Optimal Auction Design under Non-Commitment

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December 2007

Abstract

This paper characterizes revenue maximizing auctions for a finite horizon version of the standard IVP model of Myerson (1981) for a seller who cannot commit not to propose a new mechanism, if previously chosen ones fail to allocate the object. We show that a revenue maximizing mechanism in the first period assigns the good to the buyer with the highest virtual valuation, provided that it is above a buyer-specific reserve price. If no buyer obtains the good in the first period, the same procedure is repeated in the second period, where virtual valuations are calculated using the posterior distributions and the reserves prices are lower, and so forth, until we reach the last period of the game. This is the first paper that characterizes optimal mechanisms in a multi-agent environment where the designer behaves sequentially rationally. The characterization procedure can be applicable to other multi-agent mechanism design problems "with limited commitment." *Keywords: mechanism design, optimal auctions, limited commitment. JEL Classification Codes: C72, D44, D82.*

1. Introduction

The classical works on optimal auctions (see Myerson (1981) and Riley and Samuelson (1981)) characterize the revenue-maximizing allocation mechanism for a risk-neutral seller

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[†] I am grateful to Masaki Aoyagi and Philip Reny for all their help and support. I had inspiring discussions with Andreas Blume, Roberto Burguet, Ignacio Esponda, Patrick Kehoe, Ellen McGrattan, Ennio Stacchetti, Balázs Szentes and Charles Zheng. Many thanks to the seminar participants at the University of Chicago, Northwestern University,, University of Pennsylvania,Yale University and University of Western Ontario for numerous suggestions and for very illuminating questions. I would also like to thank the Department of Management and Strategy, KGSM, Northwestern University and the Institute of Economic Analysis, UAB, for their warm hospitality; the Andrew Mellon Fellowship, the Faculty of Arts and Sciences at the University of Pittsburgh and the TMR Network Contract ERBFMRXCT980203 for financial support. Financial support from the National Science Foundation, Award # 0451365 is gratefully acknowledged.

who owns one object and faces a fixed number of buyers whose valuations are private information. An important assumption in these papers is that the seller can commit to withdraw the item from the market in the event that it is not sold. In other words, the seller is free to employ any mechanism to sell the object, but she should respect its outcome forever. This commitment assumption is far-fetched and often not met in reality. Auction houses very seldom remove from the market items that remain unsold. For instance, Christies in Chicago auctions the same bottles of wine that failed to sell in earlier auctions. The US government re-auctions properties that fail to sell: lumber tracts, oil tracts and real estate are put up for a new auction if no bidder bids above the reserve price.[1] As Porter (1995) reports, 46.8 percent of the oil and gas tracts with rejected high bids were put up for a new auction. The mean time elapsed between the first and the second auction is 2.7 years.

The inability of a seller to commit to a given institution in the event that it fails to realize all gains of trade, has been studied extensively in the durable good monopolist literature, (Bulow (1982), Gul-Sonnenschein and Wilson (1986), Stokey (1981)), and by a more recent paper in an auction set-up by McAfee and Vincent (1997). A crucial assumption in these papers is that the seller's action in each stage is restricted to be out of a *specific class*. The seller chooses *prices* in the durable goods case, and *reservation prices* in McAfee and Vincent (1997).[2] Skreta (2006b) examined the case where the seller faces one buyer whose valuation is private information in a finite horizon model (essentially the durable good case) and showed that in fact posted prices are revenue maximizing among all possible conceivable mechanisms. It is interesting and relevant to characterize what is the revenue maximizing procedure when the seller faces *many* buyers. This is what the present paper does.

We consider the following problem. There is a risk neutral seller who owns a single object and faces I risk neutral buyers, whose valuation is private information. Valuations are private, independently distributed across buyers, and they remain constant overtime. The buyers and the seller interact for a maximum of $T < \infty$ periods, and discount the future with the same discount factor. At the beginning of each period the seller proposes a mechanism to sell the object. If the object is sold, the game ends, otherwise, the seller returns the next period and offers a new mechanism. The game ends after T periods even if the object remains unsold. We show that the seller will maximize expected discounted revenue by running at $t = 1$ a 'Myerson' auction with buyer-specific cutoffs. A 'Myerson' auction assigns the object to the buyer with the highest virtual valuation if it is above a cut-off. A buyer can either claim a type above his/her cut-off or wait until next period. If no bidder claims a value above his/her cut-off, no trade takes place in the first period and the seller runs a 'Myerson' auction in the second period, and so forth, until we reach the final period. When buyers are ex-ante symmetric this procedure is equivalent to running

[1] These examples are also mentioned in McAfee and Vincent (1997).

[2] Other papers that study reserve price dymanics without commitment are Burguet and Sakovics (1996) who examine cases of costly bidding and Caillaud and Mezzetti (2004) who look at sequential auctions of many identical units.

either a sequence of second- or first price auctions with optimally chosen reservation prices in each period. The reservation prices depend on the discount factor and on the distribution of valuations and they are decreasing overtime.

When the designer is free to choose new rules or change the pre-existing ones in each period we talk about mechanism design with no- or with limited commitment. This paper is the first work that characterizes optimal mechanisms under non-commitment in a multi-agent asymmetric information environment. Mechanism design under non-commitment is notoriously difficult even in single agent environments. Also, as we discuss below, in mechanism design without commitment going from a single- to a multi-agent environment introduces new challenges and conceptual difficulties. This is in contract to the standard mechanism case with commitment, where the single- and multi-agent cases are similar. We first discuss the challenges that are present both in single- and multi-agent environments and then we discuss the new issues that arise explicitly because of the presence of multiple agents.

Mechanism design with no- or limited commitment is quite difficult, even in single agent environments, because, as it was first observed by the literature on the ratchet effect (Freixas, X., R. Guesnerie, and J. Tirole (1985) and Laffont and Tirole (1988)) one cannot use the revelation principle. One of the main assumptions of the revelation principle is that the choice of mechanism(s) is done once and for all. This assumption implies that the designer can never change the rules in the future, even though it might be then obvious that there exist better ones. When this commitment assumption fails, there is no generally applicable canonical class of mechanisms. A very important step towards the direction of providing a "canonical" class of mechanisms for environments of limited commitment is the paper by Bester and Strausz (2001). That paper establishes that for *single agent* and *finite type* models it is without loss of generality to restrict attention to mechanisms with message spaces that have the same cardinality as the type space and where the agent reports his true type with strictly positive probability. However, as Bester and Strausz (2000) show this result *fails* with *multiple* agents.

Another set of complications that is both present in single- and multi agent environments arise because we cannot impose any assumptions on the distributions of types. This is because in all but the first period of the game, distributions arise endogenously by updating the prior given the information released up to that stage. Then, one does not have the luxury of imposing assumptions on the distributions without loosing generality. Our analysis does not rely on the existence of densities (we allow for discrete and/or mixed distributions) nor on other convenient properties such as the monotone hazard rate.[3]

The most important conceptual challenges of multi-agent environments are two. The first is related to the fact that what buyers observe at each stage, that is the *transparency*

[3] Skreta (2007b) characterizes revenue maximizing auctions *with commitment* allowing for general distributions.

of the mechanisms, critically affects their beliefs about each other. For example, competing in a sealed-did auction versus an open outcry auction has different implications about what buyers learn about each other, which affects their future behavior. The issue of transparency is not captured by the classical definition of mechanisms, which are defined as game forms, because two institutions that are very different in terms of transparency can have the same game form representation. In order to address this new issue we propose here an alternative definition of mechanisms that captures also the degree of transparency of mechanisms. The second difference is that the mechanism designer may become privately informed overtime. This is because it is possible that the seller (mechanism designer) observes more than what the other buyers (agents) observe about the behavior of their competitors; think for instance sealed bid auctions. Then the seller becomes endogenously an *informed principal*, since she possesses information that is not available to the agents.[4] These two issues do not arise when there is a single buyer since trivially the buyer does not need to worry about the private information of his competitors - there are none . To summarize, the multi-buyer problem is different from the single buyer one, because how transparent a mechanism is matters, and because the seller may become endogenously privately informed overtime.[5]

The aforementioned phenomena are exclusive to the multi-agent problem and compound the difficulties arising from the lack of an appropriate "revelation principle." Fortunately, the solution approach developed in Skreta (2006b) can be adjusted and enriched to address these new issues. That method does not rely on restricting attention to some "canonical" class of mechanisms, rather it relies on characterizing equilibrium outcomes. Outcomes is all that matters for payoffs and in mechanism design we care primarily for payoffs.

The ideas and techniques developed in the present paper have a large set of potential applications. One area where the designer (in that case the buyer) is choosing mechanisms sequentially is the area of government procurement and in particular defense procurement. There, there are typically multiple stages until the final winner is determined, and in each of these stages sellers submit bids, and based on the bids a subset of them advances to the next stage. Even though this problem is admittedly quite different from one analyzed in this paper, the designer and bidders face similar issues: namely if a bidder signals too much about his private information early on, his rents may be reduced at a later stage. Also, how bidders compete at each stage may depend on what information they obtain about their competitors in earlier stages, so the issue of transparency arises here too.

Another area of possible applications is related to ability of sellers to track the interactions with various buyers. Nowadays, keeping track of buyers has become easy and

[4] For a brief account of the literature on informed principal problems see Skreta (2007).

[5] Even without these two conceptual challenges the analysis of our multi-agents problem is still quite more involved because at each point the seller has to choose an optimal mechanism as a function of her posterior beliefs, anticipating that this choice itself affects the future set of possible posterior beliefs. When there are many buyers optimal mechanisms as functions of posterior beliefs are significantly more complicated objects than what they are in a the single-agent version of the problem.

inexpensive, and the old theories that had firms or sellers treat buyers as being anonymous are enriched with ones where the sellers do not only keep track of the buyers that they are dealing, but they also design their pricing schemes (coupons, loyalty programs) based on the information that they have obtained thus far. In such problems, the ratcheting and transparency issues present in our analysis are central.[6] Our techniques could be applicable in situations where other issues relating to transparency, for example privacy, are important.[7],[8]

Coming to some more related work, no sale is not the only form of inefficiency of the classical optimal auction. Sometimes is allocates the object to a buyer other than the one with the highest valuation, thus leaving open resale opportunities for the new owner. Zheng (2002) studies optimal auctions allowing for resale. With an impressive construction, that paper derives conditions under which the optimal allocation derived in Myerson (1981) can be attained also by a seller who cannot prevent resale. In Zheng (2002) there is no discounting. Here we look at a complementary problem and we allow for discounting. We characterize what mechanisms maximize revenue for a seller who cannot prevent *herself* from re-auctioning the good.

Other works study sellers that have even less commitment than the seller in this paper, in the sense that she does not even commit to carry out the rules of the current auction. McAdams and Schwarz (2006) consider a symmetric IPV environment without discounting, where the seller after observing the bids cannot commit not to ask for other rounds of bids. Only the seller incurs a cost of asking for a new round of offers, and the offers are only made by the buyers. If the cost of negotiating is very low or prohibitively high, the seller does not suffer from her inability to commit, and her revenue is the one corresponding to an efficient mechanism. Otherwise, there are multiple rounds of offers because buyers initially hesitate to make serious offers and this reduces seller's revenue significantly. The current paper in contrast, assumes that the seller *commits* at each stage to carry out the mechanism for that stage. Moreover the seller is choosing *revenue maximizing* auctions. Vartiainen (2007) examines again a symmetric model, where buyers' types are finite and there is no discounting. He allows both forms of no commitment (no commitment to current nor future mechanisms) and asks what mechanism leads to a sustainable outcome given complete lack of commitment. He shows that essentially only the English auction achieves sustainability. Again, this paper is different from Vartiainen (2007) because the interest is in designing

[6] See for instance Acquisti and Varian (2001) and Fudenberg and Villas-Boas (2005) for an excellent and comprehensive survey of the work in this area.

[7] The pioneering papers on the economics of privacy are Hirshleifer (1980), Posner (1981) and Stigler (1980). For a recent survey see Hui and Png (2006). For a paper contributing to the policy debate on privacy issues see Varian (1996).

[8] Some recent work on mechanism design that addresses these issues marries mechanism design theory with cryptography. Two recent contributions are Izmalkov, Lepinski and Micali (2005) and Izmalkov, Lepinski, Micali and Shelat (2007).

optimal mechanisms and there is discounting.[9] We also allow for general uncertainty and for non-transparent mechanisms. Vartiainen (2007) allows only for public mechanisms.

The paper is structured as follows. The environment under consideration is described in Section 2. Section 3 outlines our method for characterizing the optimal mechanism under non-commitment. The main analysis and results of this work can be found in Section 4, which examines in great detail the case where the game lasts for two periods. Section 5 discusses the case where $T > 2$. Concluding remarks are in Section 6.

2. The Environment

A risk neutral seller, indexed by zero, owns a unit of an indivisible object, and faces I risk neutral buyers. We use the female pronoun for the seller and male pronouns for the buyers. The set of all players, (buyers and the seller) is denoted by $\bar{I}$. The seller's valuation is denoted by $v_0 \equiv 0$ and is common knowledge, whereas that of buyer i, denoted by v_i, is *private information* and it remains *constant overtime*. It is distributed on V_i according to F_i. The convex hull of V_i is $[a_i, b_i]$ with $-\infty < a_i \leq b_i < \infty$. Buyers' valuations are private and are independently distributed across buyers. We use $F(v) = \times_{i \in I} F_i(v_i)$, where $v \in V = \times_{i \in I} V_i$ and $F_{-i}(v_{-i}) = \times_{\substack{j \in I \\ j \neq i}} F_j(v_j)$. Time is discrete and the game lasts T periods, $t = 1, 2, ..., T$. The buyers and the seller discount the future with the same discount factor $\delta \in [0, 1]$. All elements of the game except the realization of the buyers' valuations are common knowledge. The seller's goal is to maximize expected discounted revenue. The buyers aim to maximize expected surplus.

We now describe the timing of the game.

Timing

- At the beginning of period $t = 1$ nature determines the valuations of the buyers. Subsequently, the seller proposes a mechanism. The mechanism is played, and if one of the buyers obtains the object the game ends, else we move on to period $t = 2$.
- At $t = 2$ the seller proposes a mechanism. The mechanism is played and if one of the buyers obtains the object the game ends, else we move on to period $t = 3$.

....

- At $t = T$ the seller proposes a mechanism. The mechanism is played and the game ends at the end of period T, irrespective of whether trade takes place or not.

[9] In contrast, to Vartiainen's (2007) problem, our problem without discounting is trivial: in that case the seller would wait until the last period of the game and offer the mechanism described in Myerson (1981).

We will use the variable χ^t to keep track of whether trade takes place at t or not; in particular $\chi^t = \begin{cases} 1 \text{ if trade takes place at } t \\ 0 \text{ otherwise} \end{cases}$.

We now define mechanisms.

Mechanisms

Usually a mechanism is defined as a *game form.*

A *game form* $G^t = (S^t, g^t)$ consists of a set of actions for each player $S^t = S_0^t \times S_1^t \times S_2^t \times \ldots \times S_I^t$ and an outcome mapping $g^t : S^t \to [0,1]^{\bar{I}} \times \mathbb{R}^{\bar{I}}$, which determines the probability of obtaining the good, and an expected payment for each player. That is,

$$g^t(s^t) = (q_0^t(s^t), q_1^t(s^t), \ldots, q_I^t(s^t), z_0^t(s^t), z_1^t(s^t), \ldots, z_I^t(s^t)), \tag{1}$$

with the properties that

$$\begin{aligned} \sum_{i=0}^{\bar{I}} q_i^t(s^t) &= 1,\ q_i^t(s^t) \geq 0; \\ z_i^t(s^t) &\in \mathbb{R} \text{ for } i \in I \text{ and } z_0^t(s^t) = -\Sigma_{i \in I} z_i^t(s^t). \end{aligned} \tag{2}$$

The equality $z_0^t(s^t) = -\Sigma_{i \in I} z_i^t(s^t)$ holds because the expected payments incurred by the buyers are received by the seller. Notice also that by definition we have that

$$q_0^t(s^t) \equiv 1 - \Sigma_{i \in I} q_i^t(s^t),$$

which is the probability that the seller keeps the good, or put differently, the probability that no trade occurs at period t, when the vector of actions chosen is s^t.

A buyer can always choose not to participate, in which case he does not get the object and he does not pay anything. We include the choice of non-participation in the definition of each game form by assuming that it contains an action for each buyer, call it s_i^{NP}, such that when i chooses it he does not get the object and he does not pay anything irrespective of what the other buyers do.

Note that in our definition of game forms we also allow the seller to choose actions. We do so, because, as we explain in what follows, the seller may obtain private information as the game progresses and in those cases the seller's "reports" or actions matter.

When the choice of the mechanism(s) is done once and for all, as in a one-shot scenario, or a multi-period environment with commitment, defining a mechanism as a game form is the most general definition, since the object of interest is to map *preferences to outcomes.* However, in general an institution or a set of rules, can also affect what *information* players obtain during play, and this can be crucial in a dynamic setup. This is especially true when, as in the environment under consideration, more than one party has private information. In these set-ups modelling a mechanism simply as a game form does not capture the fact that two different institutions may lead to the same current outcome, but provide their participants with different information. For instance, a second price auction and an English auction allocate the good in the same way (when agents use their weakly dominant actions to bid their value), but at the end of the second price auction a buyer will have not observed his opponents bids, whereas at the end of the English auction he will have observed everyone's drop-out prices. What buyers' observe determines what they know and think about their competitors which may affect their future behavior, and hence the set of future equilibrium outcomes. From these considerations it is clear that the complete description of an institution in a dynamic setup should consist (i) of the economic outcomes that result from the interaction of agents, that is, the allocations, and (ii) of the information that its participants obtain during play. We model this second feature of institutions by endowing the game form with an *information disclosure policy.*

An *information disclosure policy* is a mapping from the vector of actions chosen, to a vector of messages, one for each buyer, or $D^t : S^t \to \Delta(\Lambda^t)$ where $\Lambda^t := \times_{i \in I} \Lambda_i^t$ and Λ_i^t is the set of messages that the seller can send to buyer i at date t.

Each buyer i observes only the message disclosed to him, that is λ_i^t. However, buyers know the rule D^t.

This modelling of information revelations is very general. It encompasses, private and public disclosure, no disclosure, full disclosure and partial disclosures. A *fully revealing* information disclosure policy is one where $\lambda_i^t = s^t$, for all i and for all $s^t \in S^t$, think of open outcry auctions. An information disclosure policy that reveals *no information* is one where $\lambda_i = s_i$, for all i; think for instance sealed bid auctions. A disclosure policy that reveals some *partial information* is one where, for instance, $\lambda_i^t = \begin{cases} s_{-i}^t \text{ with probability } 0.5 \\ \hat{s}_{-i}^t \text{ with probability } 0.5 \end{cases}$.

We are now ready to define mechanisms in our setup.

Definition 1 *A* ***mechanism****, M^t, consists of a game form G^t and an information disclosure policy D^t.*

The set of all possible mechanisms is denoted by $\mathcal{M}$.

We now proceed to specify what the seller and the buyers observe during play. We assume that *all* players observe the mechanism proposed by the seller as well as whether trade takes place or not.[10] Hence a *public history* at t contains all the mechanisms proposed up to t, denoted by $M^{(t)} = \{M^1, ..., M^t\}$, and whether trade has taken place up to t, $\chi^{(t)} = \{\chi^1, ..., \chi^t\}$. The *seller* observes (i) the *actions* chosen by the buyers at each stage, that is he knows the vector $(s_0^t, s_1^t, s_2^t, ..., s_I^t)$, as well as the message that each buyer receives by the information disclosure policy, that is $\lambda^t = (\lambda_1^t, \lambda_2^t, ..., \lambda_I^t)$. An information set of the seller at the beginning of period t, ι_0^t, is of the form $\iota_0^t = \{M^{(t-1)}, \chi^{(t-1)}, s^{(t-1)}, \lambda^{(t-1)}\}$,

[10] The information about whether trade has taken place or not, can be inferred by the buyers from the seller's behavior, since the seller continues to propose mechanisms only if no trade has taken place

where $s^{(t-1)} = \{s^1, ..., s^{t-1}\}$ and $\lambda^{(t-1)} = \{\lambda^1, ..., \lambda^{t-1}\}$. The set of all information sets of the seller is denoted by I_0. Buyer i observes at each stage the action that *he* chooses, as well as the message that *he* receives from the information disclosure policy. Hence, we have that an information set of buyer i at the beginning of period t, after the seller proposed M^t, is given by $\iota_i^t = \{v, M^{(t)}, \chi^{(t-1)}, s_i^{(t-1)}, \lambda_i^{(t-1)}\}$, where $s_i^{(t-1)} = \{s_i^1, ..., s_i^{t-1}\}$ and $\lambda_i^{(t-1)} = \{\lambda_i^1, ..., \lambda_i^{t-1}\}$. The set of all information sets of buyer i is denoted by I_i.

Assessments: Beliefs and Strategies

What players observe determines their *beliefs* about the parameters that are not commonly known. What the seller observes is particularly crucial because it determines, is some sense, her *commitment power.*[11] The beliefs of the buyers and the seller comprise the games' *belief system* which we denote by μ. A *strategy profile* $\sigma = \{\sigma_i\}_{i=0,1,...,I}$ specifies a strategy for each player. A strategy for the seller, σ_0, is a mapping from I_0 to the set of mechanisms $\mathcal{M}$.[12] A *behavioral* strategy of the buyer i, σ_i, consists of a mapping from his information sets I_i to a probability distribution over actions. An *assessment* (σ, μ) denotes the pair of strategies and beliefs. We will require (σ, μ) to form a *Perfect Bayesian Equilibrium* of the game, (PBE).

This completes our description of the game and of our solution concept. We move on to describe our solution procedure.

3. The Procedure

Our objective is to find an assessment that is a PBE, and guarantees highest expected revenue for the seller among all $PBE's$. Finding, which among all $PBE's$, is the one that maximizes the seller's expected revenue is quite complicated given the complexity of the strategy spaces of the game under consideration. As we mentioned in the introduction, and is well understood by now,[13] when the mechanism designer behaves sequentially rationally, one cannot apply the standard revelation principle. Moreover, the extended revelation principle for environments with limited commitment by Bester and Strausz (2001), is not applicable in multi-agent environments, see Bester and Strausz (2000).

Without the help of a "revelation principle," any a priori restrictions on the class of mechanisms that the seller is allowed to employ and on the buyers' behavior may be with loss of generality. For this reason we consider mechanisms with arbitrary message spaces. But then how can one write down the seller's optimization problem, when she can employ arbitrary mechanisms? The idea was developed in Skreta (2006b) and it amounts to ex-

[11] If the seller does not observe anything, then all $BNE's$ are $PBE's$ and the "commitment solution" is sequentially rational.

[12] To be more precise, we have to account also for the fact that the seller herself maybe submitting reports in the mechanism, but soon in our analysis, we will conclude that it is without any loss to have the seller just choose mechanisms, where only the buyers make reports.

[13] See Laffont and Tirole (1988), Salanie (1997), or for a more recent treatment Skreta (2006b).

amining equilibrium *outcomes* rather than strategies. To be more precise, we search for a social choice function that maximizes expected revenue among all social choice functions that are implemented by assessments that are $PBE's$. We now define what we mean by an *outcome of an assessment*; a *social choice function* and *implementation*.

Outcomes from Assessments

The outcome of assessment (σ, μ), (not necessarily an equilibrium), is an *allocation rule* $p(\sigma, \mu)$ and a *payment rule* $x(\sigma, \mu)$.[14] The rule $p_i(\sigma, \mu)(v)$, $i \in \bar{I}$ is the expected, discounted probability that player i will obtain the object, and $x_i(\sigma, \mu)(v)$, $i \in \bar{I}$ is the expected, discounted payment that player i will incur given the assessment (σ, μ) when the realized vector of buyers' valuations is v. These expectations are taken from the *ex-ante* point of view. Then, buyer $i's$ expected discounted payoff given (σ, μ) is

$$U_i(p(\sigma, \mu), x(\sigma, \mu), v_i) = P_i(\sigma, \mu)(v_i)v_i - X_i(\sigma, \mu)(v_i),$$

where $P_i(\sigma, \mu)(v_i) = \int_{V_{-i}} p_i(\sigma, \mu)(v_i, v_{-i})dF_{-i}(v_{-i})$ and $X_i(\sigma, \mu)(v_i) = \int_{V_{-i}} x_i(\sigma, \mu)(v_i, v_{-i})dF_{-i}(v_{-i})$.

Since $x_0(\sigma, \mu)(v) = -\Sigma_{i \in I} x_i(\sigma, \mu)(v)$, the seller's expected payoff, given (σ, μ), is

$$\int_V \Sigma_{i \in I} x_i(\sigma, \mu)(v)dF(v).$$

Note that it is possible that different strategy profiles lead to the same allocation and payment rules.

Social Choice Functions

A social choice function specifies for each vector of valuations v and each period t, a vector of probabilities $q^t(v) = (q_0^t(v), q_1^t(v),...., q_I^t(v))$, with $q_i^t(v) \geq 0$ and $\Sigma_{i \in \bar{I}} q_i^t(v) = 1$, and a vector of expected payments $z^t(v) = (z_0^t(v), z_1^t(v),...., z_I^t(v))$, with $z_i^t(v) \in \mathbb{R}$. Given a social choice function $\{q_i^t(v), z_i^t(v)\}_{t \in T, i \in \bar{I}}$ we can define for all $i \in \bar{I}$

$$\begin{aligned} p_i(v) &= q_i^1(v) + q_0^1(v)\delta \left[q_i^2(v) + q_0^2(v)\delta [.....] \right] \text{ and} \\ x_i(v) &= z_i^1(v) + q_0^1(v)\delta \left[z_i^2(v) + q_0^2(v)\delta [.....] \right]. \end{aligned}$$

There are many different social choice functions that lead to the same $p_i(v)$ and $x_i(v)$, and hence to the same ex-ante payoffs for the buyers and for the seller. All such social choice functions are equivalent for our purposes and hence when we talk about a social choice function we will simply mean its "reduced version" given by p and x. Now that we have specified what we mean by a social choice function we can talk about implementation.

[14] We need to include the belief system in the arguments of p and x because it is part of our equilibrium concept.

Implementation

An assessment (σ, μ) implements the (reduced) social choice function p and x if for all $v \in V$ and $i \in \bar{I}$ we have that $p_i(\sigma, \mu)(v) = p_i(v)$ and $x_i(\sigma, \mu)(v) = x_i(v)$.

The set of implementable social choice functions depends on the solution concept. A solution concept imposes restrictions on (σ, μ), which in turn, translate to restrictions on p and x. Our objective is to identify a Perfect Bayesian Equilibrium, PBE, that implements p^* and x^*, that maximize expected discounted revenue among all allocation and payment rules implemented by a PBE of the game. To simplify the notation we will often omit (σ, μ) from the arguments of p_i, x_i, P_i, X_i.

Analogously, but with more involved notation, we can define allocation and payment rules implemented by continuation strategy profiles.[15]

With the help of these definitions we can express various restrictions dictated from our solution concept into properties of allocation and payment rules. This allows us to formulate a maximization problem without the need to restrict attention to some canonical class of mechanisms. We proceed as follows. First we solve the problem when the game lasts two periods. Most of the difficulties of the problem are already present when the game lasts merely two periods, so examining the $T = 2$ case in detail allows one to see the issues that arise in the simplest possible setup. The $T = 2$ result is then used as a basis for the characterization of revenue maximizing mechanisms when $T = 3$. Finally, by induction, and by following an analogous procedure we argue how one can show that the characterization extends for any $T < \infty$.

4. Analysis of the Problem when $T = 2$

In this section analyze the case where the game lasts two periods. The analysis of the $T = 2$ case proceeds as follows. We first formulate the seller's problem for finding the revenue maximizing PBE in terms of a maximization problem where the seller chooses allocation and payment rules. Then we establish that the transparency of first period mechanisms is irrelevant, in the sense that it does not affect the optimal mechanisms at $t = 2$. This observation allows us to model first period mechanisms simply as game forms and to simplify greatly the problem we formulated initially. Then we move on to solve the problem ignoring the sequential rationality constraints. As we explain, this problem is non-standard because we allow for general distributions. In the last and longest part of analysis we solve the problem taking explicitly into account the sequential rationality constraints.

[15] For more details see the longer working paper version, available at the author's website.

4.1 Formulating the Seller's Problem

In this section we first formulate the seller's problem for finding the revenue maximizing PBE in terms of a maximization problem where the seller chooses allocation and payment rules.

The seller seeks an allocation and a payment rule that maximize expected discounted revenue among all allocation and payment rules implemented by a PBE of the game. In other words, the seller seeks:

$$\max_{p,x} \int_V \Sigma_{i\in I} x_i(v) dF(v) \tag{3}$$

subject to p, x being PBE implementable.

Our first goal is to translate the requirement that p and x be PBE implementable into properties of p and x. In order to do so, we express p and x in terms of first and second period mechanisms.

Fix an assessment (σ, μ) and suppose that the seller at $t = 1$ employs a mechanism M that consists of a game form G and an information disclosure policy D. Recall that the game form consists of a set of vectors of actions S and a mapping g from S to outcomes that satisfies (1) and (2). Essentially then a game form can be described by the menus $\{q_i(s), z_i(s); i \in \bar{I}\}_{s\in S}$. Now, the probability that buyer i is choosing action s_i when his type is v_i is denoted by $m_i^{s_i}(v_i) \in [0, 1]$ for all v_i. We assume that $m_i^{s_i}$ is a measurable mapping of v_i. The set $V_i(s_i)$ contains all $v_i's$ for which $m^{s_i}(v_i) > 0$. Its convex hull is denoted by $\bar{V}_i(s_i) \equiv [\underline{v}_i(s_i), \bar{v}_i(s_i)]$.[16] Also define $\bar{V}_{-i}(s_{-i}) = \times_{\substack{j\in I \\ j\neq i}} \bar{V}_i(s_i)$ and $\bar{V}(s) = \times_{i\in I} \bar{V}_i(s_i)$. Let

$$m^s(v) = \times_{i\in I} m^{s_i}(v_i) \tag{4}$$

denote the probability that the vector of actions chosen at $t = 1$ is s when the buyers' valuations are v.

Recall that the *disclosure policy* is a mapping from S messages, to Λ. We use $d(\lambda\,|s)$ to denote the probability that the vector of messages received by the buyers is λ, when the vector of actions chosen is s. Let $\Lambda_i(s_i)$ denote the set of messages that buyer i may be receiving given the disclosure policy D when he has chosen s_i at $t = 1$. In other words, a message λ_i is in $\Lambda_i(s_i)$ if $d(\lambda_i, \tilde{\lambda}_{-i}\,|s_i, s_{-i}) > 0$, for some $s_{-i} \in S_{-i}$ and $\lambda_{-i} \in \Lambda_{-i}$.

Since the second period is the final period of the game, the seller's problem at each continuation game at $t = 2$ is like a static auction problem hence we can use the *revelation principle*. However this problem is quite more complicated, than the classical optimal auction problem. The main differences are two. The first one, is that the seller has private information as well, since she knows s, λ, whereas each buyer knows only s_i, λ_i. The second difference, is that buyer $i's$ type now consists of a payoff-relevant part v_i, and a belief-

[16] It is very well possible that $m_i^{s_i}(v_i) = 0$ for some v_i in $[\underline{v}_i(s_i), \bar{v}_i(s_i)]$.

relevant part s_i, λ_i.[17] The set of types for the seller, denoted by Θ_0, consists of vectors $\theta_0 = (s, \lambda)$, where $d(\lambda\,|s) > 0$ and $s \in S$, and the one for buyer i, is denoted by Θ_i, and it consists of triplets (v_i, s_i, λ_i), where $v_i \in V_i(s_i)$ and $\lambda_i \in \Lambda_i(s_i)$ and $s_i \in S_i$. A vector of types θ is

$$\theta = (\theta_i; \theta_{-i}; \theta_0) = \left(\underbrace{v_i, s_i, \lambda_i}_{\text{buyer } i\text{'s}} \; ; \; \underbrace{v_{-i}, s_{-i}, \lambda_{-i}}_{\text{buyers } -i} \; ; \; \underbrace{s, \lambda}_{\text{seller}} \right),$$

and the set of all vector of types is denoted by Θ. In what follows θ will be abbreviated to v, s, λ.

By *the revelation* and *the inscrutability principles*, (Myerson (1979), (1981) and (1983)), we can without loss of generality restrict attention to mechanisms where both the seller and the buyers report their types and they do so truthfully.

A *direct revelation mechanism, (DRM)*, $M = (p^2, x^2)$ consists of an *assignment rule* $p^2 : \Theta \longrightarrow \Delta(\bar{I})$ and a *payment rule* $x^2 : \Theta \longrightarrow \mathbb{R}^{\bar{I}}$. The assignment rule specifies a probability distribution over the set of buyers given a vector of reports. We denote by $p_i^2(\theta)$ the probability that i obtains the good when the vector of reports is θ. Similarly, $x_i^2(\theta)$ denotes the expected payment incurred by i, given θ.

After these definitions we are ready to obtain more precise expressions of the allocation and the payment rules.

Consider an assessment of the two-period version of our game. Then, the allocation and payment rule implemented by it have the following form:

$$\begin{aligned} p_i(v) &\equiv \Sigma_{s\in S} m^s(v) \left[q_i(s) + q_0(s)\delta\Sigma_{\lambda\in\Lambda} d(\lambda\,|s) p_i^2(v, s, \lambda) \right] \\ x_i(v) &\equiv \Sigma_{s\in S} m^s(v) \left[z_i(s) + q_0(s)\delta\Sigma_{\lambda\in\Lambda} d(\lambda\,|s) x_i^2(v, s, \lambda) \right], \end{aligned} \tag{5}$$

where $m^s(v)$ is defined in (4). We also let P_i, X_i (respectively P_i^2 and X_i^2) denote the expectations of p_i and x_i (respectively p_i^2 and x_i^2) from $i's$ perspective, that is

$$\begin{aligned} P_i(v_i) &= \int_{V_{-i}} p_i(v) dF_{-i}(v_{-i}),\ X_i(v_i) = \int_{V_{-i}} x_i(v) dF_{-i}(v_{-i}) \\ P_i^2(v_i, s_i, \lambda_i) &= E_{v_{-i}, s_{-i}, \lambda_{-i}} \left[p_i^2(v, s, \lambda)\,|v_i, s_i, \lambda_i \right],\ X_i^2(v_i, s_i, \lambda_i) = E_{v_{-i}, s_{-i}, \lambda_{-i}} \left[x_i^2(v, s, \lambda)\,|v_i, s_i, \lambda_i \right]. \end{aligned} \tag{6}$$

With the help of (5) and (6) we now translate the requirement that an assessment be a PBE, into properties of the allocation and the payment rule it implements.

First of all, the allocation rule p, as well the allocation rules implemented by continuation equilibria p^2, have to satisfy *resource constraints,* (**RES**). Second, at a PBE buyer i's and the seller's strategy must be best responses at each information set. For buyer i, this

[17] Each buyer's beliefs about another buyer are private information, because they depend on the information that the buyer has received from the disclosure policy which is not publicly available. Since the disclosure policy affect buyers' beliefs. it may create correlation in types, (types consist by valuation plus beliefs because s_i and λ_i are private information), even though the buyers' initial information was statistically independent.

implies at the very least, that there is no type of i that can strictly benefit by behaving as another type of i does at each information set. We call these the *incentive constraints* for i. The overall constraints are denoted as $\mathbf{IC}_i$ and the ones at the information set s_i, λ_i[18] are denoted by $\mathbf{IC}_i(s_i, \lambda_i)$. Also, from the fact that buyer i can always choose not to participate in a mechanism, we get that buyer i's expected payoff at each information set must be non-negative. We call these *participation constraints* and we denote them by $\mathbf{PC}_i$ and $\mathbf{PC}_i(s_i, \lambda_i)$ respectively. For the seller now, the requirement that her strategy is a best response in each information set, implies that at each information set, her strategy specifies an optimal sequence of mechanisms employed in the remainder of the game. Then given a first period mechanism, as each information set of the seller (s, λ), the mechanism p^2 and x^2 maximizes the seller's revenue among all feasible mechanisms given her posterior beliefs at that information set. We call these *the sequential rationality constraints* and denote them by $\mathbf{SRC}(s, \lambda)$. Finally the seller's beliefs can be derived from the buyers' strategies with the help of Bayes' rule as follows. If the seller observed that buyer i chose action s_i and $i's$ strategy is such that $\int_{V_i(s_i)} m_i^{s_i}(t_i) dF_i(t_i) > 0$, then the *seller's* posterior beliefs about v_i conditional on s_i, $i \in I$, are given by[19]

$$f_i(v_i \,|s_i\,) = \begin{cases} \frac{m_i^{s_i}(t_i) f_i(t_i)}{\int_{V_i(s_i)} m_i^{s_i}(t_i) dF_i(t_i)}, \text{ for } v_i \in V_i(s_i) \\ 0 \text{ otherwise} \end{cases}. \tag{7}$$

Because buyers behave non-cooperatively, they choose their actions at $t = 1$ independently from one another. Then, upon observing the vector of actions s, with $\int_{\bar{V}(s)} m^s(t) f(t) dt > 0$, the seller's posteriors beliefs about the buyers' valuations are given by

$$f(v\,|s\,) \equiv \begin{cases} \frac{m^s(v) f(v)}{\int_{\bar{V}(s)} m^s(t) f(t) dt} = \times_{i \in I} f_i(v_i \,|s\,), \text{ for } v \in \bar{V}(s) \\ 0 \text{ otherwise} \end{cases}. \tag{8}$$

Putting the pieces together, in order for p and x, defined in (5), to be PBE implementable, they must at the very least, satisfy the constraints of the following problem:

$$\max_{p,x} \int_V \Sigma_{i \in I} x_i(v) dF(v)$$

Subject to:

$\mathbf{IC}_i$ "incentive constraints,"

$P_i(v_i)v_i - X_i(v_i) \geq P_i(v_i')v_i - X_i(v_i')$, for all $i \in I$, $v_i, v_i' \in V_i$

$\mathbf{PC}_i$ " participation constraints," $P_i(v_i)v_i - X_i(v_i) \geq 0$, for all $i \in I$, $v_i \in V_i$

[18] With some abuse of terminology, we summarize an $i's$ information set by s_i, λ_i. By doing so we omit the first period mechanism and whether trade took place or not, which are parts of the public history. Analogously we summarize an information set of the seller by (s, λ).

[19] Densities can be written down also for discrete or mixed measures with the help of Dirac's Delta function, see for example Skreta (2007b).

RES "resource constraints," $0 \leq p_i(v_i, v_{-i})$ for all $i \in \bar{I}$ and $p_0(v_i, v_{-i}) = 1 - \Sigma_{i \in I} p_i(v_i, v_{-i})$ and for all $v \in \times_{i \in I} V$

SRC(s, λ) "sequential rationality constraints," for all $t = 2$, and for each pair (s, λ) the seller chooses a mechanism that maximizes revenue:

$$\max_{p^2, x^2} \int_{V(s)} \Sigma_{i \in I} x_i^2(v, s, \lambda) dF(v|s) \tag{9}$$

Subject to:

I. The constraints for the buyers:

IC$_i(s_i, \lambda_i)$: $P_i^2(v_i, s_i, \lambda_i)v_i - X_i^2(v_i, s_i, \lambda_i) \geq P_i^2(v_i', s_i', \lambda_i')v_i - X_i^2(v_i', s_i', \lambda_i')$, for all $\theta_i, \theta_i' \in \Theta_i$.

PC$_i(s_i, \lambda_i)$: $P_i^2(v_i, s_i, \lambda_i)v_i - X_i^2(v_i, s_i, \lambda_i) \geq 0$, for all $\theta_i \in \Theta_i$.

II. The constraints for the seller:

IC$_S(s, \lambda)$: $\int_{V(s)} \Sigma_{i \in I}[\Sigma_{\lambda \in \Lambda} d(\lambda\,|s) x_i^2(v, s, \lambda) dF(v\,|s) \geq \int_{V(s)} \Sigma_{i \in I}[\Sigma_{\lambda \in \Lambda} d(\lambda\,|s) x_i^2(v, \hat{s}, \hat{\lambda}) dF(v\,|s)$, for all $\theta_0, \hat{\theta}_0 \in \Theta_0$.

PC$_S(s, \lambda)$: $\int_{V(s)} \Sigma_{i \in I} \Sigma_{\lambda \in \Lambda} d(\lambda\,|s) x_i^2(v, s, \lambda) dF(v\,|s) \geq 0$, for all $\theta_0 \in \Theta_0$.

III. Resource Constraints are given by

RES : $0 \leq p_i^2(v, s, \lambda)$ and $p_0(v, s, \lambda) = 1 - \Sigma_{i \in I} p_i^2(v, s, \lambda)$, for all $\theta \in \Theta$.

Seller's Beliefs: posterior beliefs are given by (7) whenever possible; where p and x, defined in (5), and P_i, X_i and P_i^2, X_i^2 are defined in (6)

Remark 1 *Note that the problem stated in SRC contains also incentive and participation constraints for the seller, because she also has private information.*

We call this problem **Program A**. Before proceeding any further, it is important to relate the value of Program A with what the seller can achieve at a PBE. This is done in the Proposition that follows.

Proposition 1 *The value of Program A is an upper bound for what the seller can achieve at a PBE.*

Proof. First, note that the revelation principle applies at $t = 2$, because it is the last period, hence the conditions we impose on each p^2, x^2 are *necessary* and *sufficient* for what the seller can achieve at the given continuation game. However, the conditions stated in IC and PC are only *necessary*. This means that the feasible set of Program A is a superset of the set of $PBE-$implementable allocation and payment rules and the desired conclusion follows. ■

From Proposition 1 we see that a solution of Program A *need not* be $PBE-$implementable and the revenue generated by such a solution is an upper bound for the revenue that the seller can generate at a PBE. In what follows we obtain a solution of Program A and we construct an assessment that is a PBE and it implements it. Hence the upper bound is indeed achieved.

Our first step towards finding a solution amounts to showing that optimal mechanisms after a vector of actions s are independent from the disclosure policies, irrespective of how complicated they are. In other words, we show that the transparency of first period mechanisms is irrelevant, since it does not affect the mechanisms that the seller employs at $t = 2$. This is established in the subsection that follows.

4.2 The Irrelevance of Transparency of First Period Mechanisms

We now discuss why disclosure policies, irrespective of how sophisticated they are, and irrespective of the correlations that they induce in types, do not have any impact on what mechanisms the seller will choose at $t = 2$. In particular, we discuss[20] why no matter what disclosure policy the seller employs, the revenue generated by an optimal mechanism at $t = 2$ is always, meaning for all possible vectors of actions that could have been chosen at $t = 1$, that is for all $s \in S$, equal to the revenue she can achieve when s is observed by everyone. Put differently, the degree of "transparency" (what buyers observe) of the first period mechanism has no effect on expected revenue generated from an optimal mechanism at $t = 2$, nor on the buyers' payoffs at $t = 2$.

Formally, the result states that an optimal p^2,x^2 is independent of λ conditional on s and that for all v, and λ, it holds that

$$p_i^2(v, s, \lambda) = p^{2(s)}(v) \text{ and } x_i^2(v, s, \lambda) = x^{2(s)}(v),$$

where $p^{2(s)}, x^{2(s)}$ is optimal for the seller when the vector of actions s is *common knowledge.*

This result is established in Theorem 1 Skreta (2007). Before we explain the forces behind this result we argue why the current problem fits the formulation of Skreta (2007). That paper considers a problem analogous to the problem stated in $\mathbf{SRC}(s, \lambda)$ given some exogenous process that generates signals "$s = (s_1, .., s_I)$." For each i signal s_i is correlated with v_i, but $s_j's$ are independent across buyers. In the current problem the vector of signals is endogenously generated and it is the vector of actions chosen at $t = 1$. This difference is irrelevant for the result, as the issue of interest is whether the information disclosed by the seller affects the *subsequently* chosen revenue maximizing allocation. What is important are the statistical properties on the vector of actions and these indeed coincide with the assumptions in Skreta (2007).

To see this, note that since each buyer's strategy is a mapping from his valuations to actions at each period, s_i is correlated with buyer $i's$ valuation. Moreover, since each buyer chooses his action at $t = 1$ independently from the other buyers as a function of his valuation, and because we consider an independent private value environment, the choices of actions across buyers are *independent* conditionally on the public history, which at that

[20] Full details can be found in Skreta (2007).

point contains only the mechanism that the seller has proposed at $t = 1$. Therefore, the signals s_i are statistically independent across buyers. Hence the vector of first period actions, s, corresponds to the information that the seller has in Skreta (2007). We conclude that Theorem 1 of Skreta (2007) applies for the problem stated in $\mathbf{SRC}(s, \lambda)$. We restate this result in the Proposition that follows and proceed to briefly discuss what are the forces behind it, as well its consequences for the problem that we have in mind here.

Proposition 2 *Revenue maximizing mechanisms at* $t = 2$ *are independent from the disclosure policy and coincide with the ones that are optimal when the vector of actions chosen at* $t = 1$, *namely* s, *is common knowledge.*

The proof of Proposition 2 can be found in Skreta (2007), which corresponds to Theorem 1 of that paper.

There are three main forces behind our results. First, disclosure policies, irrespective of how sophisticated they are, essentially have no impact on a buyer's information rents. The reason for this is that a buyer can still "mimic" the behavior of the same set of valuations, as in the case where all the information that the seller has were public. Moreover, the expected payments that a seller can extract from "selling" to the buyers (agents), information about their competitors are always equal to zero. This is because there is a common prior, which implies that the side contracts written between the seller and a buyer that have positive expected value for the seller have negative expected value for the buyer, so they are never accepted. The second force is the fact that the seller is not penalized from having private information. Formally, this tells us that the seller's incentive constraints are not binding. This happens because the seller's information is non-exclusive: what the seller knows about a buyer, is also known to that particular buyer himself. The third force is related to how disclosure policies affect the set of incentive compatible mechanisms. In the case of independent private values disclosure policies do not enlarge the set of incentive compatible mechanisms in any relevant way. The reason is that in the IPV case, even when beliefs are part of buyers' types, an optimal Bayesian incentive compatible mechanism is also dominant strategy incentive compatible. This is not true in general, however.

Proposition 2 allows us to conclude that it is without loss of generality to remove the disclosure policies from our analysis. From now on, when we write 'mechanism' we will simply mean a game form, and we will be taking the vector of actions chosen at $t = 1$ to be public information. Then, throughout the game the only piece of information that is known to buyer i, but not known to the other buyers, is his valuation, and hence buyers' types coincide with their valuations. Also, when s is observed by all players, the seller has no private information, and it is without loss to have the seller simply choose game forms where only the buyers' choose actions. With these considerations, given a mechanism at $t = 1$, all histories at $t = 2$ are summarized by the vector of actions chosen by the buyers at $t = 1$, namely s. For this reason we will, with some abuse of notation, be indexing the $t = 2$ histories by s.

The $t = 2$ allocation and payment rule chosen by the seller at $t = 2$ are denoted by $p^{2(s)}, x^{2(s)}$ and we use $P_i^{2(s)}(v_i) = \int_{V_{-i}} p_i^{2(s)}(v) f_{-i}(v_{-i} \,|s_{-i}) dv_{-i}$, $X_i^{2(s)}(v_i) = \int_{V_{-i}} x_i^{2(s)}(v) f_{-i}(v_{-i} \,|s_{-i}) dv_{-i}$ to denote their corresponding expectations from $i's$ perspective.

Given that now we can take first period mechanisms to consist merely of game forms, the previous program simplifies to:

$$\max_{p,x} \int_V \Sigma_{i \in I} x_i(v) dF(v) \tag{10}$$

subject to:

IC_i: $P_i(v_i)v_i - X_i(v_i) \geq P_i(v_i')v_i - X_i(v_i')$, for all $i \in I$, $v_i, v_i' \in V_i$

PC_i: $P_i(v_i)v_i - X_i(v_i) \geq 0$, for all $i \in I$, $v_i \in V_i$

RES : $0 \leq p_i(v_i, v_{-i})$ and $p_0(v_i, v_{-i}) = 1 - \Sigma_{i \in I} p_i(v_i, v_{-i})$, for all $i \in \bar{I}$ and $v \in \times_{i \in I} V$,

$SRC(s)$ for all $s \in S$, such that $q_0(s) > 0$,

$$\max_{p^{2(s)}, x^{2(s)}} \int_{V(s)} \Sigma_{i \in I} x_i^{2(s)}(v) dF(v|s)$$

subject to:

$IC_i(s)$: $P_i^{2(s)}(v_i)v_i - X_i^{2(s)}(v_i) \geq P_i^{2(s)}(v_i')v_i - X_i^{2(s)}(v_i')$, for all $v_i, v_i' \in V_i(s_i)$

$PC_i(s)$: $P_i^{2(s)}(v_i)v_i - X_i^{2(s)}(v_i) \geq 0$, for all $v_i \in V_i(s_i)$

$RES(s)$: $0 \leq p_i^{2(s)}(v_i, v_{-i})$ and $p_0^{2(s)}(v_i, v_{-i}) = 1 - \Sigma_{i \in I} p_i^{2(s)}(v_i, v_{-i})$, for all $i \in \bar{I}$ and $v \in \times_{i \in I} V(s)$,

$Beliefs$ posterior beliefs are given by (7).

From now on, when we refer to Program A we refer to its simplified version just stated.

Still Program A is quite complicated. The primary difficulties arise from the sequential rationality constraints. However, because of the generality of the distributions that we allow, (possibly zero and/or discontinuous densities, or even complete lack of densities), even the solution of the problem without sequential rationality constraints is non-standard. We start by solving first the problem ignoring the sequential rationality constraints. This solution, serves two roles. First it serves as benchmark allowing us to evaluate the impact of the sequential rationality constraints on the optimal mechanisms. Also, its solution is itself a building block for the characterization of sequentially rational revenue maximizing auctions, since when the seller chooses an auction procedure optimally in each period her problem at the beginning of the *last period* of the game is isomorphic to an optimal auction problem without sequential rationality constraints.[21] We solve this problem in the subsection that follows.

[21] This is the reason why we need our analysis to be valid for general distributions.

4.3 Optimal Auctions without Sequential Rationality Constraints

In this section we describe revenue maximizing auctions ignoring the sequential rationality constraints allowing for general distributions of the buyers' valuations. This problem is addressed in Skreta (2007b), who generalizes Myerson's (1981)[22] results to distributions that can be discrete, mixed, or continuous, but without necessarily strictly positive densities. We now summarize findings of that paper, repeating some intermediate results that are also useful in our later analysis.

The problem without sequential rationality constraints amounts to choosing an allocation rule $p : \times_{i\in I} V_i \to [0,1]^I$ and a payment rule $x : \times_{i\in I} V_i \to \mathbb{R}$ that maximize (10) subject to IC, PC and RES. One of the difficulties of this problem is that the spaces of valuations, that is V_i and $V = \times_{i\in I} V_i$ need not be convex. Proposition 1 in Skreta (2006) establishes that it is without loss of generality to replace V_i and V with their corresponding convex hulls, namely $\bar{V}_i = [a_i, b_i]$ and $\bar{V} = \times_{i\in I}[a_i, b_i]$. In particular, we show that the program of interest cannot have a strictly higher value than the artificial program obtained by replacing the space of valuations with its convex hull.[23] This is despite the fact that the actual program is less constrained, since IC and PC are imposed only on V and not on $\bar{V}$, which implies that $value(actual) \geq value(convexified)$. The equality is established by contradiction as follows. We suppose that $value(actual) > value(convexified)$. Then we extend a hypothetical solution of the actual program on $\bar{V}$ and establish that the extension is feasible for the extended program as well, implying that $value(actual) = value(convexified)$.

This extension can be based on the observation that whenever there are gaps in V_i, say there exist v_i^L, v_i^H on the boundary of V_i, such that $(v_i^L, v_i^H) \cap V_i = \{\emptyset\}$, then it must be the case that at an optimal mechanism the incentive constraint for type v_i^H is binding. This is established in the following Lemma:

Lemma 1 *Suppose that there exist v_i^L, v_i^H on the boundary of V_i, such that $(v_i^L, v_i^H) \cap V_i = \{\emptyset\}$. Then if p, x is a solution of Program A when we ignore the sequential rationality constraints, it must hold that*

$$P_i(v_i^H)v_i^H - X_i(v_i^H) = P_i(v_i^L)v_i^H - X_i(v_i^L). \tag{11}$$

Proof. Suppose that p, x solves Program A when we ignore the sequential rationality constraints. From the incentive compatibility of (p, x) on V it follows that

$$P_i(v_i^H)v_i^H - X_i(v_i^H) \geq P_i(v_i^L)v_i^H - X_i(v_i^L). \tag{12}$$

We now demonstrate that the above inequality must hold with equality. To see this, we argue by contradiction. Suppose that

[22] Myerson's (1981) analysis assumes distributions that have strictly positive and continuous densities. These conditions guarantee that distributions are invertible, which is employed for his "ironing" technique.

[23] This modification leaves the objective function unchanged, and changes only the constraints.

$$P_i(v_i^H)v_i^H - X_i(v_i^H) > P_i(v_i^L)v_i^H - X_i(v_i^L).$$

Then, let ΔX_i be such that

$$P_i(v_i^H)v_i^H - X_i(v_i^H) - \Delta X_i = P_i(v_i^L)v_i^H - X_i(v_i^L). \tag{13}$$

Now consider the following modification of p, x. For types $v_i > v_i^H$ increase the payment of buyer i by a constant ΔX_i, that is

$$\hat{x}_i(v_i, v_{-i}) = x_i(v_i, v_{-i}) + \Delta X_i, \text{ for all } v_{-i} \in V_{-i}.$$

Given this modification, it is straightforward to see that the resulting mechanism satisfies IC, PC and RES. Moreover, it generates strictly higher revenue for the seller, contradicting the fact that p, x solves Program A. Hence (11) holds. ■

Then we can extend the mechanism for all $v_i \in (v_i^L, v_i^H)$ by setting it equal to its value at v_i^L for all v_{-i}. The extension is performed as follows. Consider a $v_i \in \bar{V}_i \backslash V_i$ and define $v_i^L(v_i) = \max\{v_i' \in V_i : v_i' \leq v_i\}$ and $v_i^H(v_i) = \min\{v_i' \in V_i(s_i) : v_i' \geq v_i\}$, (these maxima and minima exist because without loss V_i can be taken to be closed).[24] Let $v_i^{Ind}(v_i) \in [v_i^L(v_i), v_i^H(v_i)]$ denote the type for which the following is true:

$$P_i(v_i^H)v_i^{Ind}(v_i) - X_i(v_i^H) = P_i(v_i^L)v_i^{Ind}(v_i) - X_i(v_i^L). \tag{14}$$

By Lemma 1 we have that $v_i^{Ind}(v_i) = v_i^L(v_i)$.

Consider the following extension of p_i, x_i, call it $\bar{p}_i, \bar{x}_i$ on $\bar{V}$

$$\bar{p}_i(v_i, v_{-i}) = p_i(\tilde{v}_i(v_i), \tilde{v}_{-i}(v_{-i})) \text{ and } \bar{x}_i(v_i, v_{-i}) = x_i(\tilde{v}_i(v_i), \tilde{v}_{-i}(v_{-i})),$$

where $\tilde{v}_i(v_i) = \left\{ \begin{array}{c} v_i \text{ if } v_i \in V_i \\ v_i^L(v_i) \text{ if } v_i \in \bar{V}_i \backslash V_i \end{array} \right.$ and where $\tilde{v}_{-i}(v_{-i}) = (\tilde{v}_1(v_1), ..., \tilde{v}_{i-1}(v_{i-1}), \tilde{v}_{i+1}(v_{i+1}), ..., \tilde{v}_i(v_i))$.

Proposition 1 in Skreta (2006) shows that indeed this extension is feasible for the artificial program, allowing us to conclude that the actual and the extended program is the same. This, in turn, allows us to conclude that a solution of the actual problem can be obtained by solving the artificial one and by restricting its solution on the actual space. For the artificial problem we have as usual a "revenue equivalence theorem" and we use standard arguments to rewrite the objective function simply as a function of the allocation rule, as follows:

$$\max_{p,x} \int_{\bar{V}} \Sigma_{i \in I} p_i(v_i, v_{-i}) \left[v_i f_i(v_i) - (1 - F_i(v_i)) \right] f_{-i}(v_{-i}) dv - \Sigma_{i \in I} U_i(p, x, a_i)$$

[24] If V_i were not closed, we can very easily extent p, x on its closure by making it constant at the limiting vectors that are not in V_i.

subject to:

$$\begin{aligned} & P_i(v_i) \text{ increasing in } v_i \\ 0 \;\leq\; & p_i(v_i, v_{-i}) \leq 1 \text{ and } \Sigma_{i \in I} p_i(v_i, v_{-i}) \leq 1 \; v \in \times_{i \in I}[a_i, b_i]. \end{aligned}$$

Notice that this writing is valid (with slight modifications) for measures that have atoms. Skreta (2007b) shows how one can do so with the help of Dirac's Delta function.

In the standard problem distributions have strictly positive densities, that is $f_i(v_i) > 0$, for all v_i, and we usually factor $f_i(.)$ out, by dividing by it. Then, each buyer's virtual valuation is weighted with the same number, namely $f(v)$. Here $f_i(.)$ can be zero and so we cannot divide by it.[25] As is shown in Skreta (2007b), the way to proceed is to prioritize buyers according to their *extended* virtual valuations $J_i(v_i)$, which are defined as follows:

$$J_i(v_i) = \begin{cases} v_i - \frac{(1-F_i(v_i))}{f_i(v_i)} \text{ for each } v_i \in [a_i, b_i] \text{ s.t. } f_i(v_i) > 0 \\ \tilde{v}_i(v_i) - \frac{(1-F_i(\tilde{v}_i(v_i)))}{f_i(\tilde{v}_i(v_i))} \text{ where } \tilde{v}_i(v_i) = \sup\{t_i \in [a_i, b_i] \text{ s.t. } t_i \leq v_i \text{ and } f_i(t_i) > 0\}. \end{cases} \tag{15}$$

If $J_i(v_i)$ is increasing for all $v_i \in [a_i, b_i]$ and $i \in I$, the problem is regular, meaning that the "pointwise optimum," (here we use quotations because the objective function we optimize, is actually different from the actual objective function), is incentive compatible.[26] If not, we replace them with their "ironed" versions according to a procedure described in Skreta (2007b).[27] In what follows in order to avoid extra notation, when we write J_i, we will mean its *extended and ironed* version. Once we replace virtual valuations with their extended and ironed versions, the resulting problem has the same structure as the familiar problem in Myerson (1981).

Its solution is as follows:

$$p_i(v_i, v_{-i}) = \begin{cases} \frac{1}{\#\mathcal{I}(v)} \text{ if } i \in \mathcal{I}(v) \\ 0 \text{ otherwise} \end{cases}, \tag{16}$$

where $\mathcal{I}(v)$ denotes the set of buyers that have maximal virtual valuations when the vector of valuations is equal to v, and it is given by

$$\mathcal{I}(v) \equiv \{i \in I, \text{ s.t. } i \in \arg\max_{i \in I} J_i(v_i), \text{ and } J_i(v_i) \geq 0\}. \tag{17}$$

Observe that ties can occur for regions of valuations that have strictly positive measure, and for this reason ties have to be broken in a consistent way to avoid obtaining an allocation

[25] Moreover, as we explain in Skreta (2007b), it is NOT possible to prioritize buyers according to "weighted" virtual valuations $\mathcal{J}_i(v_i, v_{-i}) = [v_i f_i(v_i) - (1 - F_i(v_i))] f_{-i}(v_{-i})$, even though each buyer is assigned the same weight in the objective function namely 1, and these quantities are well defined also for vectors of valuations.

[26] For more details about the validity of this approach, see Skreta (2007b).

[27] The reason we need to modify Myerson's (1981) ironing procedure, is because it requires distributions to be invertible, which is not the case here.

rule that violates incentive compatibility. A consistent way of breaking ties could be to randomize with equal probability among all buyers who have the highest virtual valuation,[28] or to assign the good to the buyer that has the lowest index, among the set of buyers who tie. The way ties are broken is inconsequential, since the seller is indifferent between assigning the good to any buyer in $\mathcal{I}(v)$.

As usual, the payment rule is constructed from the allocation rule as follows:

$$x_i(v_i, v_{-i}) = p_i(v_i, v_{-i})v_i - \int_{a_i}^{v_i} p_i(t_i, v_{-i})dt_i. \tag{18}$$

From the above considerations it follows that a revenue maximizing mechanism without sequential rationality constraints is described by boundaries that characterize the areas where a buyer or the seller is awarded the good with probability one (the areas of indifference can be thought of as "fat" boundaries). The boundary between two buyers is determined by the equality $J_i(v_i) = J_j(v_j)$ and the boundary between the seller and i is determined by the equality

$$J_i(v_i) = 0 \tag{19}$$

and it is denoted by ξ_i.

Before we move on to the problem with the sequential rationality constraints, we apply this solution to describe an optimal mechanism at the beginning of the final period of game (here $t = 2$), at a given information set of the seller, which as we have already argued it can be summarized by the vector of actions chosen at $t = 1$ namely s.

An optimal auction $p^{2(s)}, x^{2(s)}$ is described by (16) and by (18), when we replace the prior with the posterior extended virtual valuations (which we denote by $J_i(v_i\,|s_i\,)$) and it is given by

$$\begin{aligned} \text{for each } s \ &\in \ S,\ p_i^{2(s)}(v) \ = \begin{cases} \frac{1}{\#\mathcal{I}(v|s)} \text{ if } i \in \mathcal{I}(v\,|s\,) \\ \quad 0 \text{ otherwise} \end{cases}, \\ \text{where } \mathcal{I}(v\,|s\,) \ &\equiv \ \{i \in I,\ \text{s.t. } i \in \arg\max_{i\in I} J_i(v_i\,|s_i\,),\ \text{and } J_i(v_i\,|s_i\,) \geq 0\} \end{aligned} \tag{20}$$

and

$$x_i^{2(s)}(v_i, v_{-i}) = p_i^{2(s)}(v_i, v_{-i})v_i - \int_{a_i}^{v_i} p_i^{2(s)}(t_i, v_{-i})dt_i. \tag{21}$$

The posterior virtual valuation of buyer i is obtained from (15) by replacing f_i with $f_i(.|s_i)$ described in (7).

However, there is a difference here compared to the case without sequential rationality constraints. The difference is that whenever indifferences arise at $t = 2$, the way that ties are broken matters, in the sense that the seller's revenue from the $t = 1$ perspective can be strictly higher for a given tie-breaking rule, versus another. The reason for this is that when ties occur at $t = 2$, the seller is indifferent from the $t = 2$ perspective, but she may

[28] This is the tie-breaking rule employed in the general case in Myerson (1981).

not be indifferent from the $t = 1$ perspective. This is because it is possible that *posterior* virtual valuations of a certain set of buyers are the same over a region of valuations, while their corresponding prior virtual valuations are different over the exact same region. Later, we will see that at a revenue maximizing PBE, ties are broken in the way that maximizes revenue of the seller from the ex-ante perspective.

We now proceed to characterize revenue maximizing auctions when the seller and the buyers behave sequentially rationally.

4.4 Optimal Auctions with Sequential Rationality Constraints when $T = 2$

In this section we characterize revenue maximizing auctions when the seller behaves sequentially rationally. Sequential rationality constraints create distortions, because at $t = 2$ and after each vector of actions s according to which no trade takes place at $t = 1$, the seller ranks various alternatives, (that is whether to give away the object and, if yes, to which buyer), according to the posterior virtual surpluses instead of the prior virtual surpluses, which is what would have been ex-ante optimal. Here the analysis is lengthier and more involved, so before we begin let us describe briefly the four main blocks involved. First we show (analogously as we did in the case where we ignored the sequential rationality constraints) that it is without any loss to convexify the problem. We also discuss the main sources of its intricacies. Then we obtain an artificial solution by ignoring some of those intricacies. In the penultimate step, we establish that our "quasi solution" is in fact a real solution of the program of interest. Finally, we show that the solution is indeed attainable by a PBE (remember our maximization problem gives us an upper bound for what the seller can achieve at a PBE) by constructing a strategy profile and a belief system that is a PBE and implements the solution.

4.4.1 Convexifying the Problem and Intricacies of the Resulting Program

As we did in the case without the sequential rationality constraints, we first argue that it is without any loss to replace the spaces of valuations with their corresponding convex hulls. In particular, we show that it is without loss of generality to solve an artificial problem which we call Program B, which is exactly the same as Program A, but with V_i and $V_i(s)$ replaced by their corresponding convex hulls, $[a_i, b_i]$ and $\bar{V}_i(s)$. This is established in the Proposition 3.

Proposition 3 *A solution of Program B restricted on V solves Program A.*

We now sketch why Proposition 3 is true. This result is analogous to the result in Proposition 1 in Skreta (2006), but now we have to take into account the sequential rationality constraints. By applying that result, we know that we can obtain every solution of $SRC(s)$ by extending $p^{2(s)}$ and $x^{2(s)}$ on $\bar{V}(s) = \times_{i \in I} \bar{V}_i(s)$ and by imposing $IC_i(s)$ and $PC_i(s)$ on $\bar{V}_i(s_i)$. Therefore, essentially nothing changes on the set of constraints described in $SRC(s)$.

Then, using exactly the same arguments as in the proof of Proposition 1 in Skreta (2006), one can show that the values of Programs A and B are the same by establishing that a solution p and x of Program A appropriately extended on $\times_{i\in I}[a_i, b_i]$ satisfies all the constraints of Program B. Then Proposition 3 follows.

As before, using standard arguments, we can rewrite Program B more compactly as follows:

$$\max_{\{m^s,q(s)\}_{s\in S}} \int_{\bar{V}} \Sigma_{i\in I}\Sigma_{s\in S} m^s(v)\left(q_i(s)+q_0(s)\delta p_i^{2(s)}(v)\right)[v_i f_i(v_i)-(1-F_i(v_i))]\, f_{-i}(v_{-i})dv$$
$$-\Sigma_{i\in I}U_i(p,x,a_i) \tag{22}$$

subject to:

$$\begin{aligned} & P_i(v_i) \text{ increasing in } v_i \\ 0 \;\leq\; & p_i(v_i,v_{-i}) \leq 1 \text{ and } \Sigma_{i\in I} p_i(v_i,v_{-i}) \leq 1 \;\; v \in \times_{i\in I}[a_i,b_i] \end{aligned}$$

and where $p^{2(s)}$ is given by (20) for each vector of actions s that is chosen with strictly positive probability.

From now on, when we write Program B we refer to its rewriting just stated. In what follows we first lay out what are the specific intricacies of the problem at hand, as well as an outline of the solution approach.

In order to understand the source of the main intricacies of Program B, it may be worthwhile to go through a few preliminary steps. There are $I+1$ options that the seller can do with the good, namely give it to one of the I buyers or keep it herself. Of course there are also all possible randomizations among these options, and the incentive, the participation and the sequential rationality constraints that one needs to worry about. First let us examine how a solution of Program B would look like, if we assume that it is not an option that the seller keeps the good.

Once it is not an option for the seller to keep the object, things become quite trivial indeed. This is because the sequential rationality constraints are always (trivially) satisfied by all allocation rules that never assign positive weight to the seller keeping the object. Then an optimal allocation rule can be obtained via pointwise maximization of (22) ignoring all the constraints, and it assigns for each v probability one to the buyer with the highest (ironed) virtual valuation. Using the usual arguments, one can show that this allocation rule satisfies incentive, participation and resource constraints. From this observation, it also immediately follows that even when the seller can be keep the object, if the highest virtual valuation is always, (that is for all realizations of the buyers' valuations), greater than the seller's valuation for the good, then the allocation rule resulting from pointwise maximization is also sequentially rational and thus satisfies all the complicated constraints of Program B. However, if sometimes the highest virtual valuation falls short of the seller's valuation, the proposed allocation rule violates sequential rationality constraints, because for all these histories where the seller keeps the good at $t = 1$, she will propose another mechanism at $t = 2$, which must be optimal given her posterior beliefs. Then, what is an optimal allocation rule when we do take into account the sequential rationality constraints?

In order to find an optimal allocation rule p when the sequential rationality constraints are present we have to identify what is the nature of period one mechanisms that achieve the best in generating revenue at $t = 1$, but in the event that trade does not take place, they simultaneously minimize the cost of sequential rationality constraints. In order to do so it can be helpful to address questions of the following form. Is it better to have many different vectors of actions s where the seller keeps the good with strictly positive probability at $t = 1$? Would such a possibility lead to posteriors that support an expected second period allocation rule that is better from the ex-ante point of view? It is also conceivable that it turns out to be beneficial for the seller to sacrifice slightly first stage optimality (say, by increasing the probability of trade with very low valuations from zero to ε), in order to induce posteriors that lead to second period mechanisms that perform better from the ex-ant point of view. Is there a trade-off between generating as much revenue as possible at $t = 1$ versus guaranteeing the least costly posteriors for $T = 2$?

These questions are quite delicate to address given the complicated fixed point nature of a solution. By this we mean the following. Fix a first stage menu. This consists of various vectors of actions, the $s's$, and each one of these different vector's of actions determines a first period menu, eg. $\{q_i(s), z_i(s)\}_{i\in\bar{I}}$, $\{q_i(\hat{s}), z_i(\hat{s})\}_{i\in\bar{I}}$ etc. The regions of types that choose, say, the vector of actions s has convex hull of the form $\bar{V}(s)$. The mapping $m^s(v)$ (together with the prior) determines the seller's posterior beliefs, which are given by (8), which in turn determine the $t = 2$ menu that satisfies $SRC(s)$, (that is, the $t = 2$ menu that is optimal for the seller after s is chosen), the $\{p_i^{2(s)}, x_i^{2(s)}\}_{i\in\bar{I}}$. Additionally, the $t = 1$ and $t = 2$ menus together determine the feasible $m^s(v) = \times_{i\in I} m^{s_i}(v_i)$. The reason for this is that in order for buyer i to be choosing action s_i with probability less than one, that is $m^{s_i}(v_i) < 1$, it must be the case that i with valuation v_i is indifferent between action s_i and some other action, call it $\hat{s}_i$. Summing up, the problem is complicated because we have the following interrelations:

$$\{q_i(s), z_i(s)\}_{i\in\bar{I}}, \text{ and } m^s(v),\ s \in S \Longrightarrow \text{feasible } p^{2(s)}, x^{2(s)} \text{ for } s \in S$$
$$\{q_i(s), z_i(s)\}_{i\in\bar{I}}, \text{ and } p^{2(s)}, x^{2(s)},\ s \in S \Longrightarrow \text{feasible } m^s(v),\ s \in S.$$

These interrelations also make the problem non-linear.

In order to solve Program B we have to specify optimally

(i) The structure of S, (how many different vectors of actions are available at $t = 1$).

(ii) The $\{q_i(s), z_i(s)\}_{i\in\bar{I}}$ (how many different $t = 1$ menus are available)

(iii) The way buyers randomize between different vectors of actions, that is the $m^s(v)'s$.

These steps are delicate, given all the above mentioned interdependencies. If we want to consider the effect of changing $\{q_i(s), z_i(s)\}_{i\in\bar{I}}$ slightly, (while keeping $\{q_i(\tilde{s}), z_i(\tilde{s})\}_{i\in\bar{I}}$, $\tilde{s} \in S\backslash\{s\}$ fixed), we have to take into account that the previously feasible $m^{s\prime}s$ may be no longer feasible, which may have, a dramatic, and a hard to assess, effect on the optimal second period mechanisms. This kind of possible dramatic effects of slight perturbations lie at the heart of the intricacies of the problem. Another delicate issue, is that it is conceivable that at a solution there are more than one vectors of actions associated with the same menu $\{q_i(s), z_i(s)\}_{i\in\bar{I}}$, that is $q_i(s) = \tilde{q}_i(\hat{s})$, and $z_i(s) = \tilde{z}_i(\hat{s})$ for all $i \in \bar{I}$, and that the possible randomizations lead to posteriors that induce period two mechanisms $p^{2(s)}, x^{2(s)}$ that create less distortions than the period 2 mechanism that would have been optimal if there is only one vector of actions that leads to the menu $\{q_i(s), z(s)\}_{i\in\bar{I}}$.

The complications of this problem reflect the simultaneous presence of counteracting forces. All else equal, it is good for the seller at $t = 2$ to have more information. This occurs at a strategy profile where many different vectors of actions are chosen at $t = 1$. However, the ability to be able to condition on more information at $t = 2$ is costly at $t = 1$ because the seller has to compensate the buyers for the information that they are willing to release through their actions at $t = 1$. Moreover, adding many different options at $t = 1$ means that there will be many different options where the probability of trade is less than one. Such options are costly for the seller because they generate less expected surplus for the buyers, which consequently implies that the seller can extract less payments. And as already discussed the formal trade-offs and analysis can be even more delicate.

Despite the complicated nature of Program B, it turns out that it has a surprisingly simple solution. In what follows we will establish that at a solution of Program B there is only one vector of actions s for which the seller keeps the good at $t = 1$ with strictly positive probability, that is, for which $q_0(s) > 0$. In fact, we show that s is chosen with probability one when $v \in \times_{i\in I}[a_i, \bar{v}_i(\delta)]$, that is, $q_0(s) = 1$ and $m^s(v) = 1$ for $v \in \times_{i\in I}[a_i, \bar{v}_i(\delta)]$. This implies, that after the history where the seller observes s at $t = 1$, the posteriors are truncations of the priors, and are given by

$$f_i(v_i|\bar{v}_i) = \begin{cases} \frac{f_i(v_i)}{F_i(\bar{v}_i)}, \text{ for } v_i \in [a_i, \bar{v}_i] \\ \quad 0 \text{ otherwise} \end{cases}, \text{ for all } i \in I. \tag{23}$$

This solution is derived in two-steps. First we derive a quasi-solution by ignoring the indirect effects of the first period menus on the set of $m^{s\prime}s$ that can be supported. Then we show that even if we were to take these effects into account the solution would not change, establishing that our "quasi solution" is a real solution.

4.4.2 A Quasi-Solution

First, we solve the problem ignoring the indirect effects that the first stage menus, namely the $\{q(s)\}_{s\in S}$, on the set of second period mechanisms. This significantly simplifies matters because ignoring the aforementioned effects, we see that the seller's objective function becomes a linear function of the first period menus. This immediate observation is stated in the following Lemma:

Lemma 2 *The seller's expected revenue is linear in the first period menus* $\{q_i(s)\}_{i\in \bar{I}}$ *for all* $s \in S$.

Lemma 2 allows us to conclude that at a solution the optimal menu's $\{q_i(s)\}_{i\in \bar{I}}$ are of the form:

$$q^0 = \underbrace{1,0,...,0}_{I+1 \text{ components}} \quad ; q^1 = \underbrace{0,1,...,0}_{I+1 \text{ components}} \quad ; ...; q^I = \underbrace{0,0,...,1}_{I+1 \text{ components}} \quad ,$$

where the first position of each one of these vectors stands for the probability that the seller gets the good, the second component the probability that buyer 1 gets good, and so on, until the $I+1$ position which stands for the probability that buyer I gets the good. We call Q^* the menus $q^0, ..., q^I$. In the next step we find the regions where each one of these vectors is implemented and establish that the resulting allocation and payment rules are feasible for Program B.

By the monotonicity of the buyers' (ironed) virtual valuations the region where the seller gets to keep the good (if non-empty) consists of an area of valuations where all buyers $v_i's$ lie below some cut-off, $\bar{v}_i$. Without sequential rationality constraints the cut-off for a buyer i is a v_i where his virtual valuation is equal to the seller's value. With sequential rationality constraints the cut-off depends on the discount factor and is (weakly) larger than the "static" cut-off. This is because the seller anticipates at $t = 1$ her temptation at $t = 2$ to trade with some of those buyers with small and possibly negative (ex-ante) virtual valuation. In other words, the seller at $t = 2$ "overassigns" the object compared to what is revenue maximizing from the ex-ante perspective. The reason for this overassignment is that at $t = 2$ the seller perceives $i's$ valuation to belong in $[a_i, \bar{v}_i]$, which implies that compared to the ex-ante perspective $i's$ virtual valuation is overestimated by $\frac{[1-F_i(\bar{v}_i)]}{f_i(v_i)}$.[29]

In an equilibrium the buyers and the seller at $t = 1$ anticipate the seller's behavior at $t = 2$, and at the revenue maximizing equilibrium the seller adjusts the first period cut-off upwards in anticipation of her behavior at $t = 2$. The optimal level of $\bar{v}_i$ depends on the discount factor δ, which also determines how large the cost of the sequential rationality constraints is. For the cases of extreme discount factors, namely for $\delta = 0$ and for $\delta = 1$ the

[29]This is because the prior virtual valuation is $v_i - \frac{[1-F_i(v_i)]}{f_i(v_i)}$, whereas the posterior virtual valuation is equivalent to $v_i - \frac{[F_i(\bar{v}_i)-F_i(v_i)]}{f_i(v_i)}$.

sequential rationality constraints are always non-binding and for these cases the optimal vector of cutoffs $\bar{v}$ can be easily described for any set of prior distributions: when $\delta = 0$ the future does not matter at all, so the sequential rationality constraints disappear and the optimal vector of cutoffs is given by the vectors of valuations where all buyers' virtual valuations are equal to zero (the seller's valuation), namely $\bar{v}(0) = (\xi_1, ..., \xi_I)$, where for all i, ξ_i satisfies (19). When $\delta = 1$, there is absolutely no cost to waiting, so the seller can "wait" until the last period of the game and then offer an optimal mechanism without sequential rationality constraints. This "waiting" can be achieved by selecting $\bar{v}$ to be equal to the vector of the highest possible valuations of all buyers, that is $\bar{v}(1) = (b_1, ..., b_I)$. For intermediate discount factors, an optimal vector of cutoffs is somewhere between $\bar{v}(0)$ and $\bar{v}(1)$, which means that that for all $\delta > 0$ the seller keeps the object at $t = 1$ for a larger area of valuations compared to the case without sequential rationality constraints.

From these considerations it follows that for valuations in the region $\times_{i\in I}[a_i, \bar{v}_i]$ the menu q^0 is implemented at $t = 1$, which simply means that the seller keeps the good with probability one at $t = 1$. Then, at $t = 2$ the seller's posterior beliefs about the valuation of buyer i, $i \in I$, are given by (23) and she maximizes revenue by assigning probability one to the buyer with the highest posterior virtual valuation provided that it is non-negative. We use $\mathcal{I}(v\,|\bar{v}\,)$ to denote the set of buyers that have maximal non-negative *posterior* virtual valuations when the vector of valuations is equal to v, that is

$$\mathcal{I}(v\,|\bar{v}\,) \equiv \{i \in I, \text{ s.t. } i \in \arg\max_{i\in I} J_i(v_i\,|\bar{v}_i\,), \text{ and } J_i(v_i\,|\bar{v}_i\,) \geq 0\}. \tag{24}$$

For vectors of valuations where ties occur, the set $\mathcal{I}(v\,|\bar{v}\,)$ contains more than one buyer. From the $t = 2$ perspective the seller is just indifferent between assigning the good to any of the buyers in $\mathcal{I}(v\,|\bar{v}\,)$. However, as we have already discussed, the seller may not be indifferent from the $t = 1$ perspective, in which case she can increase revenue from the $t = 1$ perspective by assigning the good to the subset of buyers in $\mathcal{I}(v\,|\bar{v}\,)$ that have maximal *prior* virtual valuations. We call this subset of buyers $\mathcal{I}^{prior}(v\,|\bar{v}\,)$ and it is given by:

$$\mathcal{I}^{prior}(v\,|\bar{v}\,) = \{i \in \mathcal{I}(v\,|\bar{v}\,), \text{ s.t. } i \in \arg\max_{i\in\mathcal{I}(v|\bar{v}\,)} J_i(v_i)\}. \tag{25}$$

For the region of valuations $\bar{V}\setminus \times_{i\in I} [a_i, \bar{v}_i]$ the best that the seller can achieve is to assign the good to the buyer with the highest virtual valuation among the buyers that have valuations above the cut-off $\bar{v}_i$. This is because in the region $\bar{V}\setminus \times_{i\in I} [a_i, \bar{v}_i]$ the highest virtual valuation is above the seller's value. We use $I^1(v)$ to denote the set of buyers that have valuations above the cut-off $\bar{v}_i$ when the vector of realized valuations is v, that is $I^1(v) = \{i \in I : v_i \in (\bar{v}_i, b_i]\}$. We also use $I^*(v)$ to denote the set of buyers that have maximal virtual valuations among $I^1(v)$, that is

$$I^*(v) = \{i \in I, \text{ s.t. } i \in \arg\max_{i\in I^1(v)} J_i(v_i)\}. \tag{26}$$

If there are ties $I^*(v)$ contains more than one buyers. But here because the seller is indifferent from the ex-ante point of view the way ties are broken is inconsequential. However, some caution must be exercised in the way ties are broken in order to guarantee that the resulting allocation rule is indeed incentive compatible. One way to do so is to always assign the good to the buyer with the lowest index among all the buyers that tie, both in cases where ties occur at $t = 1$ and at $t = 2$.

Summarizing, our "quasi solution" consists of a $t = 1$ menu where the seller keeps the good with probability one for vectors of valuations in $\times_{i\in I}[a_i, \bar{v}_i]$, and for the remaining vectors of valuations the good is assigned with probability one to the buyer with the highest virtual valuation. The overall allocation rule arising from this first period menu is given by:

$$\begin{aligned} \text{for } v &\in \bar{V}\setminus \times_{i\in I}[a_i, \bar{v}_i],\ p_i(v_i, v_{-i}) = \begin{cases} 1 \text{ if } i \in I^*(v), \text{ and } i < j \text{ for all } i, j \in I^*(v) \\ 0 \text{ otherwise} \end{cases} \quad (27) \\ \text{for } v &\in \times_{i\in I}[a_i, \bar{v}_i],\ p_i(v_i, v_{-i}) = \delta p_i^2(v), \end{aligned}$$

where

$$p_i^2(v) = \begin{cases} 1 \text{ if } i \in \mathcal{I}(v\,|\bar{v}) \text{ and } i < j \text{ for all } i, j \in \mathcal{I}^{prior}(v\,|\bar{v}) \\ 0 \text{ otherwise} \end{cases}, \qquad (28)$$

and where $\bar{v}_i \in [a_i, b_i]$ for all $i \in I$; $I^*(v)$ is given by (26), $\mathcal{I}(v\,|\bar{v})$ is given by (24) and $\mathcal{I}^{prior}(v\,|\bar{v})$ is given by (25).

The allocation rule given by (27) will be a "solution" of Program B if it satisfies all the constraints. This turns out to be true and we establish it in the Proposition that follows.

Proposition 4 *The allocation rule in (27) is feasible for Program B.*

Proof. First observe that p in (27), satisfies resource constraints. Moreover, p^2 in (28), is optimal given posterior (23), hence it satisfies SRC. In order to establish feasibility of p it remains to establish that P_i is increasing in v_i.

For $v_i \in [a_i, \bar{v}_i]$ $P_i(v_i) = \delta P_i^2(v_i)$, and by standard arguments one can see that in fact $p_i^2(v_i, v_{-i})$ is increasing in v_i, which immediately implies that $P_i^2(v_i)$ is increasing in v_i as well. So for $v_i \in [a_i, \bar{v}_i]$ P_i is increasing in v_i. Along the region $(\bar{v}_i, b_i]$ the monotonicity of P_i follows using standard arguments. It remains to check that P_i does not drop at $\bar{v}_i$. Buyer i with valuation slightly below $\bar{v}_i$ can only obtain the object at $t = 2$. We reach $t = 2$ if $v_{-i} \leq \bar{v}_{-i}$, which happens with probability $F_{-i}(\bar{v}_{-i})$. Otherwise, the object is assigned with probability one to some other buyer. So even if he gets the object with probability 1 at $t = 2$, from the ex-ante point of view it must hold $P_i(\bar{v}_i) \leq \delta F_{-i}(\bar{v}_{-i})$. In other words, the upper bound for $P_i(\bar{v}_i)$ is $\delta F_{-i}(\bar{v}_{-i})$. Now type $\bar{v}_i + \varepsilon$ where $\varepsilon > 0$ gets the object with probability one, **at least** when $v_{-i} \leq \bar{v}_{-i}$. This occurs with probability $F_{-i}(\bar{v}_{-i})$. Hence it holds that $P_i(\bar{v}_i + \varepsilon) \geq F_{-i}(\bar{v}_{-i})$, and because $\delta F_{-i}(\bar{v}_{-i})$ is an upper bound for $P_i(\bar{v}_i)$, we get that $P_i(\bar{v}_i+\varepsilon) \geq P_i(\bar{v}_i)$. Therefore P_i does not drop at $\bar{v}_i$, and hence it is increasing in v_i on $[a_i, b_i]$. ■

Remark 2 *Actually, one can show an even stronger version of monotonicity. It turns out that $p_i(v_i, v_{-i})$ is increasing in v_i for all v_{-i}. From our arguments it is easy to see that $p_i(v_i, v_{-i})$ is increasing in v_i for all v_{-i} and for $v_i \in [a_i, b_i] \backslash \{\bar{v}_i\}$. Hence it remains to show that it is increasing at $\bar{v}_i$. When at least one $v_j > \bar{v}_j$ we have that $p_i(\bar{v}_i - \varepsilon, v_{-i}) = 0$ whereas $p_i(\bar{v}_i + \varepsilon, v_{-i})$ is either 0 or 1. In both cases it is increasing. Now when $v_{-i} < \bar{v}_{-i}$ we have that $p_i(\bar{v}_i - \varepsilon, v_{-i})$ is either equal to 0 or δ, whereas $p_i(\bar{v}_i + \varepsilon, v_{-i}) = 1$; hence again p_i is increasing. From these observations we can conclude that our "solution" is dominant strategy incentive compatible. This observation is employed later to establish the irrelevance of disclosure policies when we examine situations where the game lasts longer than two periods.*

Proposition 4 establishes that (27) is feasible for Program B. Varying $\bar{v}$ we get different allocation rules all of whom satisfy (27). At a solution of Program B $\bar{v}$ must be determined optimally. Put differently, after substituting (27) in the objective function, Program B reduces to the problem of finding $\bar{v} = (\bar{v}_1, ..., \bar{v}_I)$, with $\bar{v}_i \in [a_i, b_i]$, that is:

$$\max_{\bar{v} \in \times_{i \in I}[a_i, b_i]} R(\bar{v}). \tag{29}$$

This program is tremendously simpler than the one we set out to solve. Instead of maximizing over an infinite dimensional space, we are choosing the vector $\bar{v}$, which is a finite dimensional object, out of a compact set $\times_{i \in I}[a_i, b_i]$.

We now proceed to illustrate how to obtain a revenue maximizing vector $\bar{v}$ in a simple example.

Example 1 *Suppose that there are 2 buyers, whose valuations are drawn both from the uniform distribution on [0,1] and that the seller's valuation for the good is zero. For this example the commitment benchmark is*

$$\bar{v}_1^c = \bar{v}_2^c = 0.5 \textit{ and Revenue}{=}0.416.$$

For a pair $\bar{v}_1$ and $\bar{v}_2$ we have that the posterior virtual valuations for 1 and 2 are respectively given by $J_1(v_1 \,|\bar{v}_1) = 2v_1 - \bar{v}_1$ and $J_2(v_2 \,|\bar{v}_2) = 2v_2 - \bar{v}_2$. At a revenue maximizing mechanism at $t = 2$ the following holds

$$\begin{aligned} \textit{if } v_1 \quad &\geq \quad v_2 - \frac{(\bar{v}_2 - \bar{v}_1)}{2} \textit{ and } v_1 \geq 0.5\bar{v}_1 \textit{ 1 obtains the object} \\ \textit{if } v_2 \quad &> \quad v_1 - \frac{(\bar{v}_1 - \bar{v}_2)}{2} \textit{ and } v_2 \geq 0.5\bar{v}_2 \textit{ 2 obtains the object} \\ &\quad \textit{otherwise the seller keeps the object.} \end{aligned}$$

At $t = 1$ if $v_1 \geq v_2$ buyer 1 gets the object and buyer 2 gets it otherwise. Then the seller's expected revenue is given by:

$$\begin{aligned} R(\bar{v}_1, \bar{v}_2) &= \int_0^{0.5\bar{v}_2} \left[\int_{0.5\bar{v}_1}^{\bar{v}_1} \delta(2v_1 - 1)dv_1 + \int_{\bar{v}_1}^1 (2v_1 - 1)dv_1 \right] dv_2 + \\ & \int_{0.5\bar{v}_2}^{\bar{v}_2} \left[\int_0^{v_2 - \frac{(\bar{v}_2 - \bar{v}_1)}{2}} \delta(2v_2 - 1)dv_1 + \int_{v_2 - \frac{(\bar{v}_2 - \bar{v}_1)}{2}}^{\bar{v}_1} \delta(2v_1 - 1)dv_1 + \int_{\bar{v}_1}^1 (2v_1 - 1)dv_1 \right] dv_2 + \\ & \int_{\bar{v}_2}^1 \left[\int_0^{v_2} (2v_2 - 1)dv_1 + \int_{v_2}^1 (2v_1 - 1)dv_1 \right] dv_2, \end{aligned}$$

which simplifies to:

$$\begin{aligned} R(\bar{v}_1, \bar{v}_2) &= 0.58333\bar{v}_2\delta\bar{v}_1^2 - 0.\,75\bar{v}_2\delta\bar{v}_1 - 1.0\bar{v}_2\bar{v}_1^2 + \bar{v}_2\bar{v}_1 \\ & +0.58333\delta\bar{v}_1\bar{v}_2^2 + 0.33333 - 0.33333\bar{v}_2^3. \end{aligned}$$

Maximizing with respect to $\bar{v}_1$ and to $\bar{v}_2$ we obtain a solution, which turns out to be symmetric. We describe it in the following table:

δ	*$\bar{v}_1$*	*$\bar{v}_2$*	*Revenue*	*Loss in Revenue in %*
0	*0.500*	*0.500*	*0.416*	*0*
0.1	*0.506*	*0.506*	*0.412*	*-0.96*
0.3	*0.525*	*0.525*	*0.404*	*-2.88*
0.5	*0.555*	*0.555*	*0.397*	*-4.57*
0.7	*0.612*	*0.612*	*0.392*	*-5.77*
0.9	*0.764*	*0.764*	*0.396*	*-4.80*
1	*1*	*1*	*0.416*	*0*

The last column of the table describes the revenue loss that the seller incurs in the best possible scenario due to her inability to commit. The loss varies non-monotonically with the discount factor.

We now proceed to establish that this is a real solution of the problem of interest.

4.4.3 The Actual Solution

The critical feature of our "quasi-solution" is that there is a single vector of actions that leads to no trade in period one; it actually leads to no trade with probability one, and moreover it is chosen with probability one when all buyers' valuations lie below a cut-off. Once the seller observes this vector of actions, the posteriors about the buyers' valuations are simply truncations of the prior. The fact that the posteriors are truncations of the priors is an almost immediate consequence of the fact that in deriving this "quasi-solution" we ignored the indirect effects of the first period menu's on the second period mechanisms. In order to establish that our quasi-solution is a real solution, we have to establish that taking into account these effects, which essentially amounts to allowing for general $m_i^{s_i}(v_i)'s$ (as opposed to ones that take the extreme values of 0 or 1), does not allow the seller to do better. General $m_i^{s_i}(v_i)'s$ arise in cases when there is an interval of valuations where buyer i is indifferent between s_i and some other action $\hat{s}_i$. Then, along the region where i is indifferent, $m_i^{s_i}(v_i)$ could a priori be any number between zero and one. Mixings and/or gaps in the support of the posteriors may help because they sustain a much richer set of posteriors, compared to just mere truncations.

In one case one can immediately conclude that complicated mixings/gaps cannot lead to second period allocations that reduce the distortions introduced by sequential rationality constraints compared to the distortions arising to truncations of the prior. When all buyers are ex-ante symmetric the ex-ante optimal boundary between two buyers i and j is given by the 45 degree line namely $v_i = v_j$. When posteriors are truncations with the same cutoff across buyers, then the buyers remain symmetric in the eyes of the seller also at $t = 2$, and at that point too, the optimal boundary is $v_i = v_j$. Hence same truncations of the prior in the symmetric case lead to *no distortions* of the boundary between two buyers, compared to ex-ante optimal boundary.

Unfortunately this conclusion is much harder to obtain when it comes to comparing (i) buyers $i's$ posterior virtual valuation with the seller's value and (ii) two buyers' posterior virtual valuations when they are not symmetric. In the longer working paper version of this paper,[30] one can find the complete proof to (i). However (ii) seems impossible to obtain via direct comparisons. The reason for this is that the ex-ante optimal boundary across buyers can be essentially anything, and not simply the 45 degree line as in the case of symmetric buyers. Also, in contrast to the case of the boundary between a buyer and the seller, where only the buyer's posterior virtual valuation depends on the posterior, here both the relevant quantities for comparisons (the posterior virtual valuations) depend on the posteriors. Then, it is possible that when the posterior about i is a truncation, the comparison of $J_i(v_i\,|\bar{v}_i)$ with $J_j(v_j\,|s_j)$ is closer to the ex-ante optimal one, than, say, the comparison arising when the posterior about i is not a truncation; whereas in the case that j chooses another action, say $\tilde{s}_j$, the reverse is true. In short, when buyers are asymmetric obtaining general conclusions about which posteriors minimize the distortions arising from sequential rationality constraints seems quite challenging, if not impossible.

Given that such "direct" comparisons do not lead to any conclusions, another way to proceed is to investigate whether the conditions necessary to support "complicated posteriors" which we describe in Proposition 5 in the working paper version, are indeed feasible or not. As we discuss there, these requirements are quite demanding, and in many cases impossible to satisfy. However, even though, complicated posteriors may be very hard to sustain, establishing their impossibility for every distribution does not seem possible. For these reasons we have to appeal to stronger, and maybe more generally applicable ideas, to

[30] Available at the author's website.

establish our result. We introduce such an idea next.

The Information Postponement Principle

The information postponement principle essentially states that it does not pay for the seller (or the mechanism designer more generally) to distort the first period allocations for the purpose of generating more information at $t = 2$. The reason for this is that the highest overall payoff that the seller can achieve is obtained when the seller restricts as much as possible the amount of information that she obtains in the future. This allows the seller to support higher ex-ante revenue because it minimizes the set of future deviations that the seller has. In short, less information available to the seller in the future translates to a smaller set of deviations, which in turn translates to higher ex-ante revenue. This is essentially the principle formulated by Myerson (1986) in his paper on multi-stage games with communication. In that paper Myerson argues (Myerson (1986), page 324) that the set of equilibria is largest when the communication devise sends to each player only the recommended action at each stage and it does not release any additional information about past or current actions for the other players; nor for future actions recommended to that player himself. The reason is that any additional information makes the set of all possible deviations available to a player larger, which makes the set of equilibria smaller. Our set-up shares some similarities with the set-up in the multi-stage games with communication, however there are clearly important differences: for one, in this paper we are not looking for correlated equilibria. Still, the same idea applies: more information increases the possible deviations and results in a smaller number of equilibria, hence smaller achievable equilibrium payoffs.

With the help of these observations we establish that our quasi-solution is a real solution next.

Proposition 5 *Our quasi-solution is a real solution for the general case where buyers can be asymmetric.*[31]

Proof. In the problem under consideration, the seller has more information to condition on at $t = 2$ the more vectors of actions that lead with positive probability to no trade are chosen at $t = 1$ there are. By the "information postponement principle" then, the least amount of information that the seller can condition on results when there is a unique vector of actions that leads to no trade at $t = 1$.

Let s denote this unique vector of actions that is associated with strictly positive probability of no trade at $t = 1$, $(q_0(s) > 0)$. The monotonicity of the (ironed) virtual valuations implies that this vector of actions must be chosen by the buyers when their valuations are

[31] A constructive (but lengthy) proof without resulting to the information postponement principle is possible for the case of symmetric buyers. See the long working paper version for details.

below some cutoff. In other words, s_i is chosen with probability one by buyer i when his valuation belongs in $[a_i, \bar{v}_i]$. Moreover, since we have argued that there should not be any other vector of actions with $q_0(\tilde{s}) > 0$, it follows that for $v's$ in the region $\bar{V} \setminus \times_{i \in I} [a_i, \bar{v}_i]$ buyers choose vectors of actions with $q_0(\tilde{s}) = 0$, that is they choose vectors of actions according to which trade takes place with probability one at $t = 1$.

Given our findings so far, the objective function of Program B can be rewritten as

$$\int_{\times_{i \in I}[a_i, \bar{v}_i]} \Sigma_{i \in I}(q_i(s) + \delta q_0(s) p_i^{2(s)}(v)) J_i(v_i) f(v) dv$$
$$+ \int_{\bar{V} \setminus \times_{i \in I}[a_i, \bar{v}_i]} \Sigma_{i \in I} p_i(v) J_i(v_i) f(v) dv,$$

where $q_0(s) > 0$ and $p^{2(s)}$ is optimal given beliefs (23). Notice now that this problem is linear in the menu $q(s)$, so at an optimum we can take $q_0(s) = 1$ without loss. For the region $\bar{V} \setminus \times_{i \in I} [a_i, \bar{v}_i]$ we have that trade takes place with probability one at $t = 1$, and the best that the seller can do is to use a first period mechanism where for each vector of actions chosen by a $v \in \bar{V} \setminus \times_{i \in I} [a_i, \bar{v}_i]$ one of the buyers with the highest virtual valuation is awarded the good with probability one. This is exactly the "quasi solution" we derived earlier. ■

Summarizing, with the help of the "information postponement principle" we were able to see that our quasi-solution is an actual solution of Program B even when buyers are ex-ante asymmetric. We have also argued that in general more direct and contractive ways are extremely messy.

So far we have obtained a solution of Program B. However, because the feasible set of Program B consists of allocation and payment rules that simply satisfy *necessary* conditions of being PBE implementable, we cannot be a priori sure that there exists a strategy profile that is a PBE and that it implements the solution we have derived. Next we show that we can actually achieve this upper bound by constructing a strategy profile that implements the allocation and the payment rule that solve Program B.

4.4.4 Implementation of a Solution of Program B

A strategy profile that implements an allocation rule of the form (27), is as follows.

Seller's Strategy: The seller at $t = 1$ proposes a "cut-off mechanism," where a buyer can either choose "wait," or to report a value above a cut-off. In particular, $M = (S^1, g^1)$ is defined as follows. The actions space for each i is

$$S_i^1 = \{wait\} \cup [\bar{v}_i, b_i], \text{ for } \bar{v}_i \in [a_i, b_i]. \tag{30}$$

The outcome function $g^1 = (q^1, z^1)$, consists of an allocation rule q^1 that is given by:

$$\begin{aligned} q_i^1(wait, s_{-i}) &= 0, \text{ for all } s_{-i} \in S_{-i}^1 \\ q_i^1(v_i, s_{-i}) &= \begin{cases} 1 \text{ if } i \in I^*(v), \text{ and } i < j \text{ for all } i, j \in I^*(v) \\ 0 \text{ otherwise} \end{cases}, \end{aligned} \tag{31}$$

where $I^*(v) = \{i \in I, \text{ s.t. } i \in \arg\max_{i \in I^1(v)} J_i(v_i)\}$ and $I^1(v)$ denotes the set of buyers that choose actions different from "wait" at $t = 1$ when the vector of realized valuations is v. This is the set of buyers that participate at $t = 1$.

In words, if buyer i chooses "wait," at $t = 1$, he never gets the good at $t = 1$, otherwise, he obtains the object with probability 1 if he has the lowest index among all buyers whose virtual valuation is highest among all the buyers in $I^1(v)$. This is a mechanism that unless *all* buyers choose to "wait" at $t = 1$, one of the buyers is awarded the good with probability one at $t = 1$. The seller at $t = 2$ after the history where all buyers choose "wait" at $t = 1$ proposes a direct revelation mechanism with an allocation rule described in (28).[32]

Buyers' Strategies: At $t = 1$ a buyer who has valuation below $\bar{v}_i$ chooses "wait" and a buyer with valuation above $\bar{v}_i$ reports the truth. At $t = 2$, buyer i reports the truth when $v_i \leq \bar{v}_i$. For $v_i > \bar{v}_i$ he reports any valuation, say for instance he reports $\bar{v}_i$.[33] More formally, $i's$ strategy, for $i \in I$, is given by:

$$\begin{aligned} t &= 1: \; s_i^1(v_i) = wait \text{ for } v_i \in [a_i, \bar{v}_i], \; s_i^1(v_i) = v_i \text{ for } v_i \in (\bar{v}_i, b_i] \\ t &= 2: \; s_i^2(v_i) = v_i \text{ for } v_i \in [a_i, \bar{v}_i], \; s_i^2(v_i) = \bar{v}_i \text{ for } v_i \in (\bar{v}_i, b_i]. \end{aligned} \tag{32}$$

Beliefs: Given the buyer's behavior specified by (32), when trade does not occur at $t = 1$ the seller's posterior beliefs are given by (23).

It is immediate to see that this strategy profile implements (27). Now we move to establish that it satisfies all the requirements of being $PBE-$implementable.

Given the buyers' strategies, the seller's strategy is a best response since it generates the highest possible revenue (that is feasible), from the buyers that participate at $t = 1$ and from the buyers that participate at $t = 2$ and, since $\bar{v}$ is optimally chosen, the seller cannot do any better by changing the regions where trade takes place at $t = 1$, versus $t = 2$.

Hence the only requirement that we still need to verify in order to establish that indeed this strategy profile is a PBE, is that the buyers' strategies are best responses at each node. However in order to do so, we need to know how payments are determined at $t = 1$ and $t = 2$. With the help of

$$x_i(v_i, v_{-i}) = p_i(v_i, v_{-i})v_i - \int_{a_i}^{v_i} p_i(t_i, v_{-i})dt_i, \tag{33}$$

[32] We postpone the description of the payment rule for after we describe the buyers' strategies. It will be constructed so that a buyer with a valuation above the cut-off has an incentive to tell the truth.

[33] Note that this is off the equilibrium path because we can only reach $t = 2$ when *all* buyers choose to wait at $t = 1$, which occurs whenever all valuations are below the cut-off $\bar{v}$.

we will construct these payments in such a way so as the previously described strategies for the buyers are best responses at each node. This is guaranteed by the incentive compatibility of p which is established in Proposition 4.

In order to get a clearer view of how the payments can be split across periods, we will rewrite (33) as:

$$\begin{aligned}
x_i(v_i, v_{-i}) &= \left[q_i(s_i(v_i), s_{-i}(v_{-i})) + \delta q_0(s_i(v_i), s_{-i}(v_{-i}))p^2(v_i, v_{-i})\right] v_i \\
&\quad - \int_{a_i}^{v_i} \left[q_i(s_i(t_i), s_{-i}(v_{-i})) + \delta q_0(s_i(t_i), s_{-i}(v_{-i}))p^2(t_i, v_{-i})\right] dt_i \\
&= q_i(s_i(v_i), s_{-i}(v_{-i}))v_i - \int_{a_i}^{v_i} q_i(s_i(t_i), s_{-i}(v_{-i}))dt_i + \qquad (34) \\
&\quad \delta \left[q_0(s_i(v_i), s_{-i}(v_{-i}))p^2(v_i, v_{-i})v_i - \int_{a_i}^{v_i} q_0(s_i(t_i), s_{-i}(v_{-i}))p^2(t_i, v_{-i})dt_i\right]
\end{aligned}$$

For the region $v \in \times_{i\in I}[a_i, \bar{v}_i]$ where ALL buyers choose "wait" at period one, we know that trade never occurs at $t = 1$, that is $q_0(s_i(v_i), s_{-i}(v_{-i})) = 1$, which implies that the first two terms of (34) are zero, and it reduces to

$$x_i(v_i, v_{-i}) = p^2(v_i, v_{-i})v_i - \int_{a_i}^{v_i} p^2(v_i, v_{-i})dt_i,$$

which is exactly the one given in (21), and by the incentive compatibility of (p^2, x^2) we can immediately conclude that the buyers strategies are best responses at $t = 2$.

Now when buyers' valuations lie in $V \backslash \times_{i\in I}[a_i, \bar{v}_i]$ at least one buyer is choosing an action different from "wait." However, matters are slightly more delicate than in the previous case, because for certain regions of valuations *all* terms of (34) are important for determining $x_i(v_i, v_{-i})$. In particular, this is true for the region where only one buyer, say i, chooses an action different from "wait," whereas all other buyers choose to wait. Then for such a $v_i \in (\bar{v}_i, b_i]$ we have that

$$x_i(v_i, v_{-i}) = q_i(s_i(v_i), s_{-i}(v_{-i}))v_i - \int_{\bar{v}_i}^{v_i} q_i(s_i(t_i), s_{-i}(v_{-i}))dt_i - \delta \int_{a_i}^{\bar{v}_i} p^2(t_i, v_{-i})dt_i \text{ for } v_{-i} \in \times_{\substack{j\in I \\ j\neq i}}[a_j, \bar{v}_j],$$

which depending on whether the v'_{-i}s are in a region where all of their *posterior* virtual valuations are below the seller's value or not, it becomes:

$$x_i(v_i, v_{-i}) = \begin{cases} q_i(s_i(v_i), s_{-i}(v_{-i}))v_i - \int_{\bar{v}_i}^{v_i} q_i(s_i(t_i), s_{-i}(v_{-i}))dt_i - \delta \int_{\xi_i(\bar{v}_i)}^{\bar{v}_i} p^2(t_i, v_{-i})dt_i \\ \qquad \text{for } v_{-i} \in \times_{\substack{j\in I \\ j\neq i}}[a_j, \xi_j(\bar{v}_j)] \\ q_i(s_i(v_i), s_{-i}(v_{-i}))v_i - \int_{\bar{v}_i}^{v_i} q_i(s_i(t_i), s_{-i}(v_{-i}))dt_i - \delta \int_{c_j^2(v_{-j})}^{\bar{v}_i} p^2(t_i, v_{-i})dt_i \\ \qquad \text{for } v_{-i} \in \times_{\substack{j\in I \\ j\neq i}}[a_j, \bar{v}_j] \backslash \times_{\substack{j\in I \\ j\neq i}} [a_j, \xi_j(\bar{v}_j)] \end{cases} \qquad (35)$$

where $\xi_i(\bar{v}_i)$ is the optimal cut-off at $t = 2$ given (23), and $c_j^2(v_{-j}) = \inf\{v_i \in [a_i, \bar{v}_i]$ such that $p_i^2(v) = 1\}$.

With the help of (31), (35) can be simplified to

$$x_i(v_i, v_{-i}) = \begin{cases} (1-\delta)\bar{v}_i + \delta\xi_i(\bar{v}_i) \text{ for } v_{-i} \in \times_{\substack{j\in I \\ j\neq i}}[a_j, \xi_j(\bar{v}_j)] \\ (1-\delta)\bar{v}_i + \delta c_i^2(v_{-i}) \text{ for } v_{-i} \in \times_{\substack{j\in I \\ j\neq i}}[a_j, \bar{v}_j] \setminus \times_{\substack{j\in I \\ j\neq i}}[a_j, \xi_j(\bar{v}_j)]. \end{cases} \tag{36}$$

Notice that (36) varies with v_{-i}. However because when $v_{-i} \in \times_{\substack{j\in I \\ j\neq i}}[a_j, \bar{v}_j]$, all buyers $j \in I$ with $j \neq i$, do not make any reports at $t = 1$, it is not possible to design any payment rule that depends on v_{-i}. Therefore it is not possible implement (36) as is. However, notice that (36) has to hold only in the cases where buyer i faces no competition at $t = 1$, that is he is the only one participating at $t = 1$. Precisely then the seller can use (36) to determine a personalized reserve price for buyer i. The reserve price for buyer i, $i \in I$, is denoted by $\mathcal{Z}_i^1$ and it satisfies

$$\begin{aligned} \mathcal{Z}_i^1 F_{-i}(\bar{v}_{-i}) &= \int_{\times_{\substack{j\in I \\ j\neq i}}[a_j, \xi_j(\bar{v}_j)]} [(1-\delta)\bar{v}_i + \xi_i(\bar{v}_i)]\, f_{-i}(v_{-i})dv_{-i} \\ &+ \int_{\times_{\substack{j\in I \\ j\neq i}}[a_j, \bar{v}_j] \setminus \times_{\substack{j\in I \\ j\neq i}}[a_j, \xi_j(\bar{v}_j)]} \left[(1-\delta)\bar{v}_i + \delta c_i^2(v_{-i})\right] f_{-i}(v_{-i})dv_{-i}. \end{aligned} \tag{37}$$

Since buyer i *does not know* v_{-i}, from his perspective he is indifferent between paying $\mathcal{Z}_i^1$ wherever all other buyers choose "wait," (which occurs with probability $F_{-i}(\bar{v}_{-i})$), or incurring payments according (36). Moreover, these two different payment methods are equivalent from the seller's perspective because they have the same expectation. This completes the description of the payment rule of the first period mechanism for the circumstances where only one buyer chooses an action different from "wait."

We continue with the description of the payment rule where there is some competition between buyers at $t = 1$. This happens when more than one buyer at $t = 1$ is choosing an action different from "wait." In that case, a buyer pays only when he wins, and his payment is equal to lowest possible valuation that would still allow him to win, for the given vector of valuations of the other buyers in $I^1(v)$, that is $c_i^1(v_{-i}) = \inf\{v_i \in S_i^1$ such that $q_i^1(v) = 1\}$.[34] The winner, as well as the payment, are calculated based on the reports of the buyers that participate that is the buyers in $I^1(v)$. We have therefore derived the following scheme for the period-1 mechanism:

[34] The payment rule in this region is very similar to the one described by the optimal mechanism in Myerson (1981). The difference is that here, not all buyers participate at $t = 1$.

$$z_i^1(v_i, v_{-i}) = \begin{cases} c_i^1(v_{-i}) \text{ if } i \in \arg\max_{i \in I_1(v)} J_i(v_i) \text{ when } v_{-i} \in \bar{V}_{-i} \backslash \times_{\substack{j \in I \\ j \neq i}} [a_j, \bar{v}_j] \\ \mathcal{Z}_i^1 \text{ when } v_{-i} \in \times_{\substack{j \in I \\ j \neq i}} [a_j, \bar{v}_j] \\ 0 \text{ otherwise} \end{cases} . \quad (38)$$

Putting all the pieces together, we have established the following Proposition:

Proposition 6 *The allocation rule (27) can be implemented by a strategy profile and a belief system that form a PBE of the game. One such strategy profile and belief system is as follows. The seller at $t = 1$ proposes a mechanism described in (30),(31), (38). The only history where trade does not take place at $t = 1$ is when all buyers choose "wait" at $t = 1$ and posterior beliefs are given by (23). After that history, at $t = 2$ the seller proposes the direct revelation mechanism given by (28) and (21). Buyer i's strategy is given by (32), for $i \in I$.*

We are now ready to state and prove the main Theorem of our paper.

Theorem 1 *Without commitment a revenue maximizing seller at $t = 1$ should employ a mechanism where each buyer can either claim a value above a buyer specific cut-off, or choose to "wait". The object is awarded with probability one to the buyers with the highest virtual valuation among all buyers who claimed a value above the cut-off. If no buyer claims a value above the cutoff no trade takes place at $t = 1$ and we move on to $t = 2$ where the seller employs a direct revelation mechanism that assigns the object to an optimally chosen subset of the buyers with the highest posterior virtual valuation, if it is non-negative.*

Proof. From Proposition 1 we know that the value of Program A is an upper bound for how much the seller can achieve. Proposition 3, tell us that we can obtain a solution of Program A by solving an artificial Program B. We described a "quasi" (myopic) solution of Program B in (27). Then we showed in Proposition 5 that this "quasi" solution is a real solution of Program B. Finally, in Proposition 6 we showed that this solution can be implemented by an assessment that is a PBE. From all these results it follows that this assessment is a revenue maximizing PBE.

At this assessment, the seller at $t = 1$ employs a mechanism where each buyer can either claim a value above a buyer specific cut-off, or can choose to "wait". The object is awarded with probability one to the buyers with the highest (extended) virtual valuation among all buyers who claimed a value above the cut-off. If all buyers wait, no trade takes place at $t = 1$ and we move on to $t = 2$, where the seller employs a direct revelation mechanism that assigns the object with probability one to a *subset* of the buyers with the highest posterior (extended) virtual valuation if it is non-negative. This subset contains the buyer(s) whose *prior* virtual valuation is highest among the group of the buyers whose posterior virtual valuation is highest. ■

We now show that in the case of ex-ante symmetric buyers our optimal auction without commitment can be implemented by running a sequence of first price auctions, FPA, with optimally chosen reserve prices in each period. The reserve prices decrease overtime. It can be also implemented by a sequence of second price auctions, SPA, with optimally chosen reservation prices.[35]

Corollary 1 *Suppose that the buyers are ex-ante symmetric. Then the symmetric equilibrium of the game where the seller runs a SPA or a FPA in each period with optimally chosen reserve prices, generates maximal revenue for the seller.*

Proof. If buyers are ex-ante symmetric, then it is easy to see that at a solution of Program B we have that $\bar{v}_i = \bar{v}_j$ for all $i, j \in I$ and $i \neq j$. The reason for this, is that when buyers are ex-ante symmetric the ex-ante optimal boundaries are simply $v_i = v_j$. This ex-ante optimal boundary can be also supported by a mechanism that is optimal at $t = 2$ when $\bar{v}_i = \bar{v}_j$, because then the buyers are still symmetric in the eyes of the seller at $t = 2$.

In this case the optimal allocation can be implemented by a symmetric equilibrium of the game where the seller runs a SPA or a FPA in each period with optimally chosen reserve prices. Consider the symmetric equilibrium of a sequence of SPA with a reservation price in each period. At $t = 1$ a SPA with a reservation price assigns the object to the buyer with the highest valuation, (which due to symmetry is the buyer with the highest virtual valuation), among all buyers that submit a bid above the reservation price that the seller has posted at $t = 1$. This follows from the fact that conditional on submitting a bid above the reserve price, it is a dominant strategy for a buyer to submit a bid equal to his true valuation. At a second price auction trade does not take place at $t = 1$ if no-one bids above the reservation price. Given ex-ante symmetric buyers, at a symmetric equilibrium, the buyers are symmetric in the eyes of the seller at the beginning of $t = 2$ as well. At $t = 2$ a SPA assigns the object to the buyer with the highest valuation, (who due to symmetry is also the buyer with the highest posterior virtual valuation), if his valuation is above the reservation price posted at $t = 2$. If ties occur, either in period one or in period two, the lowest index buyer among the ones who tie is assigned the good with probability one.[36] Similar arguments hold for a FPA. ■

Our result demonstrates that at an optimum whenever at $t = 1$ the probability of trade with some buyer is positive, than it is equal to one. This is also a feature of the "commitment" solution. Without commitment, however, there are additional reasons to separate types early on, since it may allow higher surplus extraction at a subsequent date.

[35] For an example of optimally chosen reservation prices in SPA and FPA in a dynamic framework see McAfee and Vincent (1997).

[36] This is just one of many possible ways of breaking ties.

One can think that the seller by using a mechanism at $t = 1$ that consists of lotteries, may on one hand, reduce the probability of trade at $t = 1$, but on the other hand, lead to such posterior beliefs at $t = 2$ that will allow for higher surplus extraction in the future. That is, the seller could use $t = 1$ as an experimentation stage, that would allow her to obtain sharper information about the buyers' valuations that she could, in turn, use to obtain more revenue at $t = 2$. Our results show that it is not worthwhile for the seller to do so. Trying to separate out types is too costly.

5. Analysis of the Problem when $2 < T < \infty$

The characterization of a revenue maximizing PBE for the case where $2 < T < \infty$ proceeds by induction. We start by looking at the case where $T = 3$. So far we know that if $T = 2$ the seller maximizes expected revenue by employing a "Myerson" auction in each period. This result was established for general distributions. Then, our $T = 2$ characterization can be employed to obtain what mechanisms the seller would employ at every continuation game that starts at $t = 2$ irrespective of the structure of the seller's posterior beliefs.

However some caution has to exercised, because our $T = 2$ solution was derived assuming that the seller and the buyers share a *common prior.* This will not necessarily hold at a continuation game that starts at $t = 2$ if the seller is employing general disclosure policies at $t = 1$. Fortunately, these difficulties turn out not be an issue, since as we now argue, the same forces that make the disclosure policies irrelevant in the two-period version of the game are also present in the case where the game lasts three periods.

This conclusion is, however, not immediate, since in the two-period version of the model, the problem at the beginning of $t = 2$ is a static problem (the game ends afterwards) so we can employ the revelation principle. When $T = 3$, the problem at $t = 2$ is more complicated because we have to take into account the sequential rationality constraints. To see what is going on we have to examine the effect of disclosure policies on a problem described in (9) if we add sequential rationality constraints.

First note that Proposition 2 in Skreta (2007), which is shown for the version of the problem without SRC, still holds for the same reasons as before when we add the SRC. Then in this case too, we can obtain a solution of the informed seller's problem, by solving a program analogous to Program S described on page 17 in Skreta (2007), with the modification that we add the sequential rationality constraints. This "enriched" Program S has also a solution that is dominant strategy incentive compatible. This follows from our Remark 2. Then, all the key forces necessary to prove Theorem 1 in Skreta (2007) are present and using those steps we can establish an analog of Proposition 2. From this it follows that it is without loss to assume that all buyers observe each others' actions at $t = 1$. This result allows us to apply our characterization for the $T = 2$ case and to conclude that when $T = 3$ at every continuation game that starts at $t = 2$ the seller maximizes revenue by running a

"Myerson" auction in each period.

With this result in hand, we can mimic the analysis of the $T = 2$ case. All we need to do is to reinterpret $p^{2(s)}, x^{2(s)}$ as the allocation and the payment rules that are implemented by a continuation game that starts at $t = 2$. In particular, the steps of deriving our "quasi-solutions" are identical, with the only difference that now p^2, x^2 is a solution of a two-period problem without commitment. Then, in order to establish that our quasi solution is a real solution we can again appeal to "information postponement principle" and proceed in an completely analogous way as we did for the case that $T = 2$. Given the result for $T = 3$, we can continue to get the result for $T = 4$ and so forth.

6. Robustness and Concluding Remarks

We now offer a few remarks on the generality of the solution. We have considered a multi period situation where the seller cannot commit to the future sequence of auctions that she employs. Our objective has been to characterize the PBE that guarantees the highest expected revenue for the seller among all $PBE's$. We have shown that at a revenue maximizing PBE the seller employs simple mechanisms. When buyers are ex-ante symmetric our solution reduces to a sequence of first or second price auctions with optimally chosen reservation prices.

The set of $PBE's$ of the game depends (i) on the generality of mechanisms that the seller employs (ii) on the generality of the buyers' strategy (iii) on the length of the time horizon and finally (iv) what the seller observes during play. With respect to the generality of definition of "mechanisms" we have been very general: we have assumed that a mechanism consists of some abstract game form endowed by an information disclosure policy as a way of capturing different scenaria of what buyers observe during the play of an auction. This issue of transparency does not appear in single agent environments, but it is crucial here, because what buyers observe during play affects their perceptions (beliefs) about their opponents, which determines how they behave subsequently. Regarding the generality of the strategy that buyer i maybe employing, we have not imposed any restrictions: buyer i may be employing mixed strategies and non-convex set of types maybe choosing the same actions. Finally regarding what the seller observes during play, we have assumed that she observes the vector of actions that buyers choose at each stage. This assumption makes the non-commitment constraints quite strong and it is intentionally so. The point of our analysis is what is the best that the seller can do given that she cannot commit. If we had assumed that the seller observes nothing overtime, then trivially the commitment solution is sequentially rational.

In some recent work Skreta (2007c) shows that when the seller cannot commit not to propose another auction in the event that trade does not take place, but observes only whether trade or no trade takes place in each period, then she can achieve *strictly* higher revenue than what she achieves at the optimal mechanisms we have derived here, where the seller observes vectors of actions in each period. In that case too, optimal mechanisms have a cutoff structure, however, the cutoff for a buyer depends on the other buyers' valuations.

The government in the United States and in other countries sells important assets, such oil tracts, timber tracts, spectrum and treasury bills through auctions. Optimal design is especially important for revenue generation when the number of buyers who participate in the auction is very small and there is hence little competition. This is usually the case for auctions of very valuable assets. This observation, together with the fact that a large fraction of items that remain unsold are placed back in the market, makes the characterization obtained in this paper a relevant extension of the optimal auction literature.

Moreover, this is the first paper that characterizes optimal mechanisms in a multi-agent problem when the designer behaves sequentially rationally. A methodological contribution of the paper is to develop a procedure to characterize the optimal dynamic incentive schemes under non-commitment in asymmetric information environments with multiple agents, whose types are drawn from a continuum. The assumption of commitment, which makes the characterization of the optimal incentive schemes a relatively straightforward task, implies that the principal will behave in a time-inconsistent manner and it is not very appealing for many applications. Designing multi-period incentives schemes under various assumptions of commitment is an area of great importance, since most relationships are multi-period and most parties renegotiate or change a contract if it becomes clear that there exist other that dominate it. We hope that the procedure presented here will prove useful for the characterization of the optimal dynamic incentive schemes under no or limited commitment in other asymmetric information environments.

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