the same price, so that network effects and stand-alone preferences are the only elements entering each consumer's choice. For better comparison with the dynamic model, I augment consumer payoffs $\lambda(i)$ by the factor $\frac{\eta-1}{\eta}\,\delta$.

An equilibrium consists of a threshold value $\widehat{\xi}_i$ such that consumers choose network i if and only if $\xi > \widehat{\xi}_i$. This implies that network i is chosen with

24. A potentially interesting application of the present framework is the tipping example documented and discussed in Cantillon and Yin (2007).

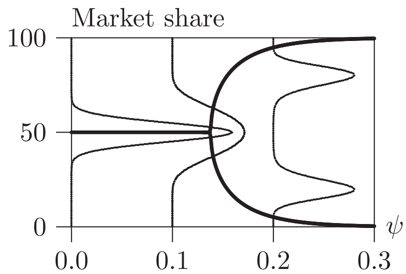

Figure 6: Static vs dynamic model. Equilibrium market share in static model (solid line), mode(s) of the stationary distribution of market shares in dynamic model (dashed line).

probability $\hat{q} = 1 - \Phi(\widehat{\xi}_i)$. This in turn implies that the equilibrium distribution over network states is binomial with mode $(\eta - 1)\left(1 - \Phi(\widehat{\xi}_i)\right)$.

Figure 6 plots the equilibrium values $\widehat{q}$ as a function of the degree of network effects, ψ. For low values of ψ, there exists a unique (symmetric) equilibrium. At a certain value of ψ, the equilibrium correspondence bifurcates. While a symmetric equilibrium still exists, this equilibrium is unstable. The only stable equilibria are asymmetric equilibria. In fact, a relatively small increase in ψ implies asymmetric equilibria that are very close to 0 and 1.[25]

For comparison purposes, I also plot the stationary distribution of market shares in the dynamic model. As can be seen, the static equilibrium predicts equilibrium values $\widehat{q}$ that are fairly close to the mode (or modes) of the dynamic equilibrium stationary distribution of market shares. In this sense, the static equilibrium does a fairly good job at approximating the outcome of the game. However, as mentioned earlier, the advantage of the dynamic approach is that we have a single symmetric equilibrium (with asymmetric outcomes).

■ **Profits and welfare.** Network effects imply that social welfare is greater the greater the asymmetry between firms. This can be seen from the bottom left panel in Figure 3. Intuitively, at an asymmetric state more value is created because of network effects. Although total welfare increases with market concentration, the same is not true for consumer surplus. If $\lambda(i)$ is relatively small, then most of the consumers' benefit is obtained in the ba-

25. Given Assumption 2, the asymmetric equilibrium values $\widehat{q}$ converge to 0 and 1 asymptotically, but are always in the $(0, 1)$ interval.

sic market. And since firms are more price competitive in the basic market when they compete neck-to-neck, consumers prefer symmetry. If however most of the consumer benefit is obtained in the aftermarket then consumers prefer asymmetric states.[26]

As far as firms are concerned, asymmetry is always payoff improving. Specifically, the firm's value function is convex, implying that joint profits are greater the more asymmetric the market structure is. This is the dynamic version of the "efficiency effect" described by Gilbert and Newbery (1982): a large firm has more to lose from decreasing its market share than a small firm has from gaining the same increment in market share. This convexity is the result of price competition (close competitors transfer value to consumers) and network effects (total value created is greater at an asymmetric state).

Regarding the impact of increasing the degree of network effects, an interesting result is worth mentioning. Figure 4 shows that, as ψ increases, the firm's value function becomes increasing and convex. But firm value does not increase uniformly in ψ: for low market shares a firm is strictly worse off when ψ is higher, even though a higher ψ increases consumer willingness to pay and firm aftermarket profits at every state. Cabral and Villas-Boas (2005) refer to this apparently puzzling outcome as a *Bertrand supertrap*.[27] The idea is that the strategic effect of an increase in ψ (lower prices) more than outweighs the direct effect (higher consumer valuations and aftermarket profits).

In the present context, I can prove the following *Bertrand supertrap* type of result.

Proposition 7 *There exist δ', λ', θ' such that, if $\delta < \delta'$, then $v(0)$ is decreasing in ψ.*

This result also suggests that network effects may create a barrier to entry. Although it is not clear in general how to define and measure barriers to entry, in the present context a reasonable measure is precisely the difference between $v(0)|_{\psi=0}$ and $v(0)|_{\psi>0}$, that is, the loss in value an entrant experiences due to the presence of network effects. I return to this issue in the next section.

26. Numerical simulations suggest that, as the share of aftermarket value captured by consumers goes from 0 to 1, the consumer welfare function changes from inverted U to a U shaped mapping.

27. In Figure 4, $v(50)$ (that is, firm value at the symmetric state) seems to be increasing in ψ. However, there can be cases when $v(50)$ decreases with ψ. Cabral and Riordan's (1994) Theorem 3.5 corresponds to a Bertrand supertrap in a symmetric state in the context of dynamic competition with learning curves.

5 Applications

The numerical solution of the model has two advantages over the analytical solution. First, by choosing reasonable numbers for the various model parameters I can go beyond qualitative analysis and actually assign numbers to the variables of interest. Second, by performing a series of comparative dynamics exercises, I can uncover qualitative results that are difficult or impossible to obtain from the model's analytical solution. In this section, I present two applications along these lines. First, I estimate the extent to which network effects create barriers to entry. Second, I extend the model to the asymmetric case and use it to estimate the effects of product improvement.

5.1 Network effects as a barrier to entry

There are several ways by which to measure barriers to entry.[28] I suggest that network effects create a barrier to entry to the extent that a new entrant's value is lower than it would be if there were not network effects. Specifically, one way to measure such decrease in entrant's value is to measure

$$\frac{v(0) \mid_{\psi=0} - v(0)}{v(0) \mid_{\psi=0}}$$

where $v(0)$ is the entrant's value (network of size zero) and $v(0) \mid_{\psi=0}$ is the entrant's value if there were no network effects.[29]

Not surprisingly, the above measure has positive sign, that is, network effects imply a barrier to entry. At this stage, the relevant question is the magnitude of the barrier to entry. For this purpose, it is useful to have an idea of what different parameter values mean. Accordingly, I construct a measure of network "die-hard" fans. Consider the following experiment: a new consumer must choose between a network of size zero and a network of size η. Suppose both networks set the same entry price. Suppose moreover

28. See Cabral (2007) for a discussion on the concept of barriers to entry. See Jullien (2001), Llobet and Manove (2006) for different perspectives on the implications of network effects for entry.

29. An alternative measure would be difference between the incumbent's value, $v(100)$, and an entrant's value, $v(0)$. This alternative is closer to Gilbert's (1989) notion of barriers to entry ("a rent that is derived from incumbency"). Notice that neither the Bain nor the Stigler definitions would indicate a barrier to entry in either case. See Gilbert (1989), Cabral (2007).

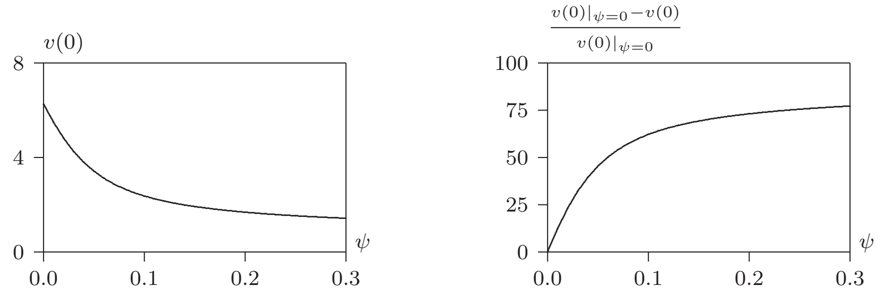

Figure 7: Network effects as a barrier to entry: how $v(0)$ depends on ψ. Parameter values: $\theta(i) = \psi\, i^2/\eta,\ \lambda(i) = \psi\, i/\eta,\ \sigma = 1,\ \delta = .9,\ \eta = 101$.

that the new consumer is myopic, that is, assumes the current network size will persist into the indefinite future. A die-hard fan is one who chooses a network of size zero in these circumstances. Given the distribution of ξ_i, the measure of die-hard fans is given by

$$\Phi\left(\frac{-\psi/\sigma}{1-\frac{\eta-1}{\eta}\,\delta}\right),$$

where $\Phi(\cdot)$ is the standardized normal cumulative distribution function.[30] The values of this expression for $\psi = 0, .1, .2, .3$ are, respectively, 50, 17.9, 3.3, and .3%.

Figure 7 plots, on the left-hand panel, $v(0)$ as a function of ψ (and for the parameter values used in the base case above). If there are no network effects, then $v(0) = 6.3$. If $\psi = .1$, which corresponds to about 18% "die hard fans," then $v(0) = 2.4$. This corresponds to a 62% loss in value. More generally, we can see that the percent loss in value changes very steeply as ψ increases away from zero: even relatively modest network effects create significant barriers to entry.

An alternative way of measuring the difficulty to enter is to compute the average time it takes for an entrant to achieve a certain market share. Let $T(i,k)$ be expected time to reach k starting from $i < k$. This can be

30. Notice that, as expected, I have a degree of freedom in determining the monetary unit in which to express consumer benefit. Different units would change both ψ and σ, and so my measure is invariant to the particular unit chosen.

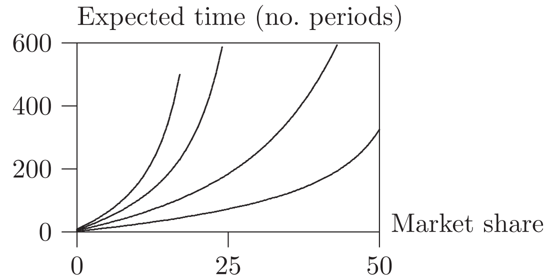

Figure 8: Expected time for an entrant to reach a given market share, assuming $\theta(i) = \psi\, i^2/N,\ \lambda(i) = \psi\, i/N,\ \sigma = 1,\ \delta = .9,\ N = 101$. Four values of ψ are considered: 0, 0.1, 0.2, 0.3

computed as follows. First notice that $T(k,k) = 0$. For $i < k$, we have

$$T(i,k) = m(i,i+1)\,T(i+1,k) + m(i,i)\,T(i,k) + m(i,i-1)\,T(i-1,k).$$

where $m(i,\ell)$ is the transition probability from i to ℓ. This forms a band linear system of k equations with k unknowns.

Solving this system I obtain the values in Figure 8. The vertical axis shows the expected time to first getting to a given state; the horizontal axis shows the specific state (market share). The qualitative conclusion from this figure is that network effects don't greatly affect the time it takes for an entrant to reach a low market share (say, 10%), but do significantly affect the time required for an entrant to achieve a 50% market share. To see this, consider the extreme cases of network effects I used before, $\psi = 0$ and $\psi = .3$. It takes an average of 25 periods for an entrant to achieve a 10% market share if there are no network effects. With strong network effects, $\psi = .3$, it takes six times as long for the entrant to achieve the same market share. If however we consider how long it takes to get to a 50% market share, then the difference between $\psi = 0$ and $\psi = .3$ is a factor of 50,200,284 (the ratio 14,909,484,327/297)!

5.2 Product improvement

Until now, I have assumed that firms are ex-ante identical. In equilibrium, firms have greater or smaller market shares, but prices and quantities are only a function of the state, not of the firm's identity. Suppose however that one of the firms improves its product. How does that change equilibrium values?

Specifically, suppose that firm A's product has a "standalone" value that is μ_A higher than firm B's. To put it differently (but equivalently), suppose that the values of ζ_i are no longer derived from the same distribution; rather, the distribution of ζ_A is shifted by μ_A with respect to the distribution of ζ_B.

Computationally, this implies that I now need to solve for equilibrium mappings $p(i)$, $q(i)$, etc., that are indexed by the firm's name. This increases the computational cost, but only slightly, since the state space is still one-dimensional (the number of states increase by two-fold). Figure 9 considers the experiment of increasing μ_A from zero to 1, keeping the remaining parameters as in the base case considered in Section 4. Solid lines represent the initial (symmetric) case; dashed lines correspond to firm A, the firm with a superior product; and dotted lines correspond to firm B, the firm with an inferior product. To get an idea of what μ_A means, notice that $\psi = .1$ and $\delta = .9$ imply that, if we expect market shares to remain constant for some time, then the difference between a large network ($i = \eta$) and a small network ($i = 1$) is approximately equal to $\psi/(1 - \delta)$, which is equal to the value of μ_A I consider. So, I am assuming that firm A's product is superior to an extent that compensates for the network effects even if firm B dominates the market.

The top left panel shows that, as firm A's product improves, firm A increases its price while firm B decreases its price. The price difference between firm A and firm B falls between zero and μ_A, firm A's product advantage. The panel on the top right side shows the value of u_A and u_B. Recall that, as a matter of convention, I am *excluding* from u_i the standalone valuation. Still, for intermediate values of i, we see that firm A enjoys a increase in u_A with respect to u_B. This is due to network effects: given that firm A has a better product, for a given market share consumers expect the future market share of firm A to be greater than that of firm B in similar circumstances.

Because of network effects and because the price difference between firms is less than μ_A, the probability that a new consumer opts for firm A is greater than in the symmetric equilibrium. This is illustrated by the middle left panel in Figure 9. Notice that it is still the case that, for a very large market share, firm A's birth rate is lower than its death rate, and the opposite holds at the other extreme value of market share. In other words, reversion to the mean at the extremes still holds.

The stationary distribution of market shares is illustrated in the right panel of Figure 9. Starting from a symmetric, uni-modal density, we now

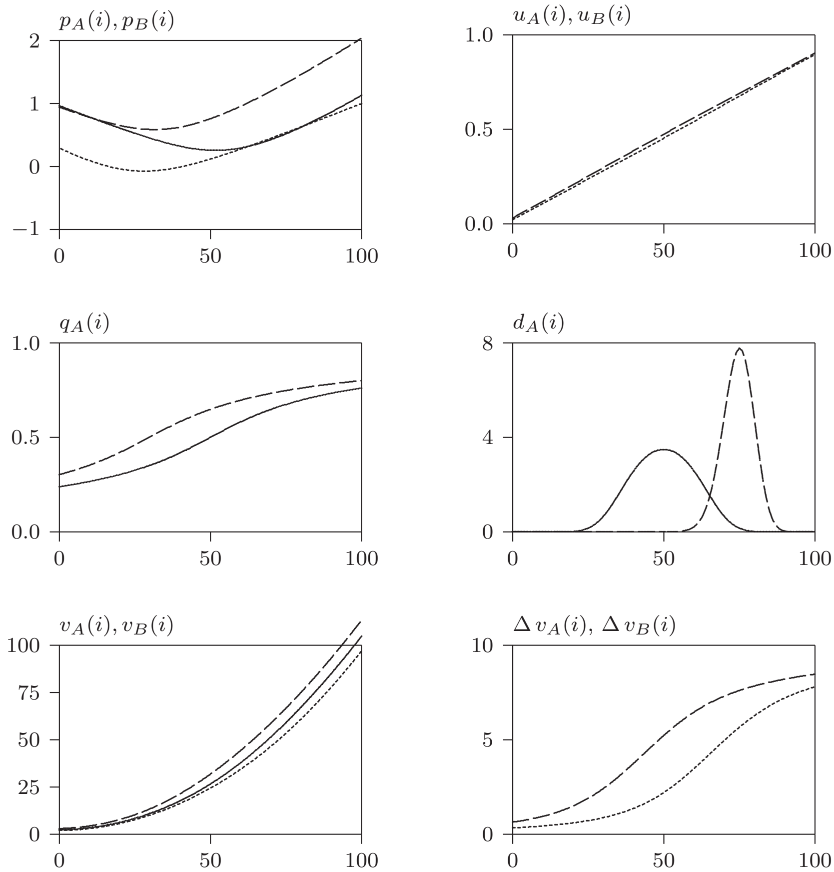

Figure 9: Effect of firm A's product improvement. Solid lines represent the symmetric case; dashed lines represent firm A's values when $\mu_A = 1$; and dotted lines represent firm B's values when $\mu_A = 1$. Other parameter values: $\theta(i) = \psi\, i^2/\eta,\ \lambda(i) = \psi\, i/\eta,\ \psi = .1,\ \sigma = 1,\ \delta = .9,\ \eta = 101.$

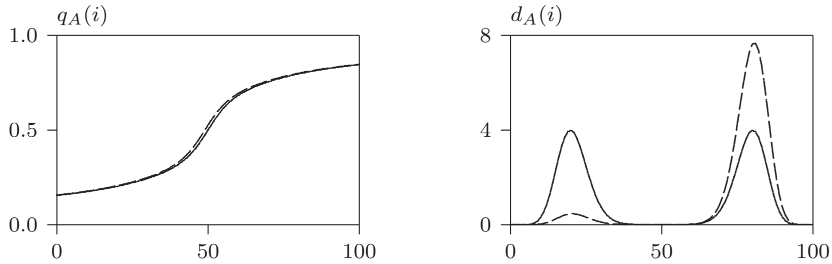

Figure 10: Effect of firm A's product improvement. Solid lines represent the symmetric case, dashed lines the case when $\mu_A = .1$. Other parameter values: $\theta(i) = \psi\, i^2/\eta,\ \lambda(i) = \psi\, i/\eta,\ \psi = .2,\ \sigma = 1,\ \delta = .9,\ \eta = 101$.

have one that, while still uni-modal, is clearly shifted to the right and has lower variance: no matter where you start from, most likely firm A will eventually become a market share leader.

In terms of firm value, the bottom left panel shows that firm A becomes better off and firm B worse off with respect to the symmetric equilibrium. Moreover, as documented in the right lower panel, (a) firm A's gain is greater than firm B's loss; and (b) firm A's gain is increasing in market share. The first result reflects the "efficiency effect" that I previously discussed in Section 4: the greater the asymmetry between firms, the greater their joint payoffs. The second result suggests that the incentives for product improvement are greater the greater a firm's market share is.

Figure 9 describes the case of product improvement with moderate network effects ($\psi = .1$), which corresponds to a unimodal stationary distribution of market shares. Figure 10 corresponds to the case of strong network effects ($\psi = .2$), when the stationary distribution of market shares is asymmetric. In Figure 9, I considered a fairly significant asymmetry in stand-alone value, $\mu_A = 1$. Now I consider a much smaller difference between firms A and B, $\mu_A = .1$. This small difference is reflected in a relatively small shift in the demand probability $q_A(i)$, as the left panel of Figure 10 shows. However, as the right panel shows, the impact of μ_A on the stationary distribution is quite drastic: from a symmetric, bi-modal distribution, we go to a very asymmetric (also bimodal) distribution.

To understand the reason for such disparity of effects, consider the following stylized stochastic model. There are N states: $i = 1, \ldots, N$. States 0 and N are absorbing. At any other state, there is a probability p of moving one step up and a probability $1-p$ of moving one step down. Let $\mathcal{P}(i)$ be the probability of hitting state N given that we are currently in state i. It can be shown that the solution to this system is a Bessel function. Moreover, the boundary conditions $\mathcal{P}(1)=0$ and $\mathcal{P}(N)=1$ pin down the exact value of the solution:

$$\mathcal{P}(i) = \frac{1-\left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^N}$$

Finally, computation establishes that

$$\left.\frac{d\,\mathcal{P}(i)}{d\,p}\right|_{p=\frac{1}{2}} = \frac{2\,i\,N - 2\,i^2}{N}$$

For example, if we consider the symmetric state $i = N/2$ (assuming N is even), then

$$\left.\frac{d\,\mathcal{P}(N/2)}{d\,p}\right|_{p=\frac{1}{2}} = \frac{N}{2}$$

In other words, even a small change in the value of p leads to a large $(\frac{N}{2})$ change in the likelihood the system will converge to the high absorbing state.

6 Final remarks

In this paper, I propose a novel framework with which to analyze the dynamics of price competition with network effects. In addition to understanding basic patterns of competitive dynamics, I believe this framework can be fruitfully applied to a variety of situations. I considered two possible applications: estimating the barrier to entry created by network effects and estimating the comparative dynamics of asymmetric product improvement.

I believe there are many other questions that my framework can address. One is the impact of aftermarket power on installed base dynamics. Antitrust cases such as *Kodak* hinge on such effects; nevertheless, most of the extant analysis is based on static models.

Another policy related question, in the specific context of competition between mobile telecommunications networks, is the establishment of access charges across networks. Many authors have looked at the issue from a static point of view. However, different regulatory policies have different impacts on the dynamics of market shares, possibly changing the ultimate impact with respect to the short-run impact.

Appendix

Proof of Proposition 1: In order to consider the possibility of asymmetric equilibria, I add a firm subscript to the relevant functions. Let k and ℓ be the firm subscripts associated to the firms with networks of size i and j, respectively.

Firm k's first-order condition (6) then becomes

$$p_k(i) = h_k(i) - w_k(i), \tag{11}$$

where

$$\begin{aligned} h_k(i) &= \frac{1 - \Phi\left(p_k(i) - p_\ell(j) - u_k(i+1) + u_\ell(j+1)\right)}{\phi\left(p_k(i) - p_\ell(j) - u_k(i+1) + u_\ell(j+1)\right)} \\ w_k(i) &= v_k(i+1) - v_k(i) \end{aligned}$$

Similar expressions hold for firm ℓ. Define

$$\begin{aligned} P_k(i) &\equiv p_k(i) - p_\ell(j) \\ H_k(i) &\equiv h_k(i) - h_\ell(j) \\ W_k(i) &\equiv w_k(i) - w_\ell(j) \\ U_k(i) &\equiv u_k(i+1) - u_\ell(j+1) \end{aligned}$$

where $j = \eta - 1 - i$. Subtracting firm j's first-order conditions from (11), I get

$$P_k(i) = H_k(i) - W_k(i) \tag{12}$$

Suppose that $\delta = 0$. Then

$$w_k(i) = \theta(i+1) - \theta(i)$$

and

$$U_k(i) = \lambda(i+1) - \lambda(j+1).$$

In other words, both $U_k(i)$ and $w_k(i)$ (and thus $W_k(i)$) are constants. It follows that (12) is an equation with one unknown, $P_k(i)$.

Specifically, the "hazard" function $H_k(i)$ is given by

$$\begin{aligned} H_k(i) &= \frac{1 - \Phi\left(P_k(i) - U_k(i)\right)}{\phi\left(P_k(i) - U_k(i)\right)} - \frac{1 - \Phi\left(P_\ell(j) - U_\ell(j)\right)}{\phi\left(P_\ell(j) - U_\ell(j)\right)} \\ &= \frac{1 - 2\,\Phi\left(P_k(i) - U_k(i)\right)}{\phi\left(P_k(i) - U_k(i)\right)}, \end{aligned}$$

since $P_\ell(j) = -P_k(i)$, $U_\ell(j) = -U_k(i)$, and $\Phi(-x) = 1 - \Phi(x)$. We can therefore re-write (12) as

$$P_k(i) + \frac{2\,\Phi\left(P_k(i) - \lambda(i+1) + \lambda(j+1)\right) - 1}{\phi\left(P_k(i) - \lambda(i+1) + \lambda(j+1)\right)} = \\ = -\theta(i+1) + \theta(i) + \theta(j+1) - \theta(j), \qquad (13)$$

Assumption 2 implies that the left-hand side of (13) is strictly increasing in $P_k(i)$, ranging from $-\infty$ to $+\infty$. The right-hand side, in turn, is a constant. The intermediate value theorem then implies that there exists a unique equilibrium value $P_k(i)$. Given symmetry of the $\theta(i)$ and $\lambda(i)$ functions, we conclude that the $P_\ell(j)$ function is identical to the $P_k(i) =$ function. Given $P_k(i)$, the values of $p_k(i)$ are uniquely determined by (11). Again, symmetry of the $\theta(i)$ and $\lambda(i)$ functions implies symmetry of the pricing function. Finally, the result follows by continuity around $\delta = 0$. ■

Proof of Proposition 2: When $\eta = 2$, the entire equilibrium is determined by the value of $x(1)$ (or equivalently $x(0) = -x(1)$). From (4),

$$\begin{aligned} u(1) &= \lambda(1) + \delta\,\frac{1}{2}\left(q(1)\,u(2) + q(0)\,u(1)\right) \\ u(2) &= \lambda(2) + \delta\,\frac{1}{2}\left(q(1)\,u(2) + q(0)\,u(1)\right) \end{aligned}$$

It follows that

$$x(1) = p(1) - p(0) - u(2) + u(1) = P(1) - \lambda(2) + \lambda(1) \qquad (14)$$

and

$$H(1) \equiv h(1) - h(0) = \\ \frac{1}{f\left(P(1) - c_1\right)}\Big(F\left(P(1) - c_1\right) - F\left(-P(1) + c_1\right)\Big) \qquad (15)$$

where $c_1 \equiv \lambda(2) - \lambda(1)$ is a constant. From (7), I get

$$\begin{aligned} w(0) &= \theta(1) - \theta(0) + \delta\,\frac{1}{2}\,\left(v(1) - v(0)\right) \\ w(1) &= \theta(2) - \theta(1) + \delta\,\frac{1}{2}\,\left(v(1) - v(0)\right) \end{aligned}$$

It follows that

$$W(1) \equiv w(1) - w(0) = \theta(2) - 2\,\theta(1) + \theta(0). \tag{16}$$

Subtracting (6) from firm j's first-order conditions ($i = 1, j = 0$), the equilibrium value of $P(1)$ is determined by

$$P(1) - H(1) = -W(1). \tag{17}$$

From (16), we conclude that $W(1)$ is a constant. From (15) and Assumption 2, we conclude that the left-hand side is continuous and strictly increasing in $P(1)$. Moreover, the left-hand side ranges from $-\infty$ to $+\infty$ as $P(1)$ varies from $-\infty$ to $+\infty$. By the intermediate value theorem, there exists a unique value of $P(1)$ that solves (17). Finally, a unique $P(1)$ determines $x(1)$, uniquely, which in turn determines $u(i)$ and $v(i)$ uniquely. ■

Proof of Proposition 3: Suppose that $\delta = 0$. Consider first the case when $\theta(i) = \bar{\theta}$. We then get $w(i) = 0$, and thus $W(i) = 0$ as well. As shown in the proof of Proposition 4, $P(i) - U(i)$ is strictly decreasing in $U(i)$. Since $\lambda(i) > \lambda(i-1)$, it follows that $U(i)$ is strictly increasing, which in turn implies that $P(i) - U(i)$ is strictly decreasing. Finally, since $w(i) = 0$, (6) reduces to

$$p(i) = \frac{1 - \Phi\left(P(i) - U(i)\right)}{\phi\left(P(i) - U(i)\right)},$$

which implies that $p(i)$ is strictly increasing in i.

Consider now the case when $\lambda(i) = \bar{\lambda}$. This implies that $U(i) = 0$. Let $H_U(i) \equiv \frac{\partial H(i)}{\partial U(i)}$ as before. Applying the Implicit Function Theorem to (19) we get

$$\frac{d\,P(i)}{d\,W(i)} = -\frac{1}{1 + H_U(i)} < 0.$$

Since $W(i) = w(i) - w(j)$, we have

$$\frac{\partial\,P(i)}{\partial\,w(i)} = \frac{\partial\,P(i)}{\partial\,W(i)}.$$

Moreover, since

$$h(i) = \frac{1 - \Phi\left(P(i) - U(i)\right)}{\phi\left(P(i) - U(i)\right)}$$

and

$$H(i) = \frac{1 - 2\,\Phi\,(P(i) - U(i))}{\phi\,(P(i) - U(i))}$$

we have

$$\frac{\partial\, h(i)}{\partial\, P(i)} = \frac{1}{2}\,\frac{\partial\, H(i)}{\partial\, P(i)} = -\frac{1}{2}\,H_U(i).$$

Differentiating the right-hand side of (6) with respect to $w(i)$, we get

$$\frac{d\,p(i)}{d\,w(i)} = \left(-\frac{1}{2}\,H_U(i)\right)\left(-\frac{1}{1 + H_U(i)}\right) - 1 < 0. \tag{18}$$

If $\delta = 0$, then $w(i) = \theta(i+1) - \theta(i)$, which is strictly increasing, by assumption. It follows by (18) that $p(i)$ is strictly decreasing in i. Finally, the results follow from continuity in δ. ■

Proof of Proposition 4: Similarly to the proof of Proposition 1, I define (now without firm subscripts)

$$\begin{aligned} P(i) &= p(i) - p(j) \\ U(i) &= u(i+1) - u(j+1) \\ H(i) &= h(i) - h(j) \\ W(i) &= w(i) - w(j) \end{aligned}$$

Subtracting the first-order conditions, I have

$$P(i) = H(i) - W(i). \tag{19}$$

By the same argument as in the proof of Proposition 1,

$$H(i) = \frac{1 - 2\,\Phi\,(P(i) - U(i))}{\phi\,(P(i) - U(i))}$$

I can thus re-write (19) as

$$P(i) + \frac{2\,\Phi\,(P(i) - U(i)) - 1}{\phi\,(P(i) - U(i))} = -W(i) \tag{20}$$

Suppose that $\delta = 0$. Suppose also that $U(i) = 0$. The left-hand side of (20) is strictly increasing in $P(i)$ and strictly positive if and only if $P(i)$ is strictly positive. It follows that, if $W(i) \geq 0$, then $P(i) \leq 0$, that is, $P(i) - U(i) \geq 0$ (as I assume $U(i) = 0$).

Let $H_U(i) \equiv \frac{\partial H(i)}{\partial U(i)}$ and $H_P(i) \equiv \frac{\partial H(i)}{\partial P(i)}$. By (13) and Assumption 2, $H_U(i) > 0$ and $H_P(i) = -H_U(i)$. Since $W(i)$ does not depend on $U(i)$, I can apply the Implicit Function Theorem to (19) to get

$$\frac{d\,P(i)}{d\,U(i)} = \frac{H_U(i)}{1 + H_U(i)} \in (0, 1).$$

It follows that

$$\frac{d\ (P(i) - U(i))}{d\,U(i)} < 0. \tag{21}$$

Since $P(i) - U(i) \leq 0$ when $U(i) = 0$ and $P(i) - U(i)$ is strictly decreasing in $U(i)$, I conclude that $P(i) - U(i) \leq 0$ if $U(i) \geq 0$. Symmetry implies that $U(i) \geq 0$ if $i > i^*$. And $q(i) \geq \frac{1}{2}$ if and only if $U(i) \geq P(i)$. Finally, the result follows from continuity in δ. ■

Proof of Proposition 5: Suppose that $q(0) > q(1)$. Since $h(i) \equiv q(i)/q'(i)$, this implies $h(0) > h(1)$ and thus $H(1) \equiv h(1) - h(0) < 0$. From (14) and $\lambda(1) \geq \lambda(0)$ (by Assumption 1), $q(0) > q(1)$ implies $P(1) > 0$. Finally, the above facts imply that the left-hand side of (17) is strictly positive, whereas by Assumption 1 the right-hand side is (weakly) negative. ■

Proof of Proposition 6: Suppose that $i^* < i < \eta - 1$. From (8), the state moves away from i^* in expected terms if and only if

$$\frac{\eta - 1 - i}{\eta}\ q(i) > \frac{i}{\eta}\ (1 - q(i)),$$

which is equivalent to

$$q(i) > \frac{i}{\eta - 1}. \tag{22}$$

In other words, the system moves away from i^* if and only if the leader's birth rate, $q(i)$, is greater that the leader's death rate, $\frac{i}{\eta-1}$.

In the proof of Proposition 4, I show that $P(i) - U(i)$ is strictly decreasing in $U(i)$. In fact, the derivative of $P(i) - U(i)$ with respect to $U(i)$ is bounded away from zero (from above). Since $q(i) = 1 - \Phi(P(i) - U(i))$ and $\frac{i}{\eta - 1} < 1$, it follows that, if $U(i)$ is high enough, then (22) is satisfied.

In the proof of Proposition 3, I show that $P(i)$ is decreasing in $W(i)$. (I then assume that $U(i) = 0$. However, all I need is that $U(i)$ does not change with $W(i)$, which follows from $\delta = 0$.) In fact, the derivative of $P(i)$ with respect to $W(i)$ is bounded away from zero (from above). It follows that, if $W(i)$ is high enough, then (22) is satisfied. ■

Proof of Proposition 7: By the same argument as in the proof of Proposition 4,

$$P(0) + \frac{2\,\Phi(P(0) - U(0)) - 1}{\phi(P(0) - U(0))} = -W(0)$$

Also following Proposition 4, I conclude that $\frac{d\,P(0) - U(0)}{d\,U(0)} < 0$. Assumption 1 and $\delta = 0$ imply that $U(0) < 0$ and $W(0) < 0$. Together, these facts imply that $P(0) - U(0) < 0$.

From (7),

$$v(0) = (1 - \delta)^{-1}(r(0) + \theta(0)),$$

where

$$r(0) = \frac{\left(1 - \Phi(P(0) - U(0))\right)^2}{\phi(P(0) - U(0))}.$$

Assumption 2 implies that $r(0)$, and thus $v(0)$, is increasing in $P(0) - U(0)$. Finally, the result follows by continuity. ■

Proof of Lemma 1: As shown in the proof of Proposition 1, the equilibrium price differences $P(i)$ are given by the solution to

$$P(i) + \frac{2\,\Phi(P(i) - U(i)) - 1}{\phi(P(i) - U(i))} = -W(i)$$

Since $U(i)$ is only a function of various values of u and $W(i)$ is only a function of various values of v, it follows that, given $\{u(i), v(i)\}$, the above equation has only one unknown, $P(i)$. The left-hand side is an increasing function of $P(i)$, ranging from $-\infty$ to $+\infty$. The intermediate value theorem implies that there exists a unique equilibrium value $P(i)$. Given $P(i)$, the values of $p(i)$ are uniquely determined by (6).

The reverse is straightforward. In fact, for given $p(i)$ and $q(i)$, (4) defines a linear system in $u(i)$; and (7) defines a linear (recursive, in fact) system in $v(i)$. ■

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