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FAIR DIVISION BY POINT ALLOCATION

BY

Steven J. Brams
and
Alan D. Taylor

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C. V. STARR CENTER FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

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Steven J. Brams
Department of Politics
New York University
New York, NY 10003

Alan D. Taylor
Department of Mathematics
Union College
Schenectady, NY 12308

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Abstract

Two fair-division procedures that are applicable to negotiations between two parties over multiple issues are analyzed. Both procedures, which involve the parties' allocating points across the issues, guarantee the envy-freeness and equitability of a settlement. The first procedure ensures that the settlement is Pareto-optimal, but it is vulnerable to strategic manipulation, whereas the second procedure is relatively invulnerable to manipulation, but it is not Pareto-optimal. Despite the vulnerability of the Pareto-optimal procedure in theory, in practice it would be difficult to exploit, though the strategically more robust second procedure could be used as a default option (if one player requests it) under a combined procedure. Possible applications of the Pareto-optimal procedure are discussed and illustrated by the Panama Canal treaty negotiations in the 1970s.

Fair Division by Point Allocation[1]

1. Introduction

A vexing problem in finding resolutions to two-party conflicts lies in the state of bargaining theory, which is notably inapplicable to the settlement of real-life disputes involving multiple issues. This is true despite the attempts by a number of theorists to demonstrate the contributions that rational-choice models have made to understanding conflicts and prescribing solutions (Sebenius, 1992; Young, 1991; Brams, 1990; Lax and Sebenius, 1986; Raiffa, 1982).

One reason for this failure, in our opinion, has been the divorce of bargaining theories—and, on the more applied side, "negotiation analysis"—from theories of fair division (some exceptions are discussed in van Damme, 1991, ch. 7). In this paper, we offer a reconciliation of these theories by combining two fair division-procedures into a practical scheme for resolving two-party conflicts over multiple issues.

We analyze these procedures with respect to normative criteria from bargaining theory, including the degree to which the procedures (1) yield settlements that are Pareto-optimal and (2) induce bargainers to be truthful, or at least almost truthful. We also apply criteria from the theory of fair division to these procedures, including the extent to which they produce settlements that are (i) envy-free and (ii) equitable.

Both procedures satisfy (i) and (ii), which we will define later. The trade-off comes between the bargaining criteria, (1) and (2): one procedure satisfies (1), the other (2), but neither satisfies both (1) and (2).

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[1]We thank Julius B. Barbanel, Robert T. Clemen, and D. Marc Kilgour for valuable comments on an earlier verion of this paper entitled "Fair Division: Pareto-Optimality Versus Strategic Robustness."

The first procedure, which we call Adjusted Winner (AW), produces a settlement that is Pareto-optimal, envy-free, and equitable with respect to the bargainers' *announced* preferences. However, because AW provides little inducement for the bargainers to be truthful in announcing their preferences, it may produce settlements that only *appear* to satisfy these criteria because (2) is not also satisfied.

For this reason, we propose a second fair-division procedure, called the Proportional Allocation procedure (PA), which could provide a "default" settlement should either party object to the settlement under AW. That is, PA could be implemented if either party, feeling that it was exploited under AW because of AW's vulnerability to false announcements, requests PA. For reasons to be discussed later, however, we think this safeguard will hardly ever be necessary and do not recommend linking the two procedures in this manner.

Like AW, PA is envy-free and equitable; unlike AW, it is extremely robust against false announcements in most situations, thereby inducing the bargainers to be truthful. However, the settlement it yields is not Pareto-optimal, so (1) is not satisfied. Nevertheless, it is substantially better for both parties than the naive fair-division procedure of splitting every issue fifty-fifty (how this might be done will be discussed later).

Keeney and Raiffa (1991), in the absence of a procedure for ensuring a Pareto-optimal settlement, proposed that the parties to a dispute first work out an "acceptable" settlement, though they leave vague what this means. They suggest that a third party ("contract embellisher") might then make adjustments in the original settlement that moves it toward Pareto-optimality (again without saying exactly how) in what Raiffa (1985, 1993) calls a "post-settlement settlement."

By contrast, AW guarantees Pareto-optimality, as well as envy-freeness and equity *at the start*; the issue for the parties is whether is is "safe" to buy into a procedure that can, in principle, be exploited. This is precisely why we raise the question of using PA as a default option, though we reject it later as unnecessary in most negotiations.

Although we think AW obviates the need for the haggling phase of negotiations—even if facilitated by a neutral third party—that Keeney and Raiffa (1991) recommend, our framework is similar to theirs. There are two parties and k issues ($k \geq 2$) that need to be resolved. Each party can quantify the relative importance of each issue to itself by distributing a total of 100 points over the k issues. Moreover, for each issue there is a set of unambiguously stipulated "resolution levels" which, as Keeny and Raiffa (1991) point out, may be either finite or a continuum.[2]

In an economic context, the problem we consider here is equivalent to that of dividing k infinitely divisible homogeneous goods between two consumers who value the goods differently. A *homogeneous good* corresponds to our earlier "issue for which a set of unambiguous resolution levels has been stipulated," and *infinite divisibility* corresponds to the resolution levels' being a continuum. Thus, "player I gets 60 percent of good i" corresponds to "issue i's being resolved 60 percent in favor of player I and 40 percent in favor of player II."

More formally, we assume that there are k goods (or issues) $G_1, \ldots, G_k$ and two parties (player I, who is male, and player II, who is female). Both players can independently assign points to the goods that indicate their

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[2]In fact, as we shall show later, AW requires that only one issue be divisible, though which one this is will not be known at the start. In the subsequent analysis, we assume that all issues are divisible, but we will revisit this assumption when we discuss how AW might be used in negotiations in which some issues are indivisible.

true values for the issue. We assume that player I's true values are $a_1, \ldots, a_k$ and player II's $b_1, \ldots, b_k$, where $a_i \geq 0$, $b_i \geq 0$, and $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i = 100$.

The a_i's and b_i's may or may not be common knowledge. In either case, we assume that $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ are the players' *announcements* of their assigned points, which may or may not be truthful. Our interest is in dividing each of the goods between the two parties so that the resulting allocation is satisfactory, according to some—if not all—the aforementioned criteria.

2. The Adjusted Winner (AW) Procedure

AW allocates k goods as follows. Let X be the sum of the points of all goods that player I announces that he values more than player II. Let Y be the sum of the values of the goods that player II announces she values more than player I. Assume $X \geq Y$. Next assign the goods so that player I initially gets all the goods where $x_i \geq y_i$, and player II gets the others. Now renumber them so that

(i) player I, based on his announcement, values goods $G_1, \ldots, G_r$ at least as much as player II does (i.e., $x_i \geq y_i$ for $1 \leq i \leq r$), where $r \leq k$.

(ii) player II, based on her announcement, values goods $G_{r+1}, \ldots, G_k$ more than player I does (i.e., $y_i > x_i$ for $r+1 \leq i \leq k$).

(iii) $x_1/y_1 \leq x_2/y_2 \leq \ldots \leq x_k/y_k$.

Thus, player I is initially given all goods 1 through r that he values at least as much as player II, and player II is given all goods r+1 through k that she values strictly more than player I.

Because $x_i \geq y_i$ for $1 \leq i \leq r$, the ratios in (iii) are at least 1. Hence, all the goods for which $x_i = y_i$ come at the beginning of the list. Player I —who, because $X \geq Y$, enjoys an advantage (if either player does) after the winner-take-all assignment of goods—is helped additionally by being assigned all goods that the players value equally, based on their announcements.

The next step involves transferring from player I to player II as much of G_1 as is needed to achieve *equity*—that is, until the point totals of the two players are equal. (Recall that equity is only apparent, not true, because we do not assume that the players' announcements of their point assignments are necessarily truthful.) If apparent equity is not achieved, even with all of G_1 transferred from player I to player II, we next transfer G_2, G_3, etc. (in that order) from player I to player II. As we will prove shortly, it is the order given by (iii) that ensures Pareto-optimality.

Example. Suppose that there are three goods for which players I and II announce the following point assignments (the larger of the two assignments is underscored):

	G_1	G_2	G_3
Player I's announced values	$\underline{6}$	$\underline{67}$	27
Player II's announced values	5	34	$\underline{61}$

Initially, G_1 and G_2 are assigned to player I, giving him 73 of his points, and G_3 is assigned to player II, giving her 61 of her points. Hence, goods must be transferred from player I to player II to create apparent equity.

Notice that $x_1/y_1 = 6/5 = 1.2$ and $x_2/y_2 = 67/34 \cong 1.97$. Even transferring all of G_1 from player I to player II, however, still leaves player I with an advantage (67 of his points to 66 of her points).

Let α denote the fraction of G_2 that will be retained by player I, with the rest transferred from player I to player II. We choose α so that the resulting point totals are equal:

$$67\alpha = 5 + 34(1-\alpha) + 61,$$

which yields $\alpha = 100/101 \cong .99$. As a consequence, player 1 ends up with 99 percent of G_1 for a total of 66.3 of his points, whereas player II ends up with all of G_1 and G_3 and 1 percent of G_2 for the same total of 66.3 of her points.[3]

AW has three compelling properties, two of which are obvious by construction and one of which is not. The properties are given by Theorem 1 and further commented upon before the proof.

Theorem 1. *AW produces an allocation of the goods, based on the announced values, that is:*

(1) <u>Pareto-optimal</u>: any allocation that is strictly better for one player is strictly worse for the other;

(2) <u>equitable</u>: player I's valuation of his allocation is the same as player II's valuation of her allocation;

(3) <u>envy-free</u>: neither player would trade his or her allocation for that of the other.

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[3]The reader might want to check that if the first transfer had been of G_2 instead G_1, both players would have received 65.0 of their points and hence have fared worse.

Comment. Note that envy-freeness and equity both address the question of whether one player believes he or she fared as well as the other player. The difference is that envy-freeness involves a comparison based on one's own valuation:

> Are you at least as well off with your allocation and, hence, would not desire to trade with the other player?

Equity, on the other hand, involves an interpersonal comparison:

> Is your valuation of what you received equal to the other player's valuation of what he or she received?

Again, what we are calling "envy-freeness" and "equity" are only "apparent envy-freeness" and "apparent equity" if the players are not truthful. When the players are truthful—$x_i = a_i$ and $y_i = b_i$ for all i—each player *assuredly* receives at least 50 points (based on his or her own valuation), and each's surplus above 50 points is the same (based on a comparison of their different valuations).

The equity adjustment, which gives each player 66.3 of its points in the example, may be interpreted as providing each player with nearly 2/3 of the total value of all three goods. This equalization of the players' utilities assumes that points (or utiles) are additive and linear. Linearity, which means roughly that x% of G_i is twice as good as (x/2)%, may not be a good reflection of players' preference functions in certain negotiations, which is a matter we return to later.

Proof. To establish the Pareto-optimality of AW, we first prove three claims:

Claim 1. Suppose we have an allocation wherein:

(i) Player I values G_i at least as much as player II does;

(ii) Player II values G_j at least as much as player I does;

(iii) Player I possesses the amount $S \subset G_i$;

(iv) Player II possesses the amount $T \subset G_j$.

Then if a trade of S for T yields an allocation that is better for one player, it is worse for the other.

Proof. Assume that a and a' are player I's values of G_i and G_j, respectively, and that b and b' are player II's values of G_i and G_j, respectively. Thus, $a \geq b$ and $a' \leq b'$. Let $|S|$ denote the fraction of G_i that S is, and let $|T|$ denote the fraction of G_j that T is. Assume that the trade strictly benefits player I. Then

[what player I gets] > [what player I has],

and so

$$|T|(a') > |S|(a).$$

But because $b' \geq a'$ and $a \geq b$, we have

$$|T|(b') > |S|(b),$$

and so

[what player II has] > [what player II gets].

Hence, the trade is strictly worse for player II. Q.E.D.

Claim 2. Suppose we have an allocation wherein:

(i) Player I's values, G_i and G_j, are a and a', respectively;

(ii) Player II's values, G_i and G_j, are b and b', respectively;

(iii) Player I possesses the amount $S \subset G_i$;

(iv) Player II possesses the amount $T \subset G_j$;

(v) $a'/b' \leq a/b$.

Then if a trade of S for T yields an allocation that is better for one player, it is worse for the other.

Proof. Suppose, for contradiction, that a trade of S for T yields an allocation that is strictly better for one player and no worse for the other. Then we have

$$[\text{what player I gets}] \geq [\text{what player I has}],$$

and

$$[\text{what player II gets}] \geq [\text{what player II has}],$$

with at least one of the inequalities strict. Thus,

$$|T|(a') \geq |S|(a),$$

and

$$|S|(b) \geq |T|(b'),$$

with at least one of these inequalities strict. Multiplying the first inequality by b on both sides, and the second by a on both sides, yields

$$|T|(a')(b) \geq |S|(a)(b),$$

and

$$|S|(a)(b) \geq |T|(a)(b').$$

Hence,

$$|T|(a')(b) \geq |T|(a)(b'),$$

and so

$$(a')(b) \geq (a)(b').$$

Moreover, this inequality is strict since one of our first two inequalities was. Consequently, $a'/b' > a/b$, in contrast to (v). Q.E.D.

Claim 3. If a given allocation is not Pareto-optimal, then there are goods G_i and G_j, and sets $S \subset G_i$ and $T \subset G_j$, such that a trade of S for T yields an allocation that dominates the given one.

Proof. Since the given allocation is not Pareto-optimal, we can choose disjoint sets S' and T' so that player I possesses S', player II possesses T', and a trade of S' for T' is better for one (say, player I) and no worse for the other. The set S', however, may not be a subset of a single good—as we want (and will now show how to obtain)—but it certainly can be written as the disjoint union $S_1 \cup \ldots \cup S_m$ of sets that *are* subsets of single goods. Player II can now split T' into disjoint sets $T_1 \cup \ldots \cup T_m$ (which are not necessarily subsets of single goods) so that a trade of T_i for S_i yields an allocation that is no worse for her than the given allocation.[4]

Because of Pareto-optimality, there must now exist at least one i so that player I finds the allocation resulting from a trade of S_i for T_i strictly preferable to the given allocation. (This also uses weak additivity of preferences.) Recall that S_i is a subset of a single good, but T_i may not be. Nevertheless, we can write T_i as the disjoint union of sets which *are* subsets of single goods and then proceed, as before, to obtain first a set $S \subset S_i$ and then a set $T \subset T_i$ so that a trade of S for T is strictly better for player I and no worse for player II. Q.E.D.

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[4]We assume here weak additivity of preferences. That is, if A and B are disjoint sets, and X and Y are disjoint, and if a player thinks A is at least as large at X, and B is at least as large as Y, then that player thinks $A \cup B$ is at least as large as $X \cup Y$.

The theorem now follows easily from the three claims. Suppose that the allocation from the scheme is not Pareto-optimal, and choose sets $S \subset G_i$ and $T \subset G_j$ as guaranteed to exist by claim 3. Assume player I had the advantage in the winner-take-all-part of the scheme, so any transference of goods in the second step of the scheme was from player I to player II. Since player I possesses S, he values good G_i at least as much as player II does (say, $a \geq b$). It now follows from claim 1 that player I values good G_j strictly more than player II does (say, $a' > b'$). Since player II possesses T, she must have received it from player I in the transfer stage of AW. However, player I still possessed part of G_i, so all of it was not transferred to player II. Thus, we must have $j < i$ and so $a'/b' \leq a/b$. This contradicts claim 2 and completes the proof of Pareto-optimality.

Equitability is clear by construction, and envy-freeness follows from Pareto-optimality and equitability. That is, if the allocation were not envy-free, then both players would receive fewer than 50 points and, hence, equal division would contradict Pareto-optimality.[5] Q.E.D.

The main drawback of AW is the extent to which it fails to induce the players to be truthful about their valuations. This is easy to illustrate, even in the case of two goods. Suppose player I values the goods equally, and player II knows that he (player I) will truthfully announce the values 50-50. Suppose player II's true values are 70-30. What should she announce? Assuming that announcements must be integers, the answer is 51-49, as we will show shortly.

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[5]It is worth noting that that if a point-allocation scheme is equitable, then either the allocation is envy-free or a trade between the two players would yield an envy-free allocation. That is, we never have a situation wherein one player wants to trade and the other does not when an allocation is equitable.

The result of this announcement will be an initial allocation of all of G_1 to player II (which she values at 70), and all of G_2 to player I (which he values at 50). Then there will be a transfer of only a trivial fraction (1/101) of G_1 to player I, since it appears that player I's initial advantage is only 51 to 50. Thereby player I will end up with a generous 70 - 0.7 = 69.3 points, but player II will realize only 50 + 0.5 = 50.5 points, based on their true valuations.[6]

Player I can turn the tables on player II if he knows her values of 70-30 and that she will announce these. If player I announces 69-31, there will be a transfer of 39/139 of G_1 from player II to player I, giving player I a total of 50 + 14.0 = 64.0 points and player II only 70 - 19.6 = 50.4 points, based on their true valuations.

Thereby one player (with complete information) can exploit another player (without such information). On the other hand, if both players were truthful in their announcements, there would be a transfer of 1/6 of G_1 from player II (70-30) to player I (50-50), giving each player 58.3 points.

If the announced and real values are restricted to integers, then optimal responses and Nash equilibria can easily be computed, with the determination of an optimal response requiring only 100 comparisons, and the testing of a potential equilibrium requiring at most 198 comparisons. Such computations are not really needed, however, because the following theorem and its corollary completely settle the question of optimal responses and Nash equilibria in the integer-valued case:

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[6]It is easy to see that if announced values are not restricted to the integers, then there is *no* optimal response for player II in this example, because the payoffs are discontinuous. That is, if $P(y)$ is the payoff to player II resulting from her announced value of $50 + y$ for G_1, then $P(y) \to 70$ as $y \to 0$, but $P(0)$ has an expected value of 50 (because it is 30 half the time and 70 half the time).

Theorem 2. *Assume there are two goods, G_1 and G_2, all true and announced values are restricted to the integers, and suppose player I's announced value of G_1 is x, where $x \geq 50$. Assume player II's true value of G_1 is b. Then player II's optimal announced value of G_1 is:*

$$\begin{aligned} x+1 &\quad \textit{if} \quad b > x \\ x &\quad \textit{if} \quad b = x \\ x-1 &\quad \textit{if} \quad b < x. \end{aligned}$$

Example. In our earlier example, x = 50 and b = 70, so player II's optimal announcement is x + 1 = 51. When the tables are turned and x = 70 and b = 50, player I's optimal announcement is x - 1 = 69.

Proof. The proof requires the following five claims:

Claim 1. Suppose player I's announced values of G_1 and G_2 are x and x', respectively, and player II's announced values are the truthful assignments b and b', respectively. Assume x is the largest of the four values. Then player I receives the fraction 100/(x+b) of G_1 and none of G_2, whereas player II receives the rest of G_1 plus all of G_2, which are the same in terms of each player's valuations.

Proof. Let α denote the fraction of G_1 that will be retained by player I. The equity adjustment requires that

$$x\alpha = b(1 - \alpha) + b',$$

so

$$\alpha = (b + b')/(x + b),$$

where b + b' = 100. Q.E.D.

Claim 2. For player II, the announced value of $x + 1$ for G_1 is optimal among all announcements $y > x$.

Proof. An announcement by player II of $y > x$ makes y larger than any of the four values posited in claim 1. The equity adjustment now requires a choice of α, as in the proof of claim 1, such that

$$y\alpha = x(1 - \alpha) + x',$$

so

$$\alpha = 100/(x + y).$$

Because player II's true value of G_1 is b, she retains only the fraction α of it, or $100b/(x + y)$ points. This is largest when y is smallest. Q.E.D.

Claim 3. For player II, the announced value of $x - 1$ for G_1 is optimal among all announcements $y < x$.

Proof. If player II announces a value of $y < x$ for G_1, then her payoff depends on whether or not $100 - y > x$. If $100 - y > x$, she receives none of G_1 and a fraction of G_2, which gives her a payoff of

$$(100 - b)[100/(100 - y + 100 - x)] = (100 - b)[100/(200 - x - y)].$$

If $100 - y \leq x$, then she receives a payoff of

$$100 - b + b[1 - 100/(x + y)] = 100[1 - b/(x + y)].$$

Either way, player II's payoff is maximized by choosing y as large as possible, which is $x - 1$ for $y < x$. Q.E.D.

Claim 4. In the case where $x = 50$ and $b \neq 50$, player II's payoff from an announcement of $x - 1$ yields a higher payoff for player II than does an announcement of $x + 1$ if and only if $b < x$.

Proof. Straightforward.

Claim 5. If $x > 50$, then an announcement of $x - 1$ yields a higher payoff for player II than does an announcement of $x + 1$ if and only if $b < x$.

Proof. Player II's payoff from an announcement of $x - 1$ is

$$(100 - b) + b[1 - 100/(2x - 1)],$$

whereas her payoff from an announcement of $x + 1$ is

$$100b/(2x + 1).$$

The former payoff is greater than the latter if and only if

$$100[1 - b/(2x - 1)] > 100b/(2x + 1).$$

Simplifying this inequality yields the following sequence of inequalities:

$$1 - b/(2x - 1) > b/(2x + 1),$$
$$(2x - 1 - b)/(2x - 1) > b/(2x + 1),$$
$$4x^2 - 2x - 2xb + 2x - 1 - b > 2xb - b,$$
$$4x^2 - 4xb > 1,$$
$$b < x - 1/4x,$$
$$b < x.$$

Q.E.D.

Claims 1 - 5, together with the trivial observation that truthfulness is the best policy if player II's true values coincide with player I's announced values, complete the proof of the theorem. Q.E.D.

Corollary 1. *Assume all true and announced values are restricted to the integers, and suppose player I's true value of G_1 is a, where $a \geq 50$. Assume player II's true value of G_1 is b, and $a \geq b$. Then the Nash equilibria are the following ordered pairs of announced values for G_1 by players I and II:*

$$(x+1, x) \quad \text{where} \quad b \leq x \leq a-1;$$
$$(a, a) \quad \text{where} \quad a = b.$$

Example. To ensure $a \geq b$ in our earlier example, let the 70-30 player be player I, so a = 70, and the 50-50 player be player II, so b = 50. Then the Nash equilibria are all ordered pairs (x+1, x) for players I and II, respectively, where $50 \leq x \leq 69$. That is, the 20 pairs

(51, 50), (52, 51) . . . (70, 69)

are precisely the announcements that players I and II can make such that neither player would have an incentive to depart unilaterally from its announcement, because it would do worse if it did.

3. The Proportional-Allocation (PA) Procedure

The PA procedure we introduce in this section comes much closer to inducing the players to be truthful. Consider again our earlier example of exploitation, wherein player I (50-50) announced his true valuation, and player II (70-30)—knowing player I's allocation—optimally responded by announcing 51-49. Thereby, player II obtained 69.3 points, compared with the 58.3 points that truthfulness would have given her (a 17.2% increase).

Under PA, as we will show, the optimal response of player II is to be nearly truthful, announcing 71-29 instead of 70-30. Her benefit from this slight distortion of the truth is only in the third decimal place, gaining her 52.087 points compared to 52.083 points (less than a 0.01% increase). But note that both players do worse, when truthful, under PA (52.1 points) than under AW (58.3 points), so PA is not Pareto-optimal.

Later we will weigh the relative nonmanipulability of PA against the Pareto-optimality of AW. But, as we shall argue, a clear-cut choice is not necessary if we use PA as a back-up to AW should one or both of the players think he or she is being exploited.

Although PA does not give a Pareto-optimal allocation, like AW it is equitable—though this is by no means obvious—and envy-free. As we just illustrated, it also comes remarkably close to inducing truthfulness, at least in situations where no good is of either negligible or of overriding value to a player.

PA, as its name implies, allocates goods proportionally. As before, assume that player I announces values of $x_1, \ldots, x_k$, and player II annou announces values of $y_1, \ldots, y_k$, for goods $G_1, \ldots, G_k$. Assume that for each i, either $x_i \neq 0$ or $y_i \neq 0$. Then player I is allocated the fraction $x_i/(x_i + y_i)$ of good G_i, and player II the fraction $y_i/(x_i + y_i)$.

Example. Consider our earlier example of three goods for which players I and II announce the following point assignments:

	G_1	G_2	G_3
Player I's announced values	6	67	27
Player II's announced values	5	34	61

Player I is awarded 6/11 of G_1, 67/101 of G_2, and 27/88 of G_3, giving him a total of 55.9 of his points. Likewise, player II receives a total of 55.9 of her points. (Recall that AW awarded both players 66.3 points when they were truthful, or 11.9% more than does PA in this example.) The equitability of PA is no accident, as we next show:

Theorem 3. *PA produces an allocation of the goods, based on the announced values, that is equitable and envy-free.*

Proof. We first show that PA is equitable. The payoffs, $P_I(\vec{x},\vec{y})$ and $P_{II}(\vec{x},\vec{y})$, of players I and II are the weighted sums of their point assignments to each good multiplied by their fractional allocations:

$$P_I(\vec{x},\vec{y}) = x_1^2/(x_1+y_1) + \ldots + x_k^2/(x_k+y_k) \tag{1}$$
$$P_{II}(\vec{x},\vec{y}) = y_1^2/(x_1+y_1) + \ldots + y_k^2/(x_k+y_k).$$

Because we assume that the players may not be truthful, these payoffs are only apparent, as we noted of the payoffs under AW as well.

Let

$$D_i = (x_1+y_1) \ \ldots (x_{i-1}+y_{i-1}) (x_{i+1}+y_{i+1}) \ \ldots (x_k+y_k) .$$

We may rewrite the payoffs as

$$P_I(\vec{x},\vec{y}) = [x_1^2D_1 + \ldots + x_k^2D_k]/ [\prod_{i=1}^{k} (x_i+y_i)]$$

and

$$P_{II}(\vec{x},\vec{y}) = [y_1^2D_1 + \ldots + y_k^2D_k]/ [\prod_{i=1}^{k} (x_i+y_i)].$$

They will be equal if $P_I(\vec{x},\vec{y}) - P_{II}(\vec{x},\vec{y}) = 0$, which is true if and only if

$$(x_1^2 - y_1^2)D_1 + \ldots + (x_k^2 - y_k^2)D_k = 0. \tag{2}$$

But since

$$(x_i^2 - y_i^2)D_i = (x_i - y_i)(x_i + y_i)D_i = (x_i - y_i)[\prod_{i=1}^{k} (x_i+y_i)],$$

equation (2) will be true precisely when

$$\prod_{i=1}^{k} (x_i+y_i)[(x_1-y_1) + \ldots + (x_k-y_k)] = 0;$$

That is,

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i.$$

Since both these sums are equal by definition, the payoffs to both players are also equal, proving that PA is equitable.

To show that PA is envy-free, we must show that if $0 \leq x_i \leq 1$, $0 \leq y_i \leq 1$, and $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i$, then

$$\sum_{i=1}^{k} x_i^2/(x_i+y_i) \geq 1/2 \qquad \text{and} \qquad \sum_{i=1}^{k} y_i^2/(x_i+y_i) \geq 1/2.$$

(That is, the payoffs to player I and player II are at least one-half the 100 points.) Because of the equitability of PA (just demonstrated), it suffices to show that

$$\sum_{i=1}^{k} (x_i^2+y_i^2)/(x_i+y_i) \geq 1. \tag{3}$$

The following demonstration, due to Julius Barbanel, improves upon what we had in the earlier version of this paper (see note 1). Since

$$\sum_{i=1}^{k} (x_i+y_i)/2 = (1/2)(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} x_i) = (1/2)(1+1) = 1,$$

it suffices to show that, for each i,

$$(x_i^2+y_i^2)/(x_i+y_i) \geq (x_i+y_i)/2.$$

Suppose this inequality fails. Then

$$2x_i^2 + 2y_i^2 < x_i^2 + 2x_iy_i + y_i^2.$$

But then

$$x_i^2 - 2x_iy_i + y_i^2 < 0,$$

so $(x_i-y_i)^2 < 0$. Contradiction; (3) is satisfied, so PA is envy-free. Q.E.D.

PA and AW do not exhaust the allocation procedures that are equitable and envy-free. For example, the naive procedure we alluded to in section 1 of splitting every good (issue) fifty-fifty gives each player exactly 50 points, so it satisfies both desiderata. Yet, not only is this allocation Pareto-inferior to that given by the Pareto-optimal AW (66.3 points to each player in our earlier example), but it is also Pareto-inferior to PA (58.3 points to each player).

The principal advantage of PA over AW is that it discourages departures from truthfulness of the kind we showed to be optimal under AW in section 2. It turns out that when there are only two goods, this discouragement will be absolute under PA if and only if (i) the preferences of the players coincide or (ii) they are diametrically opposed, as we will illustrate presently.

But what if the players are neither in complete agreement nor in complete disagreement? We next show that truthfulness is still a good—if not quite optimal—policy under PA when there are two goods and the players do not attribute overriding value (i.e., more than 80 percent) to one or the other.

To ascertain the incentive of a player to depart from truthfulness, we show in Table 1 the optimal response of player I (row), for his true

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Table 1 about here

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valuations between 20 and 80, to player II's (column's) announced valuations between 20 and 80. (We shall consider an example of more extreme valuations outside the 20-80 range shortly.) Note first that player I should be truthful if his valuation is the same as, or diametrically opposed to, player II's announced valuation.

When this is not the case, then player I's optimal response is generally very close to truthfulness. For example, if player II's announced valuation is 50, and player I's is 70, then player I's optimal response is to allocate 70.87 points to this issue. Player I's payoff will then be 52.087, compared with 52.083 when he is truthful. In other words, player I's payoff is hardly affected (only in the third decimal place) by deviating in an optimal way from truthfulness.

In Table 2 we give as entries Nash equilibrium announcements of

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Table 2 about here

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players I (row) and II (column) for true valuations between 20 and 80 points.[7] These values are rounded to the nearest integer and show that the

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[7]We do not show the valuations of player II for 60, 70, and 80 points, because the results we present can be surmised from other entries in the table. For example, if player I and player II's values for G_1 are 60 and 70, respectively, which is not shown in Table 1, they must be 40 and 30 for G_2, which is shown.

players' equilibrium announcements in the 20-80 range do not differ very much from their true values. For example, player I and player II's true values of 50 and 20 translate into 45/17 equilibrium announcements, which in relative terms is the most significant departure from truthfulness in the table. (Note that 50/20 is equidistant in the first column of Table 2 from the two truth-inducing valuations of 20/20 and 20/80.) The payoff consequences, however, are quite small: 50/20 gives both players 54.95 points under PA, whereas 45/17 gives the 50-point player 56.22 points and the 20-point player 53.60 points.

Consider a more extreme valuation than that given in Table 2—namely, 90-10 by player I, and 10-90 by player II, for G_1 and G_2, respectively. While truthfulness (90/10 and 10/90) is an equilibrium, it is nonequilibrium announcements of 100/0 by player I and 0/100 by player II that are Pareto-optimal under PA, giving them equitable and envy-free payoffs of 90 points each.[8] In fact, these are the payoffs that AW would also give: all G_1 would go to player I, and all G_2 would go to player II, with no equity adjustment necessary.

4. The Combined Procedure: A Better Alternative?

In the absence of reliable intelligence about the announced valuation of the other player (perhaps obtained by spies), it seems likely that the players will stick with their true values under PA—especially in the 20-80 range—because there are equilibrium announcements "close" to these

[8]When player II announces 0/100, player I can optimally respond with 1/99, which is why 100/0 and 0/100 does not constitute a Nash equilibrium when the true valuations of the players are 90-10 and 10-90. This optimal response on the part of player I to player II's announced valuation of 100/0 garners him nearly 95 points under PA: $(1/1)(90) + (99/199)(10) \cong 94.97$. By comparison, player II receives $(0/1)(10) + (100/199)(90) \cong 45.23$.

values. On the other hand, if a player's valuation exceeds 90 for one good, he or she may be tempted to put all 100 points on it—unless the other player has either exactly the same or exactly the opposite valuation and intends to announce it, in which case truthfulness on the part of both players is a Nash equilibrium.

But, as we just illustrated, Nash equilibria may not be Pareto-optimal; by contrast, AW always guarantees Pareto-optimality when the players are truthful. For this reason, assume AW is used initially. Its vulnerability to manipulation, however, suggests that the players might be given the opportunity to opt for the strategically more robust PA, without revising their announcements, if at least one player so chooses.

This default option might well be selected in the kind of situation we illustrated in section 2, in which player I announces 50-50 and player II announces 51-49. If player I thinks that player II really values G_1 much more than 51 points (say, at 70 points instead), then he could invoke PA. While PA would give him only 50.0 points (as opposed to 50.5 points under AW), player I may regard this sacrifice as minuscule compared with the satisfaction he derives from preventing player II's exploitation by reducing her from 69.3 points under AW to 50.0 points under PA.

We believe this kind of exploitation would be extremely rare if PA were a default option for AW, which we will refer to as the *combined procedure*. In fact, the very conditions that would allow for such exploitation—advance knowledge by one player of the other player's announcements, but not vice-versa—seem highly unrealistic, spies notwithstanding.

More probable might be a belief on the part of one player (say, I) that the other player (II) was extraordinarily lucky in just beating him out on his most important issues. PA would then permit player I to "avenge" his perceived losses. But because this may well be to the disadvantage of both players, we think that they would generally go with the AW rather than the PA allocations.

A case can be made for changing the combined procedure to allow the players to revise their point allocations if PA is selected. To be sure, allowing for revisions might make the players more willing to try to exploit their opponents, knowing that they will get a "second chance" if PA is chosen as a default option. On the other hand, if PA is invoked by one player and both players make truthful announcements—as they should—then the player who invoked PA might end up doing better than under AW, given its opponent was exploitative under AW.

A player that thinks it is being exploited under AW would probably have a greater incentive to opt for PA if revisions are allowed under the combined procedure. But this incentive on the part of the exploited player, which would presumably help to deter the exploitative player from attempting exploitation, must be weighed against the exploitative player's own greater incentive to be exploitative if it has a second chance. It is not clear where the balance of incentives lies—whether allowing or not allowing revisions under the combined procedure would induce greater truthfulness and, consequently, the use of AW. Clearly, a more refined analysis than we have space for here is needed to clarify the properties of both versions of the combined procedure.

The winner-take-all feature of AW alone has an important advantage over PA: it is applicable to indivisible goods (or issues), except on the one good on which an equity adjustment may have to be made. Thus, if one is dividing up an inheritance between two heirs, and there are several indivisible goods like a house or a work of art, then AW can more readily be applied than PA—provided, of course, that the equity adjustment is not made on one of these indivisible goods.

In fact, under AW the player could postpone determining what winning a certain percentage of a good means until the good on which the equity adjustment must be made is revealed. If, initially, the players are not told who won the larger percentage of that good, but only what this percentage (say, 60%) is, then they need only agree on what winning 60% for each player signifies. Since either player could be the 60% player, presumably they will be motivated to be impartial in translating winning (or losing) into a 60:40 breakdown that could go either way. Alternatively, if the good on which the equity adjustment is to be made is truly indivisible, the players could arrange an equivalent 60:40 trade on a divisible good or goods.

5. The Panama Canal Treaty Negotiations

If the goods are issues over which there is negotiation, there is often a mix, with some issues more divisible than others. Consider, as a case in point, the Panama Canal treaty negotiations, in which the United States and Panama agreed in June 1974, after two rounds of negotiations, on a definition of the ten major issues shown in Table 3. Raiffa (1982, pp. 176-

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Table 3 about here

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177) reports that a consulting firm then interviewed members of the U.S. negotiating team, headed by Ambassador-at-Large Ellsworth Bunker, not only about the importance that the United States attached to these issues but

also the importance, as viewed by the Americans, that Panama attached to the same issues.

These assessments are shown in Table 2. Under AW, the United States wins on issues 1, 2, 4, and 6, giving it 64 points, and Panama wins on issues 5, 7, 8, 9, and 10, giving it 53 points. On issue 3, the players tie with 15 points each, so this becomes the issue for an equity adjustment. By giving the United States 13.3% of its position (2 points), and Panama 86.7% of its position (13 points), each player receives an envy-free and equitable assignment of 66 points under AW.

This compares with the 57.9 points that PA gives the players. Thus, AW provides a 14.0% Pareto-improvement and is presumably the choice the players would make in this example. Moreover, issue 3, "land and water," is one of the more divisible issues and so, presumably, one on which the players could agree on a "resolution level" (Keeney and Raiffa, 1991, p. 132) that would give Panama about 7/8 of its way.

In fact, the only issue for which Raiffa (1982, pp. 176-177)—using slightly different terminology—indicates there is not a divisible "bargaining range" is issue 6, "expansion routes," for which there were only three possible choices. Although most issues in negotiations are likely to be more or less divisible, it would be useful if the players could agree beforehand on resolution levels—should an equity adjustment be required—to prevent quarreling on what so-and-so's winning of x% of that issue means. On the other hand, as we suggested earlier, this designation could be postponed under AW until the issue that is to be divided is known.

Perhaps a greater problem in applying any of the procedures is the implicit assumption of each that the points across all issues are additive (Keeney and Raiffa, 1991). The validity of this assumption depends on the issues' being *separable*—that is, that the amount a player wins on one issue does not depend on the amount it wins on another issue, so the issues can be treated independently.

A possible solution to this problem would be to allow the players to lump nonseparable issues together, such as issue 1 (U.S. defense rights) and issue 9 (U.S. military rights) for the United States. But the assumed linearity of points is also a potential problem. If the United States wins a particular amount on these two issues when lumped together (say, 60%), it might make more sense for it to give up more than 40% on one issue and less than 40% on the other than be forced under PA—or AW, if there is an equity adjustment on this combined issue—to give up prescribed, but non-complementary, amounts on each of the two rights issue.

An extreme form of lumping would be to have one player divide the issues into two "packages," each of which it values equally. For example, the United States values the sets {1,2,6} and {3,4,5,7,8,9,10} at 50 points each, whereas Panama values the first set at 29 points and the second set at 71 points. Even if Panama knew this was a 50-50 division for the United States, its 29-71 truthful response would be near optimal under PA (Panama's optimal response would be 28-72).

It turns out that truthfulness under PA would give each side only 52.3 points for these two packages, which is considerably below the 57.9 points that the players would receive by making truthful assignments to the ten different issues. A similar degradation occurs under AW, in which the players receive 58.7 points when there are only two packages but 66.0 points when there are ten issues.

There is not only a cost to lumping, but it also increases as the two packages approach equality for the players. For example, the packages

{1,2,8,9,10} and {3,4,5,6,7} are 50-50 for the United States and 49-51 for Panama, which yields the players only 50.5 points under AW and 50.0 points under PA. We conclude from this example that two players, in order to maximize their point totals under both AW and PA, would be well advised to apply these schemes *to as many different issues as they can reasonably make separable*.

6. Conclusions

AW and PA are two practical fair-division procedures that are envy-free and equitable, but only AW guarantees Pareto-optimality. Yet its winner-take-all feature makes it potentially vulnerable to strategic misrepresentation should one player have information about, or be able to predict, the announced point assignments of the other. PA, because it is much less vulnerable, would therefore seem useful to include as a default option should either player think that it has been exploited under AW by an opponent who misrepresents his or her preferences. We called the combination of AW and PA, with PA used only as a default option, the combined procedure.

Under this procedure, we assumed initially that the players could not revise their allocations if PA were used. Although both players would suffer if PA rather than AW were used, the player who chose the default option—assuming it is only one—would, we presume, do so primarily to punish its opponent for trying to be exploitative.

On first blush, this calculation would seem to be irrational. But it is not if the punisher attributes sufficient utility to ensuring an equitable solution, based on truthfulness. That is, by hurting itself—perhaps only slightly, as we illustrated in section 4—in order to punish its opponent severely for making a false announcement, the punisher may in fact derive a net benefit from choosing PA: letting its opponent go unpunished under AW for its false announcement may be more painful than suffering a slight loss itself under PA.[9]

Whether the combined procedure permits or does not permit the players to revise their point allocations, however, we think, for practical reaons, that the choice of PA as a default option would be a rare event. It seems highly unlikely that a player would be able to ascertain that it would benefit by exercising the default option, especially on negotiations like those on the Panama Canal treaty in which—absent spies—it seems virtually impossible to anticipate an opponent's point assignments on ten different issues.[10] AW has the additional advantage that, except on the one issue on which an equity adjustment must be made, issues may be indivisible because a player wins or loses completely on each. What "winning" and "losing" (in their entirety) mean, however, needs to be agreed to in advance by the players.

In applying AW, probably the biggest problem is identifying a set of tolerably separable issues. As we illustrated, lumping nonseparable issues together leads to lower point totals for the players. Consequently, it is in

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[9]Nuclear deterrence, interestingly enough, has a similar rationale. Suppose a country threatens to escalate a conflict to nuclear war, which could result in horrendous damage to itself as well its opponent if the threat is carried out. Just as this seemingly irrational threat is rational if it is successful in deterring a conflict that could escalate to nuclear war in the first place (Brams, 1985; Brams and Kilgour, 1988; Powell, 1989), the possibility that PA might be used under the combined procedure is rational if it deters a player from trying to be exploitative.

[10]Besides the Panama canal treaty, Raiffa (1982, ch. 10) gives a hypothetical example of negotiations involving ten issues in which he suggests that players, "thrashing around" (p. 139), might eventually find the Pareto-optimal frontier. But he also says (p. 288) that some "systems or mechanisms for conflict resolution . . . are far better than unstructured improvisation" (p. 288). We agree—and offer AW as our candidate.

the players' interest to try to identify as many issues as possible such that winning or losing on one does not greatly impinge on winning or losing on another.

A related problem is to spell out beforehand what each side obtains (e.g., in compensation, jurisdiction, or rights in the case of the Panama Canal Treaty) when it wins or loses on each issue. Only on the issue on which an equity adjustment must be made will a finer breakdown be necessary. However, this probably can be done after AW is applied if which side won the larger percentage on this issue is not revealed until the players agree on how winning by this percentage (by either player) will be translated into a settlement on this issue.

We suggest that AW first be tried out in low-level negotiations that involve easily specified issues or well-defined goods. Examples might include a dispute within a company over the division of job responsibilities, or the division of jointly owned goods in a divorce settlement. If the procedure works well in these settings, it might be used in more complex or consequential negotiations. In the political arena, for example, negotiations over arms control or border disputes often involve a plethora of issues that the AW could help to resolve.[11] In the economic sphere, negotiations between labor and management over a new contract, or between two companies over a merger, are usually sufficiently complex that a point-allocation scheme could prove extremely useful in finding a settlement that mirrors each side's priorities.

Although we believe it extremely unlikely that AW can be exploited in the absence of spies, for jittery negotiators the combined procedure

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[11]Hopmann (1991) illustrates the application of fair-division techniques to arms reductions by NATO and the former Warsaw Pact.

might still be considered as an alternative. While it may make negotiators less nervous about starting with AW, it needs more careful analysis than we have given it here before it can be recommended as a serious alternative to the simpler AW.

References

Brams, Steven J. Brams (1985). *Superpower Games: Applying Game Theory to Superpower Conflict.* New Haven, CT: Yale University Press.

Brams, Steven J. (1990). *Negotiation Games: Applying Game Theory to Bargaining and Arbitration.* New York: Routledge.

Brams, Steven J., and D. Marc Kilgour (1988). *Game Theory and National Security.* New York: Basil Blackwell.

Brams, Steven J., and Alan D. Taylor (1994a). "An Envy-Free Cake Division Protocol." *American Mathematical Monthly* (forthcoming).

Brams, Steven J., and Alan D. Taylor (1994b). "Fair Division." (forthcoming).

Hopmann, P. Terrence (1991). "'I Cut, You Choose': A Model for Negotiating Tradeoffs in Complex, Multi-Issue Negotiations." Preprint, Brown University.

Keeney, Ralph L., and Howard Raiffa (1991). "Structuring and Analyzing Values for Multiple-Issue Negotiations. In H. Peyton Young (ed.), *Negotiation Analysis.* Ann Arbor, MI: University of Michigan Press, pp. 131-151.

Lax, David A., and James K. Sebenius (1986). *The Manager as Negotiator: Bargaining for Cooperation and Competitive Gain.* Free Press.

Powell, Robert (1989). *Nuclear Deterrence Theory: The Search for Credibility.* Cambridge, UK: Cambridge University Press.

Raiffa, Howard (1982). *The Art and Science of Negotiation.* Cambridge, MA: Harvard University Press.

Raiffa, Howard (1985). "Post-Settlement Settlements." *Negotiation*

Journal 2, no. 1: 9-12.

Raiffa, Howard (1993). "The Neutral Analyst: Helping Parties to Reach Better Solutions." In Lavinia Hall (ed.), *Negotiation: Strategies for Mutual Gain.* Newbury Park, CA: Sage, pp. 14-27.

Sebenius, James K. (1992). "Negotiation Analysis: A Characterization and Review." *Management Science* 18, no. 1 (January): 18-38.

Young, H. Peyton (ed.) (1991). *Negotiation Analysis.* Ann Arbor, MI: University of Michigan Press.

van Damme, Eric (1991). *Stability and Perfection of Nash Equilibria*, 2d ed. New York: Springer-Verlag.

Table 1

Optimal Responses of Player I to Player II's Announced Valuations under PA

True Valuation of Player I	Announced Valuation of Player II						
	20	30	40	50	60	70	80
20	20	19.32	17.98	16.67	15.96	16.61	20
30	29.32	30	29.67	29.13	29.00	30	33.39
40	37.98	39.67	40	39.90	40	41.00	44.04
50	46.67	49.13	49.90	50	50.10	50.87	53.33
60	55.96	59.00	60	60.10	60	60.33	62.02
70	66.61	70	71.00	70.87	70.33	70	70.68
80	80	83.39	84.04	83.33	82.02	80.68	80

Table 2

Nash-Equilibrium Announcements under PA

		True Valuation of Player II			
		20	30	40	50
	20	20/20	19/29	18/37	17/45
	30	29/19	30/30	30/40	29/49
True	40	37/18	40/30	40/40	40/50
Valuation	50	45/17	49/29	50/40	50/50
of Player I	60	54/16	59/29	60/40	60/50
	70	64/16	70/30	71/41	71/51
	80	80/20	84/36	84/46	83/55

Table 3

Values of Issues in Panama Canal Treaty Negotiations

Issue	United States	Panama
1. U.S. defense rights	22	9
2. Use rights	22	15
3. Land and water	15	15
4. Expansion rights	14	3
5. Duration	11	15
6. Expansion routes	6	5
7. Compensation	4	11
8. Jurisdiction	2	7
9. U.S. military rights	2	7
10. Defense role of Panama	2	13

Source: Raiffa (1982, Table 10, p. 177).