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Two-Stage Auctions I: Private-Value Strategies*

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Abstract

Two-stage auctions are defined by the following rules of play:

1. 1st stage: The players submit sealed bids, all of which are then opened and made public.
2. 2d stage: Each player chooses exactly one of the 1st-stage bids, either by affirming its own bid or by usurping another player's bid.
3. Payoffs: If there is only one player who makes the highest bid at the 2d stage, that player wins the auction, regardless of what it bid at the 1st stage. If there is more than one player who makes the highest bid at the 2d stage, the player who made the highest bid at the 1st stage wins the auction.

Bidding one's true valuation in the 1st stage, and--if one is the highest 1st-stage bidder--usurping the second-highest bid, is shown to be the optimal strategy of players under different sets of plausible conditions when valuations are private. Thus, it is the product of the iterative elimination of weakly dominated strategies under one set of conditions, and it is the unique Nash/Bayesian equilibrium under another set.

Deviations from this equilibrium generate restorative forces that return players to the equilibrium. While this equilibrium is the same as that produced by a Vickrey auction, two-stage auctions have important practical advantages, including their invulnerability to cheating and the ability they give to players to revise their bids in light of knowledge gained when the other bids are revealed in the 1st stage.

Two-Stage Auctions I: Private-Value Strategies

1. Introduction

In a review article on auctions, McAffee and McMillan (1987, p. 711) remark that "the essence of the auction problem is the unobservability of bidders' valuations." We concur. To ameliorate this problem, we propose the following rules of play for a two-stage auction:

1. 1st stage: The players submit sealed bids, all of which are then opened and made public.

2. 2d stage: Each player chooses exactly one of the 1st-stage bids, either by affirming its own bid or by usurping another player's bid.

3. Payoffs: If there is only one player who makes the highest bid at the 2d stage, that player wins the auction, regardless of what it bid at the 1st stage. If there is more than one player who makes the highest bid at the 2d stage, the player who made the highest bid at the 1st stage wins the auction.[1]

In terms of the information they provide, these rules make a two-stage auction more akin to an English auction--in which bids are oral, open, and ascending--than to a sealed-bid auction, in which no

[1]In the latter case, if bidders who tie at the 2d stage also tied with the highest bids at the 1st stage. the tie might be broken randomly. But the possibility of such a double tie seems remote unless players' valuations are not private (more on this possibility later). Note that in an English auction, information about bids is revealed sequentially, whereas bidding information in a two-stage auction is revealed all at once (i.e., in the 2d stage).

information about other players' bids is revealed until the conclusion of the auction, when the winner is announced.[2]

A sincere bid is one in which a player bids its estimate of the true worth of an object, or its valuation, in the 1st stage, making it indifferent between winning at that bid and losing the auction. In a two-stage auction, as we shall show, it is rational under different sets of plausible conditions for players to bid sincerely in the 1st stage. Moreover, if it turns out that the players' valuations are private (i.e., they are completely unaffected by knowledge of the other players' valuations), all players except the highest bidder will affirm their 1st-stage bids in the 2d stage; the highest bidder will usurp the second-highest bid, winning the auction at a price lower than its reservation price and thereby ensuring itself of a profit. This outcome is identical to that in a Vickrey (1961) auction, which is a sealed-bid auction in which the player with highest bid wins but pays only the second-highest bid.

Unlike Vickrey auctions, this outcome in a two-stage auction, which we call the Vickrey outcome, is not the product of dominant strategy choices by the players. However, we show in section 2 that a few reasonable assumptions about bidding imply the Vickrey outcome. Moreover, it is a symmetric Nash equilibrium under even weaker conditions, and uniquely so (even in the Bayesian sense of a game of incomplete information) if 2d-stage bids are "ordinal," as shown in section 3. In section 4 we show that if there are only two players, one

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[2]If the purpose of the auction is to minimize the cost of a service to be performed, then "highest" would be replaced by "lowest" in rule 3. In this case, the players would be competing to make the lowest bid.

player's deviation from sincere bidding in the lst stage need not hurt it in the 2d stage, but there are restorative forces that return the players to the equilibrium if one or both deviate from it.

While two-stage auctions implement the Vickrey outcome under rather general conditions, they do so in a way that obviates certain practical difficulties of Vickrey auctions that have made them rarely used (Smith, 1987; Rothkopf, Teisberg, and Kahn, 1990). In particular, two-stage auctions eliminate the possibility of fraud in a Vickrey auction by providing a simple means--to be described later--for the players to check that there are no bogus bids that have been surreptitiously introduced in order to increase the amount that the highest lst-stage bidder must pay when it usurps the second-highest bid.

Even without bogus bids, winners in a Vickrey auction may end up paying almost what they bid, giving them only a minimal profit. By contrast, in a two-stage auction, unless everyone else bids only slightly less than the highest lst-stage bid, the highest bidder can effectively "bail out" by usurping a considerably lower bid. We demonstrate that this is not a rational strategy in a private-value auction, but in a companion paper we show that in a common-value auction (i.e., one in which everyone has the same valuation, but about which there may be incomplete information), the ability of players to "correct" their bids in the 2d stage mitigates the so-called winner's curse (Brams and Taylor, 1991b).

We believe two-stage auctions offer a persuasive and sensible alternative not only to Vickrey auctions but also to many other kinds of auctions that have been proposed or used (Smith, 1987; McAfee and McMillan, 1987; Milgrom, 1989). By making the bids common knowledge at the 2d stage, they solve the "unobservability-of-bidders'-valuations" problem alluded to earlier. In the process, they enable bidders to make more informed and, therefore, intelligent choices in the 2d stage, when these choices count. This is especially true in common-value auctions–in which players discover in the 2d stage that they may have overbid (or underbid) in the 1st stage--but in this paper we shall concentrate on the private-value case.[3]

2. The Rationality of the Vickrey Outcome in Two-Stage Auctions

We develop a model in this section that implies the Vickrey outcome in two-stage auctions, based on the iterative elimination of weakly dominated strategies. The first assumption of the model is the following:

Judicious-Bidding Assumption (JBA). In a two-stage auction, a player will never make a 1st-stage bid, or choose a 2d-stage option, such that, if it wins the auction, it might suffer a loss.[4]

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[3]This case is interesting not only as a baseline for comparison with the common-value case but also because of the surprising relationship between the outcome of a private-value two-stage auction and the outcome of a Vickrey auction. The chief practical difference between these two outcomes is that the highest 1st-stage bidder in a two-stage auction learns the second-highest bid at the 2d stage and can therefore <u>choose</u> whether to usurp it or not, whereas in a Vickrey auction the highest 1st-stage bidder <u>must</u> pay this bid.

[4]The use of the word "bid" in the 1st stage is clear; we use "option" rather than "strategy" in the 2d stage in order to reserve the latter term for player choices

An immediate implication of this assumption is that no player will ever <u>overbid</u> in the 1st stage--that is, make a bid greater than its reservation price.

A simple example illustrates why overbidding may result in a loss for a player. Assume that three players value an object at 1, 2, and 3, respectively. But they bid 4, 5, and 6 for it in order to try to ensure that their bids are the highest in the 1st stage and, therefore, will be able to "trump" the choice of a competitor, should it choose the same option in the 2d stage.

Once these bids are revealed in the 1st stage, however, this common knowledge cannot help any of the three players, if it wins, avoid a loss. In particular, the players who bid 4 and 5, by affirming and usurping 4, respectively, can guarantee that the player who bid 6 wins but suffers a loss of 4-3 = 1 (when it usurps the lowest bid of 4).

To avoid the possibility of overpaying, therefore, it seems reasonable to suppose that players will not make 1st-stage bids greater than their valuations. Otherwise, they may lose; indeed, for the highest bidder, a loss may be unavoidable, as we have illustrated. JBA also precludes players from usurping bids greater than their valuations in the 2d stage.

Our second assumption postulates that players want first to win at the minimum price, but then to be as competitive as possible in their bidding at this price. It requires the following definition: Suppose that

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in <u>both</u> stages of the auction. Thus, a strategy comprises both a bid and an option.

S and T are strategies for player P. Then we shall say that S is <u>preferred</u> to T if

- S weakly dominates T in the usual sense; or
- S and T are equivalent in the sense of yielding exactly the same outcomes, but S involves a higher stage I bid than does T, or the same stage I bid and a higher stage II bid.

Competitiveness Assumption (CA). Players choose preferred strategies by successively eliminating nonpreferred strategies.

CA can be justified in the following way. P's primary goal is to win and pay as little as possible; all strategies that satisfy this goal are then distinguished by the secondary goal, which is to be competitive by bidding higher. This lexicographic order, in which the secondary goal is used to break "ties" between bids that satisfy the primary goal, implies that P never suffers when it behaves according to CA. When all players, other things being equal, choose higher rather than lower bids (up to their reservation prices, because of JBA), the consequence is the iterative elimination of nonpreferred strategies.

Although CA seems unimpeachable as a prescription of rational behavior, it is rendered even more compelling if play is likely to be repeated. For then a player's aggressive bidding in one auction will enhance its reputation for toughness later, which may drive out potential competitors from even entering the later auctions.

Alternatively, assume that bids are solicited from only a few parties. Then to maximize one's chances of being selected, one would presumably want to be viewed as a serious player, which will be more likely if one has been competitive in the past.

We stress that the rationale of CA does not depend on the assumption of repeated play. The latter assumption simply reinforces the rationality of competitive bidding, as embodied in CA, upon which a player cannot improve, even in a one-shot game.

Given JBA and CA and two minor technical conditions, a two-stage auction turns out to be equivalent to a Vickrey auction in the sense that both elicit sincere bids and make the highest bidder the winner, paying the second-highest bid. We shall argue later that the way in which a two-stage auction implements the Vickrey outcome confers on it substantial practical advantages, but first we prove

Theorem 1. Consider a two-stage auction with n players. Assume the following are true and common knowledge:

1. Different players never have equal valuations and never make equal bids in stage I.
2. All reservation prices are in the interval [0, m], and the value of m is common knowledge.[5]
3. JBA, and that it is common knowledge.
4. CA, and that it is common knowledge.

Then iterative elimination of weakly dominated strategies leads all players to bid their valuations in stage I. In stage II, the highest bidder

[5]If the lowest bid is the winning bid (e.g., in a contracting job), then "0" would play the role of "m." In this case, 0 would be the lower bound, but there would be no upper bound on m.

will usurp the second-highest bid and thereby win the auction at the Vickrey outcome.

Proof. Order the bids, b_i, of the n players such that $b_1 < b_2 < \ldots < b_n$. The proof proceeds by a series of five claims:

Claim 1. For i = 1, 2, . . ., n-2, the bid b_i will not be usurped by those with bids $b_{i+1}, \ldots, b_n$.

Proof. We proceed by induction on i. For i = 1, notice that if anyone except the highest bidder in stage I usurps b_1 in stage II, then this will be a definite loss for that player, rendering this option less preferred than usurping a higher bid. CA thus prescribes that no one, except possibly the highest bidder, will usurp b_1. But now the highest bidder, realizing that no one else will usurp the lowest bid, knows that it will definitely lose if it usurps the lowest bid. Thus, the highest bidder will not usurp the lowest bid either. Hence, b_1 will not be usurped by anyone with a higher stage I bid.

Now assume that for each i = 1, . . ., k, where $1 \leq k < n-2$, players know that b_i will not be usurped by anyone with a higher stage I bid. Because the highest bidder, in particular, will not usurp any of the bids $b_1, \ldots, b_k$, its stage II option will be at least b_{k+1}. Thus, if any other bidder usurps b_{k+1}, it knows that it will definitely lose. Hence, CA again dictates that anyone with bids $b_{k+2}, \ldots, b_{n-1}$ will not usurp b_{k+1}. As before, the highest bidder, knowing this, will not usurp b_{k+1}, because this will be a definite loss. Thus, b_{k+1} will not be usurped by anyone with a higher stage I bid. This completes the induction.

Claim 2. Let H be the bidder with the highest valuation, h. If H bids h in stage I, then it will be the highest bidder in stage I and will usurp the second-highest bid in stage II.

Proof. Notice that if H bids h in stage I, by JBA its stage I bid will be the highest (i.e., b_n). Moreover, all its options at stage II are ruled out by claim 1 except either to affirm b_n or usurp b_{n-1}. If it affirms b_n, then it receives a payoff equivalent to losing (i.e., its profit is $b_n - h = 0$). Since usurping b_{n-1} is strictly better than this option, H will usurp b_{n-1}.

Claim 3. H can do no better than to win at the second-highest bid.

Proof. Let b denote the highest stage I bid among the players other than H. If H bids higher than b in stage I, by claim 1 the player who bid b will not usurp any stage I bid lower than b. Thus, H can do no better than to win by usurping b. If H bids lower than b in stage I, then the player who bid b is the highest bidder. It will usurp no lower than the highest bid that is less than b, so H will have to usurp b (i.e., the highest bid) to win.

Claim 4. If a player is H, then its optimal strategy is to bid h and then usurp the second-highest bid.

Proof. Bidding $b_n = h$ and then usurping b_{n-1} yields a win for H at the second-highest bid. By claim 3, no strategy can yield a better result for H. By CA, then, it will bid h and then, by claim 2, usurp b_{n-1}.

Claim 5. For j = 0, 1, . . ., m (where m is the top possible valuation), if P has valuation m-j, then it will bid it. If its bid is the highest, it will usurp the second-highest bid.

Proof. We proceed by induction on j. Suppose j = 0, so P has the highest possible valuation, m. Then P knows it is H, so claim 4 implies the desired conclusion. Now assume that $0 \leq k < m$, and we have shown that it is both true and common knowledge that

- if any P has valuation m, m-1, . . ., m-k, then it will bid it, whether or not it knows it is the highest valuation; and
- if P's bid turns out to be the highest stage I bid, it will usurp the second-highest bid.

Next assume that H has valuation m-(k+1). Then this is either the highest valuation, or it is not.

Case 1. It has the highest valuation.

In this case, claims 2 and 3 demonstrate that H can do no better than bid m-(k+1) in stage I. By JBA and CA, this will be P's stage I bid, and it will usurp the second-highest bid in stage II.

Case 2. It does not have the highest valuation.

In this case, we know, by our inductive hypothesis, that whoever has the highest valuation will bid it in stage I. Moreover, when that bid turns out to be the highest, its bidder (i.e., H) will usurp the second-highest bid in stage II and thereby win the auction. Thus, the player with valuation m-(k+1) will definitely lose; by CA, it will therefore bid its valuation. In general, every P that is not H will also bid its valuation in stage I. (What it does in stage II we do not address here.)

This completes the proof of claim 5 and, with it, the theorem. ▯

CA is not the only assumption, in combination with the other three in the statement of Theorem 1, that can be used to prove the rationality of the Vickrey outcome in a two-stage auction. As an alternative assumption, consider the following:

No-Regret Assumption (NRA). P will never choose a strategy such that, if it should turn out to be H--the player with the highest valuation--it might lose the auction.

Consider NRA in conjunction with JBA. By JBA, P will never bid more than its valuation in stage I. By NRA, it will never bid less than its valuation; otherwise, if it is H, its stage I bid might not be the highest, and it will definitely lose if the highest bidder affirms. Consequently, P will bid its valuation in stage I.

Notice that this argument for sincere bidding at stage I postulates the possibility that the highest bidder will affirm. While this possibility may not be rational (e.g., if the highest bidder bids its valuation), it induces everyone to be sincere. But if everyone is sincere, and everyone knows everyone is sincere (which they do because JBA and NRA are common knowledge), then the highest bidder will know that it is H and usurp the next-highest bid.

With NRA instead of CA, then, we do not need to rely on any inductive arguments, based on iterative weak dominance, to prove the rationality of the Vickrey outcome. It instead follows by contradiction: assuming that the highest bidder might affirm, we show that this will never be the case because

- all players will be sincere in stage I, given JBA and NRA;
- in stage II, the highest bidder will therefore know that it is H;
- knowing that it is H, it can usurp the next-highest bid, assured that it will not be usurped itself.

What are the relative merits of substituting NRA for CA in Theorem 1 to prove the rationality of the Vickrey outcome? On the plus side, NRA does not require the use of a long chain of reasoning--the five claims in the proof of Theorem 1--to yield the Vickrey outcome. Rather, the rationality of this outcome follows immediately from NRA and JBA.

On the minus side, NRA does not really justify the rationality of the Vickrey outcome. Rather, it simply asserts that players will not underbid, but only because they fear they might lose if they should turn out to be H and thereby experience regret.

But consider a player who is almost certain it is not H but thinks it is probably the player with the second-highest valuation ($\bar{H}$). Moreover, suppose $\bar{H}$ thinks that H will substantially underbid in stage I. Then if $\bar{H}$ only slightly underbids in stage I <u>and</u> turns out to be the highest bidder, it should affirm in stage II (because H will definitely usurp its stage II bid), ensuring itself of a positive, if small, profit.

Because this rational scenario for $\bar{H}$ is inconsistent with its bidding sincerely in stage I (and usurping in stage II), it casts doubt on the reasonableness of NRA. CA, by contrast, provides rational reasons for players to be sincere in stage I and, consequently, for H, necessarily the highest bidder, to usurp the second-highest bid in stage II.

We conclude that CA offers a more persuasive justification of the Vickrey outcome from "first principles": successively eliminating weakly dominated strategies enables players to maximize their payoffs should they win. While NRA ignores profit maximization--indeed, $\bar{H}$ can suffer if H underbids and it does not--NRA is nevertheless quite commonsensical. It, along with JBA, provides players who want to avoid possible regret a transparent reason to bid sincerely and then choose the Vickrey outcome.

The transparency of NRA must be weighed against the more full-fledged justification, from first principles, of CA. We do not takes sides on which assumption is "better" but simply point out that there are different paths that lead to the consequence of Theorem 1.[6] Moreover, both lead to a Pareto-efficient outcome: the bidder with the highest valuation wins the auction.

3. Equilibrium Properties of the Vickrey Outcome

In this section, we present a collection $\{S_k: 1 \leq k \leq (n-1)^{n-2}\}$ of strategies and show that, without JBA and CA (or NRA), each is a symmetric Nash equilibrium. (Roughly speaking, an equilibrium is symmetric if all players make their stage I bids the same function of their valuations, and their stage II options the same function of their

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[6]Another path is to assume a different tie-breaking rule in stage II: in the event of a tie for the highest bid, the winner is the player that has the closest but strictly higher bid in stage I. Then claim 1 of Theorem 1 would work by a different induction argument: the highest stage I bidder will not usurp the lowest bid because it will definitely lose, and hence no player will usurp this bid; proceed inductively (highest bidder will not usurp the next-to-lowest bid, etc.) until the next-highest bid is reached, which will be usurped by the highest bidder. Note that if $n = 2$, this tie-breaking rule is the same as that assumed in the text (i.e., the highest stage I bidder breaks a tie in its favor).

stage I bids and valuations; we shall introduce more precise terminology shortly.) The symmetric strategies comprising this collection are those for which each player

- bids sincerely in stage I; and
- usurps the second-highest bid in stage II if it has the highest bid in stage I; affirms in stage II if it has the second-highest bid in stage I.

The reason we present a collection of strategies rather than a single strategy is that, for players other than the highest and second-highest bidders in stage II, stage II play is left unspecified, except that none of the n-2 lowest bidders can usurp the highest bid. (If one could, then the highest bidder would lose when it usurps the second-highest bid.) Because there are n-1 bids which the n-2 lowest bidders can usurp, there are exactly $(n-1)^{n-2}$ strategies in our collection.

Theorem 2. Each of the strategies S_k is a symmetric Nash equilibrium, whether each player knows (1) only its own valuation or (2) the valuations of all the other players as well prior to the the stage I bidding.[7]

Proof. If everyone uses S_k, then the player with the highest valuation wins at the second-highest bid (which is also the second-

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[7]Indeed, if all the valuations were published prior to the 1st stage so that they are common knowledge (i.e., not only does P know them, but P and all other players know that everybody else knows them, etc.), this equilibrium, which we call <u>information-proof</u>, would not be upset. We shall say more about its "restorative" qualities in section 5.

highest valuation). Suppose now that player P knows all the other valuations. We show that it cannot gain by unilaterally deviating from strategy S_k:

Case 1. P does not have the highest valuation.

In this case, the player with the highest valuation will either affirm (if P bids higher than that player bid) or usurp the second-highest bid (if P bids lower than that player bid). Either way, the only possible win for P is at the highest valuation, and this is worse than losing the auction.

Case 2. P has the highest valuation.

Assume player Q has the second-highest valuation, which it, following S_k, bids in stage I. If P bids higher than Q, then Q will affirm, and the best that P can do is win by usurping Q's bid. (This is precisely what S_k yields.) On the other hand, if P bids lower than Q, then Q, following S_k, will usurp the bid immediately below its own. Since P is lower than Q, P can win only by usurping a bid higher than the bid usurped by Q, namely Q's Ist-stage bid. Thus, P can match, but not do better than, a win at the second-highest valuation (i.e., that of Q). □

We next provide conditions under which the strategies given by Theorem 2 are the only Nash (as well as Bayesian) symmetric equilibria in two-stage auctions. This result requires a bit more precision in specifying permissible stage I bidding and stage II options of the players.[8]

Define a strategy S_i for player i in an n-person two-stage auction to be a pair (β_i, f_i) where:

- β_i: $[0, 1] \rightarrow [0, 1]$ is the stage I component. That is, $\beta_i(v)$ is what player i bids in stage I if it has valuation v.
- f_i: $[0, 1]^{n+1} \rightarrow \{1, \ldots, n\}$ is the stage II component. That is, $f_i(b_1, \ldots, b_n, v) = j$ means that player i usurps bid b_j (or affirms its own bid if i = j), given that the stage I bids of players 1, . . . , n are $b_1, \ldots, b_n$, and player i's valuation is v.

We say that the stage I function β is increasing iff the order of the players' lst-stage bids is the same as the order of their valuations. That is, the higher a player's v, the higher, or at least not lower, is its lst-stage bid.

We use the phrase "ordinal behavior at stage II," or, more simply, ordinality, to refer to strategies in which the value of $f_i(b_1, \ldots, b_n, v)$ is determined by the order of the entries. For example (and this is the special case we shall use later in claim 2 of Theorem 3), if

$$b_1, \ldots, b_{n-2} < b_{n-1} < b_n < v \text{ and } b_{n-1} < b'_{n-1} < b_n,$$

then

$$f_n(b_1, \ldots, b_{n-2}, b_{n-1}, b_n, v) = f_n(b_1, \ldots, b_{n-2}, b'_{n-1}, b_n, v)$$

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[8]That the functions that do this are the same for all players makes the resulting equilibria symmetric.

That is, if player n makes the highest bid b_n, but it is less than its valuation v, then its decision to affirm, or its decision to usurp the second-highest bid, third-highest bid, etc., is unaffected by a change in the second-highest bid, as long as it does not "cross over" any other bid or player n's valuation.[9]

Theorem 3. The strategies S_k are the only symmetric Nash equilibria that have an increasing stage I function and ordinality at stage II. Moreover, strategies other than the S_k's fail even to be symmetric Bayesian equilibria, if β is continuous and the valuations are probabilistically distributed so that open sets have have positive measure.

Proof. Suppose we have a strategy S that is a symmetric Nash equilibrium. Assume also that S has an increasing stage I function β, and its stage II function f is ordinal. We show that S is one of the S_k's by a series of seven claims:

Claim 1. The highest bidder in stage I wins at stage II.

Proof. Suppose not, and choose a sequence $(b_1^*, \ldots, b_n^*)$ of bids produced by β for which everyone's use of strategy S in stage II results in a loss for the highest bidder. Let S* be the result of adding the following proviso to S:

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[9]Notice that ordinality implies that the stage II function f is continuous. That is, in the presence of ordinality, slight changes in the stage I bids, and a slight change in the valuation v, result in no change at all in the stage II behavior.

"If the sequence of bids is (b_1*, . . . , b_n*), and you have the highest bid, then affirm your lst-stage bid if it is below your valuation, and usurp the second-highest bid if your bid equals your reservation price."

Any highest bidder unilaterally defecting to S* will do better, because it wins (at a profit) on (b_1*, . . . , b_n*), whereas S gave it a loss for this particular sequence of bids. (Of course, S and S* are the same for any other sequence of bids.) Contradiction: a player can do better with S*, confirming the claim.

Claim 2. Suppose there exists a sequence (b_1, . . . , b_n) of lst-stage bids produced by S for which the highest bidder has bid strictly less than its valuation, but it does not affirm in stage II. Then there exists such a sequence (b_1*, . . . , b_n*) for which the highest bid is less than the second- highest valuation.

Proof. Without loss of generality, assume $b_1 < \ldots < b_n < v_n$ where v_n is player n's valuation. If we now slide player n-1's valuation v_{n-1} up until it is between b_n and v_n, then b_{n-1} does not move above b_n (since β is increasing and v_{n-1} is still less than v_n). Then, by ordinality, player n's decision not to affirm remains intact. This yields the desired sequence (b_1*, . . . , b_n*).

Claim 3. If the highest bidder in stage I bids strictly below its valuation, then it will affirm in stage II.

Proof. Suppose not, and choose a sequence $(b_1^*, \ldots, b_n^*)$ of bids as guaranteed to exist by claim 2. Let S* be the result of adding the following proviso to S:

> "If the sequence of bids is $(b_1^*, \ldots, b_n^*)$, and you have the second-highest bid, then usurp the highest bid."

Any player unilaterally defecting to S* will do better, because it wins (at a profit) on $(b_1^*, \ldots, b_n^*)$, while S gave it a loss (by claim 1) for this particular sequence of bids. (Of course, S and S* are the same for any other sequence of bids.) Contradiction: a player can do better with S*, confirming the claim.

Claim 4. $\beta(v) = v$ for every $v \in [0, 1]$.

Proof. Assume not. Let S* be the strategy in which $\beta(v) = v$ for every v, and stage II behavior is as follows:

- If you would also have bid your valuation using S, then proceed in stage II exactly as S dictates.
- If not, then usurp the second-highest bid in stage II if you are the highest bidder.

We claim that any player (say, player n for definiteness) unilaterally defecting to S* does better. To see this, consider any sequence $(v_1, \ldots, v_n)$ of valuations whereby player n wins by using strategy S. By claim 1, we know that player n has the highest valuation. If $\beta(v_n) = v_n$, then player n proceeds the same, and hence the outcome is the same, using S* as using S. If $\beta(v_n) < v_n$, then we know, by claim 3, that player n affirms in stage II and, therefore, wins at $\beta(v_n)$ using S. However, by using S*, player n wins at the second-highest bid. This second-highest bid is $\beta(v_{n-1})$, where v_{n-1} is the second-highest valuation. Because β is increasing, we know $\beta(v_{n-1}) < \beta(v_n)$. Thus, S* yields a strictly better result in this case.

Claim 5. In stage II, the highest bidder either usurps the second-highest bid or the third-highest bid.

Proof. Note, by claim 4, that we know that 1st-stage bids are sincere. Hence, if the highest bidder affirmed for some sequence of bids, it could do better by unilaterally defecting to a strategy that usurped the second-highest bid for this particular sequence (and adhered to S otherwise). On the other hand, if the highest bidder usurped some bid below the third-highest, then the second-highest bidder could do better by unilaterally defecting to a strategy that usurped the third-highest bid for this particular sequence (and adhered to S otherwise).

Claim 6. In stage II, the highest bidder always usurps the second-highest bid, or it always usurps the third-highest bid.

Proof. This is an immediate consequence of ordinality at stage II and sincerity at stage I.

Claim 7. In stage II, the highest bidder always usurps the second-highest bid.

<u>Proof</u>. Assume not. Then, by claim 6, it must usurp the third-highest bid. We claim that any player can do better by unilaterally defecting to a strategy that says:

> "Bid 0 in stage I, and usurp the third-highest valuation if you have either the highest valuation or the second-highest valuation."

Any player so defecting will do exactly as it will do using S if it has the highest valuation. (The player with the second-highest valuation will then be the highest bidder, and thus usurp the third-highest bid--which is the fourth-highest valuation.) However, if the defecting player has the second-highest valuation, then it goes from a loss with S to a win at a profit. (Again, the highest bidder will be usurping the fourth-highest valuation, making the win for the defecting player at the third-highest valuation.) Contradiction: the highest bidder can do better <u>not</u> usurping the third-highest bid (i.e., when the second-highest bid is usurped by a 0-bidder with the second-highest valuation), confirming the claim.

All that remains to be shown is the "moreover" part of the theorem. This simply amounts to the observation that the continuity of β in stage I, and the ordinality of f in stage II, allow us not only to produce a single offending sequence where needed in the claims, but an open set of such sequences. That is, if a sequence is offending in a particular way, then so is any sequence resulting from sufficiently small changes in each of the valuations in the sequence. Thus, unilateral defection produces an increase in value on a set large enough to guarantee an increase in expected value, so there cannot be a symmetric Bayesian equilibrium with strategies differerent from S_k. □

In section 4 we shall consider possible deviations from the symmetric equilibrium S_k, demonstrating that a single player's deviation from sincere bidding in the lst stage need not adversely affect its payoff. However, the outcome resulting from such a deviation will not be in equilibrium--other players will then have an incentive to depart. Indeed, as we shall demonstrate in the two-person case, nonequilibrium strategies are rapidly "restored" to symmetric equilibrium strategies when the deviating player responds optimally to the other player's deviation.

4. Nonequilibrium Behavior

Curiously, despite the stability of strategies S_k, a single player can defect from S_k with impunity. To see this, let S denote the <u>specific</u> symmetric equilibrium strategy in which all players are sincere in stage I, and the highest bidder usurps the second-highest bid in stage II while everyone else affirms its stage I bid.

Now assume that a single player Q defects from S and bids 0 in stage I, regardless of its valuation. In stage II, it usurps the highest bid less than its valuation. This strategy, which we call S_Q, yields all players (including Q) exactly the same payoffs as S. This is because Q will win the auction when its valuation is the highest and it usurps the highest bid "on the table" (which would be only the second-highest bid had it also chosen S). On the other hand, if Q's valuation is not the highest, it will usurp a lower bid and lose to the highest bidder (with the highest valuation) when this bidder usurps the second-highest bid.[10]

Although "being off the equilibrium path" will not hurt Q if the other players continue to adhere to S, their adherence will not in general be to their advantage. We not only show that the resulting outcome is not an equilibrium in Theorem 4, but we also demonstrate that there are strong restorative forces that impel players to return to the symmetric equilibrium, at least in the two-person case.

Theorem 4. Assume the valuations of P_1 and P_2 are uniformly distributed over [0, 1]. Prior to a two-stage auction, suppose P_1 and P_2 are in a game--which might be considered a thought experiment--involving iterative play governed by the following rules:

1. At move 1, P_1 defects from S to a strategy that involves bidding $\beta(v) = \alpha_1 v$ for some fixed $\alpha_1 \in [0, 1]$.

2. At move 2, P_2--knowing the value of α_1 that P_1 is using[11]--does an expected-value calculation and chooses α_2 so as to maximize its gain among all strategies of the form $\beta(v) = xv$.

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[10]If Q had bid any amount less than its valuation (but greater than 0), its choice of S_Q--which may involve affirming its valuation if there is no higher bid less than its valuation to usurp--would clearly lead to the same outcome. Hence, <u>any</u> 1st-stage deviation from sincerity by Q will have no effect on Q's payoff, as long as Q substitutes S_Q for S in the 2d stage.

[11]P_2 could determine α_1 if it knew P_1's valuation v, and its intended bid $\alpha_1 v$, at move 1.

3. At move 3, P_1--knowing the value of α_2 that P_2 is using--does an expected-value calculation and chooses α_3 so as to maximize its gain among all strategies of the form $\beta(v) = xv$. And so on.

Given that each player at each move seeks to maximize its expected payoff at stage II, then

(i) $\alpha_{j+1} = 1/[(\alpha_j-1)^2 + 1]$ for any $j = 1, 2, \ldots$

(ii) $\lim_{j\to\infty} \alpha_j = 1.$

Proof. We prove (i) first. For notational simplicity, let $\alpha_j = \alpha$. Assume P_1 has valuation v_1 and is bidding $\beta(v_1) = \alpha v_1$. Let v_2 be P_2's valuation and let $E(x)$ denote P_2's expected payoff from a bid of $x \le v_2$:[12]

$$\begin{aligned} E(x) &= [\Pr(0 \le v_1 \le x)][P_2\text{'s payoff from winning by usurping } \alpha v_1] + \\ &\quad [\Pr(x \le v_1 \le x/\alpha)][P_2\text{'s payoff from winning by affirming its bid } x] \\ &= (x)[v_2 - \alpha(x/2)] + (x/\alpha - x)[v_2 - x] \\ &= xv_2 - \alpha x^2/2 + (x/\alpha)v_2 - xv_2 - (1/\alpha)x^2 + x^2 \\ &= x^2[1 - \alpha/2 - 1/\alpha] + (v_2/\alpha)x. \end{aligned}$$

Thus,

$$E'(x) = (2x)[1 - \alpha/2 - 1/\alpha] + v_2/\alpha.$$

Setting $E'(x) = 0$, it is not difficult to show that there is an extremum iff

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[12]Note that we are not actually assuming that P_2 will choose a strategy of the form $\beta(v) = \alpha_2 v$, but this will follow from our analysis.

$$x = v_2/[(\alpha-1)^2 + 1].$$

Notice that

$$0 < (\alpha-1)^2 + 1 = \alpha^2 - 2\alpha + 2,$$

so

$$2\alpha < \alpha^2 + 2, \text{ or } 1 < (\alpha^2+2)/2\alpha.$$

Hence,

$$1 < \alpha/2 - 1/\alpha, \text{ or } 1 - \alpha/2 - 1/\alpha < 0.$$

Thus, $E''(x) < 0$ at the extremum, so it is a maximum. Consequently, P_2 maximizes its expected payoff by bidding

$$\beta(v_2) = v_2/[(\alpha-1)^2 + 1],$$

so

$$\alpha_{j+1} = 1/[(\alpha_j-1)^2 + 1],$$

which proves (i).

In order to prove (ii), we need to establish two claims. Let

$$f(\alpha) = 1/[(\alpha-1)^2 + 1]$$

for $\alpha \in [0, 1]$. Then $\alpha_2 = f(\alpha_1)$, $\alpha_3 = f^2(\alpha_2)$, etc., by part (i).

Claim 1. The function f is increasing on the interval (0, 1).

Proof. The derivative of f is

$$f'(\alpha) = [(-2)(\alpha-1)]/[(\alpha-1)^2 + 1]^2.$$

Because $\alpha < 1$, $\alpha - 1 < 0$. Thus, $f'(\alpha) > 0$ for every $\alpha \in (0, 1)$, proving the claim.

Claim 2. $f^n(0)$ is of the form $k/(k+1)$ for every $n = 1, 2, \ldots$

Proof. We proceed by induction on n. For $n = 1$, we have $f^1(0) = f(0) = 1/2$. Assume $f^n(0) = k/(k+1)$. Then

$$f^{n+1}(0) = f(f^n(0)) = 1/\{[k/(k+1) - 1]^2 + 1\} = (k+1)^2/[1 + (k+1)^2]$$

after simplification. The desired result now follows since claims 1 and 2 together immediately yield $\lim_{n\to\infty} f^n(0) = 1$. (Thus, if $\alpha = 0$, then $\lim_{j\to\infty} \alpha_j = 1$.) But because f is increasing, we have, for any $\alpha > 0$,

$$0 < \alpha \Rightarrow f(0) < f(\alpha) \Rightarrow f^2(0) < f^2(\alpha) \Rightarrow \cdots,$$

so the sequence $\langle \alpha, f(\alpha), f^2(\alpha), \ldots \rangle$ is, term by term, greater than $\langle 0, f(0), f^2(0), \ldots \rangle$. Because the latter converges to 1, the former converges to at least 1, and, hence, exactly 1. □

Notice that the restorative forces are quite powerful: if P_1 defects to bidding 0, P_2 will respond with $(1/2)v_2$. Then P_1 will respond with $(4/5)v_1$, P_2 with $(25/26)v_2$, P_1 with $(676/677)v_1$, etc. Thus, by the 5th stage, the bid is over 99.8 percent of the player's reservation price. We may summarize our findings as follows:

Corollary. If valuations are uniformly distributed over [0, 1], then the only linear Bayesian equilibrium (symmetric or not) in a two-person two-stage auction is that in which both players are sincere in stage I, and the highest bidder usurps the other's bid in stage II. (A linear strategy is one with a stage I function of the form $\beta(v) = av + b$ for some constants a and b.)

5. Conclusions: Theory and Practice

We introduced two-stage auctions by saying that they were a way of implementing the Vickrey outcome. Given the Judicious-Bidding Assumption (JBA) and the Competitiveness Assumption (CA), we showed that players would not only bid sincerely at the 1st stage but also that the highest bidder (with the highest valuation) would usurp the second-highest bid at the 2d stage.

Substituting the No-Regret Assumption (NRA) for CA makes this result more transparent. But NRA, in precluding underbidding at the 1st stage, is not rooted to fundamental rationality calculations (e.g., based on the desire of players to maximize their payoffs). Rather, it simply asserts that players will eschew the possibility of any regret that might occur if, by underbidding, they lose. Indeed, blindly following NRA could cost players profits in certain scenarios, whereas CA rules out such scenarios when the players themselves successively eliminate less preferred strategies.

As an alternative justification of the Vickrey outcome, we showed that, without JBA and CA (or NRA), it is a symmetric Nash equilibrium. Moreover, this equilibrium is invulnerable to players' knowing each others' valuations, rendering it information-proof. If, in addition, the bids of the players are increasing in v in stage I, and their behavior is ordinal in stage II, the Nash equilibrium is the unique symmetric equilibrium and, by extension, the unique symmetric Bayesian equilibrium in a game of incomplete information.

Any single player that deviates from sincerity in stage I, by usurping the highest bid less than its valuation (or affirming its own bid if there is none higher) in stage II, can do as well as S. However, the resulting outcome is not in equilibrium. We showed that when the players alternately respond optimally to each other's deviations in a two-person game--or thought experiment--played prior to the auction, they will move rapidly toward sincerity to maximize their payoff.

Compelling as our theoretical results are, do they offer more than just a mechanism to implement the Vickrey outcome? We think the answer is "yes": two-stage auctions have important practical advantages over Vickrey auctions. For one thing, they provide an easy way--by means of a simple checking procedure (described next)--to prevent the introduction of bogus bids by shils or confederates that can raise the amount that the highest bidder must pay.

Under this checking procedure, one would ask each player after the 1st stage whether its bid is one of those made public. If all the bids are different and equal the number of players, and if each bidder answers affirmatively, then one can be sure that no fraudulent bids have been inserted to bias the outcome in favor of obtaining a higher price for the bid taker. This procedure would address one of the principal objections--fear of cheating--that Rothkopf, Teisberg, and

Kahn (1990) claim bedevils Vickrey auctions and has worked against their adoption, though collusion among a ring of bidders may still be possible (Mailath and Zemsky, 1991).

Another advantage of two-stage auctions is that they allow bidders to revise their bids by usurping others' bids once the latter become common knowledge.[13] To be sure, we showed that all bidders except the highest will be motivated to affirm their 1st-stage bids. However, one can certainly imagine situations in which a player, after observing the bids in the 1st stage, decides that it wants to opt out, or at least to revise its bid in a common-value auction, which is a subject we treat in detail in Brams and Taylor (1991b).

In summary, two-stage auctions, like Vickrey auctions, induce players initially to bid sincerely. Beyond sharing this remarkable property of Vickrey auctions, two-stage auctions, in our view, offer striking practical advantages over Vickrey auctions: they can be made invulnerable to cheating by a simple checking procedure; and they allow players to revise their bids in light of knowledge gained when the other bids are revealed. Consequently, we believe two-stage auctions are worthy of experimentation and--if their advantages are borne out in the laboratory--think they should be tried out in natural settings.

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[13]We first introduced the notion of a 2d stage, in which bids could be revised (but not in auctions--in how to divide a dollar), in Brams and Taylor (1991a).

References

Brams, Steven J., and Alan D. Taylor (1991a). "Three Solutions to Divide the Dollar." Preprint, New York University.

Brams, Steven J., and Alan D. Taylor (1991b). "Two-Stage Auctions II: Common-Value Strategies and the Winner's Curse." Preprint, New York University.

McAfee, R. Preston, and John McMillan (1987). "Auctions and Bidding." Journal of Economic Literature 25 (June): 699-738.

Mailath, George J., and Peter Zemsky (1991). "Collusion in Second Price Auctions with Heterogeneous Bidders." Games and Economic Behavior 3, no. 3 (November): 467-486.

Milgrom, Paul (1989). "Auctions and Bidding: A Primer." Journal of Economic Perspectives 3, no. 3 (Summer): 3-22.

Rothkopf, Michael H., Thomas J. Teisberg, and Edward P. Kahn (1990). "Why Are Vickrey Auctions Rare?" Journal of Political Economy 98, no. 11 (February): 94-109.

Smith, Vernon L. (1987). "Auctions." The New Palgrave : A Dictionary of Economics, vol. 1. London: Macmillan, 1987, pp. 138-144.

Vickrey, William (1961). "Counterspeculation, Auctions, and Competitive Sealed Tenders." Journal of Finance 16, no. 1 (March): 87-37.