ECONOMIC RESEARCH REPORTS

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RR# 99-01 **January 1999**

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U.S. and Canadian Industrial Production Indices As Coupled Oscillators

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December 1998

Abstract

This paper explores the impact of different types of dynamical linkages (coupling) between the indices of industrial production for the U.S. and Canada. The Ozaki model provides an appropriate empirical framework for analyzing the dynamic path of each economy's productive activity because it provides an effective approximation to continuous time differential equations. We examine a combination of six different types of linkage between the indices of production. Major questions we study include whether the linkages increase or decrease the stability of the equilibrium paths, whether the linkages encourage or discourage business cycle oscillations, and whether the oscillations are synchronized.

The empirical analysis reveals that for each country the requisite lag structure extends for up to six years and that the coupling linkages are functions of lagged changes in growth rates. Incorporating these linkages leads to an increased degree of oscillation in the equilibrium paths, a marginal improvement in stability, and somewhat greater synchronization between the two series. The out-of-sample forecasts for each series are improved by the coupling.

JEL classification: C32, E32.
Keywords: Coupled dynamics, synchronization, macrodynamics, industrial production, Ozaki model, forecasting.

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1. Introduction

It is widely suspected that economic conditions in one country can be transmitted to other countries. This has led to a concern that the current problems in Asia will be transmitted to the United States and especially to Latin America. Similarly, there have been suggestions that an economic collapse in Russia might lead to a rapid deterioration in European economies. We hear much about "international repercussions" and their "consequences for the global economy". Such topics involve an implicit assumption that links through trade and capital flows have significant implications for international economic stability, although little is actually known about the dynamic processes that would be generated by these sorts of linkages. It is often thought that trade and financial linkages lead to internationally synchronized business cycles, but whether these linkages dampen or accentuate the amplitude of the cycles is not clear. Some economists believe that international linkages may lead to instability; others maintain that the global economy is inherently stable and that international trade promotes this stability.

The question of whether cross-country ties contribute towards global stability or instability is an interesting one which has not yet received much attention in the literature. Two relevant theoretical papers include work by Lorenz [22] and Puu [32], who explore the role of trade as a destabilizing linking mechanism for output dynamics. However, usually the theoretical work on international economic linkages has modeled these linkages in a very simple manner and has not addressed the issues of how different types of linkages might affect the dynamics of the global system. Similarly, much of the empirical work on economic linkages has concentrated on non-dynamic issues, such as the determination of the relative importance of openness, trade, monetary policy, technology spillover or common supply shocks.

With respect to the empirical work on international interaction that explicitly incorporates dynamics, project LINK (see the discussion in [20]), and some other analyses based on vector autoregressions are the best known. Most of the remaining empirical work on international interactions has avoided dynamic issues by focusing only on contemporaneous correlations in output. Exceptions include a small literature that models the comovements in different countries' business cycles as multivariate systems with common dynamic factors (see [9], [17], or [1]), and a few papers that ask whether the volatility of output is affected by different exchange rate regimes (see [14] or [3]). The common factor papers incorporate the notion that different country business cycles are synchronized, but none of these empirical research papers have explicitly explored the stability properties associated with linking the economies.

The issue of how economic ties between different countries affect comovement in output across countries can be recognized as a special case of the more general question of how interdependencies between markets might influence the joint behavior of these markets. Goodwin [15] studied a version of this problem in a paper in which he was the first to apply the idea of dynamic coupling to economics. Modeling two continuous time series of interrelated industries as a coupled system of industries, he found that the phase properties of the interdependent or coupled system were quite different from the phase properties of each individual industry. He also found that the dynamic coupling of two markets could decrease the stability of each market. However, since Goodwin's work, the study of coupling in economic series has tended to be sporadic, and has been mostly restricted to work which discusses the possibility of chaotic behavior in economic variables. See, for example, the references in [12], [13], and [23].

Dynamic coupling has an extensive literature in mathematics and physics, and somewhat less in biology. In these literatures, it is recognized that the dynamics of coupled (or linked) systems can be very different from the dynamics of the constituent sub-systems. Further, the properties of a coupled system depend on the nature of the coupling mechanism. Some types of coupling can lead to chaos, while other types can lead to the synchronization of dynamical components. Mode locking (or entrainment), which is the state of being locked into the forcing frequency, can sometimes occur. Given that closely connected economies might be viewed as coupled systems and that coupling can lead to many different types of joint dynamical behavior, it is clear that a useful research topic would be to explore the links between economies and the effects that these links have on both stability and coordination. Questions that deserve immediate attention include the types of linkages that occur across economies, whether or not those linkages enhance stability, and under what circumstances the linkages might lead to synchronization of industrial activity.

The discussion in this article should be regarded as an initial foray into an important, but relatively unexplored topic. We attempt to incorporate cyclical nonlinearities into a study of the interactions between the business cycles of different countries in terms of a simple bivariate model. The mathematical framework is Ozaki's (1985) variable amplitude coefficient model, see [18] and [26]. We apply this model to study the interactions between industrial production in Canada and the United States. The model specification which we estimate includes the nonlinear dynamics and a set of "link" variables that are used to capture the interactions between these two countries. Our major task is to discover empirically the type of linkage or coupling that is involved and then to derive the implications of these linkages for synchronization and instability within the linked system.

The remainder of this paper is in four parts. Our next section will briefly review the literature and elucidate some of the mathematical arguments underlying the idea of coupling. This discussion is followed by a review of the modelling structure that we choose as the mathematical framework for our analysis. The largest section of the paper is devoted to describing our statistical and simulation results. Simulation is an important tool because the complexity of the systems precludes any meaningful algebraic analysis. The paper ends with the usual summary and conclusions. In the conclusion, we try to indicate the broader implications for our research, and to give some guidance concerning the extension and development of the research implications stimulated by the idea of dynamic coupling.

2. The Concept of Dynamic Coupling

In order to illustrate some of the concepts involved we consider as an example the dynamics of a pendulum, which can be described by a simple linear unforced second order differential equation given by

$$m\ddot{x} + r\dot{x} + sx = 0, \tag{2.1}$$

where m represents "mass," s the "stiffness" of the pendulum, and r the degree of "dampening" of the pendulum's motion. Setting the dampening coefficient r equal to zero (for mathematical simplicity), the natural frequency of oscillation is given by the parameter ω (with $\omega^2 = s/m$), which summarizes the effects of gravity and the physical properties of the pendulum on the pendulum's motion.

Now suppose that we have two such pendula, with positions x and y described by 2.1 (with $r = 0$), and that these pendula are linked by a spring with stiffness, "f". The equations of motion for small oscillations in this coupled system are given by

$$\begin{aligned} \ddot{x} &= -\omega^2 x - f/m(x - y) \\ \ddot{y} &= -\omega^2 y + f/m(x - y). \end{aligned} \tag{2.2}$$

Each pendulum by itself will oscillate with a frequency of ω, but the system of two pendula has a more complex motion. We immediately recognize that the two pendula can move together in phase, in which case there is no compression or extension of the spring, or out of phase, in which case there is alternating compression and expansion of the spring.

In this simple model we can decompose the oscillations of the system into an "in phase" component and an "out of phase" component by a simple transformation. To do this, we define new coordinates by

$$\begin{aligned} X &= x + y \\ Y &= x - y \end{aligned} \tag{2.3}$$

and then derive 2.4 by adding and subtracting the constituent equations in equation 2.2 to obtain

$$\begin{aligned} \ddot{X} + \omega^2 X &= 0 \\ \ddot{Y} + (\omega^2 + 2f/m)Y &= 0. \end{aligned} \tag{2.4}$$

Equation 2.4 shows that we have been able to "decouple" the coupled system in the following sense. In terms of the new coordinates, we can analyze the oscillations separately; more importantly we can add the energy of each mode to get the energy of the system as a whole, which is constant. There are two "normal" modes of vibration. One mode, X, oscillates at a frequency of ω while the other mode, Y, oscillates at a frequency of $\sqrt{\omega^2 + 2f/m} = \omega_1$, which is a higher frequency than the first. The former mode X is termed the "in phase" mode, while the latter mode Y is termed the "out of phase mode." If Y is always zero, then the link will be ineffective since $x = y$ at all times, and there will be neither compression nor expansion in the spring; the "operating mode" is the "in phase mode." If X is always zero, then the coupling will be effective, since $x = -y$ at all times, and the spring will expand when y is positive and contract when y is negative; the "operating mode" is the "out of phase mode." The "out of phase" mode's oscillation increases with an increase in stiffness, or a decrease in the common mass. As the stiffness approaches zero, the frequencies of the normal modes converge.

To obtain a better picture of the increased complexity of the dynamics we need to examine the dynamical behavior of the actual coordinates, not that of the normal modes. From our derivation, we note that $X + Y = 2x$ and that $X - Y = 2y$. From these relations we can easily derive the solution loci for each of x and y from a displacement of size "a":

$$\begin{aligned} x &= 2a\cos\left(\frac{\omega_1 - \omega}{2}t\right)\sin\left(\frac{\omega_1 + \omega}{2}t\right) \\ \text{and } y &= 2a\cos\left(\frac{\omega_1 + \omega}{2}t\right)\sin\left(\frac{\omega_1 - \omega}{2}t\right) \end{aligned} \tag{2.5}$$

from which we see that the envelope beat oscillation given by the difference in frequencies for the two positions is out of phase by $\pi/2$ radians. From individual pendula oscillating at a harmonic frequency of ω each, we now have a pair of pendula oscillating at a frequency of $(\omega_1 + \omega)/2$ within an envelope beat frequency of $(\omega_1 - \omega)/2$.

An alternative linkage mechanism might be a coupling through the common masses; this occurs in nature when electrical oscillators couple. In this case, one can derive the pair of frequencies at which the system resonates. Sometimes this form of coupling involves a "forced oscillator", in which case precedence is given to one of the oscillators as a control and this control then determines the forcing frequency. It is well known that, as the forcing frequency approaches the natural frequency of the forced system, there is resonance so that at the common frequency the amplitude grows without bound. Further, as has been discussed in a number of papers, (see the references in Gandolfo [13] for examples), the forcing of nonlinear differential equations can easily lead to chaotic time paths.

In general the dynamics of coupled systems are very different from those of the constituent sub-systems. There are two polar cases; one in which there is synchronization of orbits and another in which there is inducement of chaos or at least the inducement of highly complex dynamical behavior. Pecora and Carroll (1990) [30] demonstrate the conditions under which synchronization occurs, while Gandolfo, [13] for example, mentions the Landau route to complexity in the coupling of more than two dimensional systems. Chernikov and Schmidt [6] show that when coupling is through the velocities of motion synchronization in frequency and amplitude is enhanced with a consequent reduction in the effective dimension of phase space. In contrast, Baesens et al. [2] find conditions that imply an increasing dimension of phase space.

There are many examples of synchronized systems in real life, as is clear from the discussion in Mirollo and Strogatz [24], and Strogatz and Stewart [35]. Examples include clocks attached to boards, fireflies signalling for mates, animal gates, cardiac rhythms, and so on. Given that dynamical sub-systems in economics are naturally linked because of trade, capital flows, technology spillover and numerous other reasons, it seems feasible that a similar sort of coupling arises in dynamic systems of interrelated economies.

Lorenz [22] has developed a very simple model which explores the effect of economic linkages on the dynamics of an aggregate economy, and to illustrate how a coupling of economic systems might arise in practice we outline the main features of this model below. Lorenz starts with three dynamic autarchic IS-LM economies described by:

$$\begin{aligned} \dot{Y}_i &= \alpha_i[I_i(Y_i, r_i) - S_i(Y_i, r_i)], \\ \dot{r}_i &= \beta_i[L_i(Y_i, r_i) - \bar{M}_i], \\ i &= 1, 2, 3;\ \alpha_i,\ \beta_i\ > 0, \end{aligned} \tag{2.6}$$

where Y_i represents real income in country i, I_i is real investment, S_i is real saving, L_i the demand for money balances, $\bar{M}_i$ the exogenously determined money supply, and r_i is the real interest rate. Prices are assumed to be fixed. Without trade, there are three independent oscillators. When the solution of each equation produces a single frequency, the motion of the whole system can be described in terms of these three frequencies.

Trade provides a coupling between the economies. With fixed exchange rates and no capital flows a trade-linked model is given by

$$\begin{aligned} \dot{Y}_i &= \alpha_i[I_i(Y_i, r_i) - S_i(Y_i, r_i) + X_i(Y_h, Y_k) - IMP_i(Y_i)], \\ \dot{r}_i &= \beta_i[L_i(Y_i, r_i) - \bar{M}_i], \\ IMP_i(Y_i) &= \sum_{j=1}^{3} Imp_{ij}(Y_i),\ j \neq i,\ X_i(Y_h, Y_k) = \sum_{k=1}^{3} Imp_{ik}(Y_i),\ k \neq i \\ i &= 1, 2, 3;\ \alpha_i,\ \beta_i\ > 0, \end{aligned} \tag{2.7}$$

which now consists of three coupled oscillators. $Imp_{ij}(Y_i)$ represents the imports from countries $j \neq i$ to country i and $X_i(Y_h, Y_k)$ represents country i's exports to the countries h and k. Total exports across the three countries equals total imports across the same three countries, and the countries are linked solely through trade imbalances. The long term effects of an imbalance in trade are ignored.

The movement in this coupled system 2.7 can be quite different to that of any constituent oscillation in 2.6. Lorenz shows that quasi-periodic motion and phase locking between the countries can occur. He also shows that, given certain parameter values, this trade linked model can lead to chaos. It is possible to transform 2.7 into three normal modes; a transformation in which the trade flows drop out consists of the sum of the real incomes in the three countries and then the interest rate equations for any two of the countries.

Notwithstanding theoretical developments such as the Lorenz model, there has been very little in the way of empirical work that concentrates on this issue, except incidentally. The closest empirical research in the spirit of examining the linkages of dynamical systems is quoted on page 157 in Granger and Teräsvirta [16], where bivariate LSTAR models of industrial production are estimated using quarterly data with lagged values of one country's output in the LSTAR model for the other. The pairs of countries which are examined include the U.S. and Canada, Germany and Belgium, and the U.S. and Japan. Granger and Teräsvirta conclude that there are significant links between each of the countries pairs, in the sense that lagged values of output of one country are significant in the dynamic equation for another. Related to this is another paper by Anderson and Vahid [1], in which they generalize the idea of common trends to common nonlinear components and find a common nonlinearity in industrial output for Canada and the U.S. These two papers emphasize common elements in output, but the emphasis in this paper is more on the linkages themselves and the role that these linkages play in modifying the dynamics of the coupled systems. Systems may have common components, but they may be linked more directly, so that what appears to be a common component is in fact the outcome of dynamic linkage.

Consequently, the objective in this paper is to use the available indices of industrial production in order to examine alternative linkages between a general exponential autoregressive model for each country's output. In part, one can interpret the empirical work in this paper in terms of the Lorenz model discussed in equation 2.7. If one were to generate a reduced form relationship from those purported behavioral equations, so that the dynamics of the subsystems could be modeled strictly in terms of the output time series, and if we summarized the complicated inter-relationships between the two countries in terms of simple functions of domestic and foreign output, then our efforts to be discussed below provide an analysis of a reduced form implementation of the Lorenz modeling concept indicated in equation 2.7. To this issue we now turn.

3. Specification of a Model with Dynamic Linkages

One problem associated with undertaking an empirical study of international comovement is that output fluctuations are not well described as linear autoregressive or ARMA processes. Business cycle asymmetries, policy regime shifts and structural change resulting from advances in technology all imply that output time series have nonlinear data generating processes, and such processes can generate very complicated behavioral patterns, even in univariate cases. See [19], [36], [31] [34], [33], [5] and associated references for an introduction to some of the empirical literature on nonlinearities in business cycles. Extensions in the direction of nonlinear interdependencies are, of course, much more complicated than the study of univariate nonlinear processes, and very little work has been done in this area. Some empirical examples are, however, provided by [1], [7], [16], [8] and [10].

The basic modeling framework that we use in this study is the nonlinear exponential autoregressive model, originally proposed by Haggan and Ozaki in 1981 [18], and discussed further in [26] and [27]. This model provides a very flexible form that can encompass discrete approximations to a wide variety of nonlinear continuous differential equations, and in particular, it can account for local instabilities within a globally stable framework. This last consideration is important, given that the primary aim of our empirical analysis is to study the stability properties of univariate and linked systems of business cycle indicators. The stability properties implied by the Ozaki model depend on the state of the system as well as on the parameters, so that the model can imply many different patterns of dynamic behavior.

Denoting y_{jt} as our business cycle indicator for country j, the Ozaki model (without allowing for linkage effects) is given by

$$\begin{aligned} y_{jt} &= \Phi_{j0} + \sum_{k=1}^{k=p} \Phi_{jk}(\, y_{jt-d_j}\,)\, y_{jt-k} + \varepsilon_{jt} \qquad (3.1) \\ \Phi_{jk}(y_{jt-d_j}) &= \Phi_{jk_0} + \pi_{jk}(y_{jt-d_j}) exp[-\gamma_j(\, y_{jt-d_j} - c_j)^2] \\ \pi_{jk}(y_{jt-d_j}) &= \pi_{jk_0} + \pi_{jk_1} y_{jt-d_j} + \pi_{jk_2} y_{jt-d_j}^2 + \pi_{jk_3} y_{jt-d_j}^3 + ...\pi_{jk_m} y_{jt-d_j}^m \end{aligned}$$

with $\gamma_j > 0$. The indicator y_{jt} is assumed to be stationary and the error process is assumed to be a zero mean i.i.d. process. In the empirical work which follows, y_{jt} will be a measure of annual growth in industrial production (defined below), with the subscripts $j = c$ and u respectively indicating the values of growth observed in Canada and the United States. The dynamic path of the two underlying raw series (which we call c_t and u_t) can be deduced from the estimated version of equation 3.1 by using appropriate integration techniques.

At first sight, the equation for y_{jt} appears to be quite complex, but actually it is reasonably simple. Line 1 indicates that we are dealing with an AR process, but with coefficients that are functions of the state space associated with y_{jt-d_j}. The lag of d_j in (y_{jt-d_j}) indicates that we allow for a variety of alternative lengths of delay in defining the relevant state space. Line two indicates the nature of the variation in the state dependent coefficients. The exponential term is key, in that it allows the weight of the polynomial terms in line 3 to vary from zero (when the exponent is approaching infinity) to a full weight of one (when the exponent is zero). The use of a polynomial in line 3 allows for many different types of nonlinear behavior and when $m > 1$, the model can capture nonlinearities in the original differential equations, beyond those which simply result from a change in state. Less complex versions of Ozaki's (1981) model have been used to study asymmetries in business cycles; some examples include the ESTAR model used by Teräsvirta and Anderson in 1992, (which imposes the restriction that $m = 0$), and the TAR model used by Potter in 1995, (in which $m = 0$, $\gamma \to \infty$, and the exponential term is a logistic function in $\gamma_j(y_{jt-d_j} - c_j)$ rather than $exp[-\gamma_j(y_{jt-d_j} - c_j)^2]$). Here, we maintain the more general format of the Ozaki formulation, so that our empirical models can capture nonlinearities other than simple state changes.

3.1. Alternative Linkage Variables

Given that equation 3.1 is sufficiently flexible to capture the univariate behavior of each business cycle indicator, the next task is to specify the coupling between these indicators. It is immediately clear that there are many different ways in which linkages between economies might be modeled. Variables measuring trade, financial flows or supply shocks could be incorporated into a joint model of two economies in a linear or nonlinear fashion, with or without lags. Here, because we want to remain within the framework of a *bivariate* coupling model, and we are not focussing on identification issues such as whether the linkages arise from trade or capital flows, we assume that all linkages affecting y_{jt} can be represented by simple (dynamic) functions which incorporate y_{it} with $(i \neq j)$. This assumption is consistent with many economic models of linkages. We denote our linkage functions for y_{jt} by $L_{jr(t-dr)}$ (with the subscript r being used to differentiate between different linkages, and the subscript $t - dr$ indicating the transmission lag), and we assume that each of these linkages has a linear effect on y_{jt}. Our "linked" nonlinear autoregressive equation is then given by

$$y_{jt} = \Phi_{j0} + \sum_{k=1}^{k=p} \Phi_{jk}(y_{jt-j_d})y_{jt-k} + \sum_{r} \varsigma_r L_{jr(t-dr)} + \varepsilon_{jt}, \tag{3.2}$$

which simply combines equation 3.1 with the link variables.

Further work is required to specify the link variables, and to do this, we work from a list of linkages which are a-priori potentially useful as explanatory variables in 3.2. We cull this list later in Section 4.2, using the data to guide our decisions. Initially, the potential linkage variables include a variety of terms meant to reflect different ways in which the growth rate of the companion country might affect growth. For Canada, these possible linkage variables are

$$\begin{aligned} &y_{ut} \\ &\Delta y_{ut} \\ &\Delta y_{ct} - \Delta y_{ut} \\ &\Delta y_{ct} * \Delta y_{ut} \\ &\frac{y_{ct} - y_{ut}}{y_{ct} + y_{ut}} \end{aligned} \tag{3.3}$$

and various lags of each are considered. For the U.S. the list is a mirror image of 3.3, with y_{ct} replacing y_{ut} and vice-versa.

The five definitions listed in equation 3.3 represent five alternative ideas for linking Canadian and U.S. output profiles. The first suggested variable follows the lead in Granger and Teräsvirta [16], in that past growth rates of U.S. production are incorporated into the Canadian growth rate equation. The next three suggestions for linking Canadian and U.S. output incorporate the idea that the link depends on the changes in growth (a second order effect which corresponds to an acceleration in the underlying levels in the output series); it might just depend on growth changes in the other countries, the difference between the growth changes in each country, or interaction between the two changes in growth rates. The fifth possibility is that links depend on the relative difference in the growth rates between the U.S. and Canada.

We also consider two other variables, labeled $plink_{ct}$ and $nlink_{ct}$, that allow for capital utilization. These variables are similar in design to the CDR (current depth of recession) variables that are now popular in the macroeconomic literature, (see [4] or [29] for some examples), and since their construction is quite complicated, we define these two variables in Appendix 1.

While the precise formulation of these links may not be correct, the range should provide insight into the generic type of linkage that may apply. Given that some linkage applies, we can then examine the dynamical implications of the linkages. In particular, we can enquire whether the linkages do, or do not, lead to greater stability. It is likely that the answer to this question depends on the state of the system. In any event whatever the answers may be, the solution depends on the empirical estimates. To this issue we now turn.

4. Empirical Models of Linked Asymmetric Business Cycles

The data used in this study consisted of 458 seasonally adjusted monthly observations on the industrial production indices for Canada and the United States. The data source was the IMF financial series data base and the available sample covered the period dating from January 1957 until February 1995. The raw data (denoted by c_t and u_t) were clearly non-stationary and were transformed into approximately stationary monthly series of annual growth rates by taking the twelfth differences of natural logarithms. This left 446 observations for the empirical analysis.

All subsequent work was based on the resulting growth rate series, y_{ct} and y_{ut}. The first 396 of these observations on growth rates were used for developing an appropriate model of the data, and the remaining 50 observations were retained for out of sample model evaluation.

4.1. The Development of Univariate Models with no Linkages

Summary statistics for the full sample of transformed data are presented in Table 1 and plots are presented in Figure 1. The plots indicate that each transformed data series is approximately stationary around a constant and the assumption that the transformation has rendered the data stationary is supported by the reported Dickey Fuller test statistics which strongly reject the null hypothesis of a unit root in either y_{ct} or y_{ut}. The plots show some clear differences between the dynamics in each of these two series, but the sample moments reported in Table 1 are remarkably similar.

The last line of Table 1 contains p-values for Teräsvirta-type LM tests of the null that the data generating processes for y_{ct} and y_{ut} are linear against STAR alternatives. These tests are based on Teräsvirta's (1994) observation that a Taylor series approximation of the $STAR(p)$ alternative is a linear combination of the $3 \times p$ variables defined by $y_{t-k}y_{t-d}$, $y_{t-k}y_{t-d}^2$ and $y_{t-k}y_{t-d}^3$, for $k = 1, ..., p$. Thus the statistical significance of these $3 \times p$ variables in an auxiliary regression of the residuals from an $AR(p)$ will support the $STAR(p)$ alternative. Because Haggan and Ozaki's (1985) specification simply generalizes the STAR model, it follows that Teräsvirta's test also has power against the type of nonlinearities incorporated in equation 3.1 . We therefore interpret a rejection of the null hypothesis of linearity as support for a variable amplitude coefficient model. As proposed by Teräsvirta in [37], the reported test statistic for linearity is the minimum of a sequence of tests in which the lag d of the transition variable in the alternative is varied. It is interesting to note that for both countries this minimum occurs when $d = 1$. The observed minimum p-values of 0.0001 for Canada and 0.0118 for USA provide strong evidence that the AR coefficients in the univariate representations of each indicator y_{jt} vary with $y_{j(t-1)}$. These results justify the use of $d_j = 1$ in the subsequent model estimation.

For fixed given values of γ and c, the Haggan and Ozaki specification described by equation 3.1 collapses to a model that is linear in simple transformations of the explanatory variables. Given the numerical analysis difficulties involved in jointly estimating these two parameters together with the other parameters in the transition function and also the relative importance of a non-singular regression matrix, the exponential component was fixed at $\exp\left[-10(y_{t-1}^2 - \mu_{y_{t-1}^2})\right]$ for each of the two countries. The parameter $\mu_{y_{t-1}^2}$ is the mean of y_{t-1}^2 obtained from the raw second moments reported in Table 1; these were $\mu_{y_{ct-1}^2} = 0.0041$ and $\mu_{y_{ut-1}^2} = 0.0042$ for Canada and the USA respectively. The use of $\mu_{y_{t-1}^2}$ as a location parameter ensures that the exponential function fluctuates around one, while the scaling parameter of $\gamma = 10$ ensures that there is sufficient variation in the exponential function so as to avoid ill-conditioning in the likelihood. Plots of the resulting exponential transition functions are provided in Figure 2 and summary statistics for these transition function are provided in Table 2.

An examination of the eigenvalues for the squared data matrix associated with $\exp\left[-10(y_{jt-1}^2 - \mu_{y_{jt-1}^2})\right] \pi_k\left(y_{jt-1}\right)$ for different powers of y_{jt-1} in $\pi_{jk}\left(y_{jt-1}\right)$ led to the specification of $\pi_{jk}\left(y_{jt-1}\right) = \pi_{k0} + \pi_{jk1}y_{jt-1}$ for each country. This choice of $m = 1$ was later supported by appropriate specification tests which were conducted once the models for each country had been developed and estimated.

Estimation of univariate nonlinear models for each country (with no linkages) revealed a relatively long memory of approximately five or six years. However, an inspection of the t-statistics associated with the estimated coefficients led to the conclusion that the statistically significant components of the lag structure were only associated with "end of year" effects. Some experimentation showed that the dynamics of each series could be adequately modeled using a skeleton lag structure which incorporated less than one quarter of the 217 parameters potentially involved in the model specified by equation 3.1 with $p = 72$, $m = 1$, and d set to 1. These general results were supported by the use of the TSMARS algorithm. The TSMARS algorithm is the time series extension of Friedman's multivariate adaptive regression splines, see [11] and [21]. Summary information relating to the chosen skeleton lag structure together with some supporting statistics are provided in Table 3. Details on all the estimated coefficients and their standard errors are shown in the Appendix Tables 2a and 2b.

We refer to these models as the nonlinear autoregressive models and use the notation $\widehat{y_{jt}} = f_{jNAR}(y_{jt-1}, ..., y_{jt-p} : \widehat{\Omega}_j)$,where $\widehat{\Omega}_j$ represents the vector of estimated coefficients for $j \in \{c, u\}$. In all, three models were examined in the empirical work. The nonlinear autoregressive model that we have just mentioned and defined in equation 3.1, the same model including links that was defined in equation 3.2, and a linear autoregressive model to serve as a benchmark in the comparison of forecasting performance. The linear autoregressive model was given exactly the same lag structure as its nonlinear counterparts.

4.2. Estimation of The Models with Links

Preliminary analysis indicated that the growth rates of the two production series have a strong contemporaneous correlation of 0.78, which is consistent with the prior that these series are related. Granger Causality tests based on a VAR which incorporates the lag structure given in equation 3.1 failed to find any evidence that the U.S. production growth index is a good linear predictor of the Canadian production growth index, or vice-versa. However, an examination of the cross correlations between Canada and the U.S. shows that the cross correlations are very strong, persistent, and asymmetric. The cross correlations switch sign at a lag of thirteen months in one direction and at a lag of eleven months in the other direction, so that the relationship between Canadian and U.S. growth is not symmetric.

4.2.1. Determining the Link Variables

The first variable to be investigated for linkages between the two countries was the simple growth rate for the companion country, which is the first variable to appear in the list given in equation 3.3. Using equation 3.2, an initial series of experiments which simply added y_{ut-k} for different k one at a time to the model for Canada and analogous experiments which added y_{ct-k} to the model for the U.S., indicated some simple dynamical linkages between the growth rates in these two countries. For example, y_{ut-1} by itself is statistically significant for Canada and each of y_{ct-1}, y_{ct-2}, y_{ct-36} and y_{ct-48} is individually significant for the U.S. Similarly, $\Delta y_{ct-23*}\Delta y_{ut-23}$ and $\Delta y_{ct-24*}\Delta y_{ut-24}$ and $plink_{ct-37}$ are individually significant in the Canadian equation, while Δy_{ct-36}, Δy_{ct-37}, $(y_{ut-1} - y_{ct-1})/(y_{ut-1} + y_{ct-1})$, $plink_{ut-2}$ and $nlink_{ut-12}$ are individually significant in the US equation.

We recognized that the linkage variables might well enter in groups, at the very least in pairs, so we investigated a variety of combinations of link variables in each equation. In this first exploratory examination of linkages we sought a combination of linkages that seemed to provide robust estimates, gave nearly, if not the very largest, values of R^2, and provided some consistency between the Canadian and U.S. results. The coefficient values, t-statistics, F test results, and the corresponding R^2 for the linkage combination which we thought best satisfied these criteria are listed in Table 4. For Canada, we see that the key links (Δy_{ut-24} and $\Delta y_{ct-24*}\Delta y_{ut-24}$) involve both the change in the U.S. growth rate and the product of the U.S. and Canadian changes in growth rates at a lag of twenty four months. Correspondingly, the U.S. model incorporates the same two variables (Δy_{ct-36} and $\Delta y_{ut-36*}\Delta y_{ct-36}$), but at a longer lag of thirty six months. This difference in lagged effects is an interesting aspect to the asymmetry between the U.S. and Canada. With the incorporation of these two pairs of variables in each model, the remaining linkage variables were discovered to be statistically insignificant.

4.2.2. Properties of the Linked Model

Summary statistics relating to the models which incorporate the linkages are presented in Table 5 and a listing of the estimated coefficients and the standard errors are provided in the Appendix Tables 3a and 3b. The diagnostics support the choice of $m = 1$, which suggests that the nonlinearity in each series has been adequately modeled. In the Canadian case, there is no evidence of misspecification of the estimated model. There is some evidence of low-order ARCH in the U.S. residuals that indicates a possible specification problem, but as is discussed below, this does not appear to affect the forecasting ability of the U.S. equation. Further, when both equations are re-estimated over different samples; March 1958 - February 1991, July 1958 - June 1991, and then March 1959 - February 1992, the estimated coefficients seem to be stable. The negative coefficient on Δy_{ut-24} in the Canadian equation and the positive coefficient on

Δy_{ct-36} in the U.S. equation initially suggests that Canadian growth is stifled when the U.S. economy grows more rapidly, whereas U.S. production is stimulated by a more rapidly growing Canadian economy. However, the positive coefficients on $\Delta y_{ut-24}.\Delta y_{ct-24}$ (for Canada), and $\Delta y_{ct-36}.\Delta y_{ut-36}$ (for the U.S.) suggest that growth in each country depends on the presence of recent similar growth trends in *both* countries. In the Canadian case, of course, the key issue is the net effect of the two linkage variables. Evaluating the net effect of the sum of two variables at their respective means yields an effect of 0.001, or 0.1% which indicates that while the effect of the U.S. on Canada is marginal, it is nevertheless positive on average.

While we do not claim to have discovered the "true" linkage, nor even to have selected from our prior alternatives the most suitable combination of variables; we have demonstrated that the linkages are more complex than merely inserting a lagged value of the corresponding country's growth rate into the model. Further, it is clear that considerably longer lags than have customarily been included should be allowed for in the formulation of such models. It is also clear that there may well be interesting asymmetries between the countries in the length of the lags that are involved in linking them together. Unfortunately, at this time we do not know why the U.S. affects Canada with a lag of only twenty four months, whereas Canada affects the U.S. with a lag of thirty six months. The explanation may lie in the difference in relative size of the industrial bases and in a difference in the composition of imports into each country. Canada still exports to the U.S. a much greater percentage of raw materials relative to the U.S.'s exports to Canada.

We henceforth refer to this linked (bivariate) model as "the linked model", and use the notation $\widehat{y_{jt}} = f_{jLK}(y_{jt-1}, ..., y_{jt-p},\ lk_{j1},\ lk_{j2,} : \widehat{\Psi_j})$ for $j \in \{c, u\}$, to describe each of these two equations.

4.3. Evaluation of the Estimated Models

The long run properties of the nonlinear autoregressive models can be studied by randomly choosing sets of consecutive starting values $y^0_{t-1}, ..., y^0_{t-p}$ from the available sample, substituting these into $\widehat{y_t} = f_{NAR}(y_{t-1}, ..., y_{t-p} : \widehat{\Omega})$ to begin the generation of predictions for $\widehat{y_t}$, $\widehat{y_{t+1}}...$, and then following the evolution of the process as $t \to \infty$. This type of dynamic simulation, which does not account for innovations which might affect the process after it has started, provides a useful indication of the inherent dynamics implied by the coefficient estimates summarized in $\widehat{\Omega}$. Ozaki (1985) discusses various behavioral patterns which such simulations can generate; the series might converge to a single point, display cyclical behavior, exhibit chaotic behavior, or diverge. In many models the observed dynamical patterns depend on the chosen initial conditions. Consequently, some exploration of the state space is needed to obtain a reasonably clear picture of the dynamic properties of the paths of the two individual country indicators, and that of the path of the joint two country system.

We assess the dynamic implications of the links by comparing simulations of f_{jLK} to those of f_{jNAR}, and by comparing forecasts of the two alternatives using data not incorporated in the estimation process. In a limited manner, we can use these techniques to obtain some idea of the effect of the links on the stability of the joint models. Further, by comparing the coherence between the two simulated series when linked and when unlinked, we can obtain some insight into the extent to which the linkage leads to synchronization. Given that the links involve the first derivative in the growth variable, there is reason to believe that the links will facilitate synchronization, but this is an issue that requires more detailed analysis.

First, we examined the nonlinear autoregressive model without links, f_{NAR}, without incorporating innovations. Using a range of initial conditions, simulations of the estimated model indicated that the Canadian version produces a time series which fluctuates between -.114 and .107 with a mean of 0.005 and a standard deviation of 0.06. The corresponding U.S. version of the model produced a series which converged to a steady state growth rate of 0.03 or 3.01%. Thus both models without linkages and without innovations appear to be inherently stable. This result was obtained notwithstanding the fact that a calculation of the minimum and maximum roots implied by the state dependent coefficients indicated global instability.

Somewhat different results were obtained when we added innovations to the simulations. The model used to produce these simulations was given by $\widehat{\widehat{y_t}} = f_{NAR}(y_{t-1}, ..., y_{t-p} : \widehat{\Omega}) + \xi_t$ where $\xi_t \sim N(0, \sigma_\xi^2)$. We found that for both countries this model could produce a diverging series, but that it will generate an oscillatory series most of the time. The specific outcome depends on the starting values, on σ_ξ^2, as well as on the precise sequence $\xi_t,\ \xi_{t+1}, \xi_{t+2}$ ... of observed innovations. For example, setting $\sigma_{c\xi} = 0.0125$, (the standard error associated with the Canadian model), and using f_{cNAR} to predict 60 periods forward for Canada, about 1% of the simulations diverge. Of those prediction series which diverge, more than 1/3 have starting values which include observations from the 1974 or 1982 recession. The vast majority of paths, 99% of them, oscillate. The corresponding U.S. model exhibits similar, but less stable behavior. Setting $\sigma_{u\xi} = 0.0081$, (the empirical standard deviation of the U.S. model residuals), and using f_{uNAR} to predict 60 periods forward for the U.S., about 20% of the simulations diverge. Of those prediction series which diverge, about half of them have starting values which include observations from the 1974 or 1982 recessions. For the U.S., the percentage of predicted series that oscillate is only 80%, as compared to the Canadian result of 99%.

The model which incorporates the linkages also produces three types of oscillation. As in the prior experiments, we can study long-run properties by randomly choosing sets of initial conditions, $y^0_{ct-1}, ..., y^0_{ct-p}$ and $y^0_{ut-1}, ..., y^0_{ut-p}$from the available sample, and then substituting these into $\widehat{y_{jt}} = f_{jLK}(y_{jt-1}, ..., y_{jt-p},\ lk_{j1},\ lk_{j2,}\ : \widehat{\Psi_j})$ for $j \in \{c,\ u\}$ in order to begin the generation of a simulated time series for the sequence of pairs $\{\widehat{y_{ct}},\ \widehat{y_{ut}}\}\ \{\widehat{y_{ct+1}},\ \widehat{y_{ut+1}}\}$.... . By following the evolution of this bivariate process as $t \to \infty$, the limiting dynamical behavior can be ascertained. However, unlike the simulations based on the nonlinear autoregressive model, the predicted series in this experiment are jointly generated and one cannot analyze one country's output without that of the other. As with the previous model, one can generate the linked model with and without innovations.

When no innovations were added to the linked model, three types of behavior were observed. The most usual result was that both series showed cyclical behavior. The Canadian series fluctuated on average between -0.128 and 0.102, with a mean of 0.004 and a standard deviation of 0.047. The corresponding U.S. series fluctuated between 0.004 and 0.047 with a mean of 0.027 and a standard deviation of 0.009. Cyclical behavior was observed for 303 out of the 324 sets of possible starting values which were considered. For 17 out of the 324 possible sets of starting values, the two series converged to a steady state. In this steady state, Canadian growth remained constant at about 0.062 (i.e. 6.2%), while U.S. growth remained constant at 0.023 (i.e. 2.3%). Four out of the possible 324 sets of starting values led to series which diverged fairly quickly. It is remarkable that all four of these instances that led to divergence were obtained from initial conditions that were drawn from the first few months of the 1974 recession.

Further simulations, which added innovations to the two linked models, produced diverging series more often. Setting the variance/covariance matrix of the innovations equal to the empirical variance/covariance matrix of the model residuals, i.e.

$$\Sigma_{NAR} = \begin{pmatrix} 1.49 \times 10^{-4} & 2.84 \times 10^{-5} \\ 2.84 \times 10^{-5} & 6.30 \times 10^{-6} \end{pmatrix}, \tag{4.1}$$

and projecting the series 60 periods ahead as with the unlinked model, diverging series are found about 17% of the time. As before, a large proportion of the diverging series (40%) have starting values which are drawn from one of the two major recessions observed in the sample. All of those simulations which do not diverge, (83% of the total), involve the oscillation of both production growth indices.

Overall, the simulations indicate that the links might be mildly stabilizing. While the linked model diverged a little more often than the individual country models when no innovations were added to the models, these models diverged slightly less often once innovations were taken into account. This suggests that the links between the two economies might allow a "good" sequence of shocks in one economy to offset the potentially destabilizing effects of a "bad" sequence of shocks in the other.

The evidence for synchronization is most clearly seen in terms of the simulations not involving innovations. Without linkages, the simulated series for the two countries are not synchronized, because the Canadian simulations oscillate, while the US simulations rapidly settle down to a fixed point. However, if we use exactly the same initial conditions, but simulate on the linked model rather than the two univariate models, then both series become oscillatory. The degree of coherence between the spectra of the two linked series increases substantially over the simulation period, particularly at frequencies lower than one year. Further, the relationship between phase and frequency becomes almost linear, with a slope that indicates a constant delay between the two series of approximately eighty months. Finally, it is interesting to note that in the linked models, with or without innovations, when there is convergence, oscillation or divergence, the two countries converge, oscillate or diverge together. This is quite strong evidence for synchronization.

In light of the forecasts that are presented below, it is interesting to note the results obtained when the starting values for the simulations are taken from the very last observations in the sample and the innovations are included as part of the simulation process. For the Canadian non-linear autoregressive, unlinked model, the series did not diverge. Correspondingly, the results for the U.S. using the unlinked model produced a diverging series in only 3% of the cases. For the linked model with innovations and using initial conditions from the very end of the estimation period, we discovered that the pair of time series diverged in about 9% of the cases.

4.4. Forecast Comparisons

The simulation techniques used to evaluate the long-run properties of the models can be used to generate forecasts. For instance, one can use the last 72 in-sample observations (which relate to January 1985 - December 1990) as the starting values in $\widehat{y_{jt}} = f_{jLK}(y_{jt-1}, ..., y_{jt-p}, lk_{j1}, lk_{j2,} : \widehat{\Psi_j})$ for $j \in \{c, u\}$, in order to predict the pairs $\{\widehat{y_{ct}}, \widehat{y_{ut}}\}$ $\{\widehat{y_{ct+1}}, \widehat{y_{ut+1}}\}$....; these are the forecasts for January 1991, February 1991, etc. The first of these predictions would be a one-step ahead forecast, the second a two step ahead forecast, and so on. We call these forecasts "dynamic forecasts." Note that these forecasts only use information that is available at the single point in time when the series of forecasts are made. This is in contrast to a sequence of forecasts in which constantly updated information is used. In the dynamic forecasts the forecast horizon grows with each forecast and the original information used to estimate the coefficients is not supplemented. In conventional practice the information is updated each forecast period and subsequent forecasts have the same forecast horizon. One problem in evaluating dynamic forecasts is that their variance increases with the horizon h. Thus, unless each forecast is appropriately weighted, average calculations such as the root mean squared error (RMSE) may be misleading.

An alternative way of calculating forecasts is to set the forecast horizon h at a given value and generate a series of h-step ahead forecasts. The first of these forecasts is the hth term in the series created above i.e. $\{\widehat{y_{ct+(h-1)}}, \widehat{y_{ut+(h-1)}}\}$. The second pair of forecasts follows the first procedure and picks off the hth term in the forecasted series as before, but the forecast uses the subsequent observations, for example the 72 observations from February 1985 - January 1991 as starting observations. The next h-step ahead forecast is generated by using the next set of 72 observations, which relate to March 1985 - February 1991 as starting values, and so on. This forecasting technique relies on the availability of observations relating to the period *after* the estimation sample. Essentially, these forecasts represent a sequence of fixed horizon forecasts using updated information about the state of the system, but without updating the parameter estimates. The more usual case for providing a sequence of forecasts is to use the out-of sample observations to re-estimate successively the forecast generating equation each time that the starting observations are moved forward. In this paper we do not examine forecasts produced by such a re-estimation procedure.

We compare forecasts produced by the linked model, the two independent nonlinear autoregressive models, and an AR benchmark model which was estimated using the same lag structure as the other models. Given that we had retained fifty observations for model evaluation, (these covered the period from January 1991 to February 1995), forecasts up to fifty periods ahead could be evaluated *using out of sample data*.

Some summary information on the forecast errors associated with the dynamic forecasts is provided in Table 6. In this table, the forecast entries are our dynamic forecasts, but in calculating the RMSE we did not weight the various forecasts by the length of the forecast horizon as mentioned above. For Canada, the linked model appears to out-perform the other two models, in that it has the smallest RMSE, and that it has the smallest forecast error (in absolute terms) for nearly half of the fifty forecast horizons. The main forecasting advantage that the linked model has over the other two models is associated with *long* forecast horizons. For the U.S. the linked model has the smallest absolute forecast error for 34 out of the fifty horizons. This result holds despite the fact that it does not have the smallest RMSE, but one that is substantially smaller than the RMSE for the linear model. As in the Canadian case, the main forecasting advantage of the U.S. linked model over the other two models is associated with *long* forecast horizons.

Summary statistics relating to the h-step ahead forecasts are provided in Table 7. These forecasts were created by the second procedure that we discussed; that is, we used the new information available at each stage, but did not re-estimate the model. In the Canadian case, the linear AR model appears to forecast best over short and medium-run horizons ($h = 1$ to $h = 24$), but for longer horizons the nonlinear models (with, and without links) seem to be a little better. On balance and especially for long horizons, the linked model is marginally better than the unlinked model. In the U.S. case the nonlinear models consistently out-perform the AR model in all but the one-step horizon, and for three of the six horizons considered both the nonlinear autoregressive model and the linked model produce forecasts which are statistically better than those produced by the AR model. Further, the linked model is consistently, but marginally, better than the unlinked model.

5. Conclusion

This paper studies the coupling of indices of industrial production for the United States and Canada. Since there can be little doubt that the output series for these two economies *are* coupled, the main questions that we ask concern the functional form of the coupling mechanism, and how the coupling affects the joint dynamics of output in these two countries. Particular issues of concern include whether the coupling contributes towards stability, whether oscillations are enhanced under coupling, and whether the coupling tends to synchronize the dynamics of the two countries.

We provide a first empirical attempt at answering these questions by estimating and studying simple adaptations of Ozaki's nonlinear autoregressive model. The feature of this model that makes it appropriate for our purpose is that it is able to provide a very effective approximation to a wide variety of underlying continuous time differential equations and because its coefficients are functions of the state space. Further, the Ozaki model is relatively easy to estimate. The functional form of the coupling mechanism is determined by experimenting with different types of linkage variables; potential linkage variables for each country include the growth rate of the other country, relative differences in growth rates, functions of changes in growth rates in the other country, and functions which incorporate capacity effects. All linkage variables are allowed to enter the nonlinear autoregressive specification for each country with a lag length determined by the data.

Our first empirical result, buttressed by analysis using TSMARS, is that the autoregressive lags involved in the Ozaki equations for each country's index of production are as long as five or six years, and that "end of year effects" appear to be important. Further, the empirically selected linkages, which for each country involve the differenced growth rate for the other country and the product of changes in both countries' growth rates, enter the specification with lags of two or three years.

These empirical findings are interesting in that they emphasize the importance of long lags, both in the formulation of the dynamical equations themselves and in terms of the linkages. Conventionally, the lags employed in the modeling of a monthly process are seldom greater than twenty four months, and if dynamical linkages are ever considered, then the lags associated with these are usually no more than one or two months. Consequently, our results cast a new light on the periods that are relevant for the modeling of dynamic macroeconomic processes. Further, the fact that the chosen linkages incorporate a differenced growth rate and the interaction between the differenced growth rates for each country, suggests that the links are "second order effects". The addition of the interaction term indicates that the linkage mechanism is essentially nonlinear.

We assessed the dynamic implications of the links by comparing simulated series based on our empirical model of the coupled economies with simulated series based on univariate unlinked models. These simulations showed that there were three classes of outcomes depending on initial conditions and the sequence of innovations; convergence, divergence, and a complicated oscillation. Adding innovations led to greater evidence of oscillations in the observed time paths and a higher probability that both the linked and unlinked series would diverge.

The evidence of the impact of linkage on stability was both mixed and weak, but on balance a case might be made for enhanced stability. One of the most interesting simulation results is that a very large fraction of the divergent cases are generated when the initial conditions incorporate observations from the '74 and '82 recessions. These results suggest that these two recessions were close to unstable areas of the state space. We might speculate that the reason unstable paths were not actually observed is that the behavioral lags in these processes are so long that the innovations beyond these recessions were able to move the processes back into more stable areas of the state space. Alternatively stated, these empirical findings indicate that had the recessionary periods been longer lived, the result may well have been a pair of collapsing economies in the absence of strenuous fiscal and monetary intervention.

The simulations lead to two results which support a hypothesis that coupling leads to increased synchronization of the two business cycles. For simulations which do not include innovations, the long- run behavior in Canadian and US growth rates implied by the univariate models differ. However, once links are incorporated into the simulations, then both series exhibit similar oscillatory behavior. The degree of coherence between the spectra of the two linked series increases substantially over the simulation period, particularly at frequencies lower than one year. Also, regardless of whether or not the simulations incorporate innovations, when there is convergence, oscillation or divergence in the linked model, then the two countries converge, oscillate or diverge together.

Finally, in terms of forecast accuracy, a comparison of the linked, unlinked, and a benchmark autoregressive model using the 50 observations set aside for this purpose, indicates that the linked model produces improved forecasts on average and that the benefits of using the linked model are greater the further out one forecasts. At one month ahead, there is very little difference between any of the forecast mechanisms; at 36 to 42 months ahead there were substantial differences.

Our results suggest that further exploration of the effects of coupling on the dynamics of interrelated economies might be worthwhile. We have explored the effects of just one set of linkage mechanisms on a single two country example. While our mechanism seems empirically appropriate for this case, other mechanisms may well fit the data better. It would be interesting to study the coupling mechanisms involved in other sets of interrelated economies, particularly members of the European Union or other larger groupings of economies. Questions of interest include whether the coupling in other examples take the same functional form as the coupling that we found in our US/Canada example and whether the coupling leads to enhanced stability and greater synchronization as we discovered.

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Appendix 1

The pair of variables that we labeled "$plink_{ct}$" and "$nlink_{ct}$" were constructed as follows. Defining c_t and u_t to be the logarithms of the (monthly) indices of industrial production for Canada and the United States, and y_{ct} and y_{ut} to be the twelfth differences of the logged series, we first constructed a variable which measured the Maximum Output To Date (MTD_t) via the relationship

$$MTD_t = \max(\{c_{t-1}, c_{t-2}, c_{t-3}, c_{t-4}, ...\}) - c_t.$$

Next, we defined a "Room To Capacity" threshold variable (RTC_t) by

$$\begin{aligned} RTC_t &= 1, \ if \ MTD_t > 0 \qquad \text{(and)} \\ RTC_t &= 0, \ if \ MTD_t \leq 0 \end{aligned}$$

and two growth index indicator variables given by

$$\begin{aligned} GI_t &= 1, \ if \ y_{ut} > 0 \\ GI_t &= 0, \ if \ y_{ut} \leq 0 \\ NGI_t &= 1 - GI_t. \end{aligned}$$

The growth indicator GI_t is positive when there is growth in the United States, while the negative growth indicator NGI_t identifies periods of non-positive growth in the United States.

From these constituent variables we constructed $plink_{ct}$ and $nlink_{ct}$ according to

$$\begin{aligned} plink_{ct} &= RTC_t * GI_t * y_{ut} \qquad \text{(and)} \\ nlink_{nt} &= NGI_t * y_{ut}. \end{aligned}$$

The intuition behind $plink_{ct}$ is that if the Canadian current output is less than its past maximum so that there is some excess productive capacity in Canadian production and if the U.S. economy is growing, then there will be a positive effect on the Canadian economy. For $nlink_{ct}$, the intuition is that if the U.S. production index is declining, then there will be a negative effect on Canada's production.

The definitions of $plink_{ut}$ and $nlink_{ut}$ are analogous to the above, with the roles of c_t and u_t reversed, and the roles of y_{ut} and y_{ct} also reversed.

Appendix 2a. Coefficients and Standard Errors for the Nonlinear Autoregressive Model of the Growth Rate of Canadian Industrial Production (1958:1 - 1990:12)

Autoregressive Coefficients (with standard errors)						
k	φ_{ck}		π_{ck0}		π_{ck1}	
Lag 1	0.519	(1.153)	0.277	(1.150)	2.664	(1.314)
Lag 2	-1.425	(1.247)	1.548	(1.233)	-0.572	(1.027)
Lag 11	4.641	(1.951)	-4.755	(1.903)	-0.195	(1.730)
Lag 12	-4.963	(2.407)	4.225	(2.332)	0.145	(2.416)
Lag 13	-0.410	(2.137)	1.156	(2.070)	1.040	(1.957)
Lag 23	5.183	(2.716)	-5.162	(2.652)	-1.832	(2.152)
Lag 24	-7.160	(2.540)	6.626	(2.478)	1.437	(2.431)
Lag 25	2.839	(2.784)	-2.298	(2.701)	-0.876	(2.275)
Lag 35	8.409	(3.100)	-8.381	(3.024)	-4.411	(2.492)
Lag 36	-8.153	(2.838)	8.027	(2.776)	2.209	(2.435)
Lag 37	-1.360	(2.426)	1.341	(2.356)	3.458	(1.942)
Lag 47	8.049	(2.342)	-7.741	(2.298)	-5.966	(2.031)
Lag 48	-8.460	(2.224)	8.125	(2.183)	6.595	(1.962)
Lag 59	5.164	(2.541)	-5.020	(2.482)	-1.842	(2.061)
Lag 60	-5.916	(2.482)	5.707	(2.430)	3.118	(1.983)
Lag 71	2.189	(1.896)	-2.106	(1.862)	-1.120	(1.433)
Lag 72	-3.909	(1.936)	3.764	(1.895)	2.564	(1.465)
Other Coefficient (with standard error)						
Constant			0.002	(0.003)		

(1) The parameters φ_{ck}, π_{ck0} and π_{ck1} relate to the variables y_{ct-k}, $y_{ct-k}.\exp[-10(y^2_{ct-1} - \mu_{y^2_{ct-1}})]$ and $y_{ct-k}.y_{ct-1}\exp[-10(y^2_{ct-1} - \mu_{y^2_{ct-1}})]$.
(2) The R^2 for this model was 0.9492, and the standard error of the regression was 0.0125. See Table 3 for other diagnostic statistics.

Appendix 2b. Coefficients and Standard Errors for the Nonlinear Autoregressive Model of the Growth Rate of U.S. Industrial Production (1958:1 - 1990:12)

Autoregressive Coefficients (with standard errors)						
k	φ_{uk}		π_{uk0}		π_{uk1}	
Lag 1	2.440	(1.450)	-1.142	(1.433)	-3.365	(1.242)
Lag 2	-1.676	(1.497)	1.439	(1.471)	3.001	(1.089)
Lag 11	2.728	(1.766)	-2.568	(1.751)	-2.364	(1.236)
Lag 12	-2.609	(2.691)	1.614	(2.652)	2.565	(2.090)
Lag 13	-0.632	(1.800)	1.380	(1.771)	0.793	(1.428)
Lag 23	5.175	(2.561)	-4.892	(2.520)	-5.113	(1.956)
Lag 24	-4.624	(3.396)	3.811	(3.332)	2.709	(2.726)
Lag 25	-0.538	(1.935)	1.046	(1.900)	2.471	(1.561)
Lag 35	1.833	(2.942)	-1.660	(2.879)	-3.900	(2.330)
Lag 36	0.935	(3.655)	-1.244	(3.577)	1.519	(2.890)
Lag 37	-1.733	(1.939)	1.877	(1.894)	1.550	(1.567)
Lag 47	-0.437	(2.276)	0.499	(2.234)	-1.903	(1.744)
Lag 48	1.311	(2.237)	-1.341	(2.195)	1.129	(1.751)
Lag 59	4.412	(1.763)	-4.147	(1.710)	-5.324	(1.705)
Lag 60	-4.204	(1.694)	3.956	(1.640)	5.006	(1.624)
Other Coefficient (with standard error)						
Constant			0.002	(0.003)		

(1) The parameters φ_{uk}, π_{uk0} and π_{uk1} relate to the variables y_{ut-k}, $y_{ut-k}\cdot\exp[-10(y_{ut-1}^2 - \mu_{y_{ut-1}^2})]$ and $y_{ut-k}\cdot y_{ut-1}\exp[-10(y_{ut-1}^2 - \mu_{y_{ut-1}^2})]$.
(2) The R^2 for this model was 0.9773 and the standard error of the regression was 0.0081. See Table 3 for other diagnostic statistics.

Appendix 3a. Coefficients and Standard Errors for the Nonlinear Model of the Growth Rate of Canadian Industrial Production (1958:1 - 1990:12)

Autoregressive Coefficients (with standard errors)						
k	φ_{ck}		π_{ck0}		π_{ck1}	
Lag 1	0.183	(1.140)	0.602	(1.136)	2.960	(1.299)
Lag 2	-0.810	(1.240)	0.949	(1.226)	-1.126	(1.022)
Lag 11	3.834	(1.933)	-3.928	(1.887)	0.110	(1.705)
Lag 12	-4.589	(2.375)	3.794	(2.302)	0.623	(2.380)
Lag 13	0.076	(2.108)	0.707	(2.042)	0.506	(1.929)
Lag 23	4.479	(2.677)	-4.440	(2.615)	-1.950	(2.118)
Lag 24	-6.492	(2.505)	5.971	(2.443)	1.260	(2.391)
Lag 25	2.682	(2.736)	-2.170	(2.654)	-0.490	(2.239)
Lag 35	8.165	(3.049)	-8.119	(2.974)	-4.453	(2.449)
Lag 36	-8.163	(2.789)	7.979	(2.728)	3.160	(2.410)
Lag 37	-1.119	(2.386)	1.147	(2.317)	2.500	(1.930)
Lag 47	8.481	(2.306)	-8.144	(2.262)	-6.571	(2.006)
Lag 48	-8.540	(2.194)	8.184	(2.152)	7.014	(1.939)
Lag 59	5.224	(2.500)	-5.037	(2.442)	-2.633	(2.048)
Lag 60	-6.025	(2.446)	5.773	(2.393)	3.821	(1.968)
Lag 71	2.004	(1.864)	-1.900	(1.831)	-1.505	(1.415)
Lag 72	-3.894	(1.904)	3.729	(1.864)	2.965	(1.449)
Other Coefficients (with standard errors)						
Constant			0.001	(0.003)		
Δy_{ut-24}			-0.157	(0.078)		
$\Delta y_{ut-24}.\Delta y_{ct-24}$			8.871	(2.870)		

(1) The parameters φ_{ck}, π_{ck0} and π_{ck1} relate to the variables y_{ct-k}, $y_{ct-k}.\exp[-10(y^2_{ct-1} - \mu_{y^2_{ct-1}})]$ and $y_{ct-k}.y_{ct-1}\exp[-10(y^2_{ct-1} - \mu_{y^2_{ct-1}})]$.
(2) The R^2 for this model was 0.9513 and the standard error of the regression was 0.0118. See Table 5 for other diagnostic statistics.

Appendix 3b. Coefficients and Standard Errors for the Model of the Growth Rate of U.S. Industrial Production (1958:1 - 1990:12)

Autoregressive Coefficients (with standard errors)						
k	φ_{uk}		π_{uk0}		π_{uk1}	
Lag 1	2.607	(1.432)	-1.287	(1.417)	-3.609	(1.226)
Lag 2	-1.764	(1.475)	1.502	(1.450)	3.248	(1.076)
Lag 11	2.938	(1.742)	-2.797	(1.728)	-2.392	(1.222)
Lag 12	-3.118	(2.657)	2.139	(2.619)	2.808	(2.072)
Lag 13	-0.395	(1.775)	1.158	(1.746)	0.480	(1.416)
Lag 23	5.634	(2.531)	-5.349	(2.491)	-5.134	(1.928)
Lag 24	-5.050	(3.348)	4.232	(3.285)	2.399	(2.689)
Lag 25	-0.503	(1.915)	1.000	(1.880)	2.905	(1.555)
Lag 35	1.719	(2.898)	-1.544	(2.836)	-3.882	(2.295)
Lag 36	1.479	(3.617)	-1.832	(3.539)	1.081	(2.855)
Lag 37	-2.138	(1.933)	2.332	(1.889)	1.873	(1.559)
Lag 47	-0.157	(2.245)	0.214	(2.204)	-1.962	(1.719)
Lag 48	0.896	(2.209)	-0.927	(2.168)	1.318	(1.727)
Lag 59	4.269	(1.738)	-4.010	(1.685)	-5.133	(1.681)
Lag 60	-4.135	(1.670)	3.891	(1.617)	4.840	(1.601)
Other Coefficients (with standard errors)						
Constant			0.002	(0.003)		
Δy_{ct-36}			0.076	(0.027)		
$\Delta y_{ut-36}.\Delta y_{ct-36}$			2.770	(1.619)		

(1) The parameters φ_{uk}, π_{uk0} and π_{uk1} relate to the variables y_{ut-k}, $y_{ut-k}.\exp[-10(y^2_{ut-1} - \mu_{y^2_{ut-1}})]$ and $y_{ut-k}.y_{ut-1}\exp[-10(y^2_{ut-1} - \mu_{y^2_{ut-1}})]$.
(2) The R^2 for this model was 0.9782 and the standard error of the regression was 0.0077. See Table 5 for other diagnostic statistics.

Table 1. Summary Properties of The Annual Growth Rates of Canadian and U.S. Indices of Industrial Production (1958:1 - 1995:2)

	Canada (y_{ct})	U.S.A. (y_{ut})
Mean	0.0359	0.0329
Median	0.0431	0.0403
Maximum	0.1741	0.1939
Minimum	-0.1638	-0.1336
Standard Deviation	0.0528	0.0558
Second Raw Moment	0.0041	0.0042
Skewness	-0.7475	-0.5610
Kurtosis	3.7549	3.6016
P-Value of Unit Root Test	< 0.01	< 0.01
P-Value of Linearity Test	0.0001	0.0118

(1) Skewness is defined by $S = \frac{\frac{1}{T}\sum_{t=1}^{T}(y_t-\overline{y})^3}{\hat{\sigma}^3}$

(2) Kurtosis is defined by $K = \frac{\frac{1}{T}\sum_{t=1}^{T}(y_t-\overline{y})^4}{\hat{\sigma}^4}$

(3) The Unit Root Test is an Augmented Dickey-Fuller τ_μ test of the null hypothesis that the growth rate y_t is a unit root process against the alternative that this growth rate is stationary around a constant. The test statistic is the t-statistic associated with y_{t-1} in the regression of Δy_t on a constant, y_{t-1} and 12 lags of Δy_t.

(4) The Linearity Test is a test of the null hypothesis that the growth rate y_t is a linear AR(p) process against the alternative that it is a STAR(p) process in which the AR coefficients vary with y_{it-d}. The test statistic is the TR^2 associated with an auxiliary regression of the residuals of an AR(p) against a constant, p lags of y_{it-k} $(k = 1, ..., p)$, p second order cross-product terms y_{it-d}. y_{it-k} $(k = 1, ..., p)$, p third order cross-product terms y^2_{it-d}. y_{it-k} $(k = 1, ..., p)$, and p fourth order cross-product terms y^3_{it-d}. y_{it-k} $(k = 1, ..., p)$, and under the null hypothesis it has a χ^2_{3p} distribution. The lag length p is chosen by AIC, and the delay parameter d is the d which maximizes the TR^2 over the series of test regressions run using $d = 1, ..., p$. See Teräsvirta (1994) for further details. The reported p-values support an alternative hypothesis in which coefficients of an AR specification vary with y_{it-1}.

Table 2. Summary Properties of The Transition Variables used for Modeling The Growth Rates of Canadian and U.S. Indices of Industrial Production (1958:1 - 1995:2)

	Canada (T_{ct})	U.S.A. (T_{ut})
Mean	1.0010	1.0001
Median	1.0131	1.0156
Maximum	1.0416	1.0428
Minimum	0.7691	0.7160
Standard Deviation	0.0429	0.0480
Skewness	-1.7184	-2.0680
Kurtosis	7.1358	9.2773

(1) Skewness is defined by $S = \frac{\frac{1}{T}\sum_{t=1}^{T}(y_t - \overline{y})^3}{\hat{\sigma}^3}$

(2) Kurtosis is defined by $K = \frac{\frac{1}{T}\sum_{t=1}^{T}(y_t - \overline{y})^4}{\hat{\sigma}^4}$

(3) The transition variables are defined by $\mathrm{T}_{jt} = \exp[-10(y_{jt}^2 - \mu_{y_{jt}^2})]$ for $j \in \{c, u, \}$.

Table 3. Statistics Relating to the Skeleton Lag Structure used to Model the Dynamics

	Canada (y_{ct})	U.S.A. (y_{ut})
P-Values for Groups of Explanatory Variables		
Constant (1)	0.4955	0.3963
Lags 1 and 2 (6)	0.0000	0.0000
Lags 11, 12 and 13 (9)	0.0000	0.0000
Lags 23, 24 and 25 (9)	0.0000	0.0000
Lags 35, 36 and 37 (9)	0.0055	0.0004
Lags 47 and 48 (6)	0.0044	0.5332
Lags 59 and 60 (6)	0.1155	0.1044
Lags 71 and 72 (6)	0.3053	-
Selected Diagnostics on the above Lag Structure (p-values for tests)		
R^2	0.9492	0.9773
Standard Error	0.0125	0.0081
H_0: Coefficients on Lags 3 and 4 are zero (6)	0.3530	0.2163
H_0: Coefficients on Lags 3 - 10 are zero (18)	0.0587	0.5525
H_0: Coefficients on All Omitted Lags are zero (165/171)†	0.3738	0.3832
H_0: No 12th order serial correlation	0.8850	0.4112
H_0: No 24th order serial correlation	0.9768	0.1771
H_0: No 36th order serial correlation	0.9186	0.1156
H_0: No 48th order serial correlation	0.8005	0.1657
H_0: No 60th order serial correlation	0.7441	0.2373

(1) There are three variables (y_{t-k}, $y_{t-k}\exp[-10(y^2_{t-1} - \mu_{y^2_{t-1}})]$ and $y_{t-k}\ y_{t-1}\exp[-10(y^2_{t-1} - \mu_{y^2_{t-1}})]$) for each lag k. See Appendix 2 for the estimated coefficients and their standard errors. The top portion of the table lists the lags which were used to model the dynamics and provides tests of the statistical significance of various subgroups of these lags. (The numbers given in the brackets count the number of restrictions involved in the relevant tests).

(2) The coefficient test marked with the † superscript provides a test of a model with all 72 lags (217 variables) against the model given in the top portion of the table. The number of restrictions involved in this test was 165 (for Canada) and 171 (for the U.S.).

Table 4a: Statistics Relating to the Link Variables which are included in the Model of the Growth Rate of the Canadian Index of Industrial Production

Link Variable				F-stat for both Links	P-Value for Excluding Both Links	R^2 for Model with Both Links
Δy_{ut-24}		$\Delta y_{ct-24}.\Delta y_{ut-24}$				
Coef	t-stat	Coef	t-stat			
-0.157	-2.01	8.871	3.09	5.86	0.0032	0.9513

Table 4b: Statistics Relating to the Link Variables which are included in the Model of the Growth Rate of the U.S. Index of Industrial Production

Link Variable				F-stat for both Links	P-Value for Excluding Both Links	R^2 for Model with Both Links
Δy_{ct-36}		$\Delta y_{ut-36}.\Delta y_{ct-36}$				
Coef	t-stat	Coef	t-stat			
0.076	2.79	2.770	1.71	5.25	0.0058	0.9782

Table 5. Model of the Annual Growth Rates of Canadian and U.S. Industrial Production (1958:1 - 1990:12)

	Canada (y_{ct})	U.S.A. (y_{ut})
P-Values for Groups of Explanatory Variables		
Constant (1)	0.7874	0.4626
Lags 1 and 2 (6)	0.0000	0.0000
Lags 11, 12 and 13 (9)	0.0000	0.0000
Lags 23, 24 and 25 (9)	0.0000	0.0000
Lags 35, 36 and 37 (9)	0.0016	0.0011
Lags 47 and 48 (6)	0.0020	0.6598
Lags 59 and 60 (6)	0.0911	0.1132
Lags 71 and 72 (6)	0.1979	-
Links (2)	0.0032	0.0058
Selected Diagnostics for the Model (p-values for the tests)		
R^2	0.9513	0.9782
Standard Error	0.0122	0.0079
H_0: $m = 0$ vs H_a: $m = 1$	0.0010	0.0000
H_0: $m = 1$ vs H_a: $m = 2$	0.0650	0.2577
H_0: Normal Residuals	0.5894	0.0000
H_0: No 12th order serial correlation	0.9128	0.4578
H_0: No 24th order serial correlation	0.9940	0.2568
H_0: No 36th order serial correlation	0.9885	0.2370
H_0: No 48th order serial correlation	0.9727	0.3215
H_0: No 60th order serial correlation	0.9643	0.3898
H_0: No 1st order ARCH	0.6877	0.0005
H_0: No 6th order ARCH	0.7447	0.0536
H_0: No 12th order ARCH	0.8806	0.3645

(1) There are three variables (y_{t-k}, $y_{t-k}\exp[-10(y_{t-1}^2 - \mu_{y_{t-1}^2})]$ and $y_{t-k}\ y_{t-1}\exp[-10(y_{t-1}^2 - \mu_{y_{t-1}^2})]$) for each lag. See Appendix 3 for the estimated coefficients and their standard errors.
(2) The test H_0: $m = 0$ vs H_a: $m = 1$ is a joint test of the significance of $y_{t-k}\ y_{t-1}\exp[-10(y_{t-1}^2 - \mu_{y_{t-1}^2})]$ for all k. The test H_0: $m = 1$ vs H_a: $m = 2$ is a joint test of the significance of $y_{t-k}\ y_{t-1}^2\exp[10(y_{t-1}^2 - \mu_{y_{t-1}^2})]$ for all k.

Table 6a: Comparison of Dynamic Forecasts Derived from Different Models of the Growth of Canadian Industrial Production (1991:1 - 1995:2)

	Linear Autoregressive	Nonlinear Autoregressive	Linked Model
RMSE (unweighted)	0.0349	0.0341	0.0276
Forecast Rankings (number of forecasts out of 50)			
Ranked First	19	8	23
Ranked Second	9	22	19
Ranked Third	22	20	8

Table 6b: Comparison of Dynamic Forecasts Derived from Different Models of the Growth of U.S. Industrial Production (1991:1 - 1995:2)

	Linear Autoregressive	Nonlinear Autoregressive	Linked Model
RMSE (unweighted)	0.0306	0.0209	0.0237
Forecast Rankings (number of forecasts out of 50)			
Ranked First	12	4	34
Ranked Second	25	19	6
Ranked Third	13	27	10

(1) The three forecasts for each of the 50 forecast horizons are ranked according to the absolute size of the forecast error (the model with the smallest absolute forecast error is ranked first).

Table 7a: Root Mean Squared Errors for h-step Ahead Forecasts of the Annual Growth Rate of the Canadian Index of Industrial Production (1991:1 - 1995:2)

Forecast Horizon (h)	Number of Forecasts	Linear AR	Nonlinear AR (no links)	Nonlinear AR (with links)
1 step ahead	50	0.0086	0.0097	0.0095
12 steps ahead	39	0.0381	0.0491*	0.0447
24 steps ahead	27	0.0259	0.0306	0.0391*
30 steps ahead	21	0.0281	0.0241	0.0345
36 steps ahead	15	0.0357	0.0403	0.0334
42 steps ahead	9	0.0457	0.0408	0.0434

Table 7b: Root Mean Squared Errors for h-step Ahead Forecasts of the Annual Growth Rate of the U.S. Index of Industrial Production (1991:1 - 1995:2)

Forecast Horizon (h)	Number of Forecasts	Linear AR	Nonlinear AR (no links)	Nonlinear AR (with links)
1 step ahead	50	0.0060	0.0061	0.0062
12 steps ahead	39	0.0254	0.0223*	0.0202*
24 steps ahead	27	0.0259	0.0246	0.0227*
30 steps ahead	21	0.0249	0.0250	0.0234
36 steps ahead	15	0.0204	0.0108*	0.0097*
42 steps ahead	9	0.0249	0.0115*	0.0110*

(1) The superscript * indicates that the forecast derived from the nonlinear model is statistically different from the forecast derived from the Linear AR model at the 5% level of significance. See Granger and Newbold (page 279) for details of the forecast comparison test.