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ECONOMIC AND FINANCIAL DATA AS NONLINEAR PROCESSES

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INTRODUCTION

The last decade has witnessed a resurgence of interest in nonlinearity as applied to a wide variety of disciplines including economics. This renewed interest has been occasioned by the development of new research tools and the growing disenchantment, in economics at least, with the forecasts that our linear models have produced. We have now come to the realization that models of money demand and supply, of the stock market, of national aggregates, such as, G.N.P., investment in plant and equipment, consumption, and savings, provide reasonable forecasts only in those cases where the models are not really needed; that is, where the next period is simply last period plus an approximately constant growth factor, or plus a random change with zero mean.

Efforts to forecast the effects of significant changes in the institutions of an economy, such as the creation of a futures market, the development of new financial instruments, or a change in tax policy; or to forecast the response to a substantial shift in industrial organization, such as the effective creation of O.P.E.C., all have failed to meet reasonable requirements of forecasting usefulness. Notwithstanding our elaborate econometric sophistication and clever modelling, our predictions, even when right in qualitative terms, are seldom more informative than intuition guided by "first principles".

What is even more embarrassing is the discovery that many economic variables are easily transformed into "uncorrelated noise" by such simple procedures as taking log first differences. Alternatively and nearly as damaging, is the fact that log first differences may produce a simple low order AR, or MA, model, but seemingly with no further structure, except for some evidence of time varying variances.

Apparently this result indicates that the only information in economic time series is contained in the first two moments and those have been milked to little effect. A major question is whether or not analysis directed to non-linear relationships can detect anything that is not revealed by the current procedures.

The situation on the face of it is depressing. Economic forces do not seem to be much in evidence in economic or financial time series. Market fundamentals do not seem to matter as the Black-Scholes model would indicate. Once common drift has been eliminated from economic or financial time series, the only possibility seemingly left for economic interaction is in the time variation of the variances of the series. Noise dominates.

As I will discuss in the body of the paper, economic data are different from the data that are analyzed in biology, or chemistry, or physics, or pure statistical time series generated by simulation with linear models. We may not yet know what they are, but we can say what they are not. Economic data are not simple ARIMA processes, they are not describable by simple linear structural models, they are not simple white noise, they are not random walks, and finally there is no evidence that there are any low dimensional attractors.

The time has come to re-evaluate the analysis of economic data and more importantly to reconsider our basic modelling procedures. A key question is what type of modelling is most relevant and productive for economic data. For some of us the search is on to discover how economic data might be modelled in generic terms.

The next section of this paper introduces some concepts of nonlinearity and tries to indicate what the potential advantages of a non-linear approach might be. In this section it is claimed that, while there may be no evidence of chaos in real economic data, the ideas and concepts in the "new" literature on qualitative dynamics is nonetheless very useful in providing insights. As a part of this discussion, system modelling is contrasted to the analysis of "isolated" time series in the following section.

The next short section identifies the economic and financial data series that have been examined as examples of the general conclusions to be drawn from this paper.

The following section discusses some of the statistical tools that can be used to address some issues of nonlinearity and their possible occurrence in economic and financial data. Some empirical examples are worked up as illustrative exercises.

The final section is important in that it contains the conclusions and speculations about the empirical results. Particular attention is paid to the problem of forecasting and some ideas are presented about what can be expected for forecast improvement from the use of non-linear analysis. The key implication in this context is that the main benefits from a non-linear analysis is in the timing of reactions to an intervention as well as the ability to detect subtle possibilities for instabilities.

I

WHY NON-LINEAR DYNAMIC ANALYSIS?

In fact, non-linear dynamic models have a long and venerable history in Economic analysis, both theoretical and empirical. The empirical literature begins with the early work on cycles with which study the name of Wesley Mitchell(1913) is justifiably linked, even if not the first. The theoretical literature begins with the early articles of Frisch(1933), Samuelson(1939), Kalecki(1943), Hicks(1950), Goodwin(1951), Phillips(1954), and finally Goodwin(1972) to mention just a few of the classic works in the field of those that were interested in more than equilibrium or balanced growth. An amusing, iconoclastic, yet insightful, review of this literature is contained in Blatt(1983). An article of interest in this connection is Baumol and Benhabib(1988).

But the analysis of non-linear dynamic models cannot be said to have dominated center stage; they have remained exceptions to the main stream of the literature, albeit often brilliant exceptions. This is surprising as the effectiveness of policy in a possibly non-linear world depends so heavily on getting the timing of policy decisions correct, not to mention the important issue of the stability of even profitable speculation, see for example, Baumol(1957,1961).

Perhaps the best way to illustrate the role of non-linear dynamics is to consider some examples. I will develop a simple model of market reaction to excess demand that is made more realistic step by step. None of the models is meant to be a serious candidate for the dynamical description of actual markets, but the discussion should be enlightening as to the potential role to be played by non-linear systems.

The objective at this time is not to present models of immediate applicability, but to stress the inherent properties of a dynamical system. Some form of oscillatory behavior is at the root of all but the most trivial of dynamical systems. Thus, the equations that we will discuss below should be regarded as fundamental prototypes that have provided, not only useful insights, but practical predictions in an incredible array of disciplines. As simple prototypes, the equations to follow should be taken seriously; they are not just yet another economist's idea of a micro-relation.

We do know that excess demand creates economic forces that, usually, tend to offset the excess demand. If that is so then the initialization of that reaction must be through the acceleration of the change in excess demand. This is the basic key to understanding the main idea underlying the discussion to follow. The idea goes back at least to Samuelson, Goodwin, Kalecki, and Phillips.

Some Simple Alternative Dynamical Models of Market Adjustment

The first simple model will do some violence to the known strong interdependencies between markets and will not stress the role played by prices in equilibrating market forces. However, its very simplicity will enhance the ideas to be illustrated.

The symbol "X" designates for some market the amount of excess demand for a given commodity at a given point in time t. The commodity could as easily be shares of some stock as a specific real commodity such as wheat or cookies. X is assumed to be a continuously differential function of time t.

$\dot{X}$ designates the time derivative of X.

The total stock, or size of the market, is designated by S; S is assumed fixed for the purposes of this analysis as it determines the scale within which the effect of excess demand, denoted by X, operates.

$\ddot{X}$ is the acceleration in the change in excess demand; $\ddot{X}$ is a continuously differential function of time t. Economists are not accustomed to think in terms of acceleration with respect to changes in excess demand, but the idea is intuitively plausible; rather the idea is fundamental to the notion of dynamics as both Samuelson and Goodwin realized so many years ago.

If economic forces are to react to an exogenous shock to restore equilibrium, then to go from a zero velocity of excess demand to some non-zero velocity, the acceleration must have been non-negative to achieve this result. What changes "first" is the acceleration in the excess demand and if the system has a first necessary condition for stability, then the direction of the accelerated reaction is to oppose the displacement.

Price is an indicator function that translates demand and supply relationships into an excess demand function. Price is implicit in the formulation of this model, but we can conclude from this formulation that:

$$\text{I.1)} \quad \dot{P} = k\dot{X};$$

And therefore that:

$$\text{I.2)} \quad P = k_0 + kX,$$

where k and k_0 are arbitrary constants. A more realistic model would recognize the lags that will exist between the occurrence of a non-negative excess demand and a change in P and between the change in P and the acceleration of the "restoring force".

If there has been a displacement in equilibrium by an amount X, the restoring force is:

$$-\sigma X,$$

where σ is the "coefficient of stiffness" in the restoring force; the greater the value of σ, the greater the economic force to restore the original equilibrium. We can now write our "equations of motion" for our economic market, namely;

$$\text{I.3)} \quad \ddot{X} = -\sigma(X/S),$$

or,

$$S\ddot{X} + \sigma X = 0,$$

which can in turn be rewritten as:

$$\text{I.4)} \quad \ddot{X} + (\kappa\eta)^2 X = 0.$$

The idea of equation (I.3) is that the restoring force accelerates the total stock against the displacement; alternatively stated the equation shows that the direction and magnitude of change is proportional to the relative size of the excess demand, relative, that is, to the size of the total market.

Equation (I.4) anticipates results later in the paper and indicates that σ/S can be re-expressed in terms of η, the frequency of oscillation of the response; that is, η is:

$$\text{I.5)} \quad \eta = 1/(2\pi)(\sigma/S)^{1/2}.$$

The constant κ in this simple case is 2π. Equation (I.5) shows that the frequency of response decreases in S, but increases in σ. This is as would be expected. The larger S, the larger the inertia of the system to a given size of displacement, X. The larger the value of σ, the larger the restoring force, and so the greater the impact on the acceleration of change. As X approaches the value of zero, the acceleration slows as shown by equation (I.3).

It is easy to show that one expression for the general solution of the differential equation (I.3) is:

$$\text{I.6)} \quad X = a\text{Sin}(\omega t + \phi); \qquad \omega = (\kappa\eta);$$

where a is the amplitude of the oscillation, ω is the natural frequency of oscillation and ϕ is the phase; a and ϕ are determined by the initial conditions.

That equation (I.6) is a solution to equation (I.3) is easy to demonstrate by taking its derivative with respect to time twice:

$$\text{I.7)} \quad \dot{X} = a\omega\cos(\omega t + \phi)$$

$$\text{I.8)} \quad \ddot{X} = -a\omega^2\sin(\omega t + \phi) = -\omega^2 X.$$

These last two equations are interesting notwithstanding the extreme simplicity of our defining differential equation. We see immediately that the first time derivative of excess demand, that is, the flow of excess demand, leads the state of excess demand, X, and the acceleration, $\ddot{X}$, by $\pi/2$ radians or 90^0; lack of phase matching is not generally given much attention by economists. While the amplitude of the excess demand oscillation is "a", that of the velocity is "aω", where ω is defined in terms of radians modulo 2π; more precisely the velocity has an amplitude of "$a(\sigma/S)^{1/2}$" and the amplitude of oscillation of the acceleration is "$a(\sigma/S)$". Thus, for a given amount of excess demand, the larger the size of the total market, the smaller the amplitude of the velocity of change. This is not surprising.

The current model is too simple even for physical applications, much less, economic ones. As our first step towards approaching reality, let us modify equation (I.3) to allow for "friction" in, or damping of, the system; that is, let us assume that the acceleration in response to the restoring force is reduced in proportion to the velocity of change. We now have:

$$\text{I.9)}\quad S\ddot{X} = -\sigma X - \beta\dot{X};$$

where $-\beta\dot{X}$ represents the effect of "friction" in the system.

Before continuing, it is useful to note that an alternative expression for the general solution to equation (I.3) is:

$$\text{I.10)}\quad a\text{EXP}\{i(\omega t + \phi)\}, \quad i = \sqrt{-1},$$

where the constants have the same interpretation as before.

The general solution of equation (I.9) is of the type $X = CEXP\{\alpha t\}$, where C is a constant to be determined by initial conditions. If we substitute this suggestion into equation (I.9) we obtain:

$$\text{I.11)} \quad CEXP\{\alpha t\}(S\alpha^2 + \beta\alpha + \sigma) = 0,$$

and solving this equation yields the result that:

$$\text{I.12)} \quad \alpha = -\beta/(2S) \pm \{\beta^2/4S^2 - \sigma/S\}^{1/2}.$$

There are, as is well known, three types of results depending on the sign of the term in curly brackets in equation (I.12). If the term in {} is positive, that is, the damping effect outweighs the restoring effect, it can be shown that the general solution is expressible in terms of hyperbolic sines and cosines. The solution path of excess demand, X, is increasing at first then decreases to produce a single humped curve as a function of time; there are no oscillations.

The case where the term in {} is zero is not of great interest in this case, being merely a boundary situation between the one just discussed and that to follow.

The last situation is one in which there are complex conjugate roots for the solution of α. The solution can be written as:

$$\text{I.13)} \quad X = AEXP\{(-\beta/2S)t\}\sin(\omega' t + \phi),$$

where ω' is given by:

$$\text{I.14)}\quad \omega' = (\sigma/S - \beta^2/4S^2)^{1/2}.$$

Clearly, this solution path is a damped oscillatory one where the damping effect depends on the damping coefficient and the size of the total market or the total volume of the stock of the commodity being traded. The rate at which the oscillations decay to zero is given by $\beta/2S$. It is also useful to note that the natural frequency of oscillation of this damped system differs from the undamped system. From equation (I.14), it is clear that the frequency of the damped system is less by $\beta^2/4S^2$ and that the larger the market the lower the frequency of oscillation. The undamped system's frequency decreases according to $\sqrt{S}$, but while the damped system's frequency also decreases according to $\sqrt{S}$, the difference between the two frequencies itself decreases for larger S; that is, for very large markets, the difference in frequency of oscillation between damped and undamped systems is small.

So far we have concentrated on linear versions of the market reaction to a single impulse shock. We might consider the effect of a series of random shocks on the solution path, but a more informative alternative to begin with is to consider a sinusoidal path of shocks to the market system. We can define the effect of the "forcing function" by:

$$\text{I.15)}\quad S\ddot{X} + \beta\ddot{X} + \sigma X = \delta \mathrm{Cos}(\psi t),$$

where ψ is the frequency of the "forcing equation" and δ is an appropriate constant. The idea is that excess demand in our market of interest is stimulated by activities elsewhere, for example, excess demand in some other competitive or complementary market, or an oscillation in the exogenous factors determining demand or supply that are not included in the formulation of our basic differential equation (I.3).

There are two components to the solution, the first is the solution to the homogeneous portion of equation (I.15), but that solution is the same as the solution to equation (I.9). This part of the solution is the transient part of the solution. Our more important concern is for the long term dynamics and that solution is given by a solution to the non-homogeneous equation.

The real part of the solution is:

I.16) $X = (\delta/(\psi Z))\sin(\psi t - \phi)$,

I.17) $\dot{X} = (\delta/Z)\cos(\psi t - \phi)$,

where:

$$Z = [\beta^2 + (\psi S - \sigma/\psi)^2]^{1/2.}$$

The quantity of excess demand and its velocity differ in phase by $\pi/2$ as before, but more importantly velocity and the forcing term also differ in phase by ϕ, where:

I.18) $\mathrm{Tan}(\phi) = (\psi S - \sigma/\psi)/\beta$.

The importance of this result is that if ϕ is zero then, but only then, are the forcing term and the velocity in phase; only when ψ satisfies the equation:

$$\psi S - \sigma/\psi = 0.$$

Otherwise, we see that for low frequencies $\dot{X}$ leads the forcing term and for high frequencies $\dot{X}$ lags the forcing term. This seemingly innocuous statement is potentially important for empirical research for it points up clearly that the timing relationship between exogenous shocks and the response in velocity may vary from leading to lagging and only in rare circumstances will the two be in phase; parenthetically we might note that economists do not often, if ever, consider the possibility that $\dot{X}$ will lead the forcing term.

Before leaving these simple linear models, we should consider one last possibility, that of the damping factor being close to zero. In this case, if the natural frequency of the harmonic oscillation, ω, in equation (I.6) is incommensurate with ψ, the frequency of the forcing term, that is, ω/ψ is not a rational, then the oscillations of the forced system, are said to be quasi-periodic. A quasi-periodic oscillation is one that returns to any neighborhood of a given point infinitely often, but never repeats its path exactly; at least in one respect we have at last achieved a result that is a characteristic of economic data.

The introduction of nonlinearity into these models can be achieved in either of two, but non-exclusive, ways. We can allow the "stiffness coefficient" to be a non-linear function of X, the quantity of excess demand, or we can recognize the importance of modifying the damping factor.

A very simple model to illustrate anharmonic oscillators is to return to the undamped model with a forcing term. The main change in the model is that now we allow the "stiffness coefficient" to be a function of X and not a constant. Let us retain the symmetry of the reaction in terms of the sign of X by setting:

$$\text{I.19)}\quad \sigma(X) = -(\sigma_1 X + \sigma_3 X^3),$$

where the basic "equation of motion" is:

$$\text{I.20)}\quad S\ddot{X} + \sigma(X) = \delta\cos(\psi t).$$

As in the previous equations, the first order response might naturally be thought to be positive, i.e., $\sigma_1>0$. But now we have a choice for the sign of σ_3. If σ_3 is positive, the non-linear term reinforces the linear part, or that the reaction to a displacement is even greater than in the linear case. But if σ_3 is negative, then the restoring force is less.

The other way in which we can obtain a non-linear model is to modify the damping term. Our basic linear differential equation that described damped harmonic motion in equation (I.9) is only an approximation for small amplitudes. The equation indicates that if β is negative, then the energy of the system would increase indefinitely. This result is not at all plausible, even if it might be for very small displacements. A simple and well known procedure to overcome this problem is to make β, the friction coefficient, a non-linear function of X, the displacement. Let us allow for the friction coefficient to be negative for very small displacements, X, and to be positive for large displacements; that is, the system generates energy for small values of X and dissipates it for large values. Let $\beta(X)$ be defined by:

$$\text{I.21)}\quad \beta(X) = -\beta_0[1-(X/_0)^2],\ \beta_0<0;$$

so that for small X β is negative and for large X β is positive; X_0 is a parameter of the system that determines what constitutes large or small. If this definition for β is substituted into equation (I.9), the resulting expression is called the Van der Pol equation. Our full equation is:

$$\text{I.22)}\quad S\ddot{X} + \beta(X)\dot{X} + \sigma X = 0.$$

The economic interpretation is that for relatively small amounts of excess demand the restoring force is offset by the velocity term, but that if the excess demand gets to be too large, then the usual process applies. Intuitive examples might be the demand for the equity stock of some firm, or the demand for a "popular" restaurant, where "popular" is defined by positive excess demand. In this formulation, the equation holds even when the excess demand is negative, that is, there is excess supply in which case the intuition may be less appealing. In both cases, small changes from equilibrium have destabilizing components.

As is perhaps well known, the solution of this equation is a limit cycle; small deviations from the unstable equilibrium, the center in fact, diverge out to the limit cycle, but large displacements converge to the limit cycle. By choosing the unit of time to be one period, i.e., $1/(\sigma/S)^{1/2}$, and by a suitable choice of units for the amplitude, we can simplify equation (I.22) to read:

$$\text{I.22')}\quad \ddot{X} - (\epsilon - X^2)\dot{X} + X = 0,$$

where ϵ is β_0/ω, $\omega^2 = \sigma/S$. For small values of ϵ, the limit cycle is nearly sinusoidal and the oscillations are nearly symmetric, but as ϵ gets bigger, the limit cycle ceases to be symmetric and the amplitude has a "saw tooth" shape.

This is the first time that we have had a need for a "phase diagram".In the previous examples the phase diagrams were trivial. A phase diagram is a diagram of the phase space; in our last example this is shown by a plot of the pairs $(X,\dot{X})$ that satisfy the equations of motion defined by equation (I.22). Phase space is the space of all possible states of the system. In our examples above, the phase space would be the space of pairs $(X, \dot{X})$ that were consistent with the flow, or solution, to the differential equations defining the dynamic system. When ϵ is small the phase diagram for equation (I.22′) is nearly a circle, but when ϵ is large then the phase diagram is approximately the shape of a tilted rectangle.

As an aside this last class of models that we have been examining can be used to illustrate the concepts of both supercritical and subcritical bifurcations; that is, qualitative shifts in behavior as one of the parameters passes a critical value; see for example, Berge, Pomeau, and Vidal(1984), or Guckenheimer and Holmes(1983). In addition, this model can also illustrate the idea of hysteresis, Berge, Pomeau, and Vidal(1984, pg.42) that will be discussed later. Hysteresis is generated by non-uniqueness in the response function so that as the unstable portion of the response function is reached the response "jumps" across the unstable portion; the point at which this occurs differs depending on the direction from which the unstable portion is reached.

Finally, we can combine our various experiments into one general, but still very simple, statement. The following equation is known as Duffing's equation; by now its constituent parts are familiar;

$$\text{I.23)} \quad \ddot{X} + \delta\dot{X} - \beta X + X^3 = \gamma\cos(\omega t).$$

Equation (I.23) has been expressed in the simplest possible manner, instead of in terms of economically meaningful coefficients in order to shorten an already lengthy discussion.

$\gamma\cos(\omega t)$ is the forcing term. $\delta\dot{X}$ is the damping factor, assumed once again to be linear, but positive. The term $-\beta X + X^3$ is the non-linear version of the "stiffness coefficient" first defined in equation (I.19). But in this version there is an important difference. The sign of the coefficient for the X term is negative in equation (I.23), while that of X^3 is positive; the opposite of the situation shown in equation (I.19). The dynamical effect is that for small changes in excess demand the market reacts in an unstable manner in that the excess demand is reinforced by the reactive force, but as the excess demand builds, the final effect is strongly in the opposite, stabilizing, direction.

Simple though this equation is, the potentially observable results are dramatically different. So far all our models have generated recognizably deterministic solution paths, even if the time paths of the solutions apparently involve discontinuous jumps. Our solution paths have all involved fixed points, periodic solutions, or the discontinuous hysteresis effects alluded to just above. The Duffing model in contrast can produce time paths of solutions that appear to be the realizations of random variables, certainly the paths are not periodic, see, for example, Kawakami(1984).

The first of the two keys to this dramatic shift in the properties of the solution path is that one requires solution paths that are characterized by exponential divergence of nearby paths; the so called "sensitivity to initial conditions". The second key is that the limit set of the long term solution path must be compact. The mathematically inevitable result is that the set of solution points is mixed and the time paths are recurrent, but the exact same path is never followed twice.

These verbal comments are illustrated in Figures 1 to 5. In Figure 1 we see the time path of the solution of equation (I.23) and Figure 2 shows the corresponding phase diagram. The parameter values chosen for this solution were $\delta = .3$, $\beta = -1$, $\gamma = .3$, and $\omega = 1$. By changing δ to .05, setting β to 0, and raising γ to 7.5, the more chaotic data shown in Figures 3 to 5 were produced.

The generation of "statistical complexity" that we have observed in this last version of our excess demand model can easily be supplemented by adding to the model specification the presence of lags, or "delays", in the reaction mechanism. As is by now well known, the addition of delays in a differential equation, essentially introduces an "infinite number of degrees of freedom" into the system, so that very complex solutions are possible. However, it is remarkable that many times the increase in degrees of freedom is quite low; see, for example, the analysis of the Mackey-Glass(1977) equation. Delays are an obvious extension to economic models as has already been mentioned. One would expect a delay between the occurrence of excess demand and the onset of the reaction, in part because the reaction must usually work through prices and one would expect a delay between the occurrence of excess demand and a consequent change in price.

So as not to elaborate endlessly, let me merely indicate some further behavioral alternatives that non-linear relationships can generate without necessarily tying them to the models discussed above. Some of the concepts that should be mentioned are amplitude-frequency dependence, frequency entrainment, and time reversibility.

Amplitude-frequency dependence is ostensibly quite a common observation in economic data. U.S. foreign exchange data certainly has the appearance of this in that the exchange rate with respect to most major currencies has the characteristic of high frequency, low amplitude, oscillations interspersed with low frequency, high amplitude, oscillations. Stock market data has a similar appearance, but not as pronounced.

Frequency entrainment, or frequency locking, is a very common phenomenon in the physical sciences; it is plausible that the idea may also have relevance in economics in so far as there exist economic phenomena that exhibit sinusoidal behavior, even if buried in noise. Imagine a non-linear market system oscillating with a frequency, ω_0, and that we now consider the effect of a complementary market that has an oscillation with a frequency of μ_1. The frequency of the combined market has a frequency of $\omega_0 - \omega_1$ called the "beat frequency" which will decrease linearly as ω_1 increases towards ω_0. However, as ω_1 approaches ω_0 the beat frequency suddenly drops to zero until the frequency ω_1 has risen substantially above the frequency ω_0, at which point the beat frequency reappears and then will increase linearly as ω_1 diverges from ω_0. The same phenomenon occurs for ω_1 approaching ω_0 from above. One interesting speculation in this regard is that if related economic markets differ in frequency by small amounts, then the frequency of the resultant joint market will not be perceptible, whereas disaggregation and separation of the markets would eventually reveal the nature of the oscillation.

The last example is the concept of "time reversibility". Time reversibility can be defined statistically by:

$$\text{I.24)}\quad F(x_{t1},\ x_{t2},\ x_{t3},\ldots X_{tn}) = F(x_{-t1},\ x_{-t2},\ x_{-t3},\ldots X_{-tn});$$

that is, if the joint probability distribution function is characterized by the statement in equation (I.24), the time series, $\{x_{ti}\}$, is said to be "time reversible" and if not, then the series is said to be "time irreversible"; the essential idea is that the "picture show does not run backwards". This idea would be unremarkable if it were not the case that virtually all economic models implicitly assume that economic time series are time reversible. One hint that one is dealing with a time irreversible process is that the time series exhibits asymmetries in the "slopes" of the upward portion of the series as compared to the downward portion of the series. By this criterion one should immediately suspect that G.N.P. statistics may be time irreversible. Further discussion of this important topic is contained in Tong(1983) and in Ramsey and Rothman(1988).

II

SYSTEM MODELLING OR THE ANALYSIS OF AN "ISOLATED TIME SERIES"

The traditional approach to economic modelling for some considerable time now has been one of analyzing contemporaneously at least a complete market, if not an entire economy. The known strong interdependencies between markets, the problem of identification of a singly specified equation, seemed to necessitate the use of multiple equations and many variables. Let me label this approach the system modelling approach. It is the basis for almost all of the literature since the beginning of the 'fifties until the introduction of the vector auto-regressive models and what might be labelled the "Sims' methodology". One advantage of the chaos literature is to add a third alternative.

The key to the modelling approach in the new qualitative dynamics literature is the "reconstruction of phase space"; for an excellent introduction to the language of dynamical systems, see Arnold(1985), or for some very brief comments see the Appendix.

Phase space is the space of all possible states of the system. In our examples above, the phase space would be the space of pairs $(X, \dot{X})$ that were consistent with the flow, or solution, to the differential equations defining the dynamic system. The graph of the flow is related to the phase space by tracing in phase space the time dependent path created by linking time successive points in phase space. The phase space summarizes the long term dynamics of the system, when the points that are due to transients created by specific initial conditions, are deleted. The set of points in phase space towards which the actual dynamical path is converging is called the (forward) limit set. The limit set might be a fixed point, a limit cycle, or a chaotic attractor, or a strange attractor.

An analysis of phase space and its changes to variations in the parameters of the dynamical system can reveal a wealth of qualitative information about the system; the presence of fixed points, of cycles, the stability of fixed points or of cycles, the existence of singularities in the system, and so on. This is why physicists pay so much attention to phase space, especially when the phase space can be portrayed graphically.

The natural coordinates to use are often those of the state of the system; its "position", its "velocity", and so on. But, the phase space of a dynamical system can also be represented in terms of the dynamical path of just one of the natural coordinates of the system, or even in terms of some monotonic, but non-constant, transformation of a natural coordinate. If the dimension, that is, "the number of essential variables, or coordinates, of the system", is κ, then the vector representation of the system in terms of one of the original coordinates must be in terms of a κ dimensional vector. For example, one might consider a sequence of κ dimensional points $\{x, D(x), D^2(X), D^3(x), \ldots D^\kappa(x)\}_t$, t indicating a sequence of observations through time in order to represent the phase space of the original system. An alternative and often equally useful, but simpler, procedure is to use $\{x_t, x_{t+\tau}, x_{t+2\tau}, \ldots X_{t+\kappa\tau}\}$ for some suitable choice of τ. This last procedure has become the standard procedure in most of the chaotic dynamics literature.

The formal justification for this approach to representing phase space is due to Ruelle and Takens(1971), but the mathematical antecedents are due to Whitney's Embedding theorem, Whitney(1944). While the theorem was proven only when the limit set of the dynamical system is a smooth manifold, it is a remarkable fact that in very many actual examples phase space reconstruction has worked for a wide variety of non-smooth, "fragmented" limit sets.

Why phase space reconstruction works is perhaps intuitively clear. The flow, or solution path, of the differential equation system is unique, so that the parametric equations of the flow, $\{x_t, y_t, z_t, w_t\}$ say, can, under suitable regularity conditions, be mapped bijectively onto the parametric equations

$\{x_t,\ D(x_t),\ D^2(x_t),\ D^3(x_t)\}$ which in turn can be mapped bijectively onto $\{x_t,\ x_{t+\tau},\ x_{t+2\tau},\ x_{t+3\tau}\}$.

This is the justification for examining, at least to begin with, a single time series in the chaotic literature. The phase space reconstruction from observations on a single variable can provide the same qualitative information as observations on the path of the complete set of natural coordinates.

Of course, this approach to modelling implies that if one were to do two reconstructions of phase space from two coordinates of a dynamical system, say x_t and w_t, then up to scale transformations and inessential rotations of axes, the reconstructions should be the same; although there is a difficulty in the appropriate choice of the delay "τ" to be chosen in each case that we are skirting for now. In terms of our previous examples, it will not matter whether we use excess demand, X, or price, P, to reconstruct the phase diagram and both reconstructions should be qualitatively the same.

This approach provides a strong test for the presence of underlying systematic variation in economic or financial data; make two phase space reconstructions and compare them. The difficulty in implementing this research strategy is that while these comments appear to be relevant to specific individual markets, the implication for aggregates is not so clear. This is especially true if there are small, but detectable, differences in the phases of the constituent time series, or if the series being considered have close, but distinct frequencies, and what we observe is the beat frequency of the combination.

Before leaving this section, these notions can be tied into the concept of identification. The conditions for the identification of a sub-system can be restated as requiring two sets of conditions. The first is that the transformation from the exogenous variables of the system to the observed endogenous variables of the system be unique up to inessential scale transformations. The second requirement, not usually emphasized in econometrics, is that the implicit dynamical system represented by the time path of the exogenous variables be the flow of a corresponding differential equation system that describes the equations of motion of the market, or economic system.

The condition in econometrics is usually stated in terms of the rank of the matrix of exogenous variables, but my previous statement emphasizes the dynamical nature of the data. Identification is in fact achieved by requiring the path of the exogenous variables to be the flow of some dynamical system.

The conditions for the identification of the equation of a single endogenous variable are less stringent, but similar in spirit to the ideas just discussed. That is, if the "identifying omitted variables" do not produce a unique path, then their omission is irrelevant to the identifiability of the equation.

The conclusion of this sub-section is that as a first step at least and as a device to learn about the qualitative properties of the dynamical system, the analysis of a single representative "variable", or coordinate, is a useful approach. Clearly, having established the nature of the dynamical system, further work would require the determination of the structural links between the various economic variables of theoretical interest. But such an examination would now be cast within the context of known properties of the dynamical system.

III

THE DATA USED FOR ILLUSTRATIVE EXAMPLES.

The data to be used for an extensive, but preliminary analysis, in order to explore these ideas to some empirical extent are described in this section. The data series include post World War II observations on the money supply, M1 and M2, weekly observations on the stock market obtained from the CRSP data, and by contrast data on monthly pigiron production since 1877.

The money supply data are the figures produced by the Federal Reserve Board, not Barnett's Divisia indices that some researchers, including myself, have used previously. The money supply observations are monthly figures from January, 1959 to November, 1987 on M1 and M2. The source is the Federal Research Report, H.6, entitled Money stock Measures and Liquid Assets.

Scheinkman and Le Baron used the Center for Research in Security Prices (CRSP) data. These data are value weighted daily stock returns, with a sample size of 5200 daily returns. Weekly returns were obtained by the simple compounding of the daily returns; the details are contained in Scheinkman and Le Baron(1986). There are 1227 weekly averaged observations that begin in July, 1962 and end in August, 1985.

For a contrast, I have also included economic data; in this case the production of pigiron since 1877 on a monthly basis to 1964. The data series are in two parts; the first part goes from 1877 to 1941 and the second series from 1941 to 1964. The first series were collected by R.F. Macaulay and are stored on the National Bureau of Economic Research (N.B.E.R.) data tapes. The second set were obtained through the N.B.E.R., but the data originated from the American Iron and Steel Institute. The two series overlap for 12 months in 1941. The series were spliced by the simple expedient of averaging the difference in the overlap values and adding the difference to the second series.

Further details can be obtained from the N.B.E.R., New York.

IV

AN EMPIRICAL DESCRIPTION OF SOME ECONOMIC AND FINANCIAL DATA.

The intent of this section is not meant to be in the least definitive, but is meant to indicate some of the tools that are available and to give some idea of the evidence in favor of non-linear models in economic and financial data. An essential element in understanding this analysis is that widely differing non-linear models may have the same auto-correlation function and as a special case may have an auto-correlation function that is identically zero. Consequently, the description of data by auto-correlation functions is irrelevant to the analysis of the non-linear structure.

However, an issue of some importance is the stationarity of the data. The usual non-linear analysis of physical or biological data starts with the assumption that the data are, or have been transformed into, stationarity; further the most common models of nonlinearity yield steady states that can be described statistically by stationarity. Economic and financial data are not by any means stationary. Economists have become used to transformations to induce first order stationarity, but until recently at least, have tended to stick to transformations that induced only first order stationarity; for example, taking log first differences. It is now common knowledge that both economic and financial data are not even second order stationary; see for example, Bollerslev and Engle with associated references in the Bibliography.

The data may be non-stationary to an even higher degree, but that is an unexplored issue at the moment.

As a consequence of this discussion, the pigiron production data were transformed to be at least second order stationary. The procedure used, in the absence of any theoretical information as to the form of the non-stationarity, was to deflate the first order stationary data by a local approximation of the standard deviation. The approximation to the local standard deviation is obtained by calculating a moving average variance of the first order stationary data and then deflating each observation by the square root of the approximate variance so found. This procedure adds to the degree of auto-correlation in the data. But in any event, the observed auto-correlation in the transformed data is allowed for in the analysis to follow.

There are two statistical tools to be discussed in the remainder of the paper. The first is the concept of "dimension", or rather, the various concepts of dimension. The second is the notion of time irreversibility and a new procedure to test for its presence.

Dimensional Analysis in Economic and Financial Data.

At this point in time some of the concepts of dimension have become familiar to economists, even if some of them are not well understood. The correlation dimension, a measure of the relative rate of scaling of the density of points within a given space, permits a researcher to obtain topological information about the underlying system generating the observed data without requiring a prior commitment to a given structural model. If the time series is a realization of a random variable, the correlation dimension estimate should increase monotonically with the dimensionality of the space within which the points are contained. By contrast, if a low correlation dimension is obtained, this provides an indication that additional structure exists in the time series; structure that may be useful for forecasting purposes. In this way, the correlation dimension estimates may prove useful to economists wishing to scrutinize uncorrelated time series or the residuals from fitted linear time series models for information on possible non-linear structure.

The Appendix contains a brief, but intuitive, discussion of the various concepts of dimension that are appearing in the literature in the physical sciences as well as in economics. The basic idea of calculating dimension from a single time series of a dynamical system is that it is, in a fundamental sense, the first piece of qualitative information about the system that is needed; namely, how many essential variables are needed to model the system, or to discover what in fact is the order of the underlying differential equations defining the dynamical system. The concept of "fractal dimension" has tended to confuse the perception of the basic idea and has added a complexity that, while important in its own right, inhibits a clear understanding of the basic notion of "dimension". Dimension, by indicating the order of the defining differential equation, when there is one, indicates the number of variables that are involved in the system. A similar result holds for maps.

The calculation of dimension, if ever it became effective for economic or financial data, would be of inestimable value in econometrics. A major difficulty in any attempt to model a market or even an economy is the fact that we have no theory to limit the number of variables and equations that must be included in any empirical analysis; current theory leaves us with an unmanageable host of variables to consider so that the decision is made ad hoc and is always treated as an empirical issue.

Whether, or not, the data have a fractal dimension is in light of this discussion irrelevant; the first step is to limit the set of possible variables. Unfortunately, in economics no one has yet been successful in isolating any low dimensional systems.

This may be due in part to the fact that economic and financial data require a different modelling approach than is true for physical or biological data. Randomness in its many forms is an obvious hazard to any formal attempts at modelling. Economic systems are very open relative even to biological systems, are subject to constant tinkering by economic agents, and whenever those agents learn more about the operation of the system, they alter their behavior and as an consequence alter the structure of the observed dynamical system. None the less, it still seems to be reasonable to assume that the system is not completely random and that once a suitable modelling approach has been acquired, that further insight can be obtained.

By this time, a substantial number of economists have attempted to discover "low dimensional attractors" in economic or financial data, Frank and Stengos(1987a,1987b), Brock and Sayers(1988), Barnett and Chen(1988), Hsieh(1987), Scheinkman and Le Baron(1986), and even this author in some unpublished earlier work. The data series that have been examined by these researchers include gold and silver prices, G.N.P., stock returns, various definitions of the money supply, work stoppages, and numerous foreign exchange rates. The list is not complete, but is illustrative.

The outcome of all this research is that as yet there is little evidence for the presence of "low dimensional attractors" in economic and financial data. This somewhat depressing claim is substantiated in Ramsey, Sayers, and Rothman(1988). Part of the difficulty is that economic and financial data contain much more noise than physical data and that the length of the data runs is minuscule by the standards in physics. Consequently, the procedures are not really adopted to the economic and financial environment. However, the results do indicate that there is some evidence of nonlinearity. The reader is referred to Ramsey and Yuan(1988) as well as Ramsey, Sayers, and Rothman(1988) for further details. Besides, the case for nonlinearity in economic and financial data has been made most ably by a remarkable series of papers by Patterson and Hinich, see the Bibliography under these two names.

There exists yet another set of problems that are particularly severe in a non-experimental discipline like economics. These problems involve the extended "maintained hypothesis" that is needed in economic analysis. In the problems examined to date in physics and chemistry, the simple dichotomy of: "either an attractor, or the data are merely (high dimensional) noise" has been considered to be appropriate. But this is not the case in economics. The extended maintained hypothesis must include as alternatives the options that the data come from ARIMA or non-linear stochastic processes.

Even more damaging to a simplistic version of dimension calculation is the realization that often researchers mistakenly perform dimensional analysis on data that are highly auto-correlated; this procedure vitiates any conclusions that might possibly be made. This is because dimension is a topological concept and at certain scales of magnification of some stochastic processes, the dimension is in fact quite low; for example, the dimension of a geometric random walk, is at intermediate scales, about 1.1: a geometric random walk can be regarded at such scales as a highly convoluted line, giving it a topological structure that is of dimension slightly higher than that of a line. What is worse is that if the data are generated by a simple ARMA process with long auto-correlation lag, that is, a long period before the auto-correlations die to zero, then dimension calculations with such data will produce, over a range of scaling values, low dimensional results.

The problem for all experimental data, even if there were a perfectly well defined and recoverable attractor, is that at small enough scales the dimension is that of noise. Thus, the practical problem of trying to distinguish between attractors, auto-correlated processes, and non-linear stochastic processes is a real one.

The conclusion is that, while the concept of dimension is of potentially great interest to econometricians, the current approach has not yet discovered how to disentangle low dimensional results from the inevitable noise and large scale aperiodic shocks that seem to beset all economic and financial data.

Indeed, the actual situation is much more of a puzzle. If the current econometric wisdom is correct, then the seemingly universal absence of economic and financial theory to relate this vast array of low order AR or MA processes, and sometimes just random walks, is the real challenge. If economic theory is to be resuscitated in terms of economic and financial time series, it will only be by pursuing alternative lines of enquiry.

Tests for Time Irreversibility in Economic and Financial Data.

The concept of time irreversibility is discussed in some detail in Ramsey and Rothman(1988). The definition of time reversibility was given above. The procedure used in Ramsey and Rothman to characterize and to test for the presence of time irreversibility is as follows. The statistics G^k_{ij} are defined by:

$$\text{IV.1)} \quad G^k_{ij} = T^{-1}\Sigma[X^i_t X^j_{t-k} - X^j_t X^i_{t-k}],$$

can be used both to characterize and to test for the presence of time irreversibilities. Under the null hypothesis that the series is time reversible, the expectation of G^k_{ij} is zero for all $\{i,j,k\}$. In practice, Ramsey and Rothman have found that $i = 2$ and $j = 1$ are sufficient for discovering time irreversibility in a wide assortment of data. As discussed in Ramsey and Rothman(1988), the statistic G^k_{ij} expressed as a function of k can be regarded in a similar light to an auto-correlation function. How large k can be allowed to be depends on the statistical properties of the time series and on sample size. The larger k the larger the corresponding standard errors. Standard errors are also increased when the data are highly correlated. As discussed in Ramsey and Rothman(1988), the shape of the plot of G^k_{ij} as a function of k is indicative of the type of time irreversibility that is exhibited in the data. For example, if the time series are characterized by "cycles" that are slow up and fast down, then the general shape of the plot of G^k_{ij} is also a cycle of the same period, but different phase. The distinction is that for a cycle that is characterized by slow up and fast down, as opposed to the opposite situation, the G^k_{21} values tend to be negative and the opposite is true for the reverse cycle.

Figure 6 shows the time series of the Ml definition of the money supply and Figure 7 shows the log first differences of that series; the period of maximum amplitude is in the middle of the "Volker experiment" during 1979 to 1981. Ml is definitely non-linear and time irreversible. From Figure 8, one sees that after an initial negative effect, the presence of time irreversibility does not reveal itself until about 62 months and the effect lasts up to 80 months; that is, the main non-linear effects for Ml seem to be concentrated in a time period of between five and six and one half years.

The confidence intervals are estimated 95% intervals based in part on the theory underlying the distribution of the G^k_{ij}, supplemented by Monte Carlo simulations to determine the effect of an auto-regressive structure on the standard errors of the estimates. The procedure is described more fully in Ramsey and Rothman(1988)

While the raw time series plot for M2 is very similar, the plot of the log first differences is quite different, see Figures 9 and 10. However, due in part to the much stronger AR effects that seem to hold for M2 as compared to that for Ml. The standard errors for the estimation of G^k_{21} are very large and seem to increase at a fast rate; there does not seem to be any strong evidence for time irreversibility in M2 data. Certainly, the presence of strong nonlinearity over a time period of 60 to 80 months is missing.

Figure 12 shows the raw time series for the Scheinkman Le Baron stock return data. The evidence for any type of ARMA process in these data is very weak, so that the nonlinearity test for time irreversibility was applied directly to these data. Figure 13 shows clearly strong evidence of time irreversibility; in this case the effect is concentrated in the first 20 weeks.

The claim is often made that financial data are different from economic data. However, both types of data evidence in their raw state exponential growth in the series. This statement is substantiated by the plot of monthly pigiron production from 1877. Within year variations of the data series had to be smoothed before any other analysis could be usefully performed. The data were subsequently differenced in the logs; the remaining steady increase in the amplitude of the oscillations is still apparent. Figure 16 shows the plot of the G_{12}^{k} function for these series from which we see that time irreversibility is clearly present. But in this case, the effect is a sharp positive peak at a lag of about 26 months.

A related, but different approach to the analysis of economic time series is discussed in detail in Ramsey and Montenegro(1988). This idea can be used whenever the time series can be represented as an MA process; but the extension to a general ARMA process has not yet been completed. The basic idea is that corresponding to any MA(q) process there exist 2^q different models with the same auto-correlation function; that is, all 2^q models belong to an equivalence class that is defined by the auto-correlation function. The alternative members of this equivalence class are defined by the set of roots $\{\lambda_i^{\pm 1}\}$, where $\{\lambda_i,\ i = 1,2,..q\}$ are the roots of any one of the alternative models, say, the invertible one. The invertible model is the only one of the set of alternative models that can be estimated by the usual procedures of either least squares or maximum likelihood. The invertible model is the one that can be expressed as a convergent sum of past values of the observed time series.

In Ramsey and Montenegro(1988) it is shown that, provided the innovations in the MA(q) process are not Normally distributed, the true model can be distinguished from the other members of the equivalence class. The approach uses the higher order cumulants, which are identically zero for the Normal distribution. Given that the actual model's coefficients can be identified, it is possible also to estimate the innovations generating the observed process.

The minimum mean square error type of forecast is not enhanced by the use of non-invertible models, although the appropriate confidence interval is modified by this approach. The real gain is in evaluating the timing of the effects of shocks and that the effects of a deliberate, non-random, intervention can be correctly assessed.

In Ramsey and Montenegro(1988) several time series were examined. Of those that were found to be low order MA, two series in particular gave clear evidence of being generated by a non-invertible process; the prime rate and expenditures on plant and equipment. Following Mehra(1986) the prime rate was included in a demand for money equation that has well recognized antecedents; the objective was to compare the effects of using the prime rate itself in the demand for money equation, the usual estimates of the innovations, and those innovations generated from the chosen non-invertible model. The results are most encouraging. While further work is needed to verify the conclusions reached so far, the clear implication is that the innovations from the non-invertible model are the preferred "explanatory" variable in the cited demand for money equation.

IV

CONCLUSIONS AND SPECULATIONS.

The main conclusion is easily stated. Economic systems are clearly nonlinear. The evidence is not only from the research cited above, but also from the extensive work of Hinich and Patterson. Even in the case of stock market returns data at low levels of time aggregation for which the case for a random walk model is strongest, the presence of nonlinearities is clear in the work of both groups of researchers.

However, the work above also indicates that the appearance of that nonlinearity may not be apparent until after a considerable lag that is measured in years. The implications for policy are serious if this result is even only partially true, because under these conditions the implementation of policy based on linear models will inevitably be in ignorance of the nonlinear effects. Worse is the fact that these circumstances will inhibit the discovery of the nonlinearity, the policy will seemingly work for awhile and then, apparently at random, break down. Imagine a myopic dog trailing a random walk rat. He will always think that he is hot on the trail and that next time he will at last catch the rat. But the next time never comes unless someone tells the dog what the underlying nonlinear mechanism is so that he can anticipate the rat's next move and the rat's reaction to his own moves.

Linear policy in a nonlinear world with delays is destabilizing; at least some of the time and probably most of the time.

A question that Professor Brock has often raised with me concerns the forecasting benefits of nonlinear models. For if nonlinear models can do no better than that which we now have, then there is no point in adding useless nonlinear burdens. The answer depends on the criterion for success and the objective.

If we retain the usual minimum mean square error criterion(mmse) and wish merely to observe the future, then nonlinear models will do little for us. The only source of improvement is a more accurate estimate of the confidence interval that we will place on our forecasts. For example, even in the case of the non-invertible MA(q) models knowledge of the correct model does not improve our mmse forecast, except in improving the accuracy of the statement of the standard errors; one might as well use the tried and true invertible version.

Further, even if there were a chaotic model underlying economic data, the forecast improvement potential is strictly limited to the very short run. Indeed, the real gain here is to know that no matter how much data one has, there are very strict limits on the forecasting period for a given level of accuracy. If in addition we add noise, as is likely to be the case, then the benefit of nonlinear models becomes problematic at best.

These highly negative comments are conditional on the choice of a mmse criterion and the objective that we wish merely to observe. If we relax either of these restrictions, the situation is changed dramatically. In order to focus attention on the essentials, let us imagine that we have a nonlinear model, or alternatively a non-invertible MA(q) model, that has the exact same auto-correlation function as an invertible MA(q) model; as a very special case, the invertible model might be simply uncorrelated random variation.

Let us change our objective. We no longer wish merely to observe, but also to implement policy. The implementation of policy in terms of our model is to assume that we will modify the path of our variable of interest, say the money supply, by imposing our own impulse onto the system in addition to whatever "random" variation occurs simultaneously. Now knowledge of the "true model" is vital if we wish to evaluate the likely effect of our policy. In effect our policy action has changed the model. Next period's output will not be the result of the same mechanism as before; we have modified it to include our own impulse to the system. The average effect will be what we would have obtained in the absence of our impulse, plus the "true model's" processing of our non-random impulse. Evaluating the policy action on the basis of the "incorrect model" will be misleading.

Let us now also change our criterion from the "averaging of squared errors" to consider the time path of the reaction of the system to any specific impulse. Knowledge of the correct model is crucial to meeting this expanded criterion function. For example, the non-invertible MA(q) model will give very different results to those predicted by the corresponding invertible model with the same auto-correlation function, indicating, perhaps, that the main response will be delayed beyond the initial period. While the dissipative systems with random shocks that economists traditionally assume may have the same expectational properties as a chaotic attractor, for example, the dynamical paths differ substantially.

Finally, if we add to our set of criteria a concern for the stability of the system and a recognition that stability may be state dependent, then once again the nonlinear model is needed and the linear approximation is no longer a useful tool. Indeed, questions of stability can only be posed in the context of nonlinear models.

Moving into the realm of speculation, the analysis above seems to lead to the following ideas about an appropriate modelling structure. Some modest, but encouraging, attempts have already been made along the lines to be suggested. The economic system must be modelled as a dynamic one, we need to begin to discover the appropriate "equations of motion". But economic systems also contain noise that is embedded in the dynamic itself, that is, the noise term enters the solution of the underlying differential equation system. There may also be observational noise as well.

What is worse is that economic systems are subject to episodic shocks that re-initialize the system. By "episodic" I mean shocks that occur every so often, but with no detectable underlying probability distribution.

It is also likely that over time the parameters of the system are changing with technology and population, but with a little bit of undeserved luck these changes can usefully be assumed to be changing slowly.

Finally, as economists well know, but tend to forget, increased understanding of the system will inevitably change the system, because we are dealing with the actions of rational optimizing agents; no matter how bounded that rationality may be.

APPENDIX

The idea of this Appendix is to provide a quick intuitive introduction to some of the language used in the chaos and qualitative dynamics literature; the various references that are cited can be pursued to good effect.

Three terms that are frequently used in this literature are "attractor", "embedding dimension", and "orbit" and should be at least intuitively defined in this paper. An "attractor" in the context of dynamical analysis is that sub-set of points towards which any dynamical path will converge; that is, the dynamical path is "attracted" to a subspace of the space containing the paths of the dynamical system from any initial condition. "Embedding dimension" is the topological dimension of the space in which the attractor is situated; loosely stated the embedding dimension is the number of axes needed to portray the attractor. Topological dimension specializes in vector spaces to the usual notion of Euclidian dimension. "Orbit" is essentially a synonym for the dynamical path, but also implies the notion that the dynamical path revisits any given part of the attractor infinitely often.

A dynamical system may be characterized as either a map, or a flow. Maps are discrete, flows are continuous. In either case, an orbit, or path, of the dynamical system is defined by the solution of the system to yield the sequence:

$\{x_1, x_2, x_3, x_4, \ldots\ldots\}$, for maps;

$\{x(t)\}$, for flows.

An attractor is a compact set, A, such that the limit set of the orbit, $\{x_n\}$, or $\{x(t)\}$, as n, or t, $\to\infty$ is A for almost all initial conditions within a neighborhood of A.

An attractor is the set of points of the path that represents the long term behavior of the dynamical system. Attractors can be very simple sets such as single points that represent equilibria, or limit cycles, such as the Cobweb cycle. But attractors can be much more complex. An attractor can be quasi-periodic, chaotic, or strange. All but the last can be defined on a manifold.

An example of a quasi-periodic attractor is an orbit on a torus, (a doughnut shape), generated by the cosines of a pair of incommensurate frequencies. A chaotic attractor is characterized by exponential divergence away from any point within the attractor. Because the attractor is compact, the exponential divergence means that the path is constantly folded over onto the attractor. Strange attractors have a fractal component, that is, at least along one axis of the attractor, the set is like a Cantor set. An attractor can be both chaotic and strange. Indeed, most of the examples with which economists are by now familiar are both strange and chaotic.

Dynamical orbits have dynamical properties and attractors have topological properties. For strange attractors we can define also measure theoretic properties. Dimension is mainly a topological concept, but some concepts of dimension have measure theoretic components.

Dimension concepts indicate:

> the amount of information needed to specify the position of a point on an attractor;
>
> the lower bound on the number of essential variables that are needed to model the attractor, or rather, the dynamical system when within the attractor;

the relative density of the points on the attractor.

The natural probability measure of an attractor is the relative frequency with which the different regions of an attractor is visited by the orbit.

There are many definitions of dimension used in the literature on nonlinear dynamics. The reader should be warned that sometimes "dimension" is used in its purely topological sense, that is, dimension is merely a generalization of the Euclidian notion with which all economists are very familiar. In this sense the dimension is always an integer and represents the "number of degrees of freedom", or "the number of axes needed to represent the attractor".

The other definitions of dimension can be put into three classes of concepts; those that are purely "metric", those based on the natural probability measure of the attractor, and those based on the dynamical properties of the orbit within the attractor. Our discussion will be restricted to the first two classes of concepts.

The purely metric concepts include the notion of capacity, d_c, and Hausdorff dimension, d_h.

The capacity measure is formally defined by:

$$d_c = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} ,$$

where $N(\epsilon)$is the minimum number of ϵ diameter cubes needed to cover the attractor. d_c is then nothing more than a measure of the relative rate of increase in the number of coverings of an attractor to the number of coverings needed to cover the unit interval. For a fixed point, an equilibrium point, $d_c = 0$; for a simple cycle $d_c = 1$; but for strange attractors d_c can be non-integer to represent the fractal structure of strange attractors. For example, the capacity measure of the Cantor set created by deleting middle thirds is $(\ln 2)/\ln 3 = 0.63$.

Hausdorff dimension is defined with respect to the concept of Hausdorff measure, which in turn is a generalization of Lebesgue measure. Hausdorff measure enables one to assign a non-negative number to many non-empty sets that under Lebesgue would have zero measure.

The Hausdorff α measure of a set A is:

$$HM_\alpha = \lim_{\epsilon \to 0} \inf_{\{A_i\}} \Sigma \delta(A_i)^\alpha , \qquad A \subseteq \cup A_i$$

where the A_i are covering sets. If a set has a non-zero Lebesgue measure, then the Hausdorff measure is the same when the choice of α is the same as the topological dimension of the set.

Hausdorff dimension is given by α_0, where α_0 is defined by:

$$\alpha_0 = \inf\{\alpha : HM_\alpha(A) = 0\}$$

α_0 is merely that scale variable such that the sum of "volumes" to cover the attractor is finite and non-zero.If $\alpha > \alpha_0$, then $HM_\alpha = 0$, and if $\alpha < \alpha_0$, then $HM_\alpha \to \infty$. If the attractor is not strange, say, for example, that it is a simple cycle, then α_0 will be an integer. Whenever Lebesgue measure is positive, Hausdorff dimension will be an integer; Hausdorff dimension will be non-integer when the attractor is strange and has Cantor set characteristics.

The problem with these purely metric concepts is that they treat all parts of the attractor equally, no matter how infrequently some part of the attractor is visited by the orbit. Such measures are enormously data extensive. Two measures that incorporate the relative frequency of visit by the orbit are information dimension, d_I, and pointwise dimension, d_p. We will assume in the sequel that there exists a natural probability measure in that any probability measure defined on the attractor is invariant to the initial conditions.

Information dimension, d_I, is defined by:

$$d_I = \lim_{\epsilon \to 0} \frac{I(\epsilon)}{\ln(1/\epsilon)},$$

where $I(\epsilon)$ is Shannon's Information measure and is formally defined by:

$$I(\epsilon) = -\Sigma_{i=1}^{N(\epsilon)} P_i \ln(P_i).$$

$N(\epsilon)$ is the same measure of the number of coverings of the attractor as occurred in the definition of capacity. If all coverings are equally likely, that is, $P_i = N(\epsilon)^{-1}$, then $d_I = d_c$; in general, d_I is not greater than d_c.

To define pointwise dimension, d_p, we have to define $\mu(B_\epsilon)$ as the natural probability measure of a "ball" of radius ϵ. Pointwise dimension is then:

$$d_p(x) = \lim_{\epsilon \to 0} \frac{\ln \mu(B\epsilon(x))}{\ln(\epsilon)}$$

If $d_p(x)$ is independent of x for almost all x on the attractor, then the common value of d_p is the pointwise dimension of the attractor. Pointwise dimension is that concept of dimension that the Grassberger-Procaccia procedure measures. Pointwise dimension measures the relative rate of scaling of the probability measure of a ball of diameter of radius ϵ as the diameter approaches zero; compare this measure with that of capacity.

Reviews of the correlation dimension procedures that are written with the economist in mind include Brock(1986), Brock, Dechert, and Scheinkman(1987), Brock and Sayers(1988), Barnett and Chen(1988a, 1988b), and a more detailed evaluation of the details with a guide to the relevant physics literature is Ramsey and Yuan(1988a). The basic idea underlying the calculation of dimension is relatively easily stated.

Any sequence of points, $\{x_t\}$, generated by some mechanism, whether random, chaotic, or otherwise, can be transformed into a sequence of d-tuples, $(x_{t1}, x_{t2}, \ldots. x_{td})$. These d-tuples, regarded as points in a d-dimensional Euclidian space, can be "plotted" and properties of the cloud of points so created examined. The choice of the value of "d" is the choice of "embedding dimension"; it is the size of the Euclidian space into which the original sequence is being fitted. If the generated points are from observations on a random variable, then as d, the embedding dimension, is increased without bound and assuming an unlimited sample size, the size of the space into which the d-tuples will fit is d for all values of d; that is, random variables are space filling. But if the points are generated by a mechanism that is deterministic, or at least one that produces a shape that requires only "k" dimensions, then as the embedding dimension is increased without limit, the dimension of the points will not increase beyond "k". Imagine, for example, an ellipse, which is an object of dimension 1, that requires at least Euclidian dimension 2 to be observed, but no more; consider embedding an ellipse in a 3 or 4 dimensional space; the dimension of the ellipse is still 1.

Unfortunately, the objects of interest to us involve more complicated structures. The simplest intuitive example is to imagine a mechanism that produces points that are best described as the Cartesian product of the unit interval and a Cantor set; a Cantor set is obtained by deleting middle thirds from the remainder of the unit interval obtained by deleting middle thirds at a previous iteration. This idea can be extended to any number of Cartesian products.

The Grassberger-Procaccia (1983 a,b,c) algorithm will be utilized throughout this paper. Let the ordered sequence $\{X_t\}$, $t = 1,\ldots,N$, represent the observed time series. Then, for a given embedding dimension d, create a sequence of d-histories,

$$\{(x_t,\ x_{t+\tau}, \ldots\ldots\ x_{t+(d-1)\tau})\}.$$

Here, τ stands for the time delay parameter. The sample correlation integral is given by,

$$C_r^N = N^{-2} \sum_{i,j} \theta(r - |X_i - X_j|),$$

$$r>0,\ X_i = (x_i, x_{i+1\tau}, \ldots, x_{i+(d-1)\tau}).$$

$\theta(.)$ is the Heaviside step function which maps positive arguments into one, and non-positive arguments into zero. Thus, $\theta(.)$ counts the number of points which are within distance r from each other. "r" is the scaling parameter. The calculation of C_r^N is useful because:

$$\lim_{\substack{N\to\infty \\ r\to 0^+}} C_r^N \to C_r,$$

and $d\ln C_r/d\ln r = D_2$,

whenever the derivative is defined, Guckenheimer(1984) ; D_2 is a member of a general class of dimensions D_q, $q = \{0,1,2,\ldots\}$, defined by:

$$D_q = -\lim_{r\to 0} K_q(r)/\ln(r)$$

$$K_q = (1-q)^{-1}\ln \Sigma_{i=1}^{N(r)} P_i(r)^q$$

where $P_i(r)$ is the probability of a point of the attractor being within r of the ith point, $N(r)$ is the number of such boxes needed to cover the attractor, and K_q is the Kolmogorov-Sinai (metric) entropy; details are in Badii and Politi, for example. In the rest of the paper, D_2 will be designated dc to stand for correlation dimension. The dc is a measure of pointwise dimension, dp. Pointwise dimension , see for example, Farmer, Ott, and Yorke(1983), measures the relative rate of change in the number of points on the attractor as the diameter of the covering sets is decreased. Pointwise dimension and related concepts differ from capacity and Hausdorff concepts in that they reflect the probability structure of the attractors; purely metric concepts, such as capacity, count all coverings equally, no matter how low the relative frequency of visitation by the orbits.

A first problem in determining the appropriate "d"-dimensional vectors to analyze is the choice of the τ, the delay parameter. Where there are, in fact, attractors, the choice is fairly simple; choose τ such that the auto-correlations are zero, or more sophisticatedly, so that the mutual information is minimized, Fraser(1986). In either case, the basic idea is the same, one seeks on average an approximate orthogonal set of basis vectors so as to provide the clearest representation of the attractor. When the data are from an ARMA process, the achievement of "zero-autocorrelation" between the d-tuples is even more important in that, if correlated vectors are used, false conclusions about the presence of low dimensional attractors can be drawn.

Correlation dimension is usually estimated from experimental data by a linear regression of the observed values of $\ln C_r^N$ on $\ln r$ over a suitably chosen sub-interval of the range of r, (0,1). The estimated slope coefficient of this regression, designated hereafter as $\hat{dc}$, is the usual estimator of correlation dimension cited in the literature and is the basic variable used in this paper.

However, there are a number of important qualifications to this seemingly simple procedure. First, an important practical issue involves the appropriate choice of the scaling region r actually used to calculate dc. While the theory discusses the properties of C_r as $r \to 0$, the reality is that the range of r used is far from zero and inevitably increases away from zero as embedding dimension is increased. Smaller values for r require substantial increases in sample size at any given embedding dimension in order to be able to determine a logarithmic linear relationship between C_r and r. In fact, the relationship between $\ln C_r$ and $\ln r$ is only approximately linear over a relatively narrow range of values for r. For large values of r, C_r saturates at unity so that the regression of $\ln C_r$ on $\ln r$ is zero. Further, as the value of r declines towards zero even with very large data sets, two complications arise; one is due to the limited precision of the data series and the other is due to the inevitable presence of noise. The former problem sets a practical lower bound on r before C_r collapses to zero and the latter difficulty offsets the decline in values of C_r when r reaches the level of the noise scales. Consequently, the negative slope of $\ln C_r$ on $\ln r$ starts at zero, increases first at an increasing rate, then may remain constant for a short range, before increasing again, and then falls very sharply.

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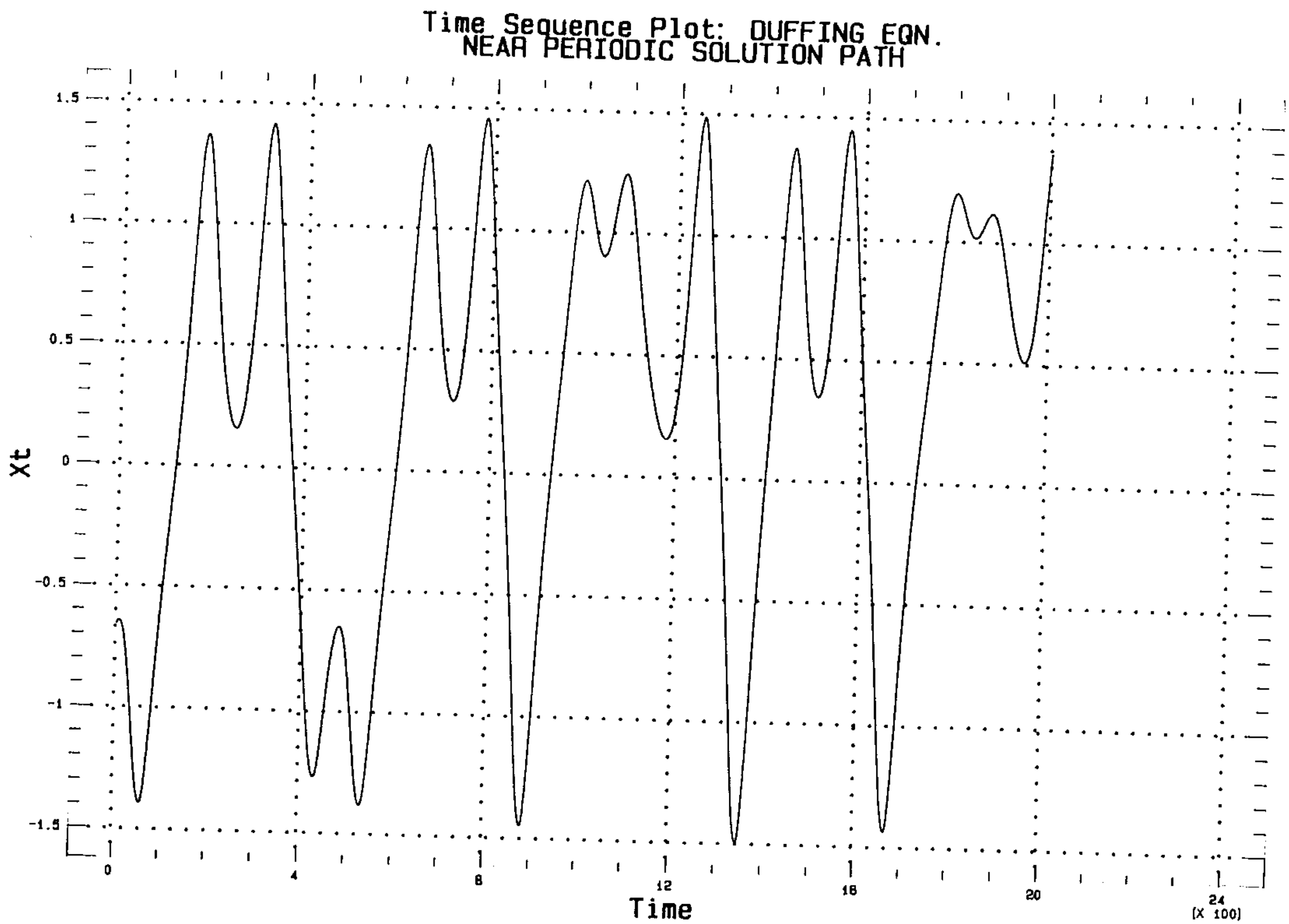
Figure 1



Time Sequence Plot: DUFFING EQN.
NEAR PERIODIC SOLUTION PATH
Xt
1.5
1
0.5
0
-0.5
-1
-1.5
0
4
8
12
16
20
24
(X 100)
Time

Figure 2

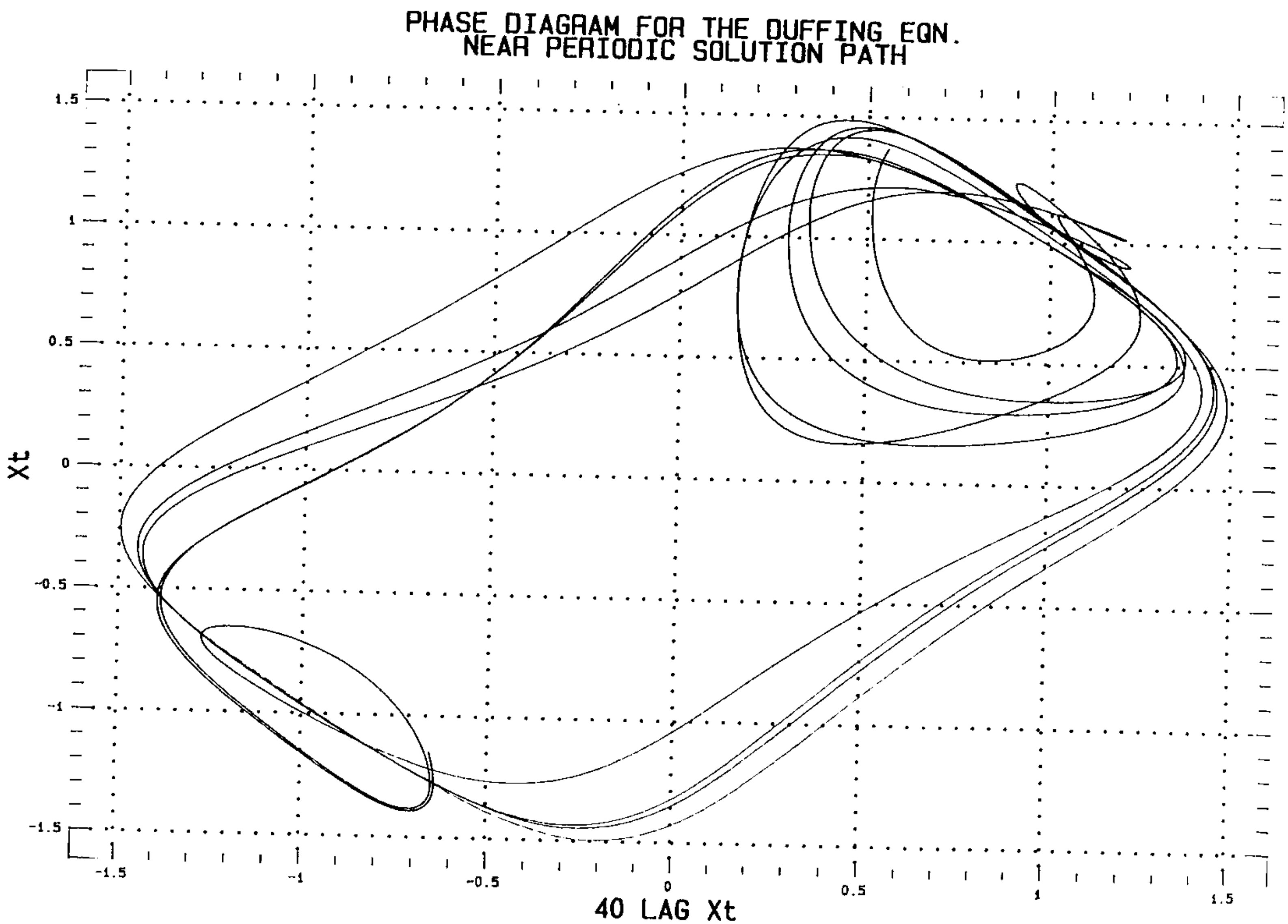
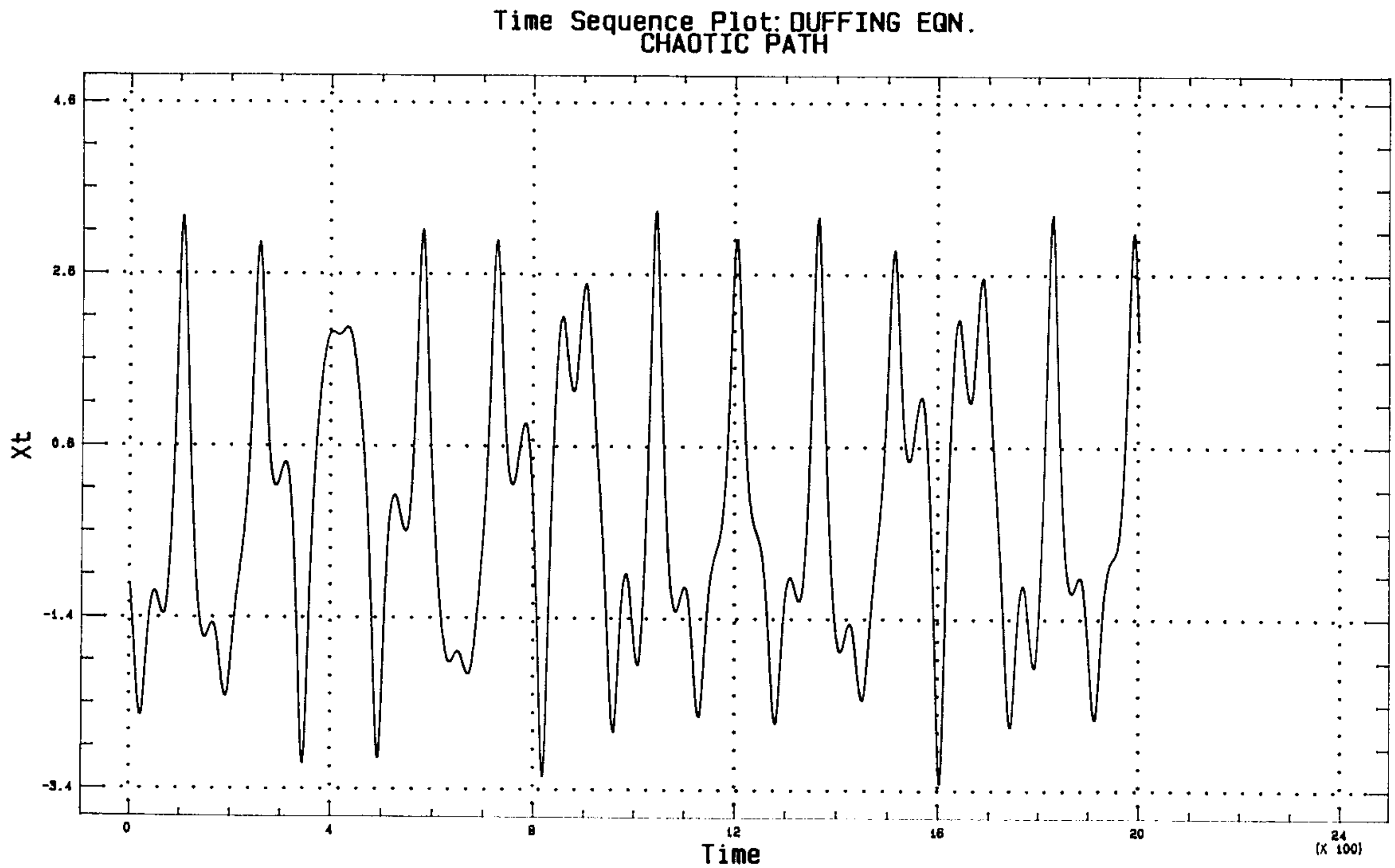

PHASE DIAGRAM FOR THE DUFFING EQN.
NEAR PERIODIC SOLUTION PATH
Xt
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
40 LAG Xt

Figure 3



Time Sequence Plot: DUFFING EQN.
CHAOTIC PATH
Xt
4.6
2.6
0.6
-1.4
-3.4
0
4
8
12
16
20
24
(X 100)
Time

Estimated Partial Autocorrelations
DUFFING EQN.; CHAOTIC PATH

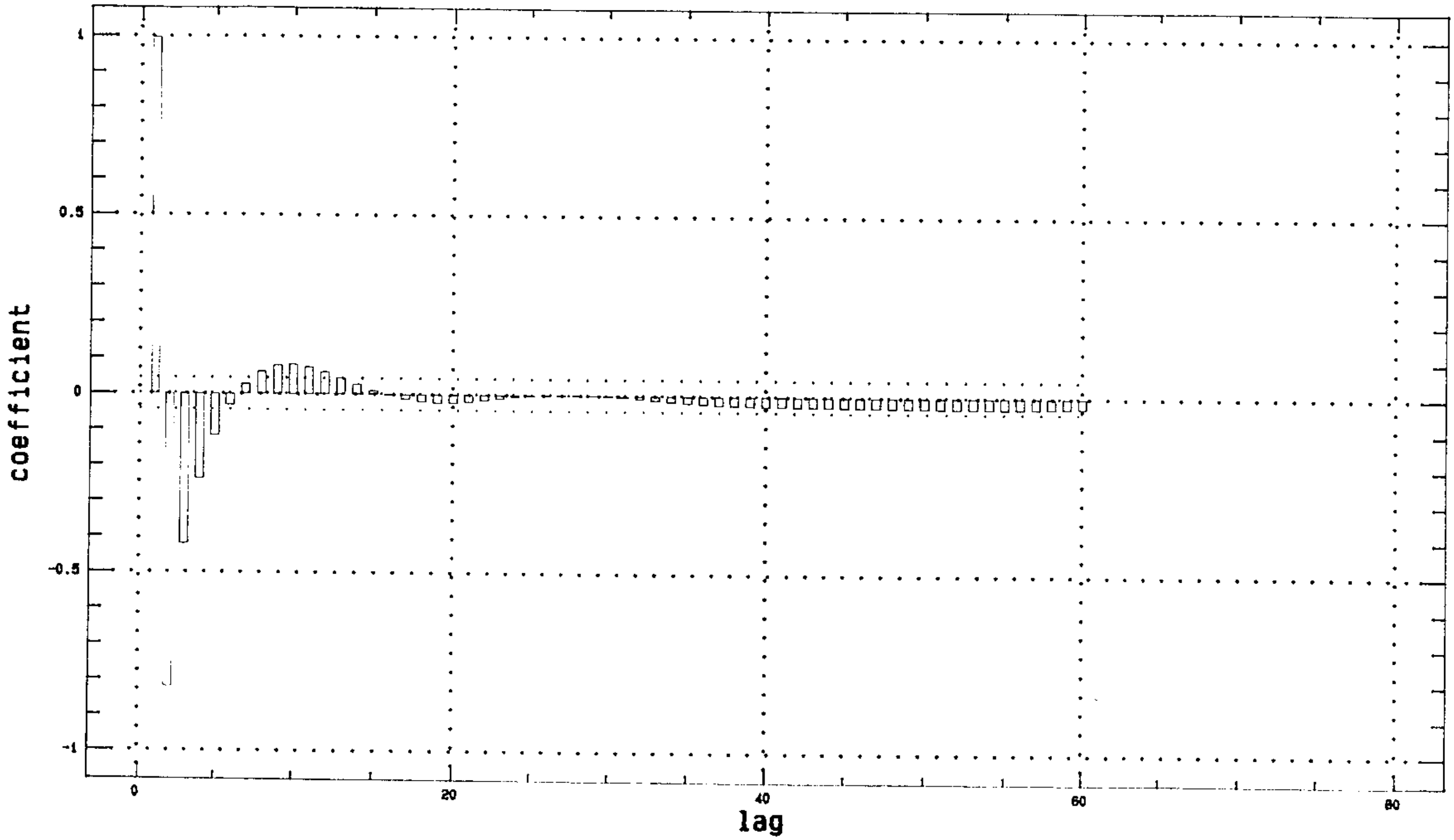


Figure 5

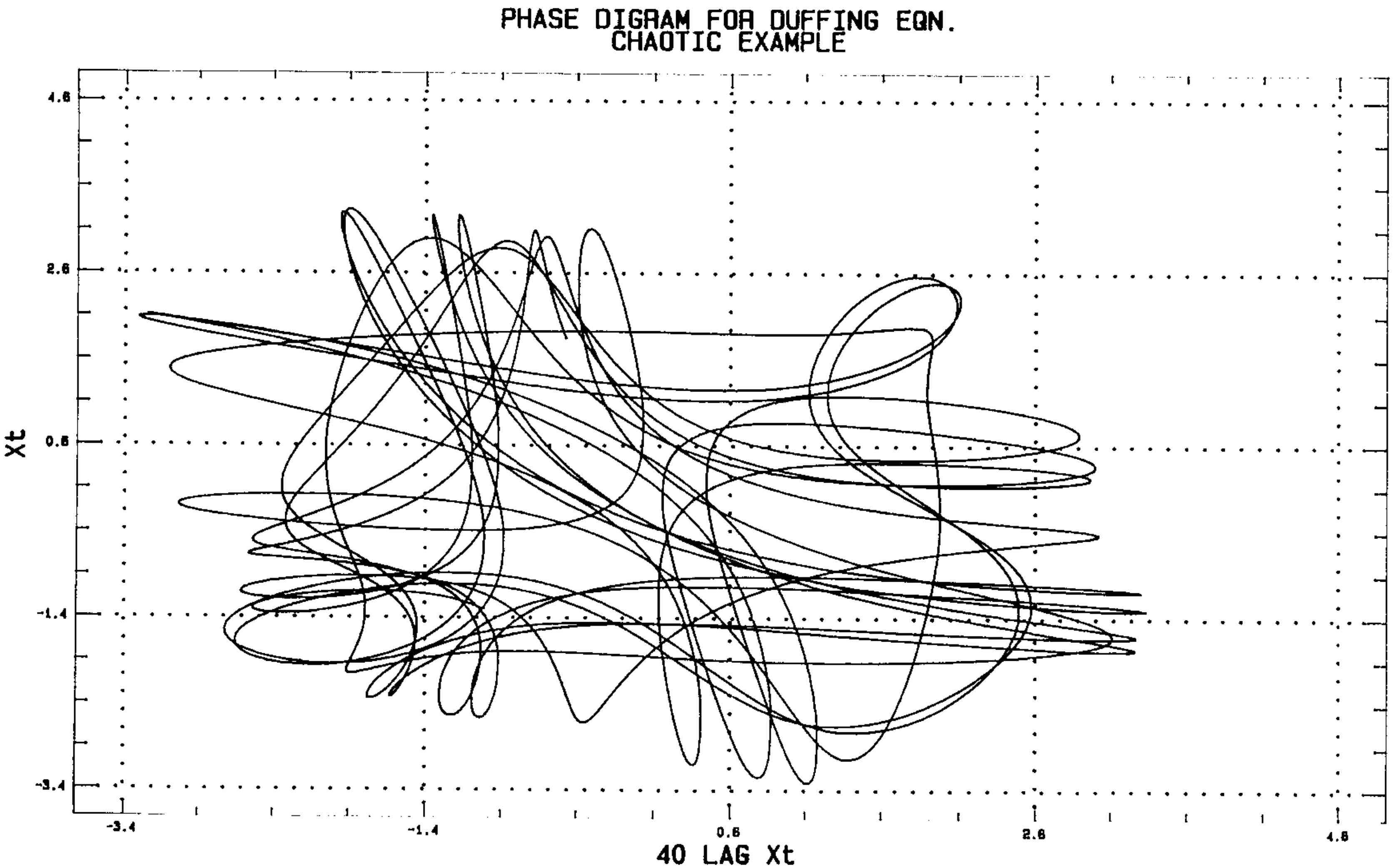

PHASE DIGRAM FOR DUFFING EQN.
CHAOTIC EXAMPLE
Xt
4.6
2.6
0.6
-1.4
-3.4
-3.4
-1.4
0.6
2.6
4.6
40 LAG Xt

Figure 6

Time Sequence Plot: Monthly M1 Data
Jan. 1959 - Nov. 1987

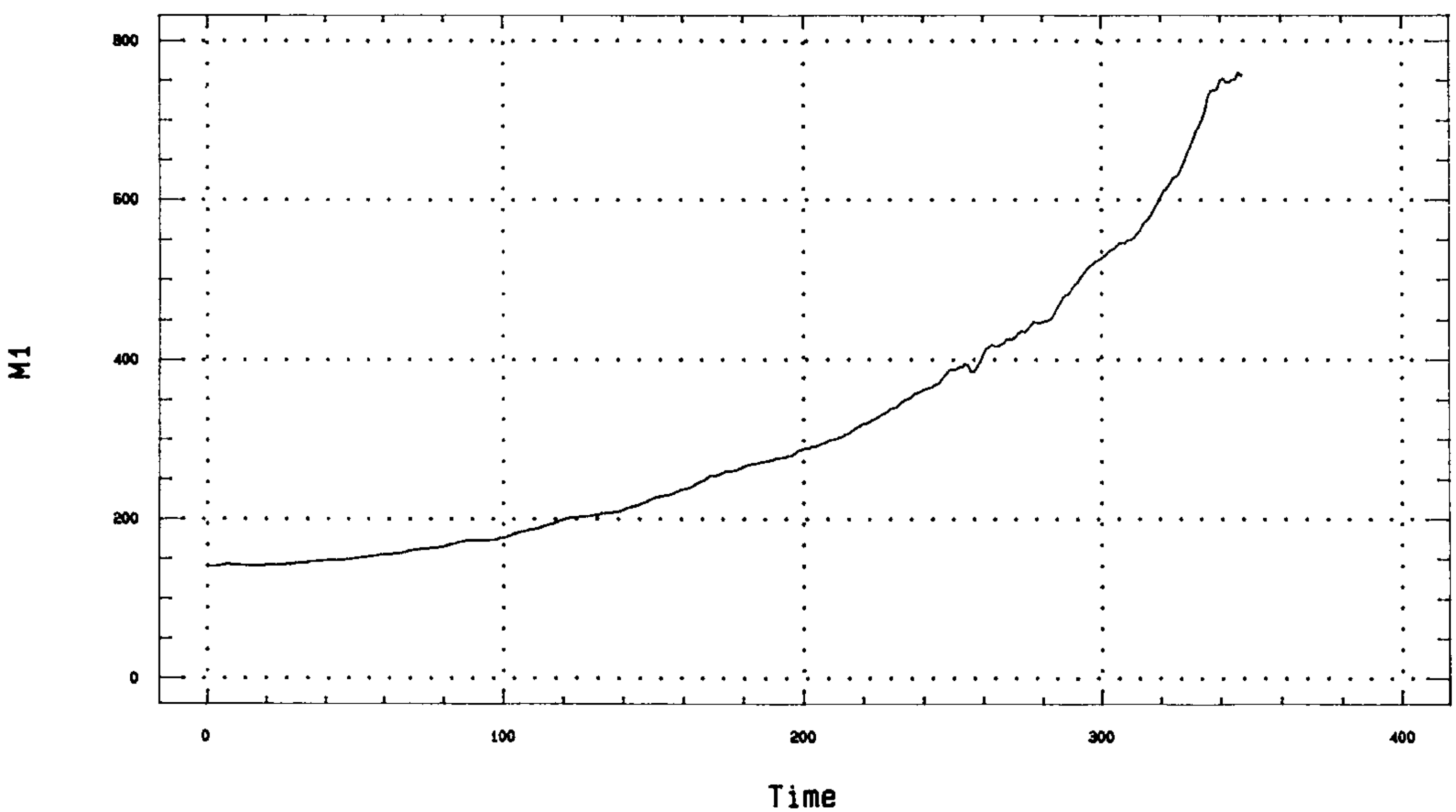


Figure 7

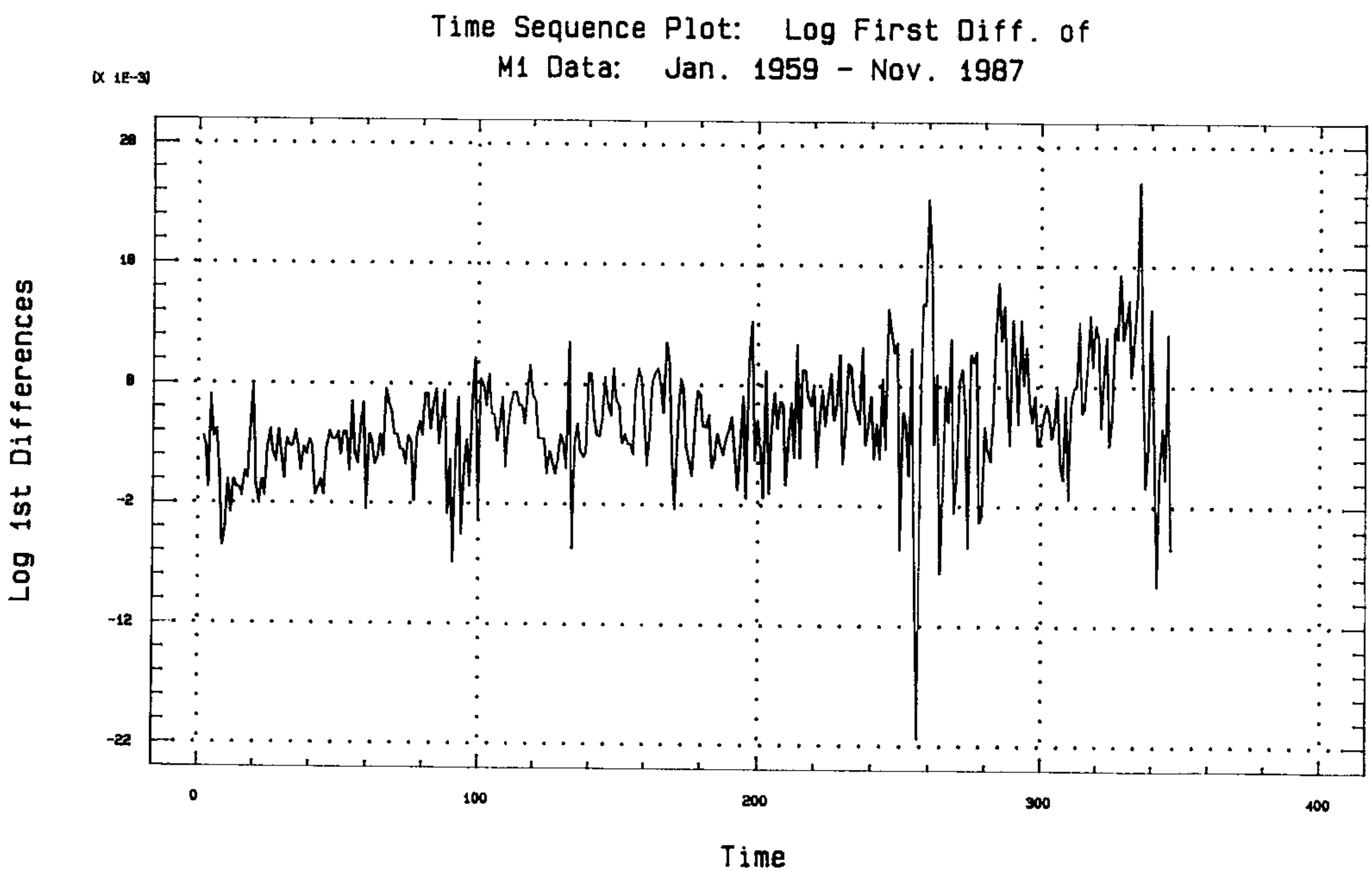

Time Sequence Plot: Log First Diff. of
M1 Data: Jan. 1959 - Nov. 1987
(X 1E-3)
20
10
0
-2
-12
-22
Log 1st Differences
0
100
200
300
400
Time

Figure 8

Plot of the Trtest Coefficients for the Log First Differences of Monthly M1 Data

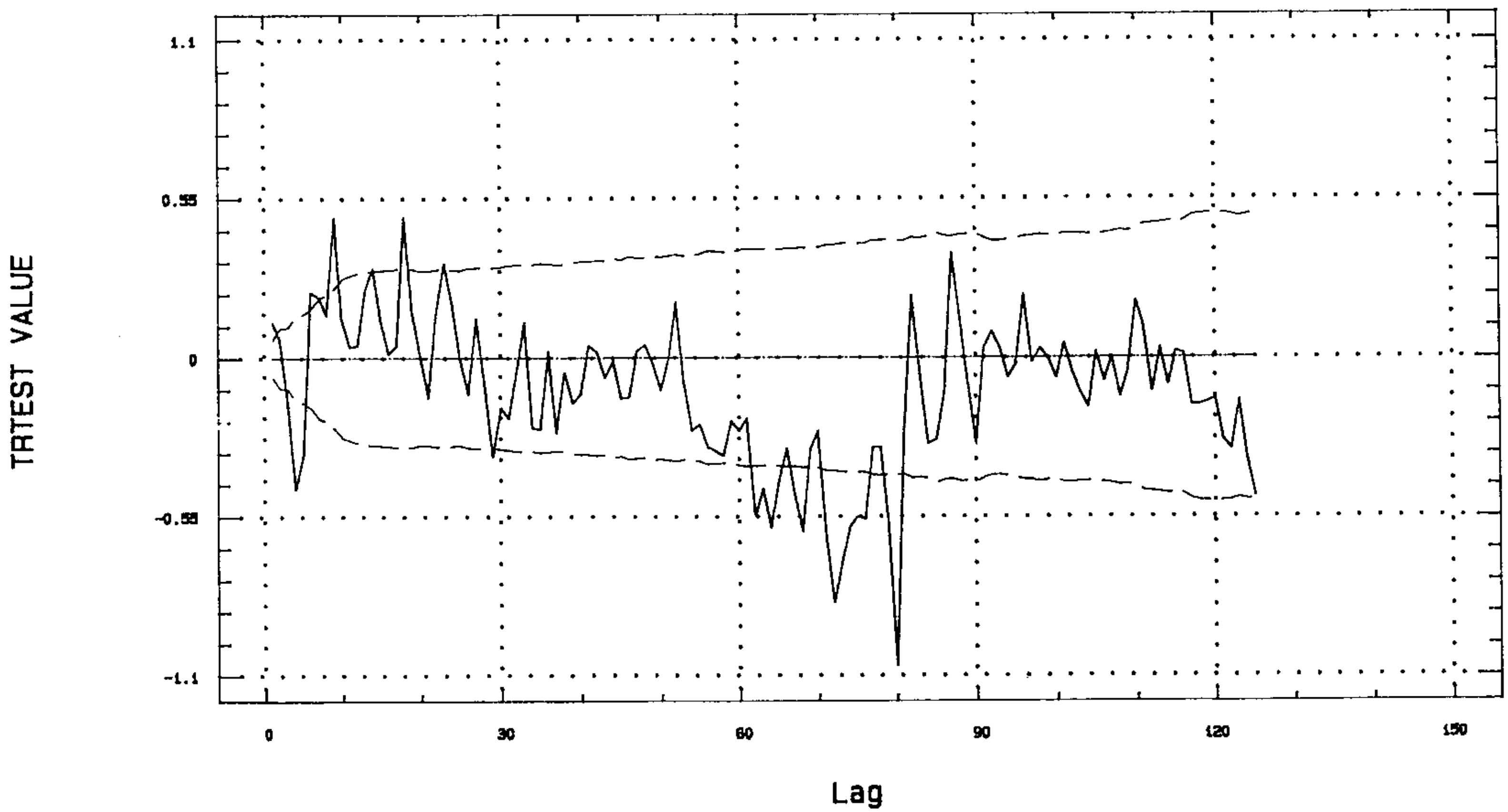


Figure 9

Time Sequence Plot: Monthly M2 Data
Jan. 1959 - Nov. 1987

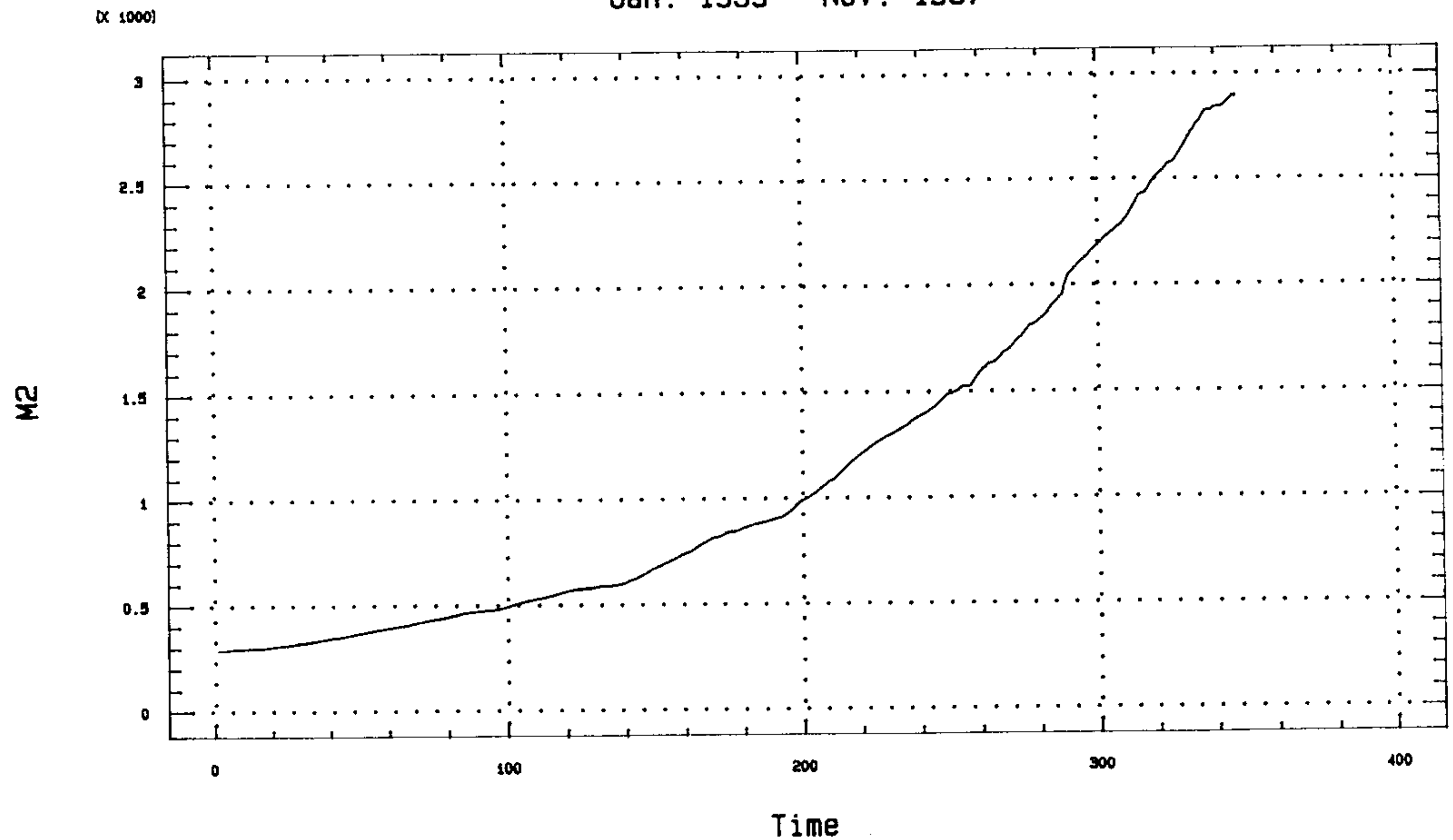


Figure 10

Time Sequence Plot: Log First Diff. of M2 Data: Jan. 1959 - Nov. 1987

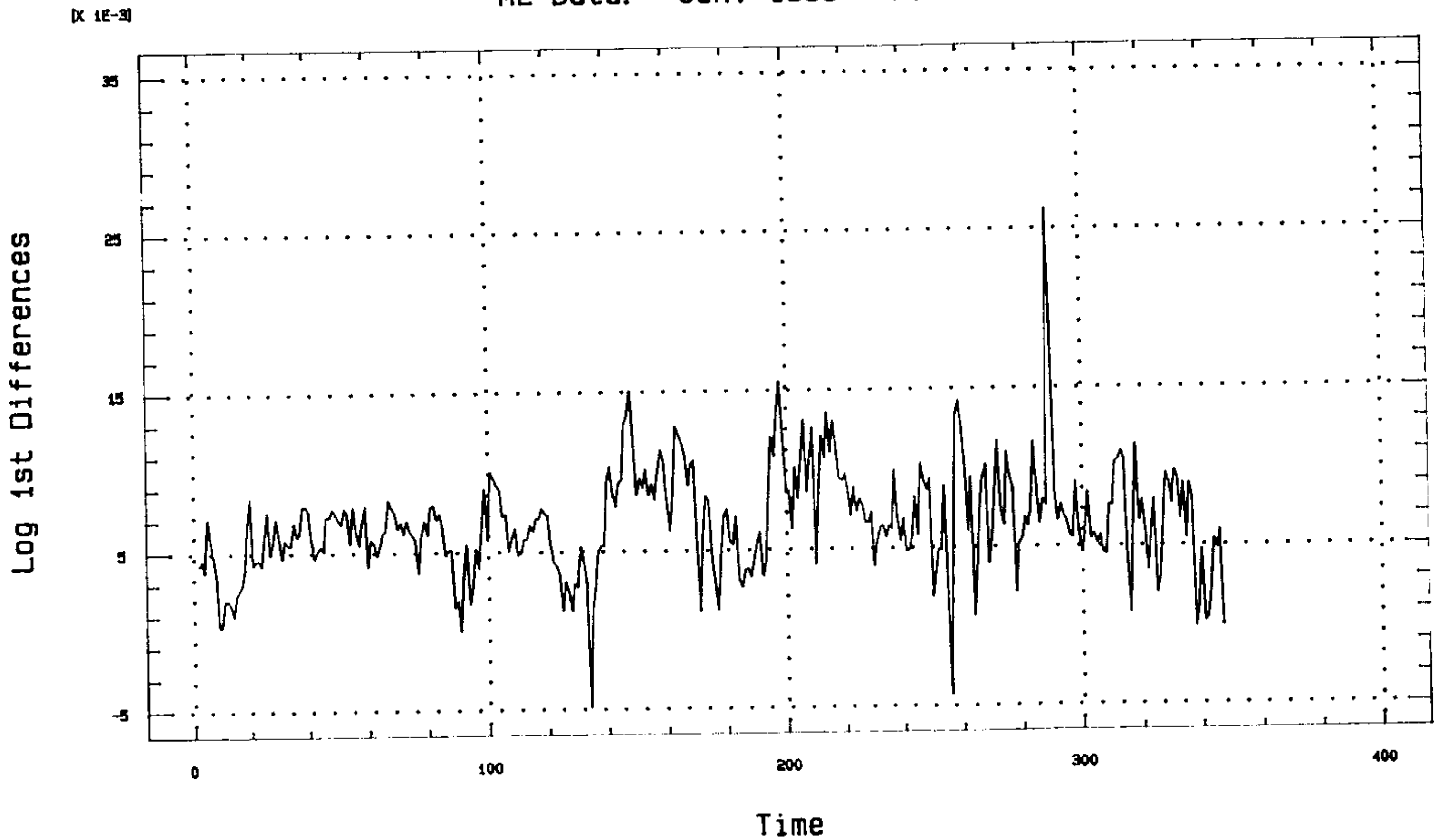


Figure 11

Plot of the Trtest Coefficeints for the
Log First Differences of Monthly M2 Data

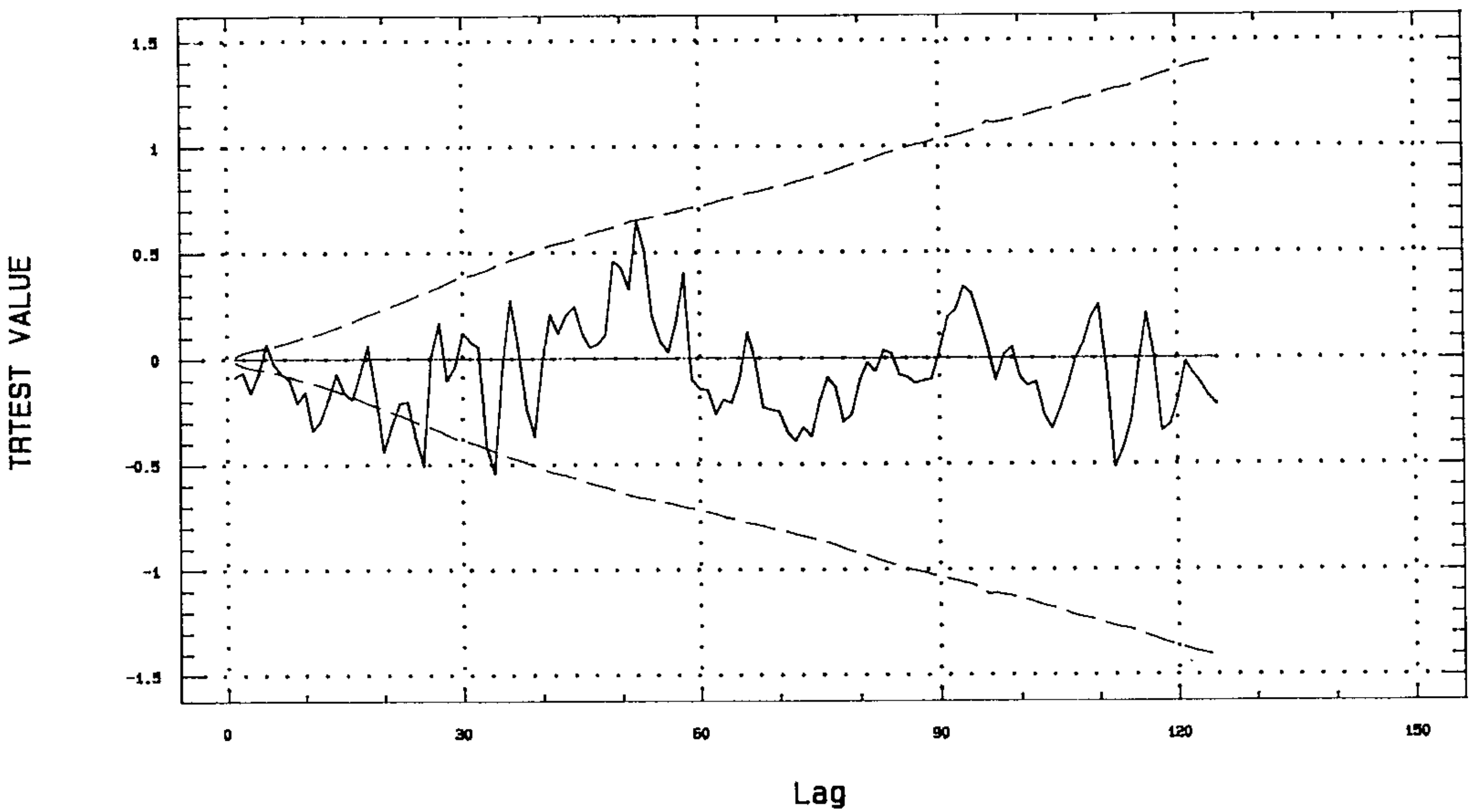


Figure 12

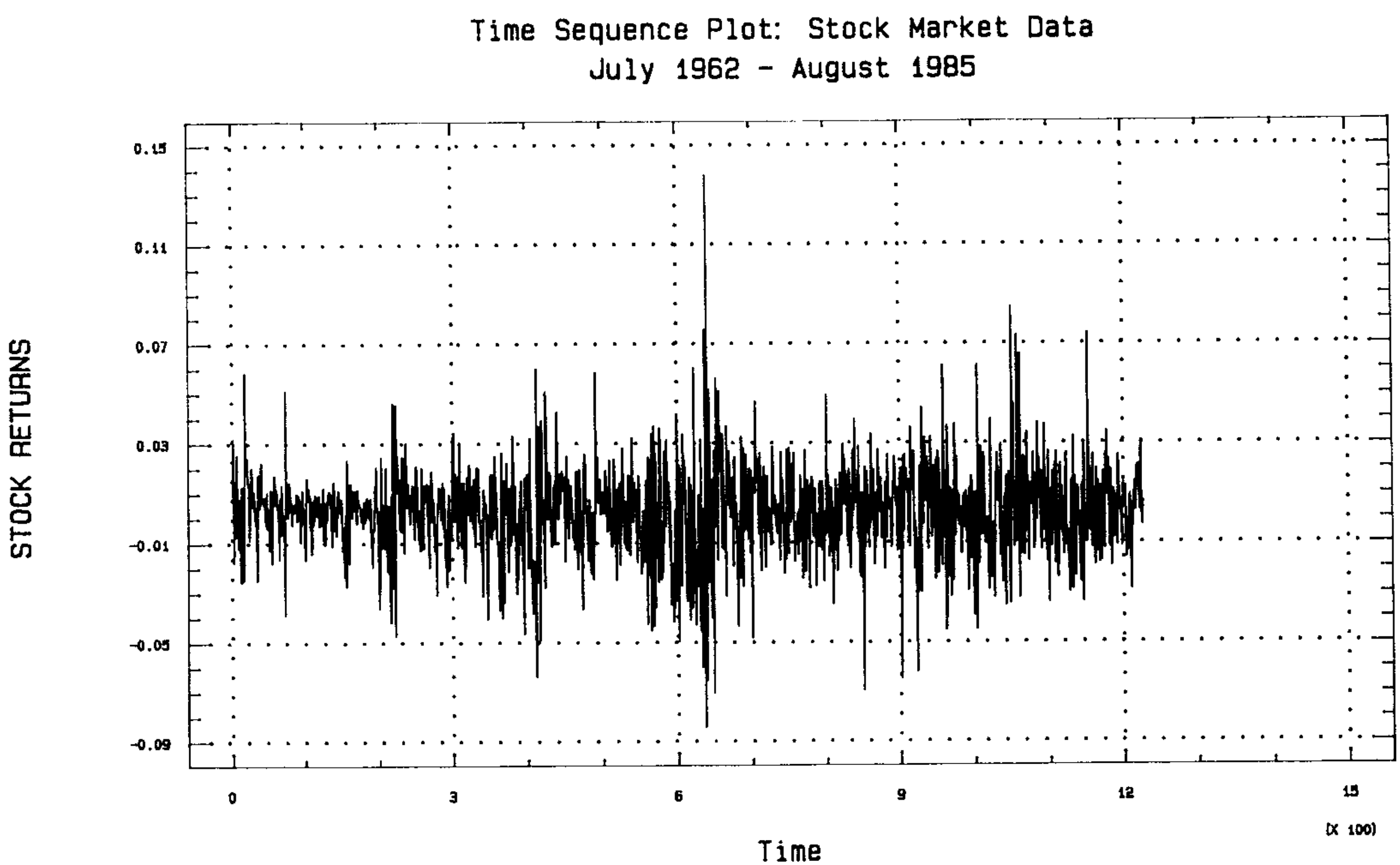

Time Sequence Plot: Stock Market Data
July 1962 - August 1985
STOCK RETURNS
0.15
0.11
0.07
0.03
-0.01
-0.05
-0.09
0
3
6
9
12
15
(X 100)
Time

Figure 13

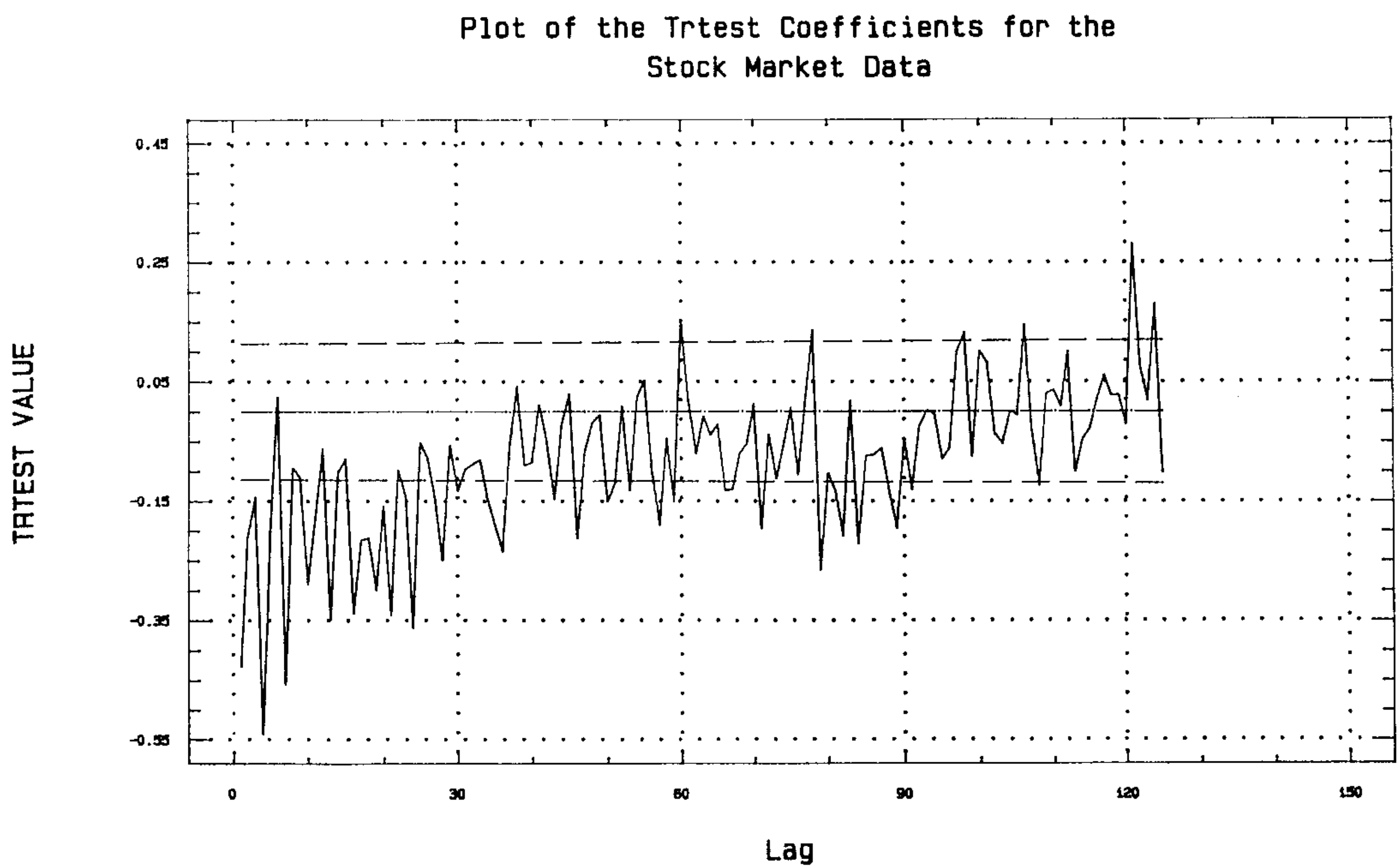

Plot of the Trtest Coefficients for the
Stock Market Data
TRTEST VALUE
0.45
0.25
0.05
-0.15
-0.35
-0.55
0
30
60
90
120
150
Lag

Figure 14

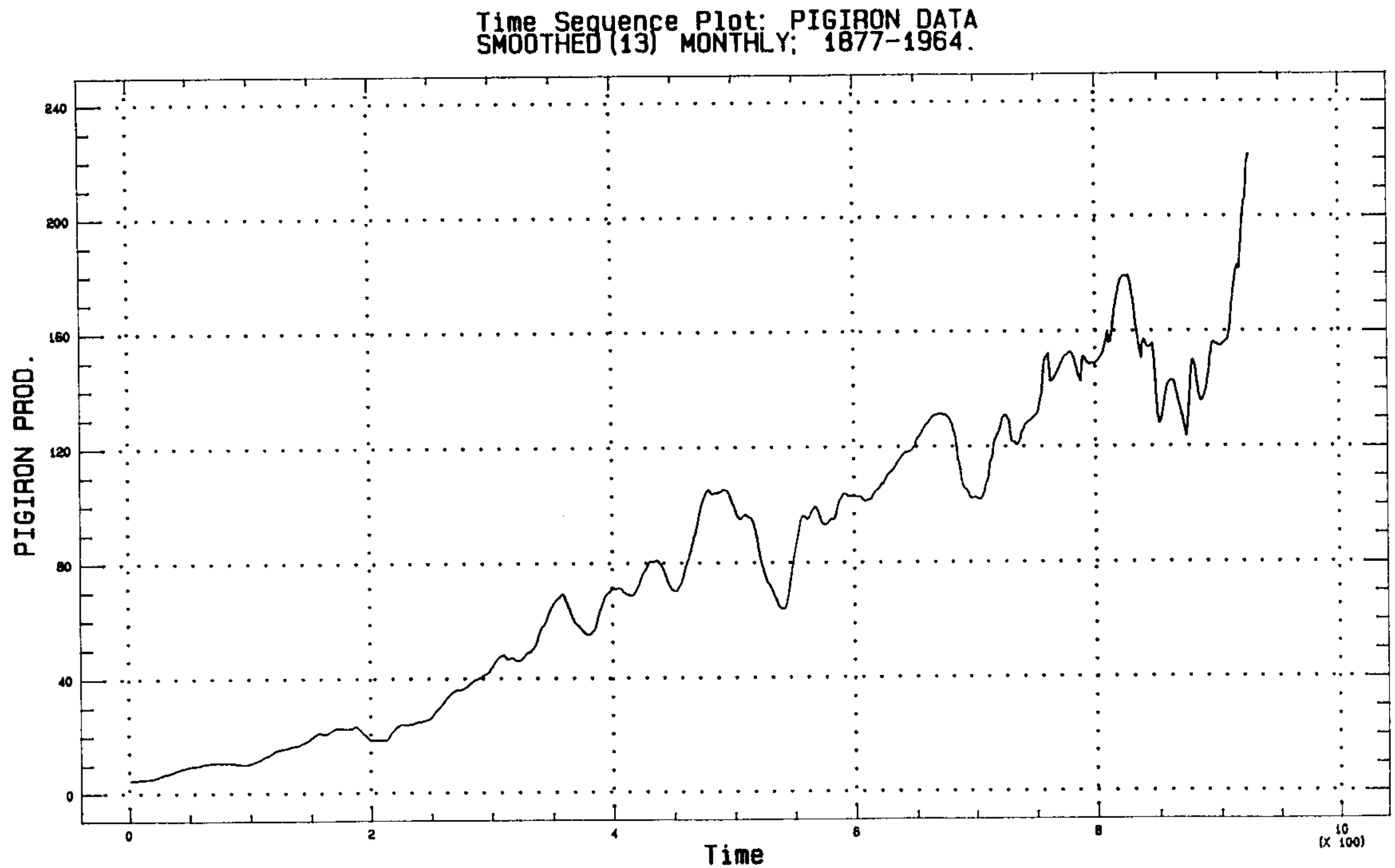

Time Sequence Plot: PIGIRON DATA
SMOOTHED(13) MONTHLY; 1877-1964.
PIGIRON PROD.
Time
(X 100)

Figure 15

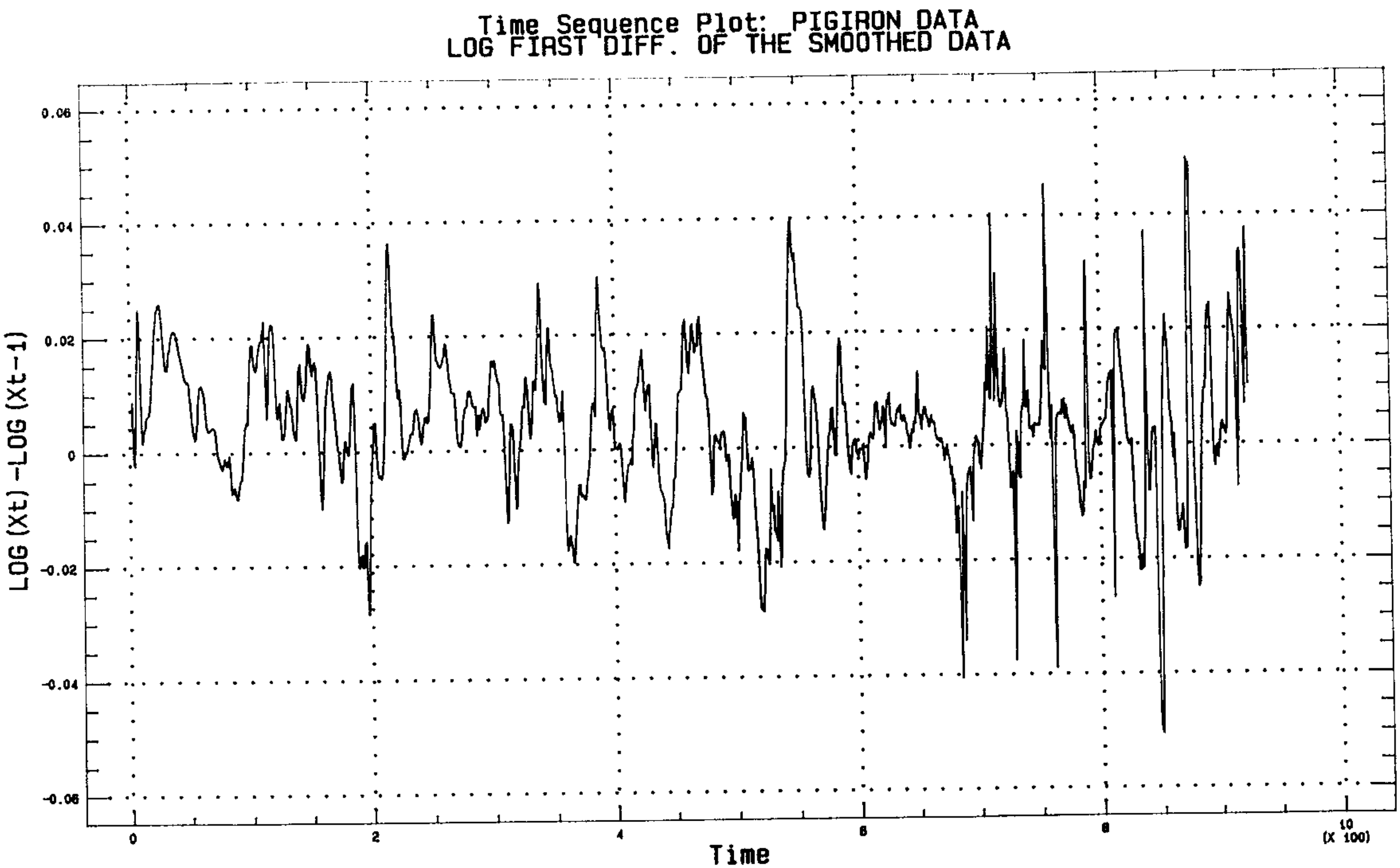

Time Sequence Plot: PIGIRON DATA
LOG FIRST DIFF. OF THE SMOOTHED DATA
LOG (Xt) -LOG (Xt-1)
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
0
2
4
6
8
10
(X 100)
Time

Figure 16

Plot of Trtest Coefficients for the Standardized Log First Diff. of Pigiron