ECONOMIC RESEARCH REPORTS

A RICARDO MODEL WITH ECONOMIES OF SCALE

BY

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RR # 92-04 January, 1992

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The following paper is included in the C.V. Starr Center Economic Research Report series as an introduction to a joint paper between the author and William Baumol that will be appearing in the series.

12/29/91

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Section 1. Introduction

This paper describes a model of international trade that resembles the classical Ricardo model, but differs from it in admitting economies of scale in production.

The 2-country model discussed here does allow both economies and diseconomies. However almost all the discussion here will be on the pure economies of scale case. The mixed economies-diseconomies case, while straightforward from a purely mathematical point of view, has independent interest because of its economic interpretation and also as a bridge to the usual pure diseconomies model. These aspects will be discussed in a forthcoming paper with William J. Baumol.

Allowing economies of scale does of course have a profound effect on the behavior of the model. One aspect of almost any production economy of scale is that it gives a production advantage to countries that are actually engaged in production of a given good, as opposed to those who are not participating at all. This barrier to entry effect tends to stabilize the production status quo, whatever it is, and leads, as we will show, to a multitude of possible equilibrium points. These different equilibrium points represent vastly different outcomes for the countries involved, either in terms of national income, or in terms of utility.

Equilibrium points in this model are virtually the same as in the ordinary Walras equilibrium model. At each of our many equilibrium points there are prices and wages at which supply equals demand for each good. The wage bill for each

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[1]A summary of some of the results of this paper appeared as Ref[1].

[2]Alfred P. Sloan Foundation. The author mentions with pleasure the many contributions of Herbert E. Scarf without which this paper would not have been written.

producer equals the value of goods produced, while for non-producers the profit for entering into production at these wages and prices, and for low levels of production, is negâtive or zero. However while in the presence of a linear model or of diseconomies of scale these conditions would provide a single equilibrium, they lead here, inherently, to many. The situation mirrors the unavoidable differences between local optimization with convexity and local optimization in the presence of non-convexity.

Integer variables enter naturally into this model through a set of 0-1 variables which determine which country is to be a producer of a given good and which is not. Finding production patterns whose associated equilibrium points have some utility maximization properties then becomes an integer programming problem.

Unlike the classical Ricardo model, the more efficient producer will not always be the one who produces in this model. An entrenched economy of scale can be a barrier that prevents effective competition from a non-producer, even one with a superior production function. However Ricardo-like concepts can be reintroduced into the model with the concepts of Ricardo Level and Ricardo Point. The Ricardo Point is one in which goods are only produced by the more efficient producer, while the Ricardo Level is the exchange rate at which this is possible. There is always a RIcardo Level but not always a Ricardo Point in the presence of economies of scale.

The outcomes from a typical two-country pure economies of scale model are illustrated by Figure 1.1 which is based on the data of Table 1.1. Fig.1.1 plots Cobb-Douglas utility versus a normalized national income Z for country 1. Each dot in the figure is an equilibrium point. The large dots are outcomes in which only only one of the two countries is a producer for each good, so these are the perfectly specialized equilibria. The exchange rate ratio w_1/w_2 corresponding to the national income is plotted on the top horizontal line, the Ricardo Level is the vertical bar decending from that line. The utility in autarky is marked by the horizontal bar on the right. This example has nine products (or nine industries.)

There are several aspects of Fig.1.1 worth noting. First there are many possible equilibrium points or equivalently many pHxsible outcomes. Second, outcomes form an array of points with a definite and characteristic shape, equilibrium points are not just anywhere. This shape recurs throughout our limited empirical experience, and a rough rationale for it will be given. Second, the upper edge of the array of outcomes is rather well defined, in figure 1.1 it is marked by a dotted line. The equilibria near this boundary are the ones that maximize utility. We will see that this boundary line, and equilibria near it, can be computed by a simple and rapid calculations without computing the assemblage of equilibrium points. Third, there is a lower boundary as well as an upper boundary to the array of perfectly specialized points, this lower boundary can also be easily computed. We will also see that as the number of industries becomes large, the entire area between the upper and lower curves fills up with equilibrium points. Fourth, we note that utility does not increase indefinitely with national income but rather decreases after a certain point, and finally we note that while most equilibria in this figure lie above the autarky level there are also a substantial number with utility below the autarky level, a feature that is far more pronounced in Fig.1.2

Each equilibrium point gives a utility to Country 2 as well as to Country 1. Figure 1.2 shows the utility of Country 2 on the left vertical axis and the utility of Country 1 on the right as before. The same collection of equilibria is shown as in Fig 1.1 but now the utility for Country 2 is plotted for each equilibrium point instead of the utility for Country 1. Each equilibrium point is represented by a gray dot. Both autarky levels are shown but only the boundary curves are shown for Country 1. The horizontal axis is still Z, and in this normalization the national income of Country 2 is 1-Z. In both figures the utility of each Country is normalized separately so that its greatest utility is 1.

In this nine industry example there happens to be a Ricardo Point, and the gray and the black squares in Fig. 1.2 show its utilities for the two countries.

Section 2. Existence of Solutions

This paper emphasizes the array of solutions rather than the existence of any particular one. Nevertheless we need an accurate statement of an existence theorem and of the conditions assumed on the production functions and utility, and we will provide that in this section.

In this model the production functions $f_{i,j}$ will always have economies of scale. $f_{i,j}(l_{i,j})/l_{i,j}$ will always be a non-decreasing function of the labor input $l_{i,j}$. Also the Cobb-Douglas utility, or its logarithm, will be used throughout, so for country j, (j=1,2)

$$\ln U_j = u_j = \sum_i d_{i,j} \ln q_i \ , \quad d_j > 0, \quad \sum_i d_{i,j} = 1$$

with q_i the quantity of the ith good. It is a well known consequence of this choice of utility function that country j spends a constant fraction $d_{i,j}$ of its national income Y_j on good i, for all prices p_i

In the classical model, the pattern of production specialization is determined by the equilibrium solution. In this formulation we will have to deal with *many* equilibrium solutions and with equilibria associated with essentially arbitrary choices of the industries that are active in each country.

For any pattern of specialization that assigns a set S_j of industries to country j, an average-cost pricing equilibrium is a price vector p_i, a set of wage rates w_j, and an allocation $l_{i,j}$ of each country's labor supply L_j among those industries in which it specializes such that

The supply of the ith good is equal to its demand

$$(2.1) \quad p_i \sum_{j, i \in S(j)} f_{i,j}(\ l_{i,j}\ ) = \sum_j d_{i,j} w_j L_j = \sum_j d_{i,j} Y_j$$

and each active industry makes a profit of zero. so

$$(2.2) \quad p_i \ f_{i,j}(l_{i,j}) = w_j l_{i,j} \quad for \ i \in S_j.$$

Many papers have been written containing existence theorems[3] for economic models in which production exhibits increasing returns to scale, for example Ref.[2]. This model is a special case of these more general models in two ways. The average-cost pricing equilibrium used here is an example of these more general pricing rules, and secondly, the production possibility sets are simple examples of the sets allowed in these more general formulations. But some of the conditions required by these existence theorems are not satisfied in this model, so we need the theorem that follows. We make two assumptions about the production functions $f_{i,j}$.

A1. Aside from a possible initial interval in which $f_{i,j}(l_{i,j})$ is zero, average productivity $f_{i,j}(l_{i,j})/l_{i,j}$ is continuous and strictly increasing.

A2. Each country in autarchy produces a positive quantity of all goods. More succinctly $f_{i,j}(d_{i,j}L_j) > 0$ for all i,j.

Theorem 2.1: Under these assumptions, there will be an average-cost pricing equilibrium for any pattern of specialization in which each of the two countries is the *sole* producer of at least one of the goods. In this equilibrium each industry assigned to each country will produce positive quantities of output.

The proof of this theorem is found in appendix A-2.1 .

Different equilibria associated with different patterns of specialization are natural in the presence of economies of scale. While the patterns of production at any one of these equilibria can not be expected to be stable against large changes that move prices, wages, and specialization close to another equilibrium point, they can reasonably be expected to be stable against sufficiently small changes. This motivates a further mild restriction on the production functions that is appropriate for economies of scale. We will assume

A3. $\lim l_{i,j} \text{->} 0\ f(l_{i,j})/l_{i,j}=0$.

This zero derivative at the origin ensures that if at a particular equilibrium point country j is a non-producer of good i, that non-producer would make a negative profit in the immediate neighborhood of the equilibrium. Stability in a

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[3]As well as on many other aspects of economies of scale in international trade. See for example Ref[3],[5],[6],[7],[8].

negative profit in the immediate neighborhood of the equilibrium. Stability in a much wider sense will be discussed in Section 9.

This condition is satisfied for all production functions of the form $f(l)=el^{\alpha}$ with $\alpha > 1$, as well as by any production function that satisfies A1 and is zero for an interval to the right of the origin. It does not hold for the Ricardo case el^{α} with $\alpha=1$, but it does hold if el is preceded by an interval of zero output.

For two countries the existence theorem provides us with $3^n-2^{n+1}+1$ equilibria, at least 2^n-2 of which, the perfectly specialized ones, are locally stable in the sense just given. We now turn to the analysis of this array of possible outcomes.

Section 3. The Array of Solutions

Dealing with the array of variables is facilitated by normalized variables that allow us to plot all the equilibria in a finite part of the plane. We also introduce variables $x_{i,j}$ that determine the pattern of production and will play a key role in the analysis.

Normalized Variables and the $x_{i,j}$

At any equilibrium point we will have (2.1) and (2.2). Together these imply that (monetary) demand equals wages, that is

$$d_{i,1}Y_1+d_{i,2}Y_2=w_1l_{i,1}+w_2l_{i,2}$$

We now define $x_{i,1}$ to be the fraction of the total demand for the ith product that is spent for product made in Country 1. Similarly $x_{i,2}$ is defined to be the fraction of the total demand for the ith product that is spent for product made in Country 2.

$$\text{(3.0)}\quad x_{i,1}(d_{i,1}Y_1+d_{i,2}Y_2)=w_1l_{i,1}$$

$$\text{(3.1)}\quad x_{i,2}(d_{i,2}Y_1+d_{i,2}Y_2)=w_2l_{i,2}$$

From the definition, $0\leq x_{i,j}\leq 1$ and $x_{i,1}+x_{i,2}=1$

Next we define the normalized national incomes of the two countries to be $Z_1=Y_1/(Y_1+Y_2)$ and $Z_2=Y_2/(Y_1+Y_2)$. Clearly $Z_1+Z_2=1$ and $0\leq Z_i\leq 1$. The ratio $Z_1/Z_2=Y_1/Y_2=(w_1/w_2)(L_1/L_2)$, so Z_1/Z_2 is proportional to the wage ratio for fixed country sizes L_i

In terms of these normalized variables (3.0) and (3.1) become

$$(3.2) \quad x_{i,1}(d_{i,1}Z_1+d_{i,2}Z_2)=l^*_{i,1}Z_1$$

$$(3.3) \quad x_{i,2}(d_{i,2}Z_1+d_{i,2}Z_2)=l^*_{i,2}Z_2$$

Here the $l^*_{i,j}$ are normalized labor variables, $l^*_{i,j}=l_{i,j}/Lj$ representing the fraction of the labor force in Country j employed in making product i, and the expression in parentheses is the normalized total demand.

In what follows we will also need to refer to the actual labor used in country j. We denote it by $l_{i,j}(Z)$, $l_{i,j}(Z)=x_j(L_j/Z_j)(d_{i,1}Z_1+d_{i,2}Z_2)$.

One of the conditions for equilibrium is that the assignment of labor provided by the $x_{i,j}$ is in fact a partition of the entire labor force, i.e that $\Sigma_i l^*_{i,j}=1$ Summing (3.2) and (3.3) over all products i gives

$$(3.4) \quad (\textstyle\sum_i d_{i,1}x_{i,1})\ Z_1+(\sum_i d_{i,2}x_{i,1})\ Z_2= Z_1$$

$$(3.5) \quad (\textstyle\sum_i d_{i,1}x_{i,2})\ Z_1+(\sum_i d_{i,2}x_{i,2})\ Z_2= Z_2$$

(3.4) and (3.5) are in fact linearly dependant and therefore equivalent. This dependence is a consequence of Walras Law, but it can also be seen directly by adding the two equations [4]. We will use (3.4) and (3.5) interchangeably and refer to either one as the zero excess labor equation.

<u>Equilibria and integer x</u>

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[4] Adding the two equations is also equivalent to adding all the terms in both (3.2) and (3.3) which yields $1=(\Sigma_i l^*_{i,1})Z_1 +(\Sigma_i l^*_{i,2})Z_2$. This implies the useful relations $\Sigma_i l^*_{i,1}=1 \iff \Sigma_i l^*_{i,2}=1$, $\Sigma_i l^*_{i,1}<1 \iff \Sigma_i l^*_{i,2}>1$, and $\Sigma_i l^*_{i,1}>1 \iff \Sigma_i l^*_{i,2}<1$.

We now look at the conditions that must be met for x to be an equilinrium point.

For any set of $x_{i,j}$, whether they give an equilibrium or not, the national incomes Z_1 and Z_2 are determined by and can be calculated from (3.4). The labor amounts $l^*_{i,j}$ can then be calculated from equations (3.2) and (3.3), and these in turn determine the amounts produced $f_{i,j}(l_{i,j})$. Note that because the zero excess labor equation (3.4) is satisfied, the total labor supply in each country is used by these $l^*_{i,j}$

For x to be an equilibrium point one more condition must hold. That condition is that for the $f_{i,j}$ and $l_{i,j}$ that have been computed using x (2.2) must hold, i.e. there is a price p_i for ith good such that, *for producers who produce at a positive level*

$$(3.P)\quad p_i=\frac{w_1 l_{i,1}}{f_{i,1}}=\frac{w_2 l_{i,2}}{f_{i,2}} \text{ or equivalently} \frac{l^*_{i,1}Z_1}{f_{i,1}}=\frac{l^*_{i,2}Z_2}{f_{i,2}}$$

This is equivalent to saying that these producers produce at equal cost. If they do we have explitly found the price p. If this condition is met (2.1) and (2.2) are satisfied and we have an equilibrium point.

While most arbitrarily chosen x do not satisfy this condition, *all integer* (i.e. 0,1) *x do*, since the entire wage bill is in one country and there is only one producer of each good who produces at a positive level. The price is then determined from (3P). So all integer x are equilibria automatically. They are of course the perfectly specialized equilibria.

We will see that the integer equilibria are the ones that largely determine the shape of the solution array.

3c. Utility

Next we write down the utility each country receives at any equilibrium point x. We will use $Z_1(x)$ and $Z_2(x)$ for the national incomes calculated from (3.4) using x. While we will derive these utility expressions for Country 1 only, the changes for Country 2 are straightforward.

The logarithm of Cobb-Douglas utility is the sum of terms involving the quantity of the ith good Country 1 receives. These quantities in turn can be written as the product $Q_i(x,Z(x))F_{i,1}(Z(x))$, where $Q_i(x,Z(x))$ is the total quantity of the ith good produced in the world and $F_{i,1}(Z)$ is the fraction used by Country 1. So the log utility can be written

$$u_1(x,Z)=\ln U_1(x,Z)=\sum_i d_{i,1}\ln F_{i,1}(Z)Q_i(x,Z)$$

Since the goods are all sold at a world price the fraction going to Country 1 is proportional to its (monetary) demand so

$$F_{i,1}(Z)=\frac{d_{i,1}Y_1}{d_{i,1}Y_1+d_{i,2}Y_2}=\frac{d_{i,1}Z_1}{d_{i,1}Z_1+d_{i,2}Z_2}$$

The quantity produced is

$$Q_i(x,Z)=q_{i,1}(x_1,Z)+q_{i,2}(x_2,Z)$$

the sum of the quantities $q_{i,j}$ produced in each country. The $q_{i,j}$ are defined by $q_{i,j}(x_i,Z)=f_{i,j}(l_{i,j})$ where the labor $l_{i,j}$ depends on x and Z and is determined from (3.2) or (3.3).

The full expression for the utility is then

$$(3.6)\quad u(x,Z)=\sum_i d_{i,1}\ln\frac{d_{i,1}Z_1}{d_{i,1}Z_1+d_{i,2}Z_2}\{q_{i,1}(x_{i,1},Z_1)+q_{i,2}(x_{i,2},Z_2)\}$$

This expression is complicated both in its dependence on the production pattern x and the normalized national incomes Z. In addition x and Z are linked to each other through (3.4). This makes it difficult to compare the many different equilibria other than by fully computing each one. Although useful and suggestive experiments along that line can be done and were done as part of this work[5] we will take a different approach in what follows.

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[5]Computer experiments played a suggestive and useful role in many parts of this paper. See Appendix C-1.

We will deal with this complexity in two ways. First, by analyzing the array of solutions and their boundary rather than individual points, and second by working with perfectly specialized solutions. The emphasis on perfectly specialized solutions will be justified retrospectively when they turn out to be the solutions that determine the upper boundary of the entire array of solutions.

Utility for Perfectly Specialized Equilibria

If x_1 and x_2 are any variables constrained to be either 0 or 1, and if $x_1=0$ implies $x_2=1$ and vice versa, then we always have for any function $g(x_1,x_2)$ the tautology $g(x_1,x_2)=x_1g(1,0)+x_2g(0,1)$. Intuitively the variables act as a switch between the two values that are the only ones possible with such restricted variables.

Letting g be successively the individual terms of the sum (3.6) yields an expression for utility that is valid for integer x only.

$$Lu_1(x,Z)=\sum_i x_{i,1}d_{i,1}\ln F_{i,1}(Z)q_{i.1}(1,Z_1)+x_{i,2}d_{i,1}\ln F_{i,1}(Z)q_{i.2}(1,Z_2)$$

For integer x we have Lu(x,Z)= u(x,Z) so for integer x *only* we can use these expressions interchangeably. The merit of Lu(x,Z) is that for fixed Z the expression is now linear in the variables x_i

Boundaries

To find the upper boundary of the array of perfectly specializes solutions we define the function $B_I(Z)$ for fixed Z to be the result of maximizing u(x,Z) subject to (3.4) and subject to the x_i being 0 or 1. Finding this maximal u(x,Z), for any given Z, is an integer programming problem, in fact a knapsack problem (Ref[4]) and we will describe it explicitly below. The $B_I(Z)$ values obtained this way will, by definition, be equal to or above the utility of any integer equilibrium point.

This maximization problem also has economic meaning. Once Z is fixed, the demand in both countries for any good is fixed. Hence the fraction $F_{i,j}$ of the total production of any good that goes to each country is also fixed. The only way to improve the utility from any *one* good is to attempt to assign the production to the more efficient producer and so increase the quantity produced. (3.4) is the labor constraint that prevents this assignment from being made in every case, and the maximization problem is to make the efficient assignments as much as possible subject to that labor constraint.

The introduction of the concept of boundary turns out to enormously simplify the task of dealing woth the array of solutions. We will be able to compare various equilibrium points to the boundary far more easily that we can compare them to each other.

Before proceeding to a boundary calculation we need one more concept.

Ricardo Level

We will see that there is only one Z for which it is even possible to assign all the production to the more efficient producer, this is the Ricardo Level mentioned in the introduction. For wage rates Z greater than the Ricardo level, were country 2 to be the sole producer of those goods it makes more efficiently at that wage rate, the demand for its labor would outstrip the supply. For Z below the Ricardo Level, the same obtains for Country 1.

More precisely, let $S_1(Z)$ be the set of goods made more efficiently by Country 1 (as sole producer) at normalized national income Z. This means $q_{i,1}(1,Z) > q_{i,2}(1,Z)$ for i in S_1. For Z_1 sufficiently small (very low wage in Country 1), S_1 will be non empty. The demand for Country 1's normalized labor *if product is always assigned entirely to the more efficient producer* is given by

$$L^*_1(Z) = \sum_{i \in S_1(Z_1)} l^*_{i,1} = \sum_{i \in S_1(Z_1)} (d_{i,1} + d_{i,2}\frac{Z_2}{Z_1})$$

This is >1 for Z_1 near 0 and then decreases monotonically with increasing Z_1 (increasing wage) and eventually becomes < 1. The steady decrease is due to two causes. First, for any fixed set S_1 the demand for County 1's labor only decreases with increasing Z_1 as we can see from the equation. Second, as Z_1 increases the set $S_1(Z)$ only loses members. Such a loss causes a discontinuous downward jump in L^*_1

We now define the Ricardo level Z_R as sup Z_1, $L^*_1(Z) > 1$. For national incomes $Z < Z_R$ the labor demanded exceeds the labor supply in Country 1. The behavior at Z_R itself and the notion of Ricardo Point, are both explained in Appendix (A3-1).

We could also have defined the Ricardo level in terms of the increasing demand for Country 2's labor. The result would be the same because of the relationships between L^*_1 and L^*_2 given in footnote (4).

The nomenclature is due to the fact that this reasoning echoes that of the usual Ricardo model.

The Boundary $B_I(Z)$

Precisely as written equation (3.4) will not usually have a solution for integer x. This reflects the economic fact that there are equilibria for certain Z only. To deal with this difficulty we relax (3.4) to an inequality and define $B_I(Z)$ by the integer programming problem,

$$(3.7)\quad B_I(Z)=Max_x\ u(x,Z) \qquad x_{i,2}\ integer,\quad 0\leq x_{i,2}\leq 1$$

$$with \quad \sum_i \{d_{i,1}Z_1+d_{i,2}Z_2\}\ x_{i,2}\leq Z_2$$

In this zero excess labor inequality we have rearranged the terms to show the $x_{i,2}$. *The inequality points as shown for Z above the Ricardo Level* ($Z_1 > Z_R$) *and is reversed for Z below the Ricardo Level*

This relaxation *allows* the underutilization of labor in the country whose labor is scarce. Consequently the maximization of utility for the given Z should push the inequality very close to equality as the attempt is made to use this valuable labor

In (3.7) we have arbitrarily chosen (3.5). Of course we could just as well have chosen (3.4). Since we will often have occasion to refer to the inequality versions of (3.4) and (3.5) we will refer to these as (3.4i) and (3.5i). It will always be assumed that these inequalities point in the proper directions.[6]

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[6]While we have given an economic meaning to the Ricardo level, it is also possible to give a purely mathematical description of this maximization problem as follows. Any equation is equivalent to two inequalities, one $\geq$ and one $\leq$. If we consider the maximization problem (for some Z) as a linear (not integer)

(3.5) only involves the variables $x_{i,2}$, but the objective function in (3.7) involves both $x_{i,1}$ and $x_{i,2}$ If we rewrite the utility in terms of the $x_{i,2}$ only, using $x_{i,1}+x_{i,2}=1$ to eliminate the $x_{i,1}$ we get

$$(3.8)\quad Lu_1(x,Z) = \sum_i d_{i,1}\ln F_i(Z)+\sum_i d_{i,1}\ln q_{i,1}(1,Z)+\sum_i x_i d_{i,1}\ln\frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)}$$

so we can put the maximization problem (3.6) in a good computational form involving the $x_{i,2}$ only.

$$(3.9)\quad Max_x\ Lu(x,Z) = P_1(Z)+\sum_i x_{i,2}d_{i,1}\ln\frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)}$$

$$with \quad \sum_i \{d_{i,1}Z_1+d_{i,2}Z_2\}\ x_{i,2}\leq Z_2 \text{ and } x_{i,2} \text{ integer}$$

With $P_1(Z)$ representing the first two sums in (3.8).

If we choose the $x_{i,1}$ instead of the $x_{i,2}$ we obtain the following formulation which we will need at times.

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programming problem, one or the other of the two inequalities will be the binding constraint. This inequality is the one that should be used for the integer maximization problem (for that Z).

$$(3.10)\quad Max_x\ Lu(x,Z) = P_2(Z) + \sum_i x_{i,1} d_{i,1} \ln \frac{q_{i,1}(1,Z)}{q_{i,2}(1,Z)}$$

$$with \quad \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}\ x_{i,1} \geq Z_1 \ and\ x_{i,1}\ integer$$

The $P_2(Z)$ is identical with $P_1(Z)$ except for the substitution of $q_{i,2}$ for $q_{i,1}$.In the objective function the q's have also been interchanged, but since this occurs in the logarithm the result is simply the negative of the corresponding term in (3.8). That is as it should be since increasing $x_{i,1}$ means decreasing $x_{i,2}$. In terms of the $x_{i,1}$ variables the problem (3.10) is to make the best assignment of producers while being obliged to *over*utilize the labor in the country whose labor is little sought after.

The formulation (3.10) uses (3.5i) while (3.10) uses (3.4i).

The $B_I(Z)$ so defined can be computed by any integer programming technique. For a single inequality problem such as this, ordinary dynamic programming is very effective. It allows the computation of the array boundary without examining the 2^n specialized solutions. Furthermore the dynamic programming problem gives actual integer solutions, and hence equilibria, which can be expected to be close to the boundary curve. (Appendix A3-2)

<u>The Boundary B(Z)</u>

In addition, there is an even easier calculation for getting a weaker boundary curve, which we will call B(Z). To get B(Z) we further relax the problem (3.6) or (3.9) by allowing *continuous* $x_{i,2}$. It is easily seen that with continuous variables the zero excess labor inequality will always be satisfied as an equality, so in fact B(Z) is given by the maximization of $Lu_1(x,Z)$ subject to (3.5).

$$(3.9a)\quad Max_x\ Lu_1(x,Z) = P_1(Z) + \sum_i x_{i,2} d_{i,1} \ln \frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)}$$

$$subject\ to \quad \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\} x_{i,2} = Z_2$$

The solution technique for such a "continuous knapsack problem"is well known and particularly simple (ref[4]). The solution procedure can be thought of as filling a space of length Z_2 with amounts $x_{i,2}$ of goods each of length

$$(3.11)\quad d_{i,1}Z_1 + d_{i,2}Z_2 \quad \textit{with value} \quad d_{i,1}\ln\frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)}$$

The solution to such a problem is to put in goods in succession in the order of their value per unit length, which we will call value density. The densest variable is used first. When its turn comes the amount $x_{i,2}$ of each good is increased until either the amount $x_{i,2}=1$, or the equation is satisfied(i.e. the space is used up), whichever occurs first. If the $x_{i,2}$ reaches 1 first, start again with the next good in order of value density. If the equation is satisfied for some value of $x_{j,2} < 1$, the current values for all $x_{i,1}$ are the solution. Note that $x_{j,2}$ is the only variable that is non-integer in this solution. The variables that preceded it are 1, and those after it are 0.

This calculation is then repeated for different Z to get the boundary curve. It is the results of these simple calculations that appear as the dotted lines in our figures. While this is a rapid and simple calculation it can be very much further refined. (Appendix A3-3))

Lower Boundaries

While the goal so far has been to find the upper boundary of the array of perfectly specialized equilibria, exactly the same methods will give us the *lower* boundary. It we minimize the objective functions in the problem (3.9) instead of maximizing we will get the lower boundaries $BL_1(Z)$ and BL(Z). This only involves changing the sign of the objective function, everything else goes forward as before. This approach produces the lower boundaries seen in figures 1.1 and 1.2 and fixes all perfectly specialized equilibria to be somewhere between these curves.

The Two Methods

The two methods of calculation we have been using, one with integer variables and the inequality form of (3.4) and one with continuous variables and the equality (3.4), are two different relaxations of the original maximization problem described in 3c. Both seem to have their advantages in thinking about boundary related problems and both will be used in the rest of this paper.

We can consider both calculations both from the point of view of generating boundary curves and from the point of view of finding actual equilibria near those curves.

Figure 3.1, which is an 8 product model based on Table 3.1, resembles Figures 1.1 and 1.2 except that the upper boundary $B_I(Z)$ has been added for Country 1. This is the jagged black line[7] under the B(Z) curve in Figure 3.1. $B_I(Z)$ does follow the location of the integer points more precisely than does B(Z). However in Fig. 3.2, which is a 17 product model based on Table 3.2 we see that the two boundary curves are much more alike.

Both calculations can also give us points near their respective boundaries, the integer calculation does this automatically while the continuous calculation does this by rounding the non-integer variable up or down. Figure 3.3 shows the B(Z) from the continuous calculation together with the integer points obtained by the integer maximization calculation. Fig 3.3 is based on Table 3.3 and represents a problem with 27 goods. From the more than 100 million specialized equilibria the calculation has produced the ones shown in the figure that are sitting virtually on top of B(Z)

While both calculations appear to be very effective in bounding the solution array in actual computation, we will also make a more precise statement about how solutions approach the boundary curve.

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[7]The data for the points on the line was obtained by using the dynamic programming calculation for a series of points from Z=.1 to Z=.9 with spacing of .005, and then using a standard plotting routine to create the line.

Section 4 : Convergence to the Boundary

To motivate what follows we will indicate very roughly one interpretation of the results. First, in this section, we will see that under certain reasonable restrictions, that, as the number of products in the model increases so that each one absorbs a decreasing fraction of the national income, every point of $B_I(Z)$ and of B(Z) is approached by equilibrium points. Then, in section 6 we will see that every point that lies between the upper and lower boundary curves is also approached by equilibria. Thus that the entire shape between the upper and lower boundary curves eventually fills in with equilibria.

Our approach will aim at proving these results with minimal complexity. As a result the estimates will be crude, and the results described here seem to occur in practice in much smaller problems that these estimates would indicate.

We will start by showing that points on B(Z) are approached by equilibria. We will first show that the there are equilibria that are not too far away in terms of it Z- coordinates, then we will show that some of these also have utility that is close to B(Z).

3a. Nearby Z.

Choosing some fixed point Z', the corresponding boundary point is (Z',B(Z')). Let us assume that we are above the Ricardo Level, ($Z' > Z_R$). Relative to this choice of Z' we define a near equality (n.e.) integer equilibrium point x to be an x satisfying (3.5i) but such that increasing some component $x_{k,2}$ from 0 to 1 would result in a new x that does not satisfy (3.5i).

That there are always n.e. equilibria for any choice of Z' follows from the fact that the optimal integer solution to (3.9) must be n.e. as is the x obtained from rounding down the non-integer component of the knapsack problem solution.

For n.e. equilibria we can state:

interval of interest by:

$$\Gamma = \frac{1}{Z_{\min}(x)}\{2+\frac{\delta L_{\max}\mu}{Z_{\min}(x)}\}$$

We can summarize this in the following lemma:
Lemma 4.2: If x is n.e. then

$$| u_1(x,Z') - u_1(x,Z(x)) | \leq \frac{\Gamma\delta}{1-\tau}$$

Step 2.Since Z'is above the Ricardo level the variables appearing at a positive level in the solution to (3.9a) will have positive coefficients in the objective function. If we round the one non-integer variable in that solution down to obtain an integer solution x_r we have

$$0 \leq B(Z') - u_1(x_r,Z') \leq \delta\beta_1(Z')$$

where

$$(4.2) \quad \beta_1(Z) = \max_i \ln\frac{q_{i,2}(1,Z(x))}{q_{i,1}(1,Z(x))} = \max_i \ln\frac{f_{i,2}(l_{i,2}(1,Z(x)))}{f_{i,1}(l_{i,1}(1,Z(x)))}$$

β_1 will be large if for the given Z, in some industry Country 2 can produce a much greater quantity as sole producer compared with Country 1 as sole producer. The argument appearing in the production functions is the (normalized) total demand divided by the wage rate and hence the amount of labor bought by the total demand.

Now consider any integer x that is as good or better an integer solution to the maximization problem (3.9) as is x_r. For such an x.

$$(4.3) \qquad u_1(x_r,Z') \leq u_1(x,Z') \leq B_1(Z') \text{ so } \quad 0 \leq B(Z') - u_1(x,Z') \leq \delta\beta_1(Z')$$

Both the maximizing integer solution and the rounded solution x_r itself are examples of such x. This completes the elements of the proof for Z' above the Ricardo level.

For Z' below the Ricardo Level we need only switch to using the $x_{i,1}$ and the argument will proceed in exactly the same way. However we will have a β_2 which is the maximum of the negatives of the terms appearing in (4.2). If we define $\beta = \mathrm{Max}(\beta_1, \beta_2)$ and put together the two preceding lemmas and (4.3) we can state the following theorem.

Theorem: If B(Z'),Z' is any point of B(Z) then, provided τ is not 0, there is an perfectly specialized equilibrium point x with $0 \leq Z' - Z_1(x) \leq \delta/1-\tau$ and $0 \leq B(Z') - u_1(x, Z(x)) \leq \delta(\beta + \Gamma)$.

This theorem relates any boundary point to a nearby integer solution in terms of the parameters of the given problem. However it can also tell us what happens as problems get large under reasonable circumstances.

Consider a sequence of problems with increasing numbers of goods, each of which absorbs a decreasing fraction of the national income. Let us denote the various parameters appearing in the theorem as it is applied to the nth problem by $\delta_n, \beta_n, \tau_n$ and μ_n.(When these are known and Z is specified we have Γ_n)

Then we have the following Corollary:

Corollary: Let P_n be a sequence of problems with bounded parameters $\beta_n, 1/(1-\tau_n), \mu_n$, and Γ_n, and with $\delta_n \to 0$. Then, for any Z', and any ϵ there is an n sufficiently large that the point $Z', B_n(Z')$ on the nth boundary curve will have an integer equilibrium point within ϵ in both coordinates.

Let us discuss briefly whether the parameters of a sequence of reasonable models would remain bounded as assumed in the corollary. We will simply assume that the models do not converge on "orthogonal demands" so that $1/(1-\tau_n)$ remains bounded. μ and β depend on the production functions, and if they are bounded and δL_{max} is bounded, so is Γ. If we assume that the production functions that appear with increasing n are not radically different from those before them, we would expect the various production ratios that make up the parameters to vary in value but remain bounded unless they are being evaluated at ever increasing labor levels. However the largest possible labor input into any one production function is, from (3.2) and (3.3) $L_{max,n}\ \delta_n/\min(Z'_1, Z'_2)$. So if $L_{max,n}\ \delta_n$ is kept bounded as n increases, i.e. industry sizes remain bounded, the conditions of the corollary will be met.

For example if the sequence of problems was produced by adding new industries one at a time to an existing model, the new industries being roughly the scale of those that preceded, and also enlarging the labor force at the same time by adding the labor for the new industry, the conditions of the corollary would be met, and we would see integer points approach every point of the boundary as n increased.

Section 5. Geometry and Linear Programming

What we have done so far is also capable of a geometrical interpretation.

We have seen that any x, $0 \leq x_{i,1} \leq 1$ determines a Z(x), and hence a labor allocation $l_{i,j}$, and production quantities $q_{i,j}$ of each good and hence utility. Geometrically we have a mapping from the n-dimensional cube C_n, $0 \leq x_{i,1} \leq 1$, into (Z,U) space. In terms of this geometry we saw in Section 4 that the mapping sends all the vertices of C_n, except (0,0,..0) and (1,1,....1), into the space between the two boundary curves in (Z,U) space.

The restriction provided by the zero excess labor equation (3.4) is linear in x once Z is fixed, so the condition Z=Z' specifiesHxn n-1 dimensional linear space cutting through C_n, with the intersection forming a polyhedron P(Z'). It is on this polyhedron that we have maximized the linearized utility by solving the knapsack problem.

The knapsack problem is of course, in its continuous form, a linear programming problem with upper bounds on the $x_{i,1}$ or the $x_{i,2}$. Its maximum value over P will always be obtained at a vertex of P(Z'), or equivalenesttly on a 1-dimensional edge of C_n. Since we are solving a linear programming problem with only one constraint (and upper bounds) we can expect, from ordinary linear programming considerations, at most one non-integer variable.

We will also obtain ordinary linear programming prices, the price associated with the zero excess labor equation being precisely the density of the non-integer variable.

By way of motivation for the next section we also observe that the polyhedron P(Z') has a vertex which minimizes the linearized utility. This point is mapped onto a point on the *lower* bounding curve. There is also a path of 1-dimensional edges of P(Z') leading from the maximizing vertex to the minimizing one. Along this path all intermediate values of the linearized utility are obtained.

Section 6. Filling In

With this background we will now show that under the same circumstances as in Section 4 the various perfectly specialized equilibria not only approach the upper boundary but entirely fill out the space between the upper and lower boundaries as the number of goods grows.

We start the proof with the following lemma which is immediate from the linear programming point of view. As usual we will assume $Z' > Z_R$

Lemma 5.1: Let B(Z') and BL(Z') be the values of the upper and lower boundary curves for some Z'. Then for any value v, $BL(Z') \leq v \leq B(Z')$, there is a feasible (non-maximizing} solution x to (3.9), with at most two non-integer components, for which the value of the objective function is v.

Proof: Let us add to the maximization problem (3.9) the linear constraint $Lu_1(x,Z') \leq v$. The problem now has two constraints and upper bounds, so the x that attains the linear programming maximum, which is v. will have at most two variables that are neither 0 or 1.

Let x' now be that solution with its two non-integer components $x'_{j,2}$ and $x'_{k,2}$. x' satisfies (3.5) as an equality so that the integer point obtained by rounding up both $x'_{j,2}$ and $x'_{k,2}$ to 1 can not satisfy (3.5i), while the x obtained by rounding them both down clearly does. It follows that x or one of the x's obtained by rounding one component up and one down has the n.e. property, and Lemma(4.1) applies to that x, as does Lemma(4.2)

Consequently we have for this x

$$|Z_1 - Z_1(x)| \leq \frac{\delta}{1-\tau} \quad \textit{and}$$

$$|u_1(x,Z') - u_1(x,Z(x))| \leq \frac{\Gamma\beta}{1-\tau}$$

To bound the difference between v and $u_1(x,Z')$ we simply observe that the difference made by changing each term can not exceed β so the total change in Lu_1 from the value of the linear programming solution, which is v, can not exceed 2β.

Putting together these three elements we have proved the following theorem. Theorem. If (v,Z') is any point between B(Z') and $B_L(Z')$, then, provided τ is not 1, there is an integer equilibrium point x with $0 \leq Z_1(x)-Z'_1 \leq \delta/1-\tau$ and with $|v-u_1(x,Z(x))| \leq \delta(2\beta+\Gamma)$.

And we have a similar corollary:

Corollary: Let P_n be a sequence of problems with bounded parameters β_n,$1/(1-\tau_n)$,μ_n, and Γ_n, and with δ_n->0. Then, for any Z', and any ϵ there is an n sufficiently large that any point Z,v between $B_n(Z)$ and $B_{L,n}(Z)$ will have an integer equilibrium point within ϵ in both coordinates.

The fill in effect is already visible in Fig. 6.1 which is a 13 product model based on the data of Table 6.1. In Fig. 6.1, as in Fig 1.2, we plot the utility values for Country 2 of various equilibria. However unlike Fig 1.2, the only points plotted are perfectly specialized equilibria.

Section 7:Non-Specialized Equilibria

So far we have worked entirely with integer solutions, that is to say with perfectly specialized solutions. Some justification for this approach can be seen from the following theorem.

Theorem 7.1: Let x be any equilibrium solution, whether specialized or not. Let Z(x) be the corresponding Z and $u_1(x,Z)$ the utility of x to Country 1, then

$$u_1(x,Z) \leq B_1(Z) \leq B(Z)$$

So all the equilibrium points, not just the specialized ones, lie under the boundary curves.

Since x is not specialized we can not use the linearized utility so we will need the following lemma

Lemma 7.1: Let $q_{i,1}(x_{i,1},Z(x))$ and $q_{i,2}(x_{i,2},Z(x))$ be the quantities of the ith good produced at national income Z(x) at an intermediate equilibrium point x. Then $q_{i,1}(x_{i,1},Z(x)) + q_{i,2}(x_{i,2},Z(x)) \leq \mathrm{Min}(\, q_{i,1}(1,Z(x)),\, q_{i,2}(1,Z(x))\,)$

This lemma states that either country, as the sole producer of good i, *at the demand and wage levels of the equilibrium point*, will produce more than the two countries together at the equilibrium point x.

The lemma does not assert that more would be produced if one country were in fact to be the sole producer. For if that were to happen, we would have a normalized national income different from Z(X), with different wages and therefore possibly a different outcome. It also does not assert that for any $0 \leq x_{i,1} \leq 1$ the inequality holds but only for those $x_{i,1}$ that come from an equilibrium x. Without that restriction the result would not be truHx

Proof:

At equilibrium we have for each i

(7.1) $\quad p_i f_{i,1} = w_i l_{i,1}$ and $p_i f_{i,2} = w_i l_{i,2}$

At x, $f_{i,1}$ is $q_{i,1}(x_{i,1},Z(x))$ and $l_{i,1}=x_{i,1}(L_l/Z_l)(d_{i,1}Z_1+d_{i,2}Z_2)$ with similar expressions for $f_{i,2}$and $l_{i,2}$. If we divide the two expressions in (7.1) and use these relations we obtain

$$q_{i,1}(x_{i,1},Z(x))/q_{i,2}(x_{i,2},Z(x))= x_1/x_2 \quad \text{or equivalently}$$

(7.2) $$q_{i,1}(x_{i,1},Z(x))/x_{i,1}=q_{i,2}(x_{i,2},Z(x))/x_{i,2}=R$$

Since $x_{i,1}+x_{1,2}=1$

$$q_{i,1}(x_{i,1},Z(x)) + q_{i,2}(x_{i,2},Z(x)) = R$$

Since the q's are the quantities produced and the x's the corresponding wage bills, the production economies of scale conditions assert that the first ratio in (7.2) grows with $x_{i,1}$ and the second with $x_{i,2}$ so

$$q_{i,1}(1,Z(x))/1 \geq q_{i,1}(x_{i,1},Z(x))/x_{i,1} = R = q_{i,1}(x_{i,1},Z(x)) + q_{i,2}(x_{i,2},Z(x)).$$

As the same relation holds for $q_{i,2}$ the lemma is established.

In the remaining part of the proof we assume, as usual, that Z(x) is above the Ricardo level.

x satisfies (3.5). Let us consider the integer equilibrium point x' obtained from x by rounding down all the non-integer $x_{i,2}$ to 0. Since all the coefficients in the inequality are nonnegative, x' satisfies (3.5i) and therefore is a feasible solution to (3.9). We will compare $u_l(x',Z(x))$ with the utility of x. Note that $u_l(x',Z(x)$ is *not* the utility of x'(which is $u_l(x',Z(x'))$) but it *is* the value that x' would give to the objective function in (3.9) whose maximization produces $B_l(Z(x))$ and B(Z(X)).

If we can show that $u_l(x',Z(x))$ is $\geq u_l(x,Z(x))$, we would know that the maximum value of the objective function in (3.9) is larger yet, so we would have $B_l(Z(x)) \geq u_l(x,Z(x))$ which would prove the theorem.

To compare the values of $u_l(x,Z(x))$ and $u_l(x',Z(x))$ we look at the individual terms in the two u_l expressions. Using z as a dummy variable, the terms are of the form

$$d_{i,1}\ln F_i(Z(x))Q_i(z,Z(x)) \quad with$$

$$F_i(Z(x))=\frac{d_{i,1}Z(x)}{d_{i,1}Z(x)+d_{i,2}Z(x)} \quad and$$

$$Q_i(z,Z(x))=q_{i,1}(z_{i,1},Z(x))+q_{i,2}(z_{i,2}Z(x))$$

The F term is the same for both expressions. However if we compare the results of putting the components of x and of x' into the z of the second term we will always get a larger result from the x' component. This is because the conditions of lemma 7.1 are fulfilled, the x' components are always 0 or 1 while the x' components come from an equilibrium point.

So $u_1(x',Z(x))$ is $\geq u_1(x,Z(x))$ which establishes the theorem.

Non-specialized Equilibria

Non-specialized equilibria are harder to analyze than are the specialized. Fortunately they are connected by the theorem we have just stated. In addition, our empirical work, of which Fig (1.1) and Fig (1.2) are examples, shows the generally lower utility of non-specialized equilibria.

However, mixed equilibria exist, they are numerous, and they have their own interesting properties.

The simplest case of a mixed equilibrium is the case in which only one good, say x_1 is produced in both countries. Let $x(x_{1,1})=(x_{1,1},x')$ where x' is a fixed m-1 vector of 0's and 1's representing the $x_{i,1}$. If we take $x_{1,1}$ as a parameter which varies from zero to 1, then for each value of $x_{1,1}$ we have an x and therefore can compute the national incomes from (3.4) and then the utilities. The result will be a curve connecting the perfectly specialized points x(1) and x(0). An example is the dashed line in figure (7.1).[8]

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[8]This figure is based on Table (7.1).It is a six product model, and the treaty curve connects the points $x_1=(1,0,0,1,0,1)$ and $x_1=(1,1,0,1,0,1)$. One effect of using a small model is to make the figures more readable, another is that the small number of goods gives large swings in national income as a result of the shift of

With isolated exceptions the points along this curve are *not* equilibrium points since the two producers are not producing at the same unit cost. However the points can be given an economic interpretation. Imagine that the two countries sign a treaty agreeing to pool their output of the good represented by $x_{1,1}$. They agree to sell this pooled output at the resulting world price, and to split the proceeds to cover their respective wage bills. This arrangement will correspond to the computed points, and we will therefore refer to them as treaty points or treaty solutions. While such a treaty might stabilize these points and allow them to persist, in the absence of such a treaty, they are non-equilibrium points and subject to market forces that will be described below and that will generally cause movement away.

For each Z along such a treaty curve one country or the other will be producing at a lower unit cost. In figure (7.2) the curves of figure (7.1) have been redrawn with the dark part of each curve showing the more efficient producer. The point at which the curve switches from dark to light is a point of equal unit production costs and hence an equilibrium point.

In Fig (7.2) there is a single transition point, and hence a single equilibrium point along each curve. This is what one might expect intuitively since as x_1 increases from 0 toward 1 Country 1 would generally become a more efficient producer of good 1 because of economies of scale, while Country 2 becomes less efficient, and this is in fact the commonest case.

If we denote the unit production costs in each country by $1/p_{1,1}$ and $1/p_{1,2}$ respectively we would expect a plot of the p's versus x_1 to look something like fig (7.3) which is in fact that plot for the treaty curve of Fig.(7.1). For $x_{1,1}$ near zero $p_{1,1}$ should be very large, and for $x_{1,1}$ near 1 $p_{1,2}$ will be very large. Also $p_{1,1}$ should generally (but not always) decrease with increasing labor to produce a single crossing point, $p_{i,1}=p_{i,2}$ where we would have equilibrium and a price $p=p_{i,1}=p_{1,2}$

The single equilibrium would in fact be the only case if the quantities

---

a single industry. Note that the production functions used here, and described in Table 7.1, are different from those of previous tables. They have a slower build up of production.

produced were given by $q_1(x_{1,1},Z')$ and $q_2(x_{1,2},Z')$ for some fixed Z' as in the lemma above. Here however we are dealing, not with a fixed Z',but with a varying $Z(x(x_{1,1}))$. Or equivalently we are dealing with a wage rate that varies as $x_{1,1}$ varies. Because of this the possible outcomes are more complex and multiple intersections and multiple equilibria can occur. Fig.(7.4) and its p plot (7.5) provide an example[9].

In any case the behavior of the $p_{1,1}$ and $p_{1,2}$ in each coming down from unboundedly large values at one end of the interval to finite ones at the other, does force the number of intersections of the price curves, and therefore of equilibrium points, to be odd in every case.

We can also associate a rough dynamics with Figs (7.3) and (7.5). To the left of the intersection in figure (7.3) Country 2 is the lower cost producer. Country 2 can therefore cover its labor costs and more, and still sell at a price lower than County 1, which at that price can not cover the wages of all its workers. This creates a situation where Country 2 will be motivated and able to increase production and get a larger share of the demand, (increase $x_{1,1}$), while Country 1, which can not even pay all its work force, must reduce it and lose share, (decrease $x_{1,2}$,increase $x_{1,1}$). These directions of change are shown by the arrows in Fig.(7.3).

These conventions for dynamics can be applied in the same way to the general situation. We will assume that at each Z if producer j has the lower price curve will increase $x_{1,j}$ while the other producer is obliged to decrease his $x_{1,j}$. A handy result of this convention is that direction arrows on a curve reverse as the curve passes through a (simple) equilibrium.

In fig (7.3) this convention for dynamics gives the intuitively plausible result. The Country 1 producer will not cover his labor costs until he is operating at the scale that gives the equilibrium point, but thereafter he can profitably increase, The Country 2 producer does well until his production is brought down to the equilibrium point, after which these dynamics would cause collapse. We will

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[9]This figure is based on Table 7.2, and the curve connects $x_1=(1,1,0,1,0,1)$ and $x_1=(1,1,1,1,0,1)$. Table 7.2 differs from Table 7.1 only in the data on the third product. There is been a change in demand for this product but more significantly the production exponent has been reduced to near 1.

call an equilibrium, such as that of figure (7.3), in which all arrows point away from the equilibrium point, an *unstable* equilibrium.

Unstable equilibria such as the one illustrated in (7.2) and (7.3) play a role in measuring how large a scale Country 1 must reach before it can compete with Country 2.

However in Fig (7.5) with its triple of equilibria, the first and third points are unstable but at the second one the arrows all point toward the equilibrium point. We will call such a point *stable*. From these conventions and our previous remarks we have at once the following theorem.

Theorem 7.2: The number of intermediate equilibria is always odd. If the intersections are numbered in order of increasing $x_{1,1}$ the odd numbered ones are unstable and the even numbered ones stable.

Some light on when the different cases occur is given by the following pair of theorems.

For production functions of the form $e_{i,j}l^{\alpha_i}$, $\alpha_i > 1$, the following holds.

Theorem 7.3: If $q_{1,1}(1,Z(x)) \geq q_{1,2}(1,Z(x))$ for both x(0) and x(1). or if $q_{1,1}(1,Z(x)) \leq q_{1,2}(1,Z(x))$ for both x(0) and x(1), then there is only one intermediate solution $x(x_1)$, $0 < x_i < 1$, and it is unstable.

In words, the theorem asserts that if one country or the other is the more efficient producer of the entire demand over the range $0 \leq x_{1,1} \leq 1$, we have the simple outcome.

That the condition on the efficiencies and the wage rates can not be wholly dispensed with, and that in its absence there are instances of multiple equilibria, is shown by the following partial converse:

Theorem 7.4: If the condition of Theorem 7.3 is not met, there are always multiple equilibria for values of the production exponent α_i sufficiently close to 1.

This implies that in these cases there are stable equilibria.

Both theorems are proved in Appendix A7-1.

Section 8.:Special Cases

While there are several interesting special cases we will confine ourselves to discussing three.

Case 1 - Symmetrical Demands.

When we calculate the boundary curves, whether these are the B(Z) or the $B_1(Z)$ we maximize an objective function

$$(8.1)\quad \sum_i d_{i,j} \ln \frac{d_{i,j} Z_j}{d_{i,1} Z_1 + d_{i,2} Z_2} Q_i(x,Z)$$

For j=1 this is the utility for Country 1 and for j=2 this is the utility for Country 2. If we have symmetrical demands, i.e. $d_{i,1}=d_{i,2}$ the functions are identical except for the first term in the logarithm which becomes Z_1 in Country 1's utility function and Z_2 for Country 2. Since this term does not enter into the optimization calculation, the objective functions to be maximized are exactly the same for both countries. Therefore we obtain the boundary curves for both countries with a single calculation and for each Z we have the same maximizing x. This is true both for the integer programming and linear programming (i.e. continuous knapsack) calculations.

The fact that the same x maximizes for both countries strongly suggests that *for symmetrical demands* a solution (for fixed Z) that is good for one country is good for the other. In fact this is true.

From the observations just made about the terms in the utility functions it follows that the ratios of the utilities (not the log utilities) of the two countries are, for any x, $U_1/U_2=Z_1/Z_2$. This applies to any x, whether integer or not and therefore the boundary curves themselves obey this ratio as do all the equilibrium solutions. Therefore if x is an integer or non-integer equilibrium near the B(Z) of Country 1 the utility of x will be near the boundary of Country 2 since the utility value of both boundary and point for Country 2 are derived by multiplying the Country 1 utility by Z_1/Z_2

This benign property of the symmetrical demands does not carry over to the non-symmetrical demand case. There countries will put different weights on different elements of the objective functions and the production plan (even for fixed Z) that is best for one is generally not best for the other

Case 2. We next consider symmetrical demands and production function $p_{i,j}(l)=e_{i,j}$ l^{α} for a *fixed* exponent α.

As usual the utility consists of two parts, one independent of x and one dependent. The independent part is

$$\sum_i d_{i,1}\ln F_i q_{i,1}=\sum_i d_{i,1}\ln Z_1+\sum_i d_{i,1}\ln e_{i,1}\left(\frac{L_1 d_{i,1}}{Z_1}\right)^{\alpha}$$

$$=K+(\alpha-1)\ln\frac{1}{Z_1}$$

$$\textit{with}\quad K=\sum_i d_{i,1}\ln\, d_{i,1}e_{i,j}L_1^{\alpha}\quad \textit{which is } u_1 \textit{ in autarchy.}$$

while the x dependant part is

$$(8.2)\qquad \sum_i x_{i,2}d_{i,1}\ln\frac{q_{i,2}}{q_{i,1}}=$$

$$\sum_i x_{i,2}d_{i,1}\ln\frac{e_{i,2}L_2^{\alpha}}{e_{i,1}L_1^{\alpha}}+\left(\sum_i x_{i,2}d_{i,1}\right)\,\ln\frac{Z_1^{\alpha}}{Z_2^{\alpha}}$$

However for symmetric demands and an optimizing x the part in parentheses is from (3.5) exactly $(1-Z_1)$ so (8.2) becomes

$$Knap(Z_1)+\alpha\ln\frac{Z_1}{1-Z_1}$$

where $\mathrm{Knap}(Z_1)$ denotes the solution to the knapsack problem with length $(1-Z_1)$.

$$\textit{objective function coefficients} \quad d_{i,1}\ln\frac{e_{i,2}L^{\alpha_2}}{e_{i,1}L^{\alpha_1}}$$

$$\textit{and length} \quad d_{i,1}Z_1+d_{i,2}Z_2=d_{i,1}$$

So the utility simplifies to

$$u(x,Z)=K+(\alpha-1)\ln\frac{1}{Z}+\ \alpha(1-Z)\ln\frac{Z}{(1-Z)}+Knap(Z)$$

Since in Knap(Z) the coefficients are independent of Z we can solve this once for all Z, either in the dynamic programming case to get $B_I(Z)$, or in the continuous knapsack case to get B(Z). It is also true that the integer solutions that are obtained for various Z in the course of the calculations all lie directly on the respective boundary curves.

Since we are solving a knapsack problem of length $(1-Z_1)$ we can express the result as a density $d(Z_1)$ times the length. $d(Z_1)$ will increase monotonically and be bounded above and below by constants representing the greatest and least possible densities. Upon rearranging terms we get for the utility,

$$(8.4) \qquad U(Z)=KZ_1\ \{(\frac{1}{Z_1})^{Z_1}(\frac{1}{(1-Z_1)})^{(1-Z_1)}\}^{\alpha}\exp(1-Z_1)d(Z_1)$$

If $d(Z_1)$ varies slowly as it often should, this is close to a simple formula giving the boundary shape.

To take it one step further we note that competition among identical countries does make sense in this model, and of course can have many different outcomes. In the identical country case with production functions $e_{i,j}l^{\alpha}$, the ratios appearing in the objective function term will be all be 1's, so their logarithms will be 0, and d(Z) will be 0. The resulting boundary curve

$$U(Z)=KZ\ \{(\frac{1}{Z})^{Z}(\frac{1}{(1-Z)})^{(1-Z)}\}^{\alpha}$$

which is given by an explicit formula, is plotted in Figure 8.1 for $\alpha=1.5$. It exhibits the characteristic boundary shape, one from which (8.4) can also deviate only slightly.

In this very special situation the lower boundary curve calculation is the same as the upper, they both have objective function 0. It follows that the upper and lower boundary curves coincide, so the curve of Figure 8.1 must have all the integer equilibria directly on it.

Case 3. Exclusionary Model.

Next we take the production function of case 2 with $\alpha=1$, but modify it for small values of the labor l to guarantee stability at the origin. This gives a case that may be called exclusionary competition. There are no economies of scale but whichever producer is actually in business can to some extent exclude the non producer. The degree of exclusion (or effort required to enter) is determined by the details of the low l production functions, but the boundary curve is the same in all cases and is therefore characteristic of the exclusionary competition situation.

Section 9. Other Aspects of the Model

One significant aspect of this model is the array of solutions it presents, and the fact that, in the model, pure market forces will allow many equilibrium points covering a wide area. In addition there are other properties of the model that are worth mentioning.

9a. Changing the Production Functions.

Economies of scale can be thought of as having two distinguishable effects.

The first in what might be referred to as a "barrier to entry" effect, it acts to give a producer an advantage as compared to a nonproducer. In this model this shows itself in the low end of the production curves as little output for the labor input. It is this aspect of the curves that forces a high level of activity before the non-producer can hope to compete with the producer, and also, through the condition A3 of Section 2 eliminates the possibility of incremental entries.

Although this theory uses the term production function to describe the translation of input labor into output, that term should not be taken to literally refer to a manufacturing plant. True barriers to entry come from many sources aside from the obvious possibility of economies of scale in manufacturing. Examples are knowledge and expertise in the manufacturing process, the largely experience born ability to design a manufacturable product, knowledge of and experience with marketing channels, knowledge of customer needs, and even knowledge of and being known to particular customers. Much knowledge can only be obtained by doing, and there will be period of doing poorly through inexperience for any new entrant. In addition, especially in the case of industries in different countries,there is the question of infrastructure. If one industry is flourishing in Country 1, and non-existent in Country 2, a large part of the difficulty in entry will be to find the people or companies who can build plants of the proper type, and supply parts, specialized instruments, and specialized support services. While some of this can be imported, some cannot, and working at a distance is often not the same as working close by.

All of these factors and many more often make entry into a new industry a large commitment now to a return that is both distant and inherently uncertain, And that uncertainty too is part of the barrier to entry. All these factors should be thought of as contributing to the shape of the low end of the production function.

The second aspect of economies of scale is the advantage that large scale may give one producer over another when *both* are active in the industry. In this model these aspects are reflected in the production functions for larger labor inputs.

Keeping these points in mind we will state and then interpret the following theorem which at this point is quite straightforward.

Theorem 9.1: Let M_1 and M_2 be two n-industry models with identical demands $d_{i,j}^1+d_{i,j}^2$ and with production functions $f_{i,j}^1(l_{i,j})$ and $f_{i,j}^2(l_{i,j})$ that are identical above the autarchy level, i.e. for $l_{i,j}>d_{i,j}Y_j$. Then the integer equilibria and the boundary curves are the same for both models.

Proof, for any integer x the resulting Z(x) will be the same in both models, since the equation (3.4) only involves the demands and not the production functions. Thus the integer equilibria are the same pairs (x,Z(x)) in both models, and these x in turn determine the labor levels $l_{i,j}$. These labor levels are above the autarchy level, so for perfectly specialized equilibria, the output is also the same in both models.

An immediate consequence is that the coefficients in the linearized utility function are the same for both models, and therefore so are all the boundary curves. This completes the proof.

Although the points and the boundaries are the same what *does* change as the production functions change from model 1 to model 2 in this limited way is the barrier to entry. If the change from model 1 is to new production functions that rise sharply near 0, we can make the barrier to entry as feeble as we wish. On the other hand if the new production function were zero till near the autarky level and then and then jumps rapidly back to the production curves of model 1, we would have an extremely strong barrier to entry.

. Return to Autarky.

The various figures always show the utility of each country decreasing after a certain point and in fact returning toward the autarky value. This effect we can call "return to autarky", and an economically plausible explanation for it will be given here and a proof in Appendix A9-1

For Z_1 near 1, Country 1 will be the producer of most goods in any purely specialized solution. The fraction of world production of these goods that Country 1 gets is $d_{i,1}Z_1/(d_{1,i}Z_1+d_{i,2}Z_2)$. For Z_1 near 1 and Z_2 near 0 that fraction is near 1. Therefore Country 1 is producing most things and keeping almost all of the production of those things. So, with the exception of the few goods being made in Country 2 and consumed in Country 1, Country 1 has "returned to autarky". Country 2's existence has little impact on Country 1.

To back up this scenario requires a more careful examination of the effect on Country 1's utility of the few goods that are made in Country 2. After all they are being made in large quantity as they consume all the labor of Country 2. Also we need a more rigorous statement of the connection between being the producer of most goods and Z_1->1. Both these elements are provided in Appendix A9-1.

Effect of Country Size

Let us substitute for Country 2 a different Country 2 with the same demand function, and the same production functions, but with a larger labor force. We will see that all specialized solutions for Country 1, with the exception of autarky, will improve in utility. In other words autarky becomes relatively less attractive for Country 1 as Country 2 grows. This is reflected In Fig. 1.2, The larger country, Country 2, does less well relative to autarky than does its smaller trading partner.

This effect is quite direct. If L_2 increases, equations (3.4),(3.2), and (3.3) are unchanged. Any specialized equilibrium point x will yield the same Z from (3.4) and the same $l^*_{i,j}$ from (3.2) and (3.3) as before. However the $l^*_{i,2}$, which are normalized labor variables, will be multiplied by a larger L_2 to get the actual labor $l_{i,2}$. This means larger quantities q_2 in every term of the utility representing a good made by Country 2, while the fraction $F_{i,2}$ of that good going to Country 1 remains the same since the F's depend only on $d_{i,j}$ and Z. Only autarky has no such term, so for every x except autarky Country 1's utility improves as L_2 gets bigger.

In words, if Country 1 can hold onto the industries it has as Country 2 gets bigger, it gets the same fraction of a bigger total output and it is better off.

Effect of Demands

Fig 3.3 illustrates the fact that the equilibria that are best for one country, for each assignment of national incomes, are not necessarily those that are best for the other. On the other hand, in section 8 case 1, we saw that for *symmetrical demands* an equilibrium near the upper boundary for one coumtry is near the upper boundary for the other. If we consider the knapsack calculations for the two countries we can see that similar demands mean similar objective functions when the utilities of the two countries are being maximized for any given national income Z. The maximizing solution for Country 1 will then usually not be too far from the maximizing solution for Country 3, while they will tend to be unrelated if the demands are very different. Similar demands seem to mean more equilibria that are good for both countries, dissimilar ones seem to make these outcomes less likely.

Effect of an industry changing hands.

Within this model we can look at specialized solutions that differ in only one industry, i.e. Country 1 makes product j in one solution, Country 2 makes that product in a second solution and all other products are made as before. In the notation of section 7, we can compare the national incomes and utilities of x(0) and x(1). The effect on national income is always to increase it in Country 2 and decrease it in Country 1. Increases and decreases with the gain of a single industry were illustrated in figure (7.1). However utility can either increase or decrease depending on the location of the initial equilibrium in the array of solutions. While the gain of an industry usually improves utility until one is beyond the characteristic hump of the boundary curve, there seems to be no simple general statement that can be made.

Section 10. Extensions

While almost all aspects of this model are unexplored at present there are two extensions that seem both worthwhile and straightforward.

1) Replace the Cobb-Douglas utility with a utility of the same type but with the demand constants $d_{i,j}$ replaced by $d_{i,j}(Z)$ which are functions of Z. This would allow the demand of a country to shift as its national income changes. Since the calculations are all done for a fixed Z they will go forward virtually unchanged. The proof of some of the theorems in sections 4 and 6 will become slightly more complicated.

2) The mixed economies-diseconomies case. If we divide the n goods into goods 1....r which are produced with economies of scale, and goods r+1....n which are produced with diseconomies, both the calculation for the boundaries and the calculation for all the equilibrium points go forward easily. Essentially there are two separate problems, one a pure economies problem and one a pure diseconomies problem, that interact by sharing the labor supplies in each country.

To compute the boundary curves, for example, we would start just as in the pure economies case by choosing a Z. The condition of equal marginal productivity fixes the production levels and labor employed in each of the diseconomy industries in each country. This means that the $x_{i,j}$, $i>r$, are determined, and what is left in (3.9) can then be calculated as a pure economies of scale problem and boundary points B(Z) and BL(Z) can be found as above. Intuitively what this means is that once the wage levels are set, the demand and production levels of the diseconomies industries are determined. This then leaves a smaller labor pool in each country to be employed in the remaining r industries, but what remains is the pure economies case with all its different production possibilities leading to different equilibria.

Much of the interest in the mixed case lies in its various interpretations and they will be discussed in the paper mentioned in the introduction.

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Appendix A2-1 Existence Proof

We will use the assumptions on autarky and on the production functions as given in Section 2.

Let S_j be the set of products which country j produces at a positive labor level when in autarky. Let us assume there are n products in the model, and the number of elements in S_1 is m. Let us assume also that S_1 and S_2 between them contain every good. Then

Theorem A1: In trade there is an equilibrium point in which Country 1 is the sole producer of any proper non-empty subset of products of S_1, with the others being solely produced by Country 2.

Proof: For any (w1,w2) we can determine $l_{i,1}$ from

(A1.1) $w_1L_1d_{i,1} + w_2L_2d_{i,2} = w_1l_{i,1}$

and thus have demand = total wages. Here i belongs to the set S_1 of products made exclusively by Country 1. Similarly we determine $l_{i,2}$ from

(A1.2) $w_1L_1d_{i,1} + w_2L_2d_{i,2} = w_2l_{i,2}$

Where i belongs to the set S_2 of products made exclusively by Country 2.

To satisfy the equilibrium conditions we also need a price p_i. If there is positive output of product i, we can get a price by dividing total demand (or total wages) by the number of units produced, provided this is not zero. This will yield a price with the right properties, as long as we can be sure of a positive output at labor level $l_{i,1}$

However $l_{i,1}$ cannot be less than the corresponding labor level in autarky as the expression for $l_{i,1}$ is $L_1d_{i,1} + (w_2/w_1)L_2d_{i,2} = l_{i,1}$ and in autarky it is $l^a_{i,1}=L_1d_{i,1}$. Since there was a positive output in autarky (one of the Theorem A1 assumptions) there is some output at this point as well. So there is a price for any choice of (w1,w2).

However for arbitrary w_1 and w_2 the resulting $l_{i,1}$ from (A1.1) and (A1.2) will not generally add up to L_1, so we will have to choose w properly.

If we apply (A1.1) to any good of which Country 1 is the sole producer we see that if w_2 is sufficiently large relative to w_1, $l_{i,1}$ alone will exceed L_1. Also for very small w_2/w_1 ,$l_{i,2}$ $i \epsilon S_2$,will exceed L_2.

Next, summing over all i gives

(A1.3) $$w_i d_{i,1} L_1 + w_2 d_{i,2} L_2 = w_1 L^D_1 + w_2 L^D_2$$

Where the L^D_i are the total amounts of labor demanded in each country. It follows immediately from (A1.3) that $L^D_1 > L_1$ implies $L^D_2 < L_2$, $L^D_2 > L_2$ implies $L^D_1 < L_1$, and $L^D_1 = L_1$ implies $L^D_2 = L_2$,

Since for w_1 sufficiently small $L^D_1 > L_1$, and for w_2 sufficiently small $L^D_1 < L_1$ there is some intermediate value of w_2/w_1 with $L^D_1 = L_1$. The same equality also holds in Country 2, so we have an equilibrium point. This proves the theorem.

We next extend the theorem to include the cases where there are goods produced by both countries. After an initial lemma required to show a unique labor level and price when both countries are producers, the proof will proceed in the same way as above.

Let us consider a good i for which, in the specialization being considered, both countries are producers. Let $x_{i,1}$ be the fraction of the total demand that goes as wages to labor in Country 1, while $x_{i,2}$ is the fraction going to Country 2. Clearly $x_{i,1} + x_{i,2} = 1$. Also, once $w = (w_1, w_2)$ is given, the $x_{i,j}$ uniquely determine the labor levels $l_{i,j}$ in each country.

Lemma A1: For any choice of $w = (w_1, w_2)$ there is a unique $x_i(w) = (x_{i,1}, x_{i,2})$ and p_i such that $p_i f_{i,1} = w_1 l_{i,1}$ and $p_i f_{i,2} = w_1 l_{i,2}$. Furthermore $x_i(w)$, and hence the labor levels, depend continuously on w.

For the proof we need a preliminary remark. The autarky labor levels in both countries are provided by $x^a_{i,1} = d_{i,1} w_1 L_1 / (d_{i,1} w_1 L_1 + d_{i,2} w_2 L_2)$ and $x^a_{i,2} = d_{i,2} w_2 L_2 / (d_{i,1} w_1 L_1 + d_{i,2} w_2 L_2)$. Since there is, by assumption, positive output in both countries at these labor levels, it follows that if $x_{i,1} < x^a_{i,1}$ there is positive output in Country 2 and if $x_{i,2} < x^a_{i,2}$ there is positive output in Country 1.

For any x yielding positive production levels in both countries we can obtain different candidate prices for the two producers by defining p_1 by $p_1 f_{i,1} = w_1 l_{i,1} = x_{i,1}(d_{i,1}w_1L_1 + d_{i,2}w_2L_2)$ and p_2 by $p_2 f_{i,2} = w_1 l_{i,2} = x_{i,2}(d_{i,1}w_1L_1 + d_{i,2}w_2L_2)$. These prices give zero profit to both producers. However they are not generally equal and we must show that there is an x for which they are equal.

Let $x^s_{i,1}$ be the fraction of total demand that just covers the set up cost of Country 1, and let $x^s_{i,2}$ be the fraction of total demand that just covers the set up cost of Country 2. Let $x_{i,1}$ approach $x^s_{i,1}$ from above. Certainly then p_1 is well defined and in fact as $x_{i,1}$ approaches $x^s_{i,1}$ p_i becomes arbitrarily large. However for these x values p_2 is also well defined because $x_{i,1}$, being near $x^s_{i,1}$ must be below the country 1 autarky level, and therefore the corresponding $x_{i,2}$ provides positive output. Consequently we have well defined p_1 and p_2 with $p_1 > p_2$. If we increase $x_{i,1}$ the labor and output from Country 1 increase continuously and monotonically while labor and output from Country 2 decrease continuously and monotonically. Finally for $x_{i,1}$ such that $x_{i,2} = x_{i,1}-1$ approaches $x^s_{i,2}$ from below we have p_2 becoming arbitrarily large. Hence for some unique inbetween x we have $p_1 = p_2$.

This x is the x of the lemma. Its continuous dependence on w follows directly from the continuity of the production functions, the continuity in the dependence of total demand on w, and the *monotone* behavior of the p's as functions of x.

With lemma A1 proved we can repeat the reasoning of the first theorem. A1.1 and A1.2 still hold for the good or goods that, by theorem hypothesis, is produced by each country alone. A1.3 holds because we can get it by summing the relations

(A1.1') $x_{i,1}\,(w_1L_1d_{i,1} + w_2L_2d_{i,2}) = w_1 l_{i,1}$

(A1.2') $x_{i,2}\,(w_1L_1d_{i,1} + w_2L_2d_{i,2}) = w_2 l_{i,2}$

over all i. Since the demand for labor in each country is continuous, and since for w_1 sufficiently small $L^D_1 > L_1$, and for w_2 sufficiently small $L^D_1 < L_1$ there is once again an intermediate value of w_2/w_1 with equality of labor demanded and total labor supply both countries. Since we have established the existence of prices already for each w, this proves the extended theorem.

Appendix A3-1: Ricardo Point

If we assume that the more efficient producer of each good *is* its producer we have a monotone decrease in the demand for Country 1's labor as Y_1 increases. This decrease in continuous except at the points where $q_1(1,Z)=q_2(1,Z)$. At these points S_1 loses a member and therefore $L^*_1(Z)$ takes a downward jump. At such a point $L^*_1(Z)$ can perfectly well pass from > 1 to < 1. If this happens, the Ricardo Level, Z_R is at that point.

Therefore there are two possibilities.

1) $L^*_1(Z_R)=1$. In this case $x_{i,1}=1$ for $i\epsilon S_i$, $x_{i,1}=0$ (and therefore $x_{i,2}$) otherwise, solves (3.4) and provides an equilibrium point where each good is being produced by the more efficient producer. We call this the Ricardo Point.

2) $L^*_1(Z_R)<1$. In this case there can be no equilibrium solution with the properties of the Ricardo Point. If there were a Ricardo Point x, Then Z(x) would be a point with $L^*_1(Z)=1$. But there is no Z value for which that equality holds.

Appendix A3-2 Dynamic Programming

The dynamic programming recursion for an n-piece knapsack problem with total length L, piece lengths l_i and piece values v_i is

$$\phi_0(s)=0$$

$$\phi_m(s)=\mathrm{Max}(\phi_{m-1}(s),\phi_{m-1}(s-l_i) + v_m) \qquad 0\leq s\leq L.$$

If we apply this to (3.9) the l_i for a given Y, would be $(d_{i,1}Y_1 + d_{i,2}Y_2)$, total length would be $Y_2=1-Y_1$ and the v_i would be $d_{i,1}\ln(q_{i,2}(1,Y/q_{i,1}(1,Y)$, just as in (3.11).

The condition on $\phi_0(s)=0$ simply sets the starting values at 0 for all s, and each successive ϕ_m then gives the best value that can be obtained for length s using only the first m pieces. The ϕ_m are related to the ϕ_{m-1} through the recursion which asserts that the best that can be done at length s with m pieces is either done *not* using the mth piece, this is the first term after Max, or it is done using the mth piece, the second term after Max. $\phi_n(s)$ then gives the maximizing value with n pieces.

To determine the actual x that gives that value requires recording, when calculating $\phi_m(s)$, whether or not the mth piece was used at that s. Then it is possible to backtrack from $\phi_m(L)$ and find out how the value was obtained which gave $\phi_n(L)$. $x_{i,2} = 0$ if the ith piece was not used, $x_{i,2} =1$ if the ith piece was used.

In actual calculation L is usually divided up into a uniform grid of P points s_i, $s_1=0$ and $s_P=L$. The lengths l_i must then be rounded up or down since for use in the recursion they must fit the grid. Rounding down can introduce some inadmissable combinations but it will give a $B_1(Y)$ that is either correct or too high and therefore is a valid boundary at each calculated point.

For each calculated value of Z there is an x, obtained by backtracking, and therefore a Z(x) that can easily be calculated from (3.4). These (x,Z(x)) generally give a set of points near the boundary.

Refined Calculation - Appendix A3-3

While the straightforward knapsack calculation for some finite grid of Z's gives a very rapid and simple calculation, the calculations can be very further refined. In visualizing these refinements it is useful to keep in mind a plot of value density versus Z_2 for each of the goods.

The value density is, from 3.11,

$$V_i(Z)=\frac{d_{i,1}\ln\frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)}}{d_{i,1}Z_1+d_{i,1}Z_2}$$

We can plot the various curves V_i against Z_2. If two curves intersect each other then their density order changes, otherwise it does not.

Let us imagine that for some Z_2 we have the solution and the non-integer variable is the kth one. Then Z_2 can be increased without changing the form of the solution until either $x_{k,2}$ becomes 1 (or 0) or until one of the V_i equals the ith one. For all Z_2 in this range the value of the knapsack problem is obtained with virtually no effort. Until one of these events occurs the $x_{i,2}$ that are 1 remain 1, those that are 0 remain 0, and only $x_{k,2}$ changes to maintain the equality in (3.9a).

If the event that occurs first is that $x_{k,2}$ becomes 1 (or 0), then we actually have an integer solution, and therefore an equilibrium point, lying on the bounding curve. At this point a new $x_{k,2}$ (the densest of the 0 valued variables) is introduced at a level of zero and the calculation continues with further increases in Z_2. If the first event that occurs is that another density curve crosses the current ith one then there is always one simple choice to be made and after that the calculation continues as before. We will give one illustrative example. Suppose the jth density curve crosses the kth curve in an upward direction. If we set $x_{k,2}=0$ and all the other $x_{i,2}$ as before, we will get a value for $x_{j,2}$ that would enable it to satisfy (3.9a). If $x_{j,2}$ is <1 it is the new non-integer variable and the calculation continues. If it is >1, then the calculation continues with $x_{j,2}=1$ and $x_{i,2}$ still the non-integer variable. In this manner all Z values can be exhausted using only a *finite* series of intervals within which the calculation is essentially unchanged.

Appendix A7-1.Proof of Theorems 7.3 and 7.4

Proof of Theorem 7.3

In the notation of section 7 equilibria are the points where

(A7.1.1) $q_1(x_{1,1}),Z(x))/q_2(x_{1,2},Z(x))= x_{1,1}/x_{1,2}= x_{1,1}/1-x_{1,1}$

and the x referred to in Z(x) is $x(x_{1,1})$.

From the definition of $x_{1,1}$ we always have $l_{i,j}(x_{1,1},Z)=x_{1,1}l_{i,j}(1,Z)$. So for production functions of the form $e_{i,j}l^{\alpha i}$ we have $q_{i,1}(x_{i,1},Z)=(x_{i,1})^{\alpha i}\, q_{i,1}(1,Z)$. So (A7-1.1) becomes

$$\left(\frac{x_{1,1}}{1-x_{1,1}}\right)^{\alpha_1}\frac{q_{1,1}(1,Z(x))}{q_{1,2}(1,Z(x))} = \frac{x_{1,1}}{1-x_{1,1}}$$

or equivalently

$$(A7-1.2)\ \left\{\frac{q_{1,2}(1,Z(x))}{q_{1,1}(1,Z(x)}\right\}^{1/(\alpha-1)} = \frac{x_{1,1}}{1-x_{1,1}}$$

We will refer to the left hand side in (A7-1.2) as $L(x_{1,1})$ and the right hand side as $R(x_{1,1})$ and we will plot L and R versus $x_{1,1}$, in Fig. A7-1.1. We are essentially plotting p_1 and p_2 since if we look back to (A7-1.1) we can see that $L>R$ is equivalent to $p_1<p_2$ and $L<R$ is equivalent to $p_1>p_2$.

The condition of Theorem 7.3 is that $L(x_{1,1})$ should either be below 1 throughout the interval $0\leq x_{1,1}\leq 1$ or always above it. In Fig. A7-1.1 we take the first case.

In Fig. A7-1.1 the right hand side starts at 0 with slope 1 and moves up toward infinity. Also since both $R(x_{1,1})$ and its slope are monotone increasing the intersection of the tangent line to this curve with the vertical line $x_{1,1}=1$, which is 1 for $x_{1,1}=0$, always is above 1 for $x_{1,1}>0$.

Next we need some similar statements about $L(x_{1,1})$. Now

$$\{\frac{q_{1,2}(1,Z(x))}{q_{1,1}(1,Z(x))}\}^{1/\alpha-1}=(\frac{e_{1,2}L_2}{e_{1,1}L_1})^{1/\alpha-1}(\frac{Z_1(x)}{1-Z_1(x)})^{\alpha/\alpha-1}$$

Clearly $L(x_{1,1})$ is monotone increasing because $Z_1(x)$ is. To see that the derivative of $L(x_{1,1})$ is monotone increasing as well we will explicitly solve (3.4) or (3.5) for $Z(x)/(1-Z(x))$ and establish that its derivative is monotone. If we use

$$D_{1,1}=\sum_{i>1}x_{i,1}d_{i,1} \textit{ and } D_{1,2}=\sum_{i>1}x_{i,1}d_{1,2}$$

we obtain

$$\frac{Z_1(x)}{1-Z_1(x)}=\frac{D_{1,2}+d_{1,2}x_{1,1}}{(1-D_{1,1})-d_{1,1}x_{1,1}}$$

the derivative of this is

$$\frac{D_{1,2}d_{1.1}+d_{1,2}(1-D_{1,1})}{((1-D_{1,1})-d_{1,1}x_{1,1})^2}$$

and this last is clearly positive and monotone increasing. Since $L(x_{1,1})$ is a constant times $Z(x)/(1-Z(x))$ raised to a power of one or more, its derivative has the same property.

With this preparation we can assert that only a single intersection of L and R is possible. Suppose otherwise. Then at the second intersection the derivative of $L(x_{1,1})$ must equal or exceed that of $R(x_{1,1})$. Since this derivative is monotone increasing $L(x_{1,1})$ must thereafter lie above the line tangent to $R(x_{1,1})$ at that second intersection point. Therefore its intersection with the vertical line $x_{1,1}$ is above that of the tangent line and therefore >1 contradicting the assumption $L(x_{1,1})\leq 1$. This ends the proof of Theorem 7.3

For the proof of the Theorem 7.4 assume $q_{1,2}(1,Z(0) < q_{1,1}(1,Z(01)$ but $q_{1,2}(1,Z(1) \geq q_{1,1}(1,Z(1)$. Clearly as α->1, 1/α-1, which is the exponent om the ratio of the q's becomes very large. Since there is a value of $x_{1,1}$ that makes the ratio of the q's be 1, there is a value for which $L(x_{1,1})=1$ and that value, $x_{1,1}=c$, is the same for all $\alpha>1$. In Fig. (A7-1.2), which is based on Table 7.2, α is 1.1 and we start to see the effect of letting α->1. As 1/1-α becomes very large, L will be as close to 0 as desired until near $x_{1,1}=c$. As it approaches $x_{1,1}=c$ it rapidly rises to 1 and then after that to very large values. With c fixed and α sufficiently near 1 we can be sure of a first intersection between L and R with a height near 0, and another with its $x_{1,1}$ coordinate near c. (It can be a bit before or a bit after c, in Fig. A7-1.2 it is after c). This provides two intersections for α sufficiently near 1 and establishes the theorem.

Of course, since there are an odd total number of intersections there will always be at least one more.

Appendix A9-1 Return to Autarky

We will prove that both the upper and lower boundary curves approaches the autarky level as Z_1->1.

Equation (3.5) is satisfied by the non-integer x that optimizes (3.9) for any given Z. As Z_1->1 the first term $\Sigma_i d_{i,1}x_{i,2}$, which is the coefficient of Z_1 must approach 0. The optimizing solution x consists of $x_{i,1}$ that are 0 or 1 except for one term the jth. For $\Sigma_i d_{i,1}x_{i,2}$ to approach 1 all terms except possibly $x_{j,2}$ will have to be 0 (all goods except one are made in Country 1) and $x_{j,2}$ will be given by

$$x_{j,2}=\frac{Z_2}{d_{j,1}Z_1+d_{j,2}Z_2}$$

which approaches 0 as Z_1 -> 1.

The terms in the utility involving the jth variable are

$$(A9-1.1) \qquad x_{j,1}d_{j,1}\ln F_{j,1}q_{j,1}(1,Z) \quad and \quad x_{j,2}d_{j,1}\ln F_{j,2}q_{j,2}(1,Z)$$

$$where$$

$$q_{j,1}=f_{j,1}(\frac{L_1(d_{j,1}Z_1+d_{i,2}Z_2)}{Z_1}) \quad and \quad q_{j,2}=f_{j,2}(\frac{L_2(d_{j,2}Z_1+d_{i,2}Z_2)}{Z_2}).$$

Clearly the $q_{j,1}$ term approaches the autarky quantity as Z_1->1, and the F terms approach 1. It remains to show that the second term in (A9-1.1) approaches 0.

This actually requires *some* assumption on the production functions because if the production of the jth good grows in some explosive fashion with additional labor the quantity of goods produced overwhelm their decreasing marginal utility and singlehandedly boost Country 1's utility to a very large level. However the rate of growth required to do this is quite extreme. In fact it is enough to assume that $f(l)/e^l$->0 as l grows very large. This reasonable assumption is will make the second term approach 0. This then gives the autarky value to the utility as Z_1->1.

The reasoning about the lower boundary is virtually identical.

C-1. A Note on Computations

The various computations referred to in the text were all run on the author's home computer, an IBM PS/2 Model 80. Typical run times are:

(1) for a boundary curve of a 17 industry model with a 90 point grid 2.5 minutes, 27 industries, 3 minutes.

(2) for the integer boundary and the integer points using a 90 point grid, 20 minutes, for 27 industries 30 minutes.

(3) for obtaining all the approximately 8000 perfectly specialized equilibria points in the 13 industry model in Fig. 6.1, 8 minutes.

(4) for the computation with roughly 19000 intermediate equilibria in Fig. 1.1 about 5 hours.

Computations (1) and (2) grow slowly with model size, and are linear in the number of grid points. (3) and (4) have of course exponential growth with model size.

All programs except those relating to Section 7 were written in Basic by the author and are far from optimal. Mathematica was used to plot all the figures and to compute the treaty curves shown in the figures of section 7.

TABLE 1.1[1]

PRODUCTS	1	2	3	4	5	6	7	8	9
C1 Demands	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07
C2 Demands	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20
Production Exponents	1.30	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61
C1 Efficiencies	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30
C2 Efficiencies	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50

C1 - Labor Supply 4   C2 - Labor Supply 8   Production Function $e_{ij}\, l^{\alpha_j}$

[1] The demands in all Tables are renormalized to total 1 in actual computation.

TABLE 3.1

PRODUCTS	1	2	3	4	5	6	7	8
C1 Demands	0.05	0.20	0.12	0.15	0.22	0.08	0.08	0.10
C2 Demands	0.12	0.10	0.20	0.15	0.20	0.05	0.08	0.15
Production Exponents	1.00	1.50	1.70	1.90	2.00	2.00	2.10	2.00
C1 Efficiencies	1.00	1.00	0.70	0.90	1.20	1.30	1.10	0.77
C2 Efficiences	0.50	1.00	0.60	0.90	1.00	1.20	1.30	1.02

C1 - Labor Supply 2	C2 - Labor Supply 2	Production Function $e_{ij}\, l^{\alpha i}$

TABLE 3.2

PRODUCTS	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
C1 Demands	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07	0.05	0.20	0.12	0.15	0.22	0.08	0.08	0.10
C2 Demands	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20	0.12	0.10	0.20	0.15	0.20	0.05	0.08	0.15
Production Exponents	1.30	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61	1.10	1.50	1.70	1.90	2.00	2.00	2.10	2.00
C1 Efficiencies	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30	1.00	1.00	0.70	0.90	1.20	1.30	1.10	0.77
C2 Efficiencies	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50	0.50	1.00	0.60	0.90	1.00	1.20	1.30	1.02

C1 - Labor Supply 8   C2 - Labor Supply 10   Production Function $e_{ij} l^{a_i}$

TABLE 3.3

PRODUCTS	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
C1 Demands	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07	0.05	0.20	0.12	0.15	0.22	0.08	0.08	0.10	0.12	0.14	0.08	0.19	0.15	0.13	0.14	0.19	0.22	0.11
C2 Demands	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20	0.12	0.10	0.20	0.15	0.20	0.05	0.08	0.15	0.13	0.19	0.06	0.11	0.11	0.17	0.10	0.11	0.11	0.21
Production Exponents	1.30	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61	1.10	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.40	1.70	1.80	2.00	2.10	2.00	1.90	1.80	1.90	1.95
C1 Efficiencies	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30	1.00	1.00	0.70	0.90	1.20	1.30	1.10	0.77	0.77	0.50	0.50	1.00	0.60	0.90	1.00	1.20	1.30	1.02
C2 Efficiencies	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50	0.50	1.00	0.60	0.90	1.00	1.20	1.30	1.02	1.02	0.30	1.00	1.00	0.70	0.90	1.20	1.30	1.10	0.77

Cl - Labor Supply 12   C2 - Labor Supply 15   Production Function $e_{ij}\, l^{pi}$

TABLE 6.1

PRODUCTS	1	2	3	4	5	6	7	8	9	10	11	12	13
C1 Demands	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20	0.13	0.17	0.11	0.25
C2 Demands	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07	0.12	0.05	0.09	0.17
Production Exponents	1.00	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61	1.40	1.30	1.20	2.01
C1 Efficiences	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50	1.10	0.90	1.20	1.01
C2 Efficiences	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30	1.20	0.70	0.82	1.11

C1 - Labor Supply 9 C2 - Labor Supply 5 Production Function $e_{ij}\, l^{\alpha d}$

TABLE 7.1

PRODUCTS	1	2	3	4	5	6
C1 Demands	0.10	0.20	0.20	0.15	0.25	0.10
C2 Demands	0.15	0.15	0.25	0.15	0.25	0.05
Production Exponents	1.15	1.50	1.70	1.90	2.00	2.00
C1 Efficiencies	1.00	1.00	0.70	0.90	1.20	1.30
C2 Efficiencies	0.50	1.00	0.60	0.90	1.00	1.20

C1 - Labor Supply 2   C2 - Labor Supply 1

Production Function $e_{ij}\, l^{xi}\, f_i(l)$, $f_i(l)=g(x)$, $x=l/(.6\, d_{i,j}\, L_j)$, $g(x)=1$ $x>1$, $g(x)=x^4$ $x<=1$.

TABLE 7.2

PRODUCTS	1	2	3	4	5	6
C1 Demands	0.10	0.20	0.20	0.15	0.25	0.10
C2 Demands	0.15	0.15	0.20	0.20	0.25	0.05
Production Exponents	1.15	1.50	1.10	1.90	2.00	2.00
C1 Efficiencies	1.00	1.00	0.70	0.90	1.20	1.30
C2 Efficiencies	0.50	1.00	0.70	0.90	1.00	1.20

C1 - Labor Supply 2 C2 - Labor Supply 1

Production Function $e_{ij}\, l^{\,xi}\, f_i(l)$, $f_i(l)=g(x)$, $x=l/(.6\, d_{i,j}\, L_j)$, $g(x)=1$ $x>1$, $g(x)=x^4$ $x<=1$.

Fig. 1.1

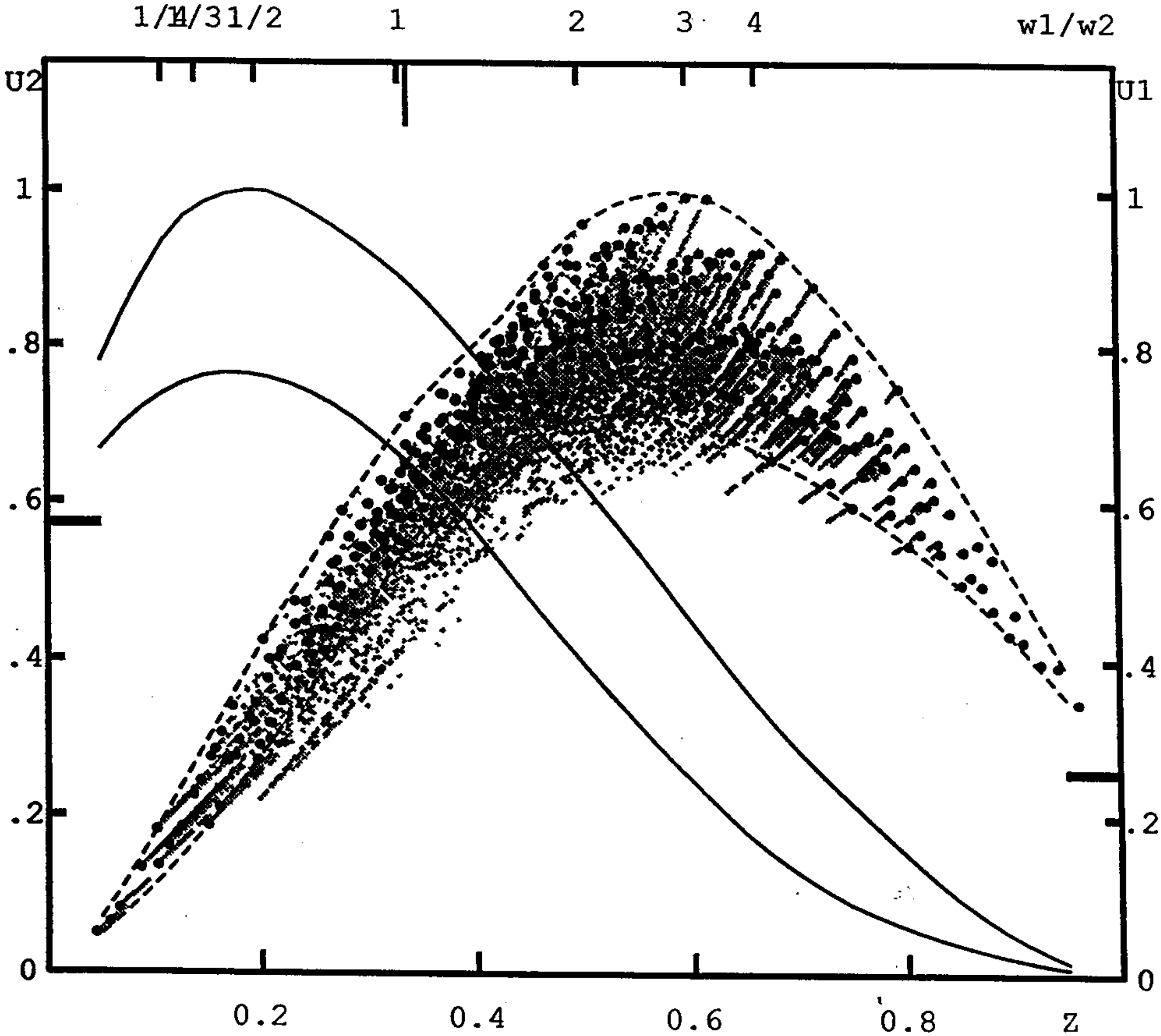

Fig. 1.2

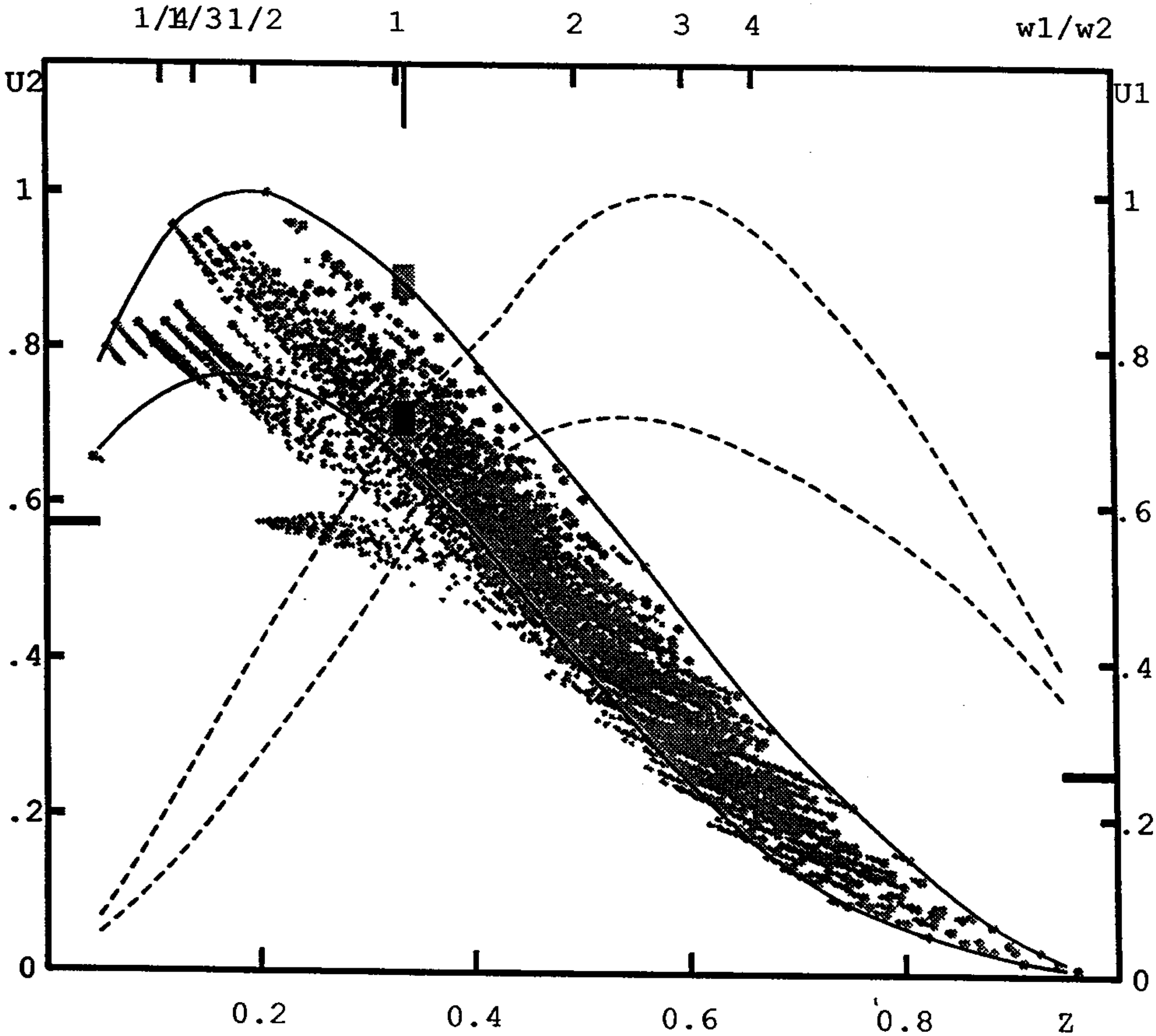

Fig. 3.1

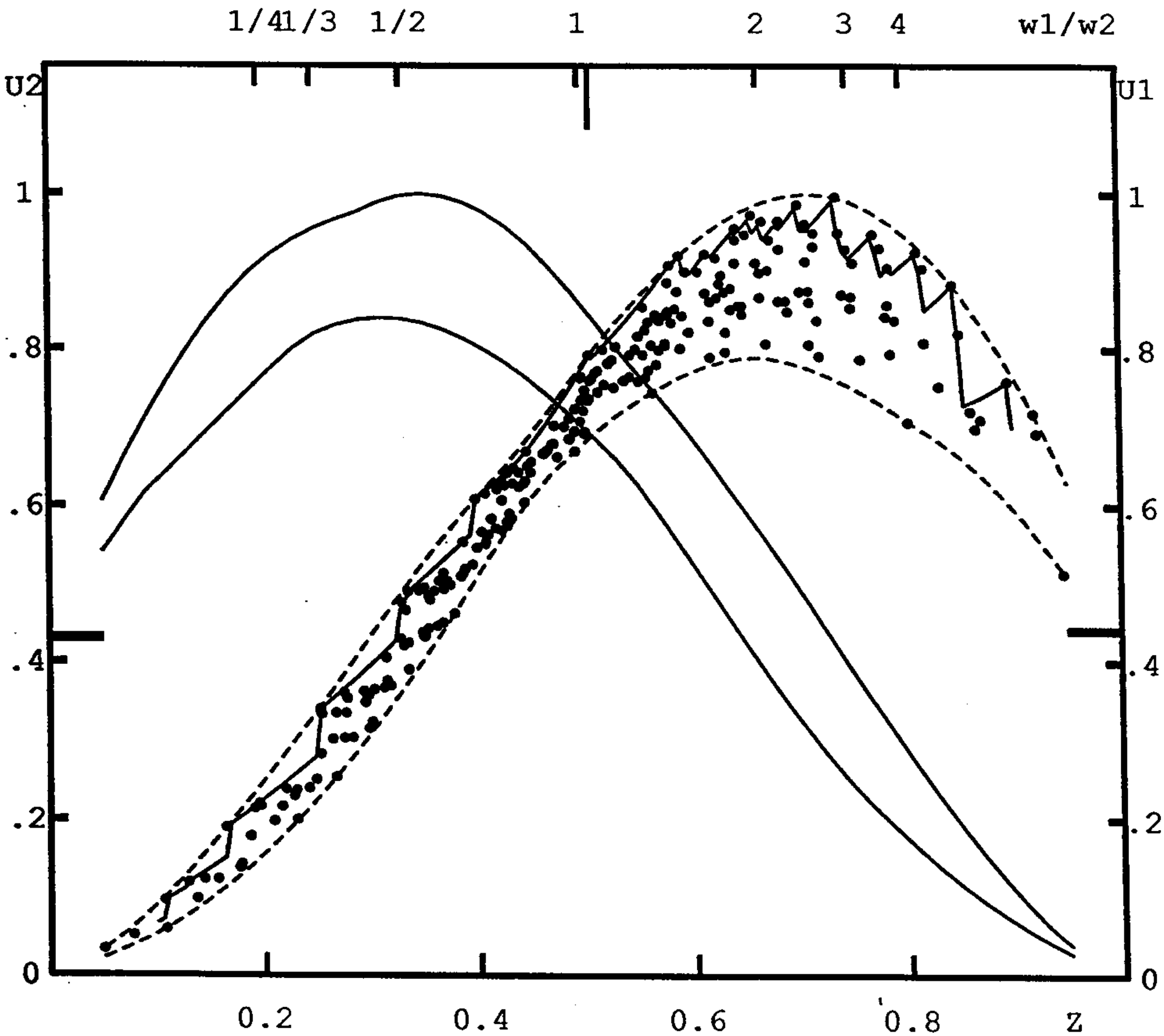

Fig. 3.2

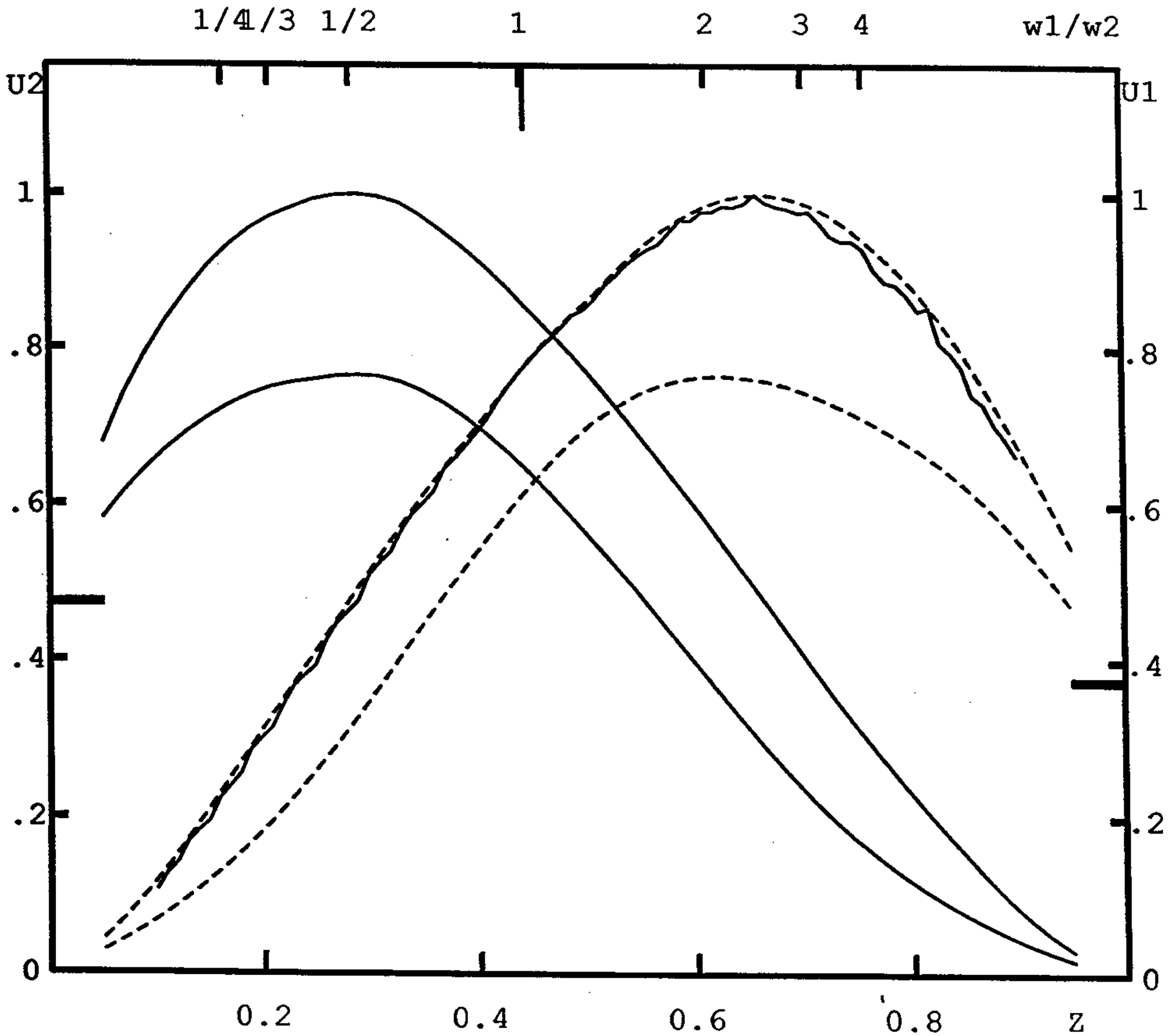

Fig.3.3

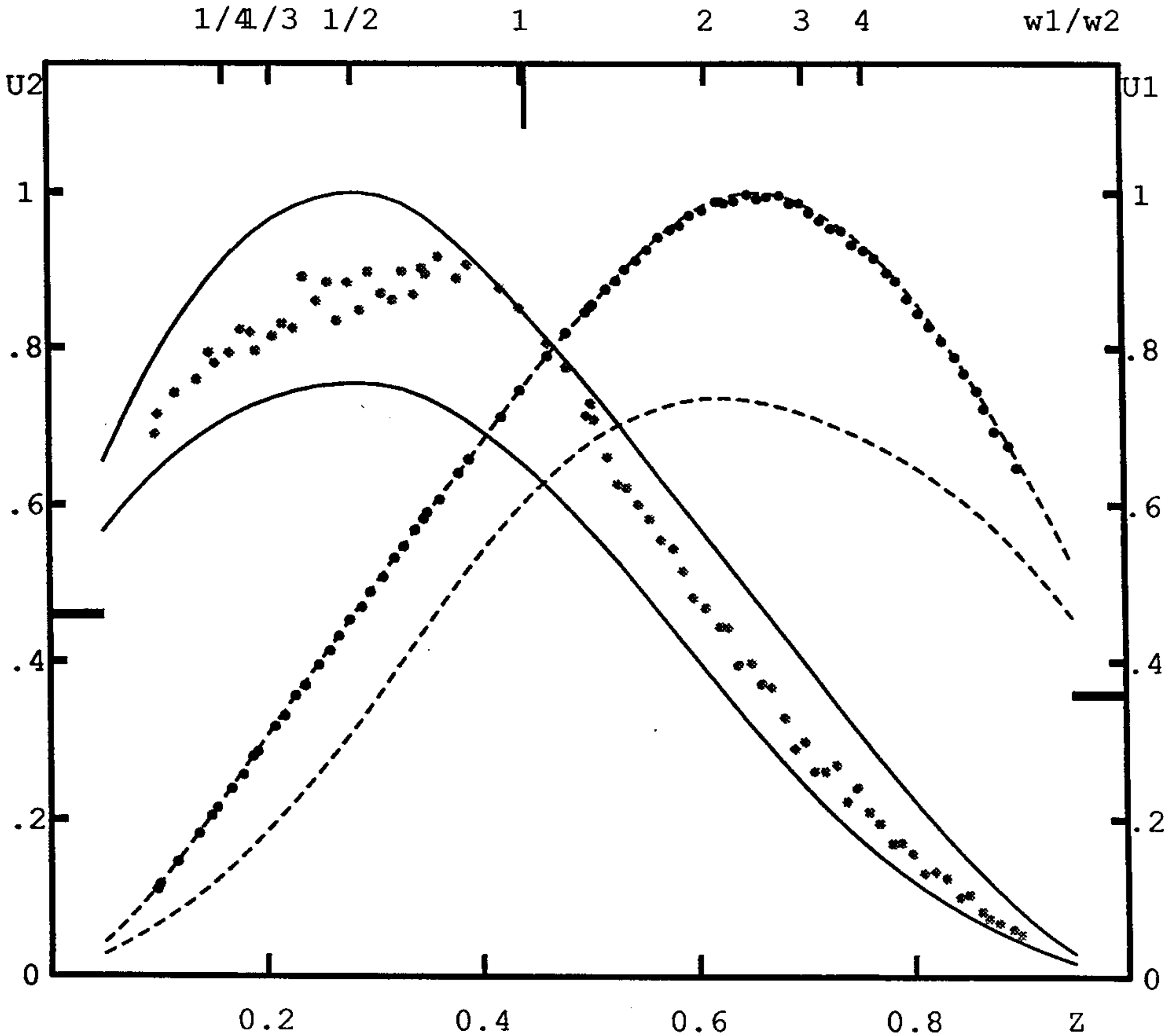

Fig. 6.1

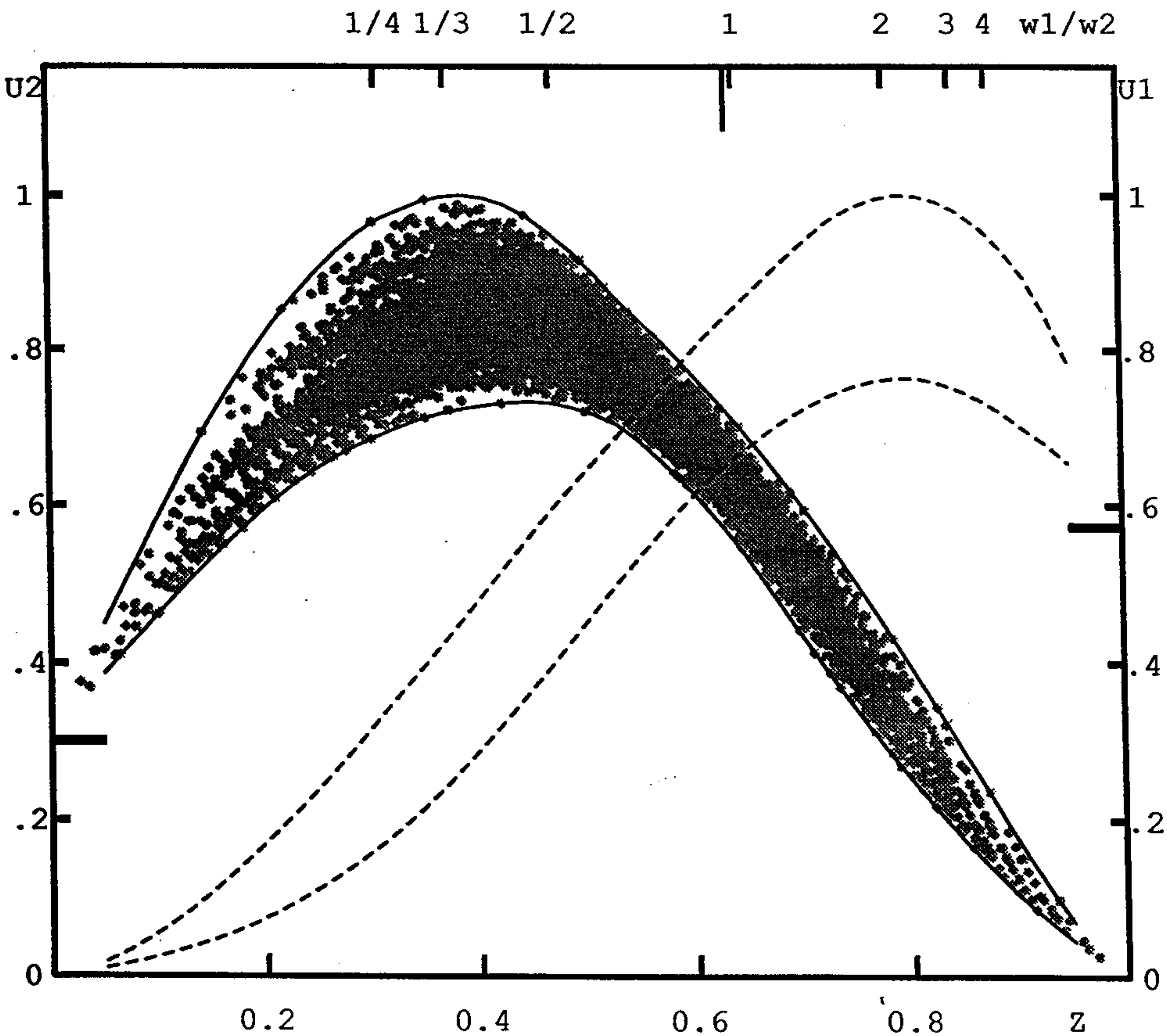

Fig.7.1

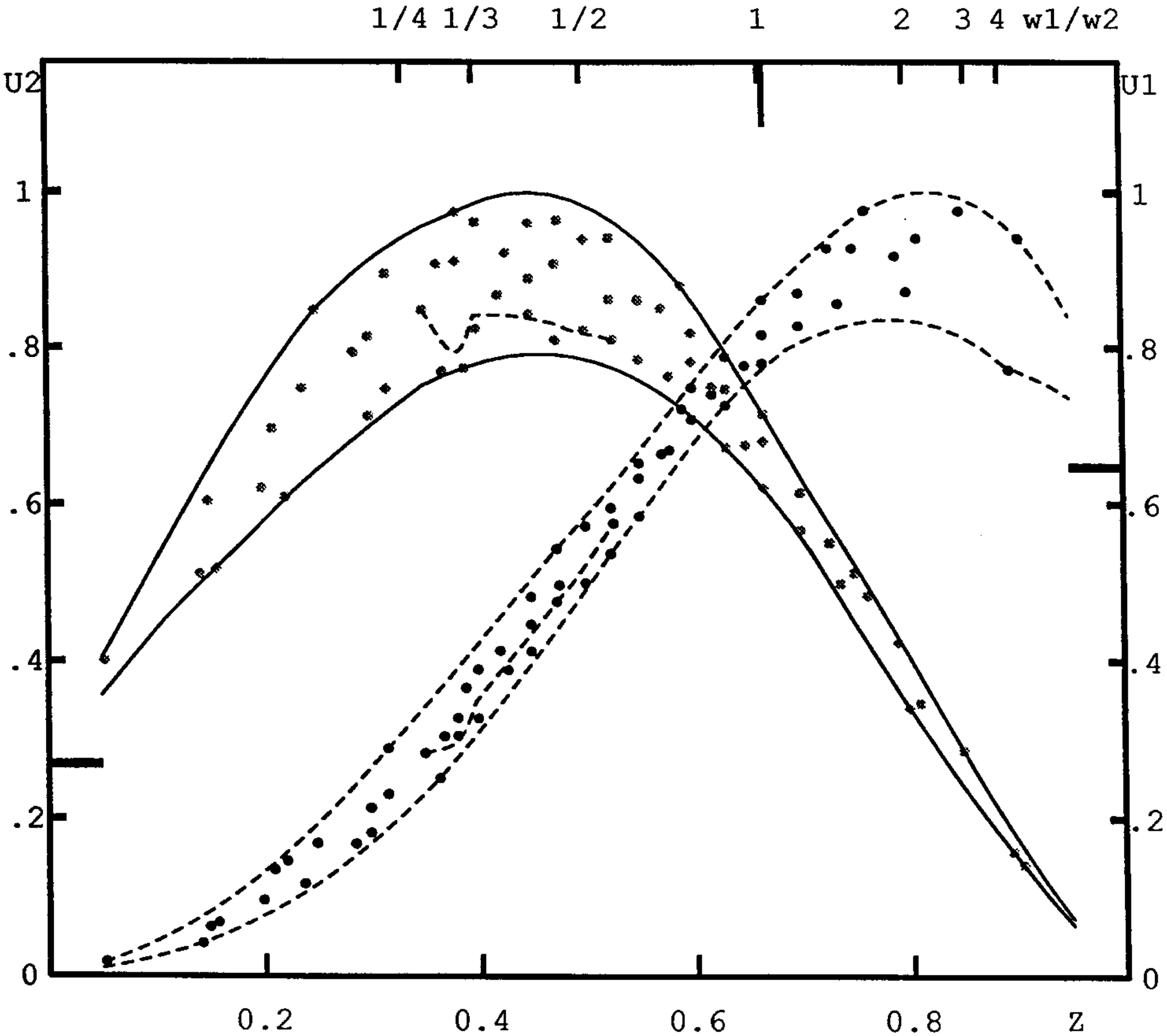

Fig.7.2

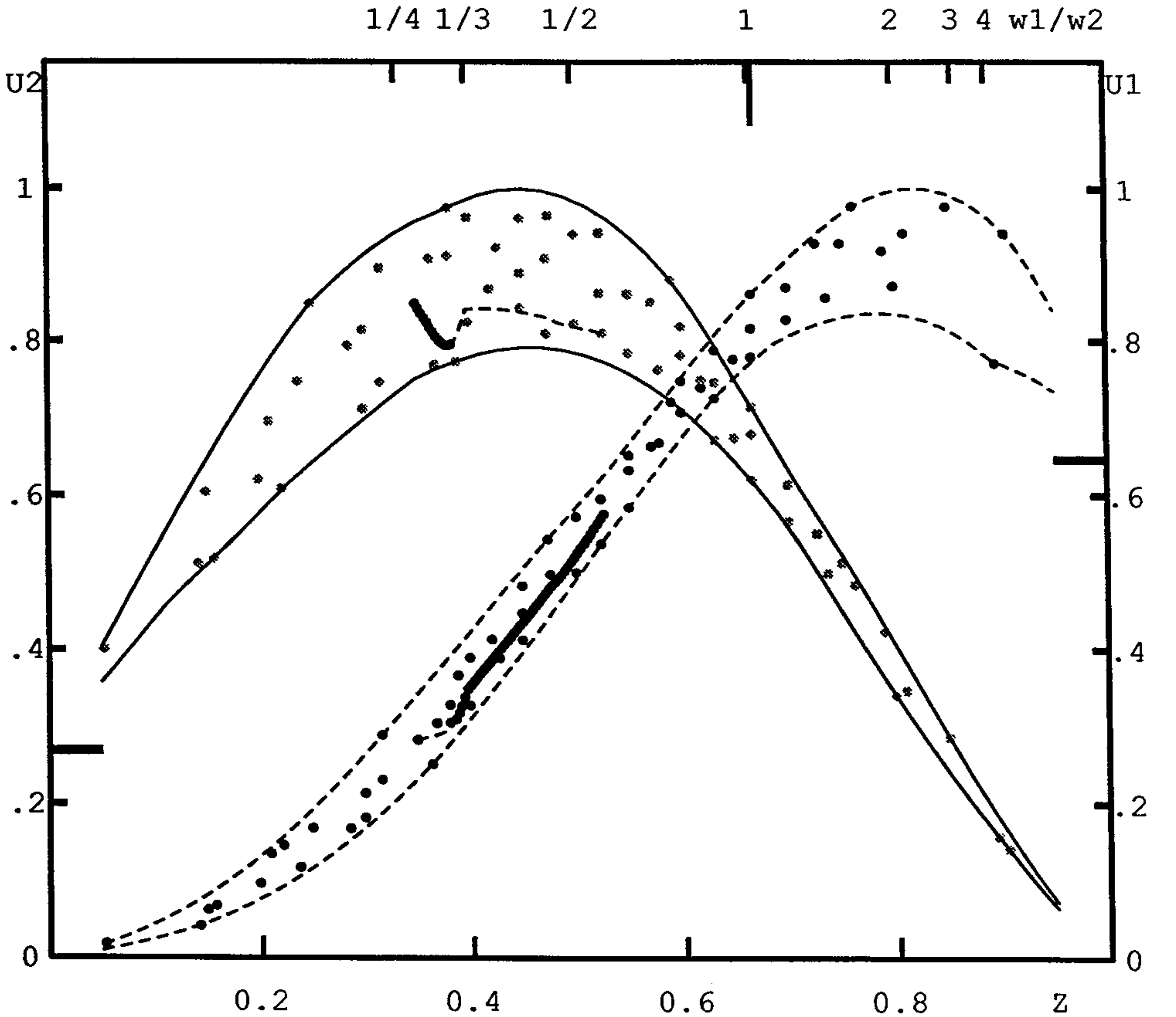

Fig . 7.3

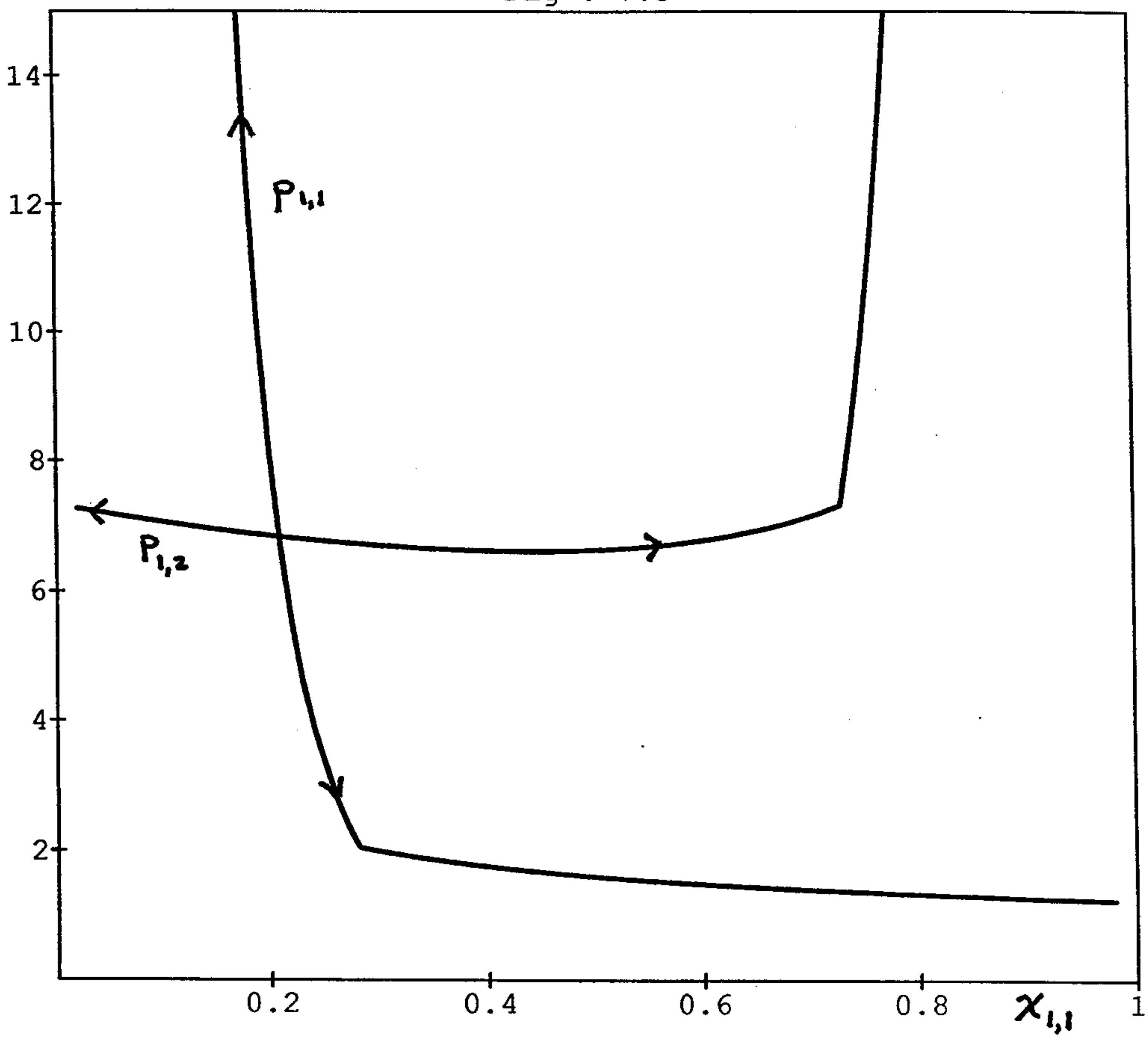

Fig. 7.4

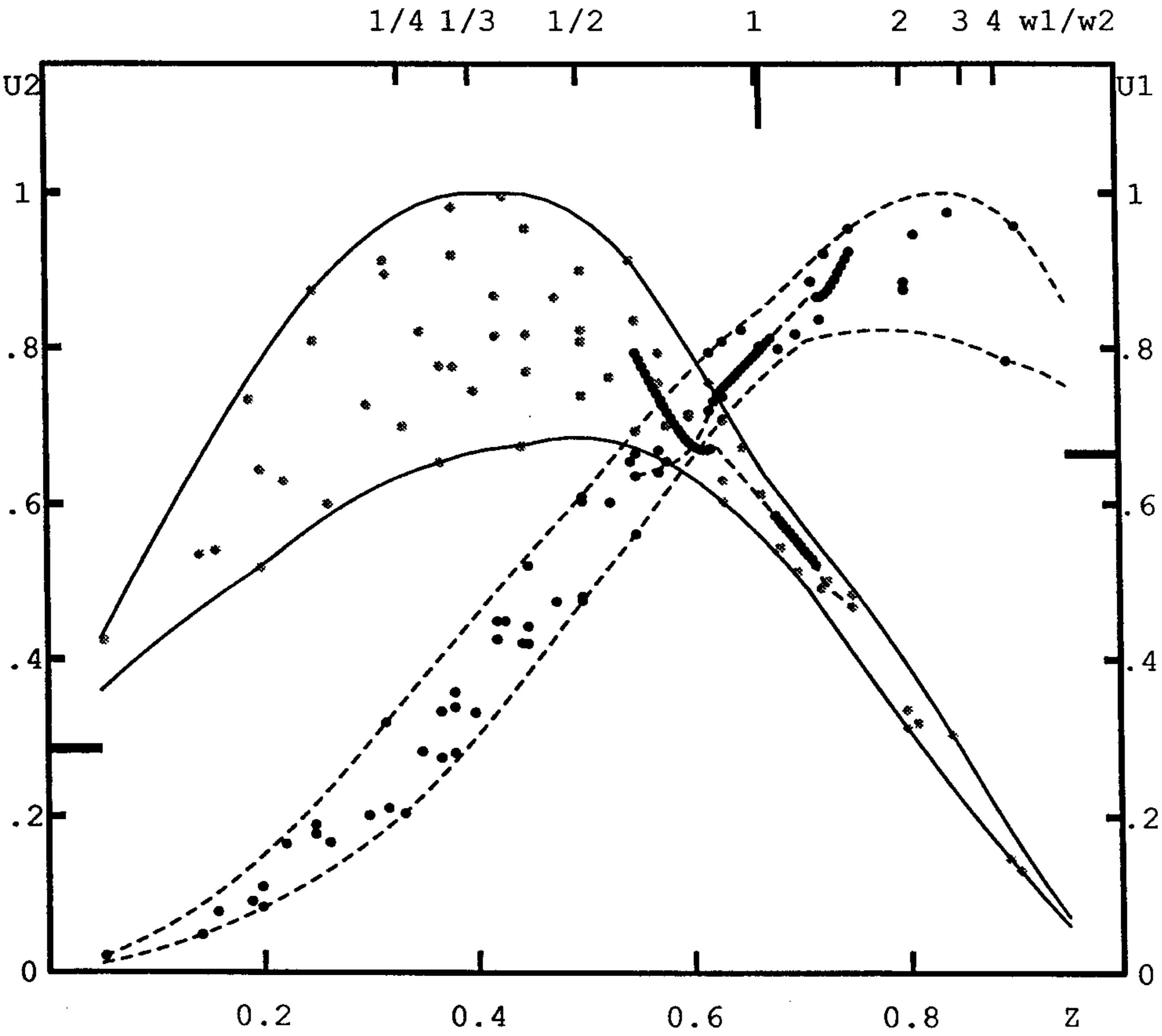

Fig . 7. 5

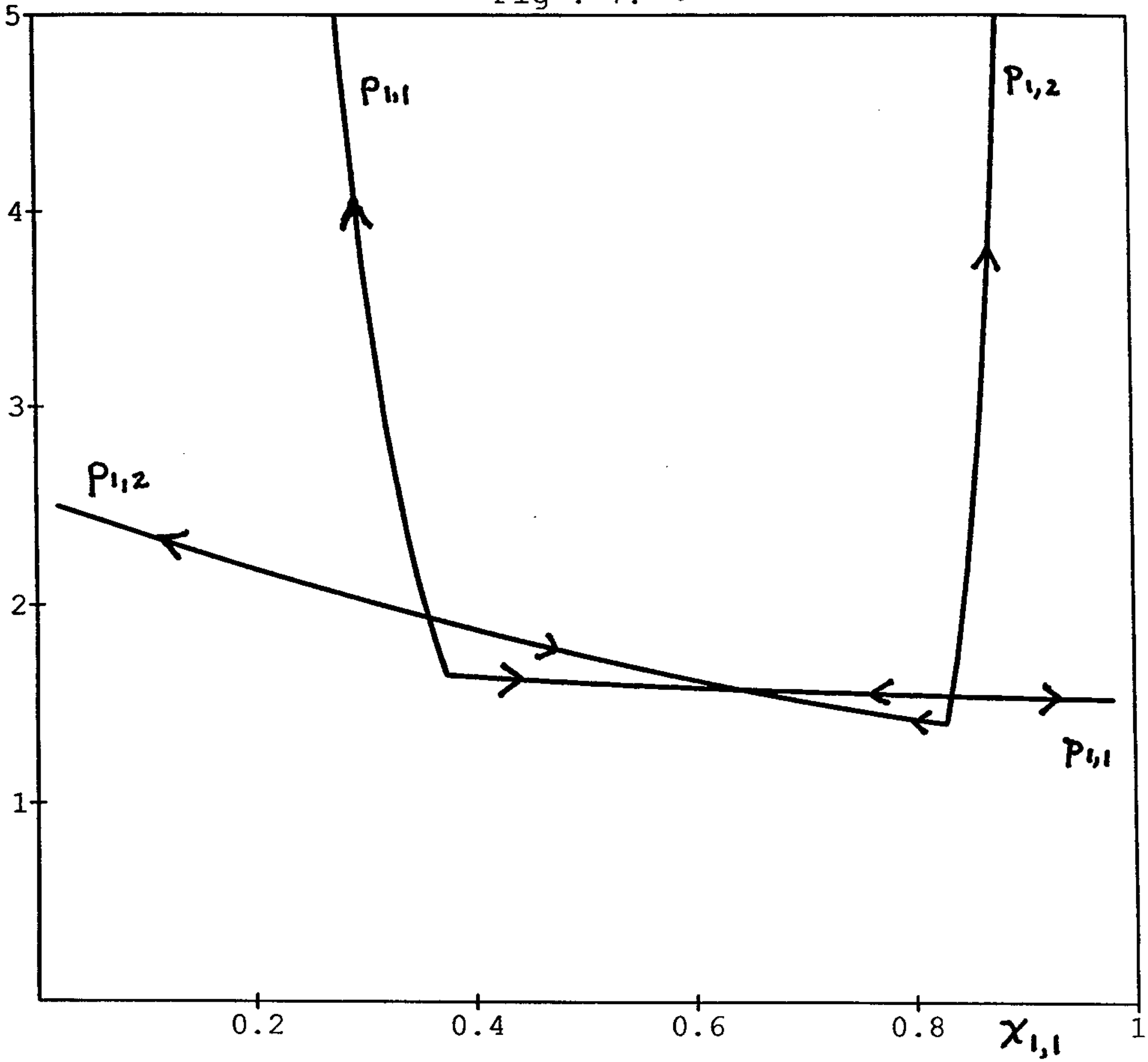

Fig . 8.1

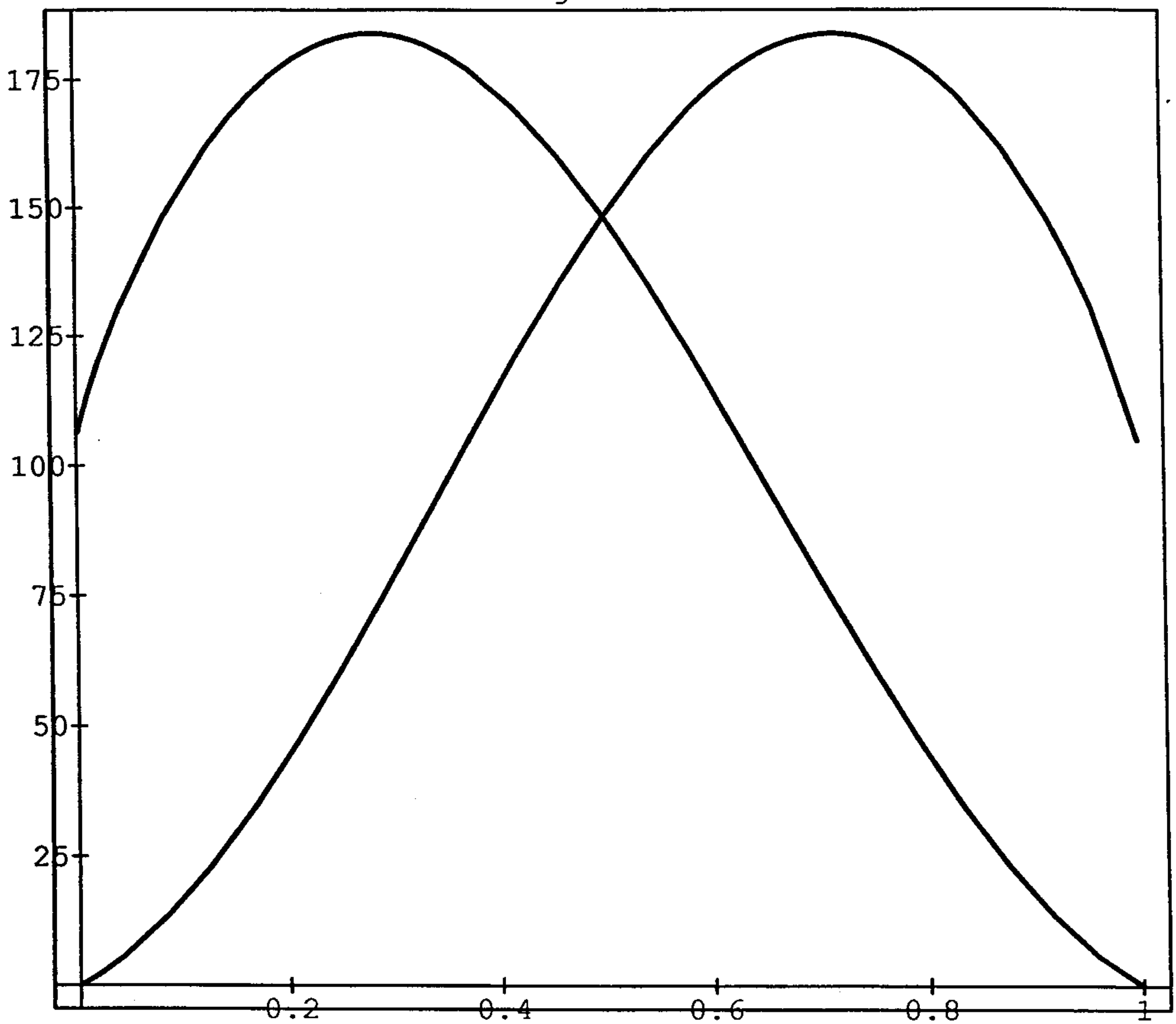

Fig. A7-1.1

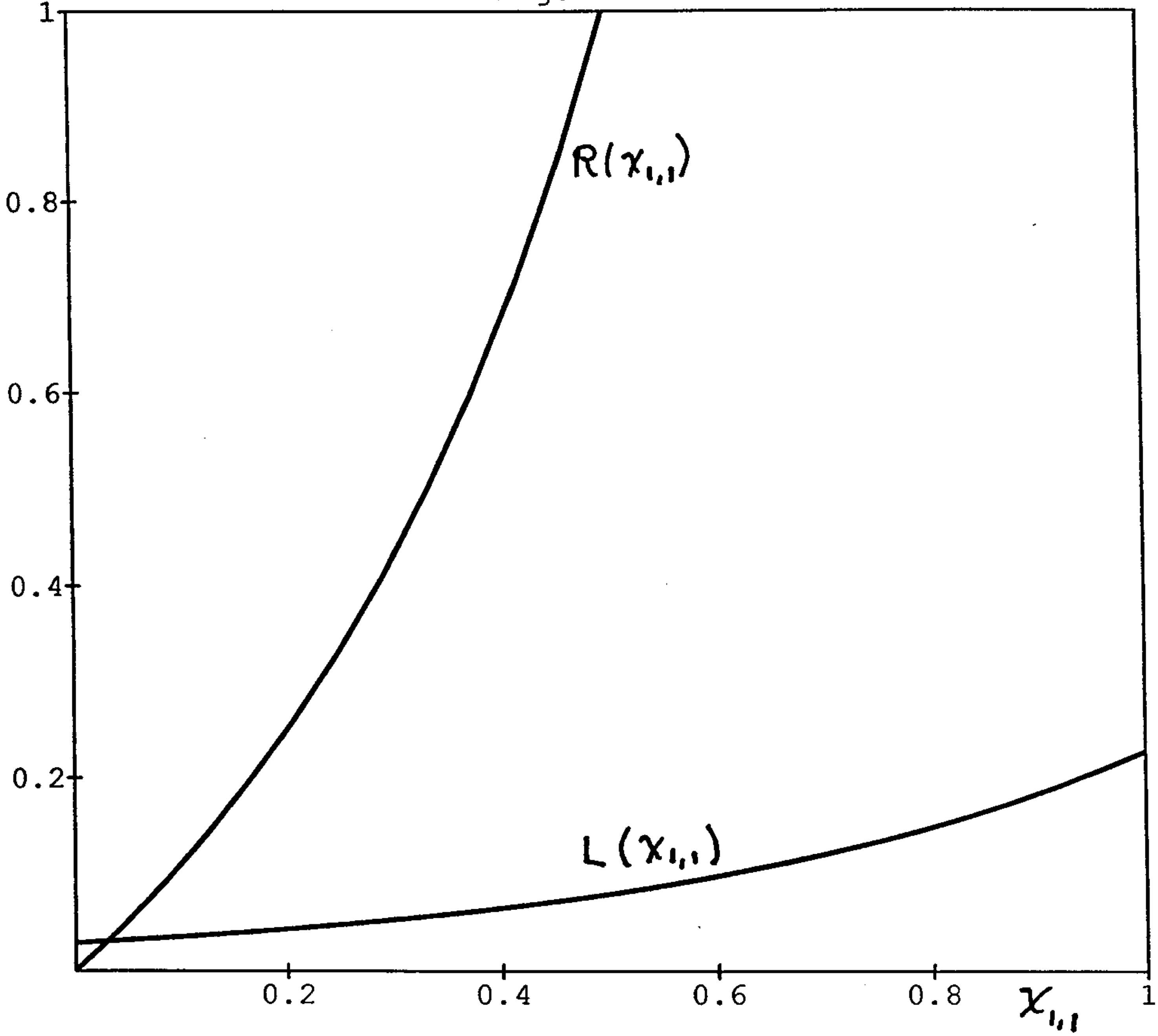

Fig. A7-1.2