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ABSTRACT

This paper examines the behavior of optimal consumption and investment policies in aggregate stochastic growth models with stock-dependent rewards. Such models arise in the study of renewable resources, monetary growth, and growth with public capital. Under certain complementarity conditions optimal policies are monotonic and converge to a unique limiting distribution. Two examples illustrate the possibility of multiple limiting distributions when these conditions are violated. It is shown that adding sufficient randomness to production transforms multiple limiting distributions into a unique limiting distribution. The paper also provides general results on the limiting behavior of Markov processes with monotone transition functions.

1. INTRODUCTION

For a number of important dynamic resource allocation problems, the reward or utility of an agent depends on the size of the resource stock as well as the amount consumed in any period. This is perhaps most evident in renewable resource models. Typically, the effort needed to obtain a given harvest level depends on the size of the resource stock. In such models harvest costs are stock-dependent. There are other problems where stock-dependent rewards are important as well. Wealth effects on consumption and savings behavior can be examined through the use of stock-dependent rewards in models of optimal growth. Kurz [1968] has shown that stock effects may lead to multiple optimal steady states, among other potential problems. Models with stock-dependent rewards have also been used to study the role of money in a growing economy (Sidrauski [1967], Brock [1974], Calvo [1979]). Feenstra [1986] shows that the incorporation of real balances in the utility function is equivalent to the appearance of liquidity costs in the budget constraint. In the area of international economics, utility of money models have been used to develop a theory of the balance of payments in dual exchange rate economies (Fried [1984], Obstfeld [1986]).

Much of the literature on stochastic dynamic resource allocation focusses on the long run behavior of optimal consumption and investment policies. For the one-sector optimal growth model with stock-independent rewards and concave production, a series of papers by Brock and Mirman [1972, 1973], and Mirman and Zilcha [1975, 1977] examine conditions under which the optimal capital stock converges to a unique limiting distribution. These results are extended to a model with non-convex technology by Majumdar, Mitra and Nyarko [1989], and to the case of irreversible investment by Olson [1989].

For stochastic models with stock-dependent rewards, the behavior of optimal policies has not been adequately characterized. In the fisheries literature, Jaquette [1972], Reed [1974, 1979] and Spulber [1982] show that an (S,s) inventory harvesting rule is optimal for the class of linear reward functions.[1] Unfortunately, such sharp results are not possible for more general models.

The purpose of this paper is to investigate the monotonicity and convergence properties of optimal policies within the context of a generic stochastic growth model with stock-dependent rewards. In our investigation we derive most results strictly from assumptions imposed on the primitive data of the model, that is, assumptions on preferences and technology. In Section 2 we show that complementarity between state and policy variables in the reward function is sufficient for optimal consumption and investment policies to be nondecreasing functions of the resource stock. Further, it is shown that optimal consumption always takes place in a region of the reward function that is increasing in c.

In Section 3 we investigate the dynamic behavior of the model. We begin by presenting some results on the convergence of Markov processes with monotone and continuous transition functions. These results are quite general and can be applied to models other than the stochastic growth model studied in this paper. In Section 3.2, we use these results to characterize the convergence of optimal resource stocks for the model of this paper. It turns out that the conditions of Section 2 are sufficient to imply that optimal resource stocks will converge to a (not necessarily unique) limiting distribution.

In Section 3.3 we investigate the existence of a unique limiting distribution for optimal resource stocks. This is an important issue since if the limiting distribution is unique then the long run behavior of optimal processes is independent of initial conditions. We show that when the reward function exhibits a form of "balanced growth complementarity," optimally managed resource stocks converge to a unique limiting distribution regardless of the discount rate. This condition is somewhat restrictive; however, we provide examples to show that multiple optimal steady states may arise if it is violated.

Finally, in Section 4 we show that a model that exhibits multiple steady states can be transformed into a model with a unique steady state merely by adding sufficient randomness to production. This suggests that highly variable economies may be less subject to dependence on initial conditions than economies exhibiting small variability. Concluding remarks are given in Section 5, while all proofs are relegated to Section 6.

There is a close similarity between our work and the analysis of renewable resource markets by Mirman and Spulber [1984] in the case of uncertainty, and by Levhari, Michener and Mirman [1981, 1982] in a deterministic setting. In this paper, we examine more closely the case where the reward function is stock-dependent. For this case, Mirman and Spulber provide no proof that the optimal consumption and investment policies are monotonic and they also assume the limiting distribution is unique. Our analysis focusses on when, in fact, these assumptions are justified. Such questions are important since work by Benhabib and Nishimura [1985, Theorem 2(ii)] can be used to construct examples where optimal investment policies are cyclic rather than monotonic, and, as already mentioned, the possibility of non-unique optimal steady states is well known for deterministic models with stock-dependent rewards.

Our work is also related to previous results obtained by Mendelssohn and Sobel [1980]. In proving the monotonicity of the optimal consumption (or harvest) policy, Mendelssohn and Sobel assume that the reward function is stock-independent. In this paper, we obtain monotonicity results for the case where rewards depend on the size of the resource stock. In addition, the convergence of resource stocks is ensured through assumptions imposed only on the primitive data of the model (ie., technology and preferences). Mendelssohn and Sobel use an alternative approach that does not rely on assumptions derived from the primitive structure of the model. Instead, their assumptions are imposed directly on the Markov transition kernel that governs the evolution of optimal resource stocks. The advantage of our approach is that the results can be interpreted in terms of complementarity conditions between resource stocks, consumption, and investment.

In the optimal growth area, our work extends the monotonicity and convergence results of Brock and Mirman [1972], and Mirman and Zilcha [1975]. In related work, Benhabib and Nishimura [1985, 1989] use a somewhat more general model to examine the occurrence of competitive and stochastic equilibrium cycles.

2. THE MODEL AND THE EXISTENCE OF AN OPTIMAL POLICY

In this section we develop a model of capital or renewable resources where rewards from consumption (or harvest) are stock-dependent and where the resource stock is subject to stochastic growth. At each date t there is a resource stock denoted by $y_t \in \Re_+$. Given knowledge of y_t, the agent determines a consumption level c_t. The resource stock left at the end of date t (after consumption) represents investment and is denoted by $x_t = y_t - c_t$. Let $\{r_t\}_{t=1}^{\infty}$ be an independent and identically distributed (i.i.d.) process taking values in some compact set Φ, where Φ is a subset of a finite dimensional Euclidean space. Growth in the resource stock is governed by the production relationship,

$$y_{t+1} = f(x_t, r_{t+1}) = f(y_t - c_t, r_{t+1}), \tag{2.1}$$

where $f: \Re_+ \times \Phi \to \Re_+$ is the production function. Let $\gamma(r_t)$ be the (common) probability measure associated with the shock process r_t. At the beginning of each period (before the consumption decision), it is assumed that the agent observes the true resource stock.

Given a resource stock y_t, and a consumption, c_t, the agent receives rewards $R(c_t, y_t)$. The agent seeks to solve the following problem:

$$\max_{\{c_t\}} E \sum_{t=0}^{\infty} \delta^t R(c_t, y_t) \tag{2.2}$$

subject to $0 \le c_t \le y_t$ where $\delta \in (0,1)$ is the discount factor.

The production and reward functions are assumed to satisfy the following restrictions throughout the paper.

A.2.1. For all r, $f(x,r)$ is strictly increasing in x.

A.2.2. f is concave in x.

A.2.3. For all r, $f(0,r) = 0$ while $f(x,r) > 0$ if $x > 0$.

A.2.4. The first and second derivatives of $f(x,r)$ exist and are continuous in (x,r).

A.2.5. There exists a $\bar{y}$ such that $f(y,r) < y$ a.s. for all $y \ge \bar{y}$.

A.2.6. $R(c,y)$ is nondecreasing in y.

A.2.7. $R(c,y)$ is concave in (c,y).

A.2.8. $R(c,y)$ is twice continuously differentiable.

A.2.9. $y_0 \in (0,\bar{y}]$.[2]

A suitable interpretation of variables allows this model to encompass the range of economic problems discussed in the introduction. Three applications are given below.

A. A General Model of Renewable Resources. In each period the agent harvests c_t from a renewable resource stock y_t. One example of such a resource is an ocean fishery. Growth in the resource stock follows the harvest-recruitment relationship $y_{t+1} = f(y_t - c_t, r_{t+1})$. In order to obtain a harvest c_t, the agent must expend some effort, e_t, that depends on the size of the resource stock (e.g., high levels of effort may be required if the resource stock is low). Effort may be random due to migration of the resource, as in a fishery. The effort function is given by $e_t = e(c_t, y_t, \epsilon_t)$, where ϵ_t is a random variable. The utility of the agent depends on consumption and the effort expended so that $U = u(e_t, c_t)$. In this case $R(c_t, y_t)$ is an expected reward function obtained by integrating over ϵ, ie. $R(c_t, y_t) = E_\epsilon u(e(c_t, y_t, \epsilon_t), c_t)$.

B. A Stochastic Monetary Growth Model Under Inflation. At the beginning of each period the agent has financial assets, $y_t = p_t x_{t-1} + w_t$, where w_t equals current period income and x_{t-1} is previous money balances. p_t can be thought of as an index that captures the change in the value of money balances due to inflation. Both p_t and w_t are allowed to be random. The agent consumes c_t, leaving new money balances of $x_t = y_t - c_t$. Financial assets evolve according to $y_{t+1} = p_{t+1}(y_t - c_t) + w_{t+1}$. Money enters the agent's utility function through liquidity or transactions costs so that $U = u(c_t, x_t)$. In this case, the reward function R is defined by the relation $R(c_t, y_t) = u(c_t, y_t - c_t)$.[3]

C. A Model of Wealth or Public Capital Effects and Consumption-Savings Behavior. If x_t = capital stock, y_t = output, and c_t = consumption then our framework fits the model used by Kurz [1968] to study wealth effects on consumption and savings behavior. Arrow and Kurz [1970] develop a similar model where publicly held capital enters the utility function.[4] In this case, our framework applies if output is produced solely by public capital.

The specification of the model satisfies the usual continuity and boundedness conditions. As a result, standard dynamic programming arguments (cf. Blackwell [1965] and Maitra [1968]) can be used to show that there exists a stationary optimal policy function of the form $c_t = C^*(y_t)$. The optimal policy function $C^*(\cdot)$ determines the optimal consumption level from any period t stock y_t. The optimal investment policy function is defined to be $X^*(y) = y - C^*(y)$. Let $\phi(y)$ be the set of optimal consumption levels from an initial stock, y. Clearly, $C^*(y)$ is a selection from $\phi(y)$. These results are summarized in the following theorem.

THEOREM 2.1. *Under A.2.1-A.2.9 there exists a stationary optimal policy function* $c_t = C^*(y_t)$. *Furthermore, the following functional equation holds:*

$$V(y) = \max_{0 \leq c \leq y} R(c,y) + \delta \int V(f(y-c,r))\gamma(dr). \tag{2.3}$$

In addition, the value function is continuous and the optimal consumption correspondence, $\phi(y)$ *is upper semicontinuous.*[5]

It is easy to show that the value function is nondecreasing and concave in y.

LEMMA 2.2. *V(y) is nondecreasing.*

LEMMA 2.3. *V(y) is concave.*[6]

If either of the following assumptions are satisfied, then $\phi(y)$ is unique-valued.

A.2.10. f(x,r) is strictly concave in x for each r and R(c,y) strictly increasing in y.

A.2.10'. R(c,y) is strictly concave.

Under Theorem 2.1 the optimal consumption correspondence, $\phi(y)$, is upper semicontinuous. Together with the additional assumption that optimal actions are unique, this implies that the optimal consumption and investment policy functions are continuous. We therefore have the following lemma.

LEMMA 2.4. *Under A.2.1-A.2.9 and either A.2.10 or A.2.10', $\phi(y)$ is unique-valued. Further, the optimal consumption policy function, $C^*(y)$, and the optimal investment policy function, $X^*(y)$, are both continuous in y.*

Throughout the remainder of this paper, we shall assume either A.2.10 or A.2.10′ holds so that the optimal consumption and investment levels are uniquely defined over the domain of resource stocks.

Define y^*_t, c^*_t, and x^*_t to be the optimal output stock, consumption level, and investment at date t. Henceforth, we assume that the optimal consumption and investment policies are strictly positive. This will hold if the usual Inada conditions are imposed on $R(\cdot)$. Under this assumption, one can show that optimal processes are characterized by the following stochastic Euler equation.

THEOREM 2.5. *Under A.2.1-A.2.10, if $c^*_t > 0$ and $y^*_t > 0$ for all t, then y^*_t, c^*_t, x^*_t satisfy*

$$R_c(c^*_t, y^*_t) = \delta E([R_c(c^*_{t+1}, y^*_{t+1}) + R_y(c^*_{t+1}, y^*_{t+1})]f'(x^*_t, r_{t+1})). \tag{2.4}$$

It should be noted that the reward function, R(c,y), is not necessarily monotone in c. This is especially typical in renewable resource models where higher consumption levels require more effort. In such models $R(\cdot)$ frequently increases with c, reaches a maximum, and then, as larger and larger effort levels are required to raise consumption, R begins to decline.

Even though R may not be monotone in c it turns out that optimal consumption always takes place in a region of R that is increasing in c. This implies that the optimal consumption level is never larger than that which would be chosen by a myopic decision-maker, ie., optimal consumption is less than or equal to that which maximizes current period rewards.

LEMMA 2.6. *If* $c < C^*(y)$ *then* $R(c,y) < R(C^*(y),y)$. *Furthermore, if* $\hat{C}(y)$ *is a solution to the one period problem*

$$\max_{0 \le c \le y} R(c,y)$$

then $\hat{C}(y) \ge C^*(y)$.

We now examine the monotonicity properties of the optimal consumption and investment policies. We shall need the following assumptions.

A.2.11. $R_{cy} \ge 0$.

A.2.11'. $R_{cy} > 0$.

A.2.12. $R_{cc} + R_{cy} \le 0$.

A.2.12'. $R_{cc} + R_{cy} < 0$.

A.2.11 and A.2.12 can be interpreted as complementarity conditions between consumption and output from production on the one hand, and investment and output from production on the other.[7]

In Theorem 2.7 below we show that under A.2.11 the optimal consumption policy function is nondecreasing in y. Then, in Theorem 2.8 we show that when A.2.12 holds, the optimal investment policy function is nondecreasing in y. Together, these two theorems show that complementarity between control (consumption and investment) and state variables is sufficient for the optimal policy functions to be nondecreasing.

THEOREM 2.7. *Under A.2.11,* $C^*(y)$ *is nondecreasing in y.* $C^*(y)$ *is strictly increasing if A.2.11′ holds.*

Recall that $X^*(y) = y - C^*(y)$, where $X^*(y)$ represents optimal investment from an initial stock y.

THEOREM 2.8. *Under A.2.12,* $X^*(y)$ *is nondecreasing in y.* $X^*(y)$ *is strictly increasing if A.2.12′ holds.*

Both A.2.11 and A.2.12 hold when the reward function depends only on consumption so that Theorems 2.7 and 2.8 apply to models with stock-independent rewards as a special case. Theorem 2.7 extends similar results obtained by Brock and Mirman [1972], Mirman and Zilcha [1975], and Mendelssohn and Sobel [1980], all of whom assume that the reward function depends solely on consumption. Theorem 2.8 is essentially a stochastic version of Theorem 2(i) of Benhabib and Nishimura [1985] and is also similar to Theorem 4.2 in Mendelssohn and Sobel [1980]. For the deterministic case, if $R_{cc} + R_{cy} > 0$, then it can be shown that optimal investment policies are cyclic (see Benhabib and Nishimura [1985, Theorem 2(ii)]).[8]

3. DYNAMICS

We now focus on the limiting behavior of optimal processes in a stochastic environment. In particular, we examine the optimal stochastic investment process governed by the transition equation

$$x_{t+1} = X^*(f(x_t, r_{t+1})).$$

We use the monotonicity of the optimal investment and consumption policy functions to prove that optimal processes converge to a limiting distribution, the stochastic analogue of a steady state. Then we give sufficient conditions for the limiting distribution to be unique. In this case there is a global convergence of the optimal process to the unique stochastic steady state. In general, however, the limiting distribution is not unique. We give examples where our sufficient conditions are not satisfied and where there are multiple steady states. For these examples, the long run behavior of optimal processes depends on the initial stock. Optimal processes from different initial stocks converge to different steady states. If the optimal policy functions are not monotone, these convergence results may not hold and there is the possibility of optimal processes exhibiting cyclical behavior (see Benhabib and Nishimura [1989] for an analysis of cycles in stochastic models).

3.1. THE GENERAL CASE: THE CONVERGENCE OF MARKOV PROCESSES WITH MONOTONE AND CONTINUOUS TRANSITION FUNCTIONS

In this sub-section we study the dynamic behavior of Markov processes generated by continuous and monotone transition functions. These Markov processes arise frequently in many economic models including models beyond the focus of this paper. Since the results of this section may be of independent interest we state them in slightly more general terms than needed for the stochastic growth model with stock-dependent rewards.

The convergence of monotone Markov processes to invariant distributions (i.e., stochastic steady states) has been discussed by Futia [1982], Hopenhayn and Prescott [1987], and Stokey, Lucas and Prescott [1989]. In the economics literature, existing results on convergence typically require conditions that are difficult to apply in practice and that also imply the uniqueness of invariant distributions. As a consequence, existing results cannot be used in many models. In particular, they cannot be applied to this paper's model which may have many invariant distributions.

In this section we use techniques developed by Dubins and Freedman [1966] to study the convergence behavior of the Markov process. In the economics literature, these methods were first used by Majumdar, Mitra, and Nyarko [1989] to study the stochastic optimal growth model with a non-convex technology. Their results rely on weak regularity conditions on the transition functions governing the evolution of optimal processes. For the model of this paper the transition equations are characterized by a stronger continuity condition. This enables us to adopt a somewhat simpler approach than that presented in Majumdar et. al. [1989].

Let the state space S be some interval $[\underline{s},\overline{s}]$. The driving stochastic process $\{r_t\}_{t=1}^{\infty}$ is assumed to be i.i.d. with support on a compact set Φ and a common marginal distribution on Φ given by γ. The stochastic process under study, $\{x_t\}_{t=0}^{\infty}$ evolves according to the relations

$$x_0 \in S \text{ and } x_{t+1} = H(x_t, r_{t+1}) \quad \text{for each } t \geq 0,$$

where for any realization r of r_{t+1}, $H(\cdot,r):S\to S$. Define $r^t = (r_1,\ldots,r_t)$ and let γ^t be the joint distribution of r^t on Φ^t. For each n and r^n, define $X^n(\cdot,r^n):S\to S$ by the relation

$$X^n(x,r^n) = H(\ldots(H(H(x,r_1),r_2,\ldots,r_n)$$

so that $X^n(x,r^n)$ is the realization of x_n when $x_0 = x$ and $r^n = (r_1,\ldots,r_n)$.

If μ is any probability on S define the probability $\gamma^n\mu$ on S to be

$$\gamma^n\mu(A) = \int_S \gamma^n(\{r^n \in \Phi^n | X^n(x,r^n) \in A\})\mu(dx),$$

where A is any (Borel) subset of S. $\gamma^n\mu$ is the distribution of x_n when the distribution of x_0 is μ. μ is an invariant probability if $\gamma^1\mu = \mu$. A subset S' of S is said to be γ-invariant if it is closed and if

$\gamma(\{r\in\Phi | H(x,r)\in S' \text{ for all } x\in S'\}) = 1$. A subset S" is a <u>minimal γ-invariant set</u> if it is γ-invariant and if any strict subset of S" is not γ-invariant.

Define

$$H_m(x) = \mathrm{Inf}_{r\in\Phi} H(x,r) \qquad \text{and} \qquad H_M(x) = \mathrm{Sup}_{r\in\Phi} H(x,r). \tag{3.1}$$

We impose the following assumptions on the transition function X:

A.H.1. $H(x,r)$ is monotone non-decreasing in x for each fixed $r\in\Phi$,

A.H.2. $H(x,r)$ is jointly continuous in x and r,

A.H.3. $H_M(x) > H_m(x)$ for all $x\in S$.

Later, we shall show that under suitable assumptions A.H.1-A.H.3 hold for the model studied in this paper.

Since we assume that H is jointly continuous and Φ is compact, the infimum and supremum in (3.1) are well-defined and attained for all x. In addition, the functions H_m and H_M are continuous. The following theorem (due to Dubins and Freedman [1966]) provides conditions under which the distribution of the x_t process converges to an invariant distribution.

THEOREM 3.1. (Dubins and Freedman [1966]) *Suppose A.H.1-A.H.3 hold and let S′ be a γ-invariant closed interval. If there exists a unique minimal γ-invariant closed interval S" in S′ then there is one and only one invariant probability μ^* in S′. Furthermore, for each probability*

μ on S', the distribution function of $\gamma^n \mu$ converges uniformly to the distribution function of μ^*.

Now, recall that the state space S is defined to be the closed interval $S = [\underline{s}, \bar{s}]$. Define the sequence $\{a_n, \hat{a}_n, b_n\}_{n=1}^{\infty}$ inductively by

$$\begin{aligned} a_n &= \text{Inf } \{x \geq b_{n-1} | H_m(x) = x\} \\ b_n &= \text{Inf } \{x > a_n | H_M(x) = x\} \\ \hat{a}_n &= \text{Sup } \{x \in [a_n, b_n] | H_m(x) = x\}, \end{aligned} \tag{3.2}$$

where $b_0 = \underline{s}$ (see Figure 1). Since H_m and H_M are continuous and S is compact, Brouwer's fixed point theorem implies that H_m and H_M have at least one fixed point. Thus, the points defined in (3.2) are well-defined for at least one n. Since H_m and H_M are continuous, whenever a_n, $\hat{a}_n$, and b_n are well-defined they are fixed points, ie. $a_n = H_m(a_n)$, $\hat{a}_n = H_m(\hat{a}_n)$, and $b_n = H_M(b_n)$.

Let N^* be the maximum over n such that a_n, $\hat{a}_n$, and b_n are well-defined.

LEMMA 3.2. *The fixed points in (3.2) are well-defined for only finitely many n, ie. $N^* < \infty$.*

Using Lemma 3.2, the continuity of H_m and H_M, and A.H.3, we obtain the following lemma.

LEMMA 3.3. *(a) For any $n \geq 1$, $H_M(x) \geq x$ for all $x \in [a_n, b_n]$ with strict inequality for $x \in [a_n, b_n)$.*

(b) For any $n \geq 1$, $H_m(x) \leq x$ for all $x \in [\hat{a}_n, b_n]$ with strict inequality for $x \in (\hat{a}_n, b_n]$.

(c) For any $n \geq 2$, $H_m(x) \leq x$ for all $x \in [b_{n-1}, a_n]$ with strict inequality for $x \in [b_{n-1}, a_n)$.

(c') $H_m(x) \geq x$ for all $x \in [b_0, a_1]$ with strict inequality for $x \in [b_0, a_1)$.

These lemmas enable us to rewrite the state space as

$$S = \cup_{n=1}^{N^*} [\hat{a}_n, b_n] \cup \cup_{n=2}^{N^*} (b_{n-1}, \hat{a}_n) \cup [b_0, \hat{a}_1) \cup (b_{N^*}, \bar{y}].$$

The following theorem characterizes the convergence of optimal processes.

THEOREM 3.4. *Suppose that A.H.1-A.H.3 hold.*

(a) Fix any integer $n \in [1, N^*]$. Then the set $[\hat{a}_n, b_n]$ is a γ-invariant interval and $[\hat{a}_n, b_n]$ is a unique γ-invariant interval in itself. Theorem 3.1 holds and there exists a unique γ-invariant distribution on $[\hat{a}_n, b_n]$. If $x_0 \in [\hat{a}_n, b_n]$ then the distribution function of x_t converges uniformly to the distribution function of the invariant distribution on $[\hat{a}_n, b_n]$.

(b) The set $T = \cup_{n=2}^{N^*} (b_{n-1}, \hat{a}_n) \cup [b_0, \hat{a}_1) \cup (b_{N^*}, \bar{y}]$ is transient. If $x_0 \in T$, then, with probability one, x_t will in finite time leave T and never return.

This theorem states that once a process enters an interval $[\hat{a}_n, b_n]$ it will remain there forever and converge to a unique limiting distribution on $[\hat{a}_n, b_n]$. Furthermore, sets of the type $(b_{n-1}, \hat{a}_n)$ are transient. The corollary below is an immediate consequence of Theorem 3.4.

COROLLARY 3.5. *Fix any distribution function F_0 for x_0 and let x_t be generated by the transition function $H(x,r)$ as defined earlier. Let F_t be the resulting distribution function of x_t. Given the transition function $H(x,r)$ define N^* as in Lemma 3.2 above. Then*

(a) F_t converges uniformly as $t \to \infty$ to the distribution function, F^*, of an invariant probability, and

(b) there is a unique invariant probability on S if and only if $N^* = 1$.

Corollary 3.5 states that optimal processes converge globally to the same (unique) invariant distribution on S if and only if there is only one interval of the type $[\hat{a}_n, b_n]$.

3.2. CONVERGENCE RESULTS FOR THE STOCHASTIC GROWTH MODEL WITH STOCK-DEPENDENT REWARDS

In this subsection we investigate the convergence of optimal processes for stochastic growth models with stock-dependent rewards. We show that the assumptions of Section 2 are sufficient to guarantee that optimal processes converge to an invariant distribution. In particular, we show that the assumptions of Section 2 are sufficient for A.H.1-A.H.3 to hold so that the results of Section 3.1 may be applied. In Subsection 3.3 we give conditions that guarantee the limiting distribution is unique.

Define

$$f_m(x) = \min_r f(x,r), \qquad f_M(x) = \max_r f(x,r)$$
$$X_m(x) = X^*(f_m(x)), \qquad X_M(x) = X^*(f_M(x)). \tag{3.3}$$

The minimum and maximum are well defined since f and X^* are continuous and defined over the compact domain $[0,\bar{y}]$, where $\bar{y}$ is given in A.2.5. Further, from the Maximum Theorem [Berge, 1963, p. 116], f_m, f_M, X_m, and X_M are all continuous in x. Define (see Figure 2):

$$z_m=\min\ \{x>0\,|\,X_m(x)=x\},\ z_M=\max\ \{x>0\,|\,X_M(x)=x\},$$
$$x_m=\max\ \{x>0\,|\,X_m(x)=x\},\ x_M=\min\ \{x>0\,|\,X_M(x)=x\}. \tag{3.4}$$

From the continuity of X_m and X_M and A.2.5, each of the above are well-defined and finite. From the definitions given in (3.4) it follows that $z_m \le x_m$ and $x_M \le z_M$.

We now impose the following assumptions.

A.3.1. There exists a $\theta > 0$ such that $X_m(x)>x$ for all $x\in(0,\theta)$.

A.3.2. If f is stochastic (ie., the distribution of r is nondegenerate) then there exists no $x > 0$ and $y \ge 0$ such that $\gamma(\{r\,|\,f(x,r)=y\})=1$.

A.3.3. $R_{cc} + R_{cy} < 0$.

Assumption A.3.1 ensures that $z_m > 0$. It holds under the following conditions on the primitives of the model.

A.3.1(a). $\lim_{x \to 0} f_x(x,r) = \infty$ for all r (Inada condition on f).

A.3.1(b). Either r is drawn from a finite set, or f(x,r) is ordered in r and the minimum shock has positive probability.

LEMMA 3.6. *Suppose A.3.1(a) and A.3.1(b) hold. Further, suppose that R(c,y) is strictly concave or that A.3.3 holds. Then, A.3.1 holds.*

A.3.1 prevents the optimal stock process from converging over time to zero even if the worst state occurs at each date. Hence, A.3.1 prevents extinction of the resource from being optimal. Mirman and Zilcha [1976] provide an example showing that for the model with stock-independent rewards if A.3.1(b) does not hold then $z_m = 0$ and there is a positive probability that the optimal stock process will come arbitrarily close to zero in the limit. It is easy to construct examples where A.3.1(a) fails and where extinction of the resource is optimal (e.g., with a linear production function).

Assumptions A.3.2 and A.3.3 are needed to rule out troublesome anomalies. Assumption A.3.2 states that if the production function is stochastic it must be sensitive to shocks to the environment for all x.

A.3.3 is identical to A.2.16′. It ensures that the optimal investment policy function is strictly increasing (see Theorem 2.8).

It turns out that these assumptions and the assumptions of the Section 2 are sufficient to guarantee that Theorem 3.1 holds and that optimal processes converge to an invariant limiting distribution.

THEOREM 3.7. *$F_t(x)$ converges uniformly in x to an invariant distribution.*

Theorem 3.7 rules out the possibility of optimal processes exhibiting cyclic behavior but leaves open questions about the number of limiting distributions and the local or global nature of convergence. These issues are now examined in more detail.

Two possible configurations for the fixed points defined by (3.4) are shown in Figure 2. Configuration A is characterized by $x_m \le x_M$, while in configuration B, $x_m > x_M$. The existence of a unique limiting distribution is guaranteed only for cases where $x_m \le x_M$ (configuration A of Figure 2). We now give two theorems that characterize the possible convergence results under configurations A and B. In the next subsection we provide sufficient conditions for configuration A to hold. Also, we give examples where there are multiple invariant distributions, ie. where configuration B applies.

Recall that F_t denotes the distribution function of x_t, where $\{x_t\}_{t=0}^{\infty}$ is the optimal investment process from the initial stock y_0. Define

$$y_m = f_m(x_m) \quad \text{and} \quad y_M = f_M(x_M). \tag{3.5}$$

THEOREM 3.8. *If configuration A holds then $F_t(x)$ converges uniformly in x to a unique invariant distribution, $F(x)$, independently of the initial stock y_0. In addition, the support of F is a subset of $[x_m, x_M]$.*

THEOREM 3.9. *Suppose configuration B holds. If $y_0 \in (0, y_M]$ (resp. $y_0 \in [y_m, \infty)$) then $F_t(x)$ converges uniformly in x to an invariant distribution $\bar{F}(x)$ (resp. $\underline{F}(x)$) whose support is a subset of $[z_m, x_M]$ (resp. $[x_m, z_M]$).*

These two theorems can be interpreted as follows. Under configuration A there is a unique limiting distribution and there is global convergence of optimal processes to this distribution from all initial resource stocks. Under configuration B there are at least two invariant distributions. From low initial stocks the optimal investment process converges to an invariant distribution in $[z_m, x_M]$, while from high initial stocks the optimal process converges to an invariant distribution in $[x_m, z_M]$. Of course, there could exist many other invariant distributions in the interval (x_m, x_M).

3.3. CONDITIONS FOR A UNIQUE INVARIANT DISTRIBUTION AND EXAMPLES OF MULTIPLE INVARIANT DISTRIBUTIONS

In this subsection we provide sufficient conditions for configuration A to hold. Under these conditions optimal policies converge to a non-trivial unique invariant distribution. We also provide examples where a violation of our sufficient conditions leads to multiple optimal steady states. We shall need the following assumptions.

A.3.4. $f(x,r)$ is strictly concave in x.

A.3.5. There exists a $\tilde{y} > 0$ such that for all $y \geq \tilde{y}$, $R_c(c,y) = 0$ implies $R_y > 0$, where $0 < c \leq y$.

A.3.6. For all $y > 0$, $0 < c < y$ and $\lambda > 1$ such that $R_c(c,y) > 0$ and $R_c(\lambda c, \lambda y) > 0$,

$$\frac{R_y(c,y)}{R_c(c,y)} \geq \frac{R_y(\lambda c, \lambda y)}{R_c(\lambda c, \lambda y)} .$$

In the standard stochastic growth model with stock independent rewards, $R_c > 0$ for all (c,y) so A.3.5 is trivially satisfied. A.3.5 is also satisfied if $R_y > 0$ for all $c > 0$ and $y > 0$. The assumption is needed to rule out the possibility of R attaining a maximum at c′ and, at the same time, being independent of y at c′. In Example 1 we show that multiple optimal steady states can result if A.3.5 is violated.

A.3.6 can be interpreted as a complementarity condition on the decision maker's preferences as consumption and output increase along a balanced growth path between c and y. $-R_y/R_c$ is the slope of indifference curves of R. Hence, A.3.6 implies that indifference curves for R have decreasing slopes as the resource stock and consumption increase along a ray through the origin in (y,c) space (see Figure 3).[9]

Assumption A.3.6 is satisfied if rewards are stock independent so that $R_y = 0$ for all (c,y). In addition, it does not rule out the possibility that $R_c = 0$ for some (c,y).[10] Later, in Example 2, we show that multiple optimal steady states may exist if A.3.6 is violated.

One class of reward functions that satisfies all of our assumptions including A.3.5 and A.3.6 is the class $R(c,y) = c^{\alpha}y^{\beta}$, where $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta < 1$.

We are now ready to state the following result.

THEOREM 3.10. *If A.3.1-A.3.6 hold in addition to the assumptions of Section 2, then optimal policies satisfy configuration A and the results of Theorem 3.8 hold, ie.* $F_t(x)$ *converges uniformly in x to a unique invariant distribution, F(x), independently of the initial stock* y_0. *In addition, the support of F is a subset of* $[x_m, x_M]$.

This theorem extends several well-known results in the literature. When viewed in the context of optimal growth, it generalizes the convergence results of Brock and Mirman [1972] and Mirman and Zilcha [1975] to the case of stock-dependent rewards. Mendelssohn and Sobel

[1980, Theorem 6.1] give conditions on the stochastic transition kernel governing the evolution of fish stocks that ensures the existence of a unique invariant distribution. In contrast, Theorem 3.10 uses assumptions imposed directly on the primitive data of the model to characterize the long run behavior of optimal policies.

It is well known that configuration B may hold and that multiple steady states may exist if the production function is not concave (see Majumdar, et. al. [1989]). We now give two examples where all our assumptions (including concavity of production) are satisfied except A.3.5 and A.3.6 and multiple optimal steady states exist. These examples indicate that even though A.3.5 and A.3.6 may be viewed as strong assumptions, it may be difficult to obtain uniqueness results if they are relaxed.

Example 1 - Violation of A.3.5 leads to multiple optimal invariant distributions.

In this subsection we construct a class of production functions such that for each production function in the class there are multiple optimal invariant distributions for all sufficiently small discount rates. This example may therefore be of independent interest.

Choose the reward function to be independent of y and strictly concave in c. In addition, let R(c,y) reach a maximum at c', with $R_c >$ 0 for c < c' and $R_c < 0$ for c > c'. Under these assumptions $R_{cc} < 0$ and $R_y = R_{yy} = R_{cy} = 0$ so that A.2.11, A.3.3 and A.3.6 hold. However, A.3.5 is violated since $R_y = 0$ at c'.

Let $g(x)$ be any function that satisfies assumptions A.2.1-A.2.5, A.3.1(a) and such that for some $x' > c'$, $g(x') > x' + c'$. Suppose further that $\lim_{x\to\infty} g'(x) = 0$. Fix any $k > \text{Max}\ \{x'/g(x'-c'),1\}$. Let $f(x,r)$ be any stochastic production function obeying all relevant assumptions of Sections 2 and 3 (ie., A.2.1-A.2.5, A.3.1(a), A.3.1(b), A.3.2., and A.3.4) with $f_m(x) = \min_r f(x,r) = kg(x)$. It should be clear that many such production functions can be constructed. Then, it follows that

$$f_m(x') > g(x') > x'+c' \tag{3.6}$$

and

$$f_m(x'-c') = kg(x'-c') > x'. \tag{3.7}$$

Given the above reward and productions functions, we shall show that configuration B holds and there are multiple optimal invariant distributions when the discount factor is sufficiently small.

First, we show that for all $y \geq x'$ optimal consumption is c'. This claim is proved by showing that it must hold for all finite horizon problems. Then, a limiting argument is used to show that c' is the optimal consumption level for the infinite horizon problem.

Consider the one-period problem

$$\max_{0 \leq c \leq y} R(c,y). \tag{3.8}$$

If $y \geq x' > c'$, then the construction of R implies optimal consumption is c'. Let $R' = R(c',y)$ and note that this is independent of y by assumption. The zero-horizon value function is $V^0(y) = R'$ for $y \geq x'$.

Next, suppose that for some arbitrary but finite T, the optimal consumption from an initial stock $y \geq x'$ is $c^*_t = c'$ for $t = 0,\ldots,T$. Then the T-horizon value function is $V^T(y) = (1+\delta+\ldots+\delta^T)R'$, for $y \geq x'$. Note that this is also independent of y. Now consider the T+1-horizon problem

$$\max_{0 \leq c \leq y} R(c,y) + \delta EV^T(f(y-c,r)). \tag{3.9}$$

$R(c,y)$ is strictly decreasing for $c > c'$ and $\delta EV^T(f(y-c,r))$ is nonincreasing in c so that the maximum of (3.9) is attained at some $c \leq c'$. Given $y \geq x'$ and $c \leq c'$, we have

$$f(y-c,r) \geq f_m(y-c') \geq f_m(x'-c') > x' \tag{3.10}$$

for all r, where the last inequality follows from (3.7).

Recall that $V^T(y) = (1+\delta+\ldots+\delta^T)R'$ is independent of y for $y \geq x'$. Since $f(y-c,r) > x'$, this implies that $\delta EV^T(f(y-c,r))$ is independent of c. Thus, (3.9) is maximized at c' for $y \geq x'$ and $V^{T+1}(y) = (1+\delta+\ldots+\delta^{T+1})R'$.

It follows by induction that, if $y \geq x'$, c' is optimal in all periods for any T-horizon problem. Taking limits as $T\to\infty$ and invoking standard dynamic programming arguments proves that c' is optimal in all periods for the infinite horizon problem.

We have now established that the optimal investment policy function is $X^*(y) = y - c'$ for all $y \geq x'$. Recall that $X_m(x) = X^*(f_m(x))$. Since (3.6) implies $f_m(x') > x'$, we obtain $X_m(x) = f_m(x) - c'$ for all $x \geq x'$. For sufficiently large x, A.2.5 implies $f_m(x) - c' < x - c' < x$. Also, from (3.6) $x \geq x'$ implies $f_m(x) - c' \geq f_m(x') - c' > x'$. Hence, $X_m(x) = f_m(x) - c'$ has a fixed point at some $x > x'$. This implies

$$x_m = \max\ \{x > 0 | X_m(x) = x\} > x'. \tag{3.11}$$

Now, note that the optimal investment policy function depends on the discount factor, δ. To emphasize this dependence we shall write X^* as $X^*_\delta(y)$. Let $X^*_0(y)$ be the optimal investment policy function for the one-period problem (3.8). Using standard dynamic programming arguments it can be shown that for each $y \geq 0$,

$$\lim_{\delta \to 0} X^*_\delta(y) = X^*_0(y). \tag{3.12}$$

By the construction of R, $R_c > 0$ for $c < c'$. For $y \leq c'$, the one-period problem (3.8) attains a maximum at $c = y$. This implies $X^*_0(y) = 0$ for $y \leq c'$. Using this fact, (3.12) implies

$$\lim_{\delta \to 0} X^*_\delta(y) = X^*_0(y) = 0 \qquad \text{for } y \leq c'. \tag{3.13}$$

Now, recall that $f_M(x) = \max_r f(x,r)$ and $X_M(x) = X^*(f_M(x))$. Again, X^* depends on δ. Fix any x" such that $f_M(x") < c' < x'$. Since $f_M(x) \geq f_m(x) \geq x'$ for $x \geq x'$, it must be that $x" < x'$. Using (3.13), it follows that for sufficiently small, but strictly positive δ, $X_M(x") < x"$. However, A.3.1 implies that for $x > 0$ sufficiently small, $X_M(x) > x$. Together, these results imply that $X_M(x)$ has a fixed point in $(0,x")$. Thus,

$$x_M = \min \{x > 0 | X_M(x) = x\} < x" < x'. \tag{3.14}$$

Combining (3.11) and (3.14) shows that $x_M < x_m$. Configuration B holds and Theorem 3.9 implies the existence of multiple optimal invariant distributions.

Example 2 - Violation of A.3.6 leads to multiple optimal steady states.

Define $R(c,y) = 16y - 1/2(c-24)^2$. Then $R_c = 24-c$, $R_y = 16$, $R_{yy} = R_{cy} = 0$, and $R_{cc} = -1$. R is nondecreasing in y and concave, as required, A.3.3 and A.3.5 are satisfied, but $R_y(\lambda c,\lambda y)/R_c(\lambda c,\lambda y) = 16/(24-\lambda c)$ so A.3.6 is violated. Assume that the production function is $f(x) = 10x^{1/2}$. Thus, A.3.1(a) is also satisfied (A.3.2 is irrelevant for nonstochastic models). One can check that when $\delta = 0.10$, the Euler equation is satisfied when (y_t,x_t,c_t) are any one of the three stationary triples given below:

a) $y_1 = 10.4695$, $x_1 = 1.0961$, $c_1 = 9.3734$

b) $y_2 = 29.2739$, $x_2 = 8.5696$, $c_2 = 20.7043$

c) $y_3 = 65.2567$, $x_3 = 42.5843$, $c_3 = 22.6723$.

Further, maintaining any of these three triples as a steady state is feasible. Since the transversality condition also holds, each of these three triples is an optimal steady state.[11]

Levhari, Michener, and Mirman [1981] have noted the fact that there may be at most one optimal steady state above the maximum sustainable output level. Our example provides evidence of this phenomenon.

4. SUFFICIENT VARIABILITY IN PRODUCTION AND THE EXISTENCE OF A UNIQUE INVARIANT DISTRIBUTION

In this section we show that even if the sufficient conditions of Section 3.2 fail, a model with a unique limiting distribution can be obtained through a sufficient "stretching out" of the randomness in the production function.

Let $\{f^k(x,r)\}_{k=0}^{\infty}$ be a collection production functions. We impose the following assumptions on the collection:

A.4.1. For each k, $f^k(x,r)$ obeys all the assumptions of Section 2 and A.3.1.

A.4.2. For each $x > 0$, $\lim_{k\to\infty} \text{Max}_r\, f^k(x,r) = \infty$.

A.4.3. For each $x > 0$, $\lim_{k\to\infty} E\partial f^k(x,r)/\partial x = \infty$.

A.4.4. For each $x > 0$ and $k \geq 0$, $f_m^k(x) \leq f_m^0(x)$.

Assumption A.4.2 ensures that the production function becomes arbitrarily large in the best state while A.4.4 ensures that in the worst state the production function is uniformly bounded above. This formalizes the notion of "stretching out" the production function. Assumption A.4.3 is an additional assumption that requires that the expected marginal product becomes arbitrarily large.

To illustrate these assumptions, suppose that the shocks to the production function enter multiplicatively so that $f(x,r) = rf(x)$ with r taking values in the set $[1,\bar{r}]$ where $\bar{r} > 1$. Further, suppose that $f(x,r)$ obeys all the assumptions of Sections 2 and assumption A.3.1. Define $f^0(x,r) = f(x,r)$ and $f^k(x,r) = r^k f(x,r)$. Then it is easy to see that the collection of production functions $\{f^k(x,r)\}_{k=0}^{\infty}$ obeys A.4.1-A.4.4.

To further illustrate the need for A.4.4, consider the class of production functions defined by $f^k(x,r) = kf(x,r)$ where $f(x,r)$ obeys all the assumptions of Section 2. This class of production functions essentially involves a simple change of units so that one does not expect the limiting behavior of optimal policies to vary with k. This class is ruled out by A.4.4.

We now state the following result.

THEOREM 4.1 *Let $\{f^k(x,r)\}_{k=0}^{\infty}$ be a collection production functions satisfying A.4.1-A.4.4 above. Suppose that the reward function obeys all assumptions of Section 2 (but not necessarily the assumptions of Section 3). Suppose, in addition, that the reward function is either strictly concave or that A.3.2. holds. Then for all k sufficiently large the model with production functions $f^k(x,r)$ has a unique non-trivial invariant distribution and the conclusions of Theorem 3.8 hold.*

This theorem implies that the convergence properties of models with monotone transition functions depend substantially on the degree of randomness in the model. In models with multiple stochastic steady states, the global convergence of optimal processes to a unique invariant distribution can be obtained merely by adding sufficient randomness to the model. In other words, enough variability in production forces the long run behavior of optimal processes to be independent of initial conditions.

5. CONCLUDING REMARKS

This paper examines the behavior of optimal consumption and investment policies in aggregate stochastic growth models with stock-dependent rewards. Such models are important in the study of renewable resources, monetary growth, and growth with public capital. The paper shows that complementarity conditions are important in guaranteeing the monotonicity and convergence of optimal policies. Further work is needed on the behavior of growth models where these conditions do not hold. In such cases optimal policy functions may be non-monotonic and cyclic behavior is possible (see Benhabib and Nishimura [1985, 1989] for important work along these lines).

6. PROOFS

Proof of Theorem 2.1. Under A.2.1 and A.2.5 it is easy to show that if $y_0 \in [0,\bar{y}]$ and $(c_t)_{t=0}^{\infty}$ is a feasible consumption process from y_0 then

$c_t \leq \bar{y}$ for all t. Hence one may define the action space to be A = $[0,\bar{y}]$. Since A is compact, one then applies standard dynamic programming arguments. //

Proof of Lemma 2.2. Assume $y > y'$. Let $\{c'_t\}$ be the optimal consumption sequence from y'. Define the sequence y''_t as follows. Let $y''_0 = y$, $y''_1 = f(y-c'_0,r_1)$, $y''_t = f(y''_{t-1}-c'_t,r_t)$ for $t \geq 2$. Using a simple induction argument it is easy to show that the sequence $\{y''_t,c'_t\}$ is feasible from y and that $y''_t \geq y'_t$ for all t, where $\{y'_t\}$ is the sequence of stocks obtained from an initial stock of y' following the consumption sequence $\{c'_t\}$. Then, $V(y) \geq \Sigma_{t=0}^{\infty} \delta^t R(c'_t,y''_t) \geq \Sigma_{t=0}^{\infty} \delta^t R(c'_t,y'_t) = V(y')$. The last inequality follows from the fact that R is nondecreasing in y and $y''_t \geq y'_t$ for all t. //

Proof of Lemma 2.3. The result follows immediately from the concavity of R(c,y). //

Proof of Lemma 2.4. Suppose that there exists $c \neq c'$ such that both c and c' are optimal from y. Define $\bar{c} = \alpha c + (1-\alpha)c'$, where $\alpha \in (0,1)$.

First, suppose that A.2.10 holds. By modifying the proof of Lemma 2.2 in the appropriate places, one can show that V(y) is strictly increasing. Specifically, using the definitions in the proof of Lemma 2.2, one can show that $y''_t > y'_t$ for all t. Then, $V(y) \geq R(c'_1,y) + \Sigma\, \delta^{t-1} ER(c'_t,y''_t) > \Sigma\, \delta^{t-1} R(c'_t,y'_t) = V(y')$, where the last inequality follows from $y''_t > y'_t$ for all t, and the assumption that R strictly increasing in y. Thus, V(y) is strictly increasing. Since c and c′ are both optimal from y, $V(y) = R(c,y) + \delta EV(f(y-c,r)) = R(c',y) + \delta EV(f(y-c',r))$. The definition of $\bar{c}$ implies $R(\bar{c},y) + \delta EV(f(y-\bar{c},r)) > R(\bar{c},y) + \delta EV(\alpha f(y-c,r)+(1-\alpha)f(y-c',r)) \geq \alpha[R(c,y) + \delta V(f(y-c,r))] + (1-\alpha)[R(c',y) + \delta V(f(y-c',r))] = V(y)$. The strict inequality follows from the strict concavity of f and V(y) being strictly increasing. The weak inequality is due to the concavity of R and V. This contradicts the optimality of c and c′.

Second, suppose that A.2.10′ holds. R(c,y) strictly concave implies V(y) is strictly concave. Note that $c \neq c'$ and A.2.1 imply that γ-a.e. $f(y-c,r) \neq f(y-c',r)$. Then, $R(\bar{c},y) + \delta EV(f(y-\bar{c},r)) \geq \alpha R(c,y) + (1-\alpha)R(c',y) + \delta EV(\alpha f(y-c,r)+(1-\alpha)f(y-c',r)) > \alpha[R(c,y) + \delta V(f(y-c,r))] + (1-\alpha)[R(c',y) + \delta EV(f(y-c',r))] = V(y)$. The first inequality follows from the concavity of R and f, while the second is due to $f(y-c,r) \neq f(y-c',r)$ and the strict concavity of V. //

<u>Proof of Theorem 2.5.</u> The proof follows the proof of Theorem 4.2 in Majumdar, Mitra, and Nyarko [1989] once it is modified to account for the dependence of R on both c and y. //

<u>Proof of Lemma 2.6.</u> Suppose the first assertion of the lemma does not hold. Then there exists some $c < C^*(y)$ such that $R(c,y) \geq R(C^*(y),y)$. But this implies $R(c,y) + \delta EV(f(y-c,r)) \geq R(C^*(y),y) + \delta EV(f(y-C^*(y),r))$ since f is strictly increasing and V is nondecreasing. Hence, c is optimal from y. This contradicts the

uniqueness of $C^*(y)$ (see Lemma 2.4). The second assertion of the lemma follows immediately from the first. //

Proof of Theorem 2.7. Let $y > y'$ and let c and c' by optimal from y and y', respectively. Suppose that $c < c'$. Then $c' \le y' < y$ and $c < c' \le y'$ so that c' is feasible from y and c is feasible from y'. $R_{cy} \ge 0$ implies

$$[R(c',y) - R(c,y)] - [R(c',y') - R(c,y')] \ge 0. \tag{2.7.1}$$

V and f concave imply that

$$\int\{[V(f(y-c',r)) - V(f(y-c,r))] - [V(f(y'-c',r)) - V(f(y'-c,r))]\}\gamma(dr) \ge 0. \tag{2.7.2}$$

Adding (2.7.1) and (2.7.2) gives

$$[R(c',y)+\delta\int V(f(y-c',r))\gamma(dr) - R(c,y)-\delta\int V(f(y-c,r))\gamma(dr)] - [R(c',y')+\delta\int V(f(y'-c',r))\gamma(dr) - R(c,y')-\delta\int V(f(y'-c,r))\gamma(dr)] \ge 0. \tag{2.7.3}$$

We have $[R(c',y) + \delta\int V(f(y-c',r))\gamma(dr) - R(c,y) - \delta\int V(f(y-c,r))\gamma(dr)] < 0$ by the optimality of c from y, and $[R(c',y') + \delta\int V(f(y'-c',r))\gamma(dr) - R(c,y') - \delta\int V(f(y'-c))\gamma(dr)] > 0$ by the optimality of c' from y'. In both cases, the strict inequality results from the uniqueness of $C^*(y)$. Combining these last two inequalities shows that (2.7.3) must be strictly negative. This is a contradiction.

If A.2.11′ holds, then the inequality in (2.7.1) is strict. When combined with the above arguments this implies $C^*(y)$ is strictly increasing. //

Proof of Theorem 2.8. Suppose that $y > y'$ and $x' > x$, where x' and x are optimal from y' and y, respectively. Clearly, x is feasible from y', while x' is feasible from y since $y-x' > y'-x'$. The functional equation (2.3) implies $R(y-x,y) + \delta\int V(f(x,r))\gamma(dr) > R(y-x',y) + \delta\int V(f(x',r))\gamma(dr)$ and $R(y'-x',y') + \delta\int V(f(x',r))\gamma(dr) > R(y'-x,y') + \delta\int V(f(x))\gamma(dr)$, where the strict inequality follows from the uniqueness of $X^*(y)$. Summing these together gives

$$R(y-x,y) - R(y-x',y) > R(y'-x,y') - R(y'-x',y') \tag{2.8.1}$$

We are now interested in seeing how the difference $R(y-x,y) - R(y-x',y)$ varies with changes in y (holding x and x' fixed). Differentiating gives $\partial[R(y-x,y) - R(y-x',y)]/\partial y = [R_c(y-x,y) + R_y(y-x,y)] - [R_c(y-x',y) + R_y(y-x',y)]$. This expression is non-positive under A.2.12. But, this implies

$$R(y-x,y) - R(y-x',y) \le R(y'-x,y') - R(y'-x',y') \tag{2.8.2}$$

which contradicts (2.8.1).

Under A.2.12′, the inequality in equation (2.8.2) is strict. When combined with the above arguments this implies that $X^*(y)$ is strictly increasing. //

Proof of Theorem 3.1. See Dubins and Freedman [1966, Theorem 5.15 and Corollary 5.5) or Majumdar et. al. [1989, Theorem 6.2]. The latter strengthens the results of Dubins and Freedman for the case where $H(\cdot,r)$ does not satisfy A.H.2. //

Proof of Lemma 3.2. Suppose that the fixed points in (3.2) are well-defined for infinitely many n. Recall that $S = [\underline{s},\bar{s}]$ so each fixed point is bounded above by $\bar{s}$. Since the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are monotone nondecreasing and bounded above they must converge to some limits a^* and b^* respectively. Since $a_n \le b_n \le a_{n+1}$, as $n\to\infty$ we get $a^* = b^*$. Taking limits in the inequality

$$H_M(b_n) = b_n \le a_{n+1} = H_m(a_{n+1})$$

implies that $H_M(b^*) \le H_m(a^*) = H_m(b^*)$. This contradicts A.H.3. //

Proof of Lemma 3.3. (a) From the definition of b_n, H_M has no fixed points in $[a_n,b_n)$. Under condition A.H.3, $H_M(a_n) > H_m(a_n) = a_n$. Therefore, if there is an x in $[a_n,b_n)$ such that $H_M(x) < x$ then the mean-value theorem implies that H_M has a fixed point in $[a_n,x)$, but this is a contradiction. Hence, $H_M(x) > x$ for all x in $[a_n,b_n)$. Since b_n is a fixed point of H_M, part (a) follows. The proofs of (b), (c) and (c′) are similar and are therefore omitted. //

Proof of Theorem 3.4. (a) Fix any integer n in $[1,N^*]$ and any x in $[\hat{a}_n,b_n]$. Then for any r in Φ,

$$\hat{a}_n = H_m(\hat{a}_n) \le H_m(x) \le H(x,r) \le H_M(x) \le H_M(b_n) = b_n$$

so for all r, $H(x,r) \in [\hat{a}_n, b_n]$. The set $[\hat{a}_n, b_n]$ is therefore a γ-invariant closed interval.

Now let $[a,b]$ be any closed interval in $[\hat{a}_n, b_n]$. If $b < b_n$, then from Lemma 3.3(a), $H_M(b) > b$, so $[a,b]$ is not γ-invariant. If $a > a_n$ then from Lemma 3.3(b), $H_m(a) < a$, so $[a,b]$ is not γ-invariant. Hence there is no closed interval $[a,b]$ strictly contained in $[\hat{a}_n, b_n]$ which is γ-invariant so $[\hat{a}_n, b_n]$ is the unique (and hence minimal) γ-invariant interval in itself.

(b) Fix any integer n in $[2, N^*]$. For such n, the set $(b_{n-1}, \hat{a}_n)$ has the two intervals $[\hat{a}_{n-1}, b_{n-1}]$ and $[\hat{a}_n, b_n]$ to its left and right respectively, and both of these sets are γ-invariant as shown in (a) above. Hence, once the x_t process leaves the set $(b_{n-1}, \hat{a}_n)$ it never returns to it.

From Lemma 3.3(a), $H_M(x) > x$ for all x in $[a_n, \hat{a}_n)$. Since H_M is continuous there exists an $\epsilon > 0$ and a $k > 0$ such that $H_M(x) > x+k$ for $x \in [a_n - \epsilon, \hat{a}_n)$. Define the sequence of numbers $\{\hat{x}_t\}_{t=0}^{\infty}$ by $\hat{x}_0 = a_n - \epsilon$ and $\hat{x}_{t+1} = H_M(\hat{x}_t)$. Let J_1 be any integer greater than $(\hat{a}_n - (a_n - \epsilon))/k$. Then it is easy to see that $\hat{x}_t$ will enter the set $[\hat{a}_n, b_n]$ in less than J_1 periods.

For any $\{x_t\}_{t=0}^{\infty}$ process generated by $H(x,r)$ define,

$$q_1 = \text{Prob}\ (\{x_{J_1} \in [\hat{a}_n, b_n] \mid x_0 = a_n - \epsilon\}).$$

Then $q_1 > 0$. Since the transition function $H(x,r)$ is monotone in x,

$$\text{Prob }(\{x_{J_1} \in [\hat{a}_n, b_n] \mid x_0 \in [a_n - \epsilon, \hat{a}_n)\}) \geq q_1. \tag{3.4.1}$$

From Lemma 3.3(c), $H_m(x) < x$ for all x in $[b_{n-1}, a_n - \epsilon)$. Hence, by an argument similar to that just used above, there exists a $q_2 > 0$ and an integer J_2 such that

$$\text{Prob }(\{x_{J_2} \in [\hat{a}_{n-1}, b_{n-1}] \mid x_0 \in (b_{n-1}, a_n - \epsilon)\}) \geq q_2. \tag{3.4.2}$$

Define $q = \text{Min }\{q_1, q_2\}$ and $J = J_1 + J_2$. Then from (3.4.1) and (3.4.2),

$$\text{Prob }(\{x_J \text{ leaves } (b_{n-1}, \hat{a}_n) \mid x_0 \in (b_{n-1}, \hat{a}_n)\}) \geq q > 0.$$

Finally, if x_0 belongs to $(b_{n-1}, \hat{a}_n)$,

$$\begin{aligned}
&\text{Prob }(\{x_t \in (b_{n-1}, \hat{a}_n) \text{ for all } t\}) \\
&\leq \text{ Prob }(\{x_{kJ} \in (b_{n-1}, \hat{a}_n) \text{ for all } k = 1,2,\ldots\}) \\
&= \Pi_{k=1}^{\infty} \text{ Prob }(\{x_{kJ} \in (b_{n-1}, \hat{a}_n) \mid x_{mJ} \in (b_{n-1}, \hat{a}_n) \text{ for all } m < k\}) \\
&\leq \Pi_{k=1}^{\infty} (1-q) = 0.
\end{aligned}$$

Hence, the set $(b_{n-1}, \hat{a}_n)$ for $2 \leq n \leq N^*$ is transient.

Using similar methods one can show that if $x_0 \in [b_n, \hat{a}_1)$ then x enters $[\hat{a}_1, b_1]$ in finite time and if $x_0 \in (b_{N*}, \bar{y}]$ then x_t enters $[\hat{a}_{N*}, b_{N*}]$ in finite time. This proves part (b). //

Proof of Lemma 3.6. We begin by proving several subsidiary lemmas. Some of these will also be used in later proofs.

LEMMA 3.6.1. *Suppose that either R(c,y) is strictly concave or that A.3.3 holds. If R(c,y) attains a global maximum on* $\Re_{++}$, *then the maximum is attained a unique value of c, ie., if R(c,y) attains its global maximum at two points (c′,y′) and (c″,y″) in* $\Re_{++}$, *then* $c' = c''$.

Proof of Lemma 3.6.1. The lemma is trivial if R(c,y) is strictly concave. Suppose instead that A.3.3 holds and suppose, per absurdum that the global maximum of R(c,y) is attained at the two points (c′,y′) and (c″,y″) in $\Re_{++}$, with c′ different from c″ and $y' \leq y''$. By assumption R(c,y) is non-decreasing in y. This means that the global maximum of R is also attained at (c′,y″). Since (c′,y″) and (c″,y″) are both global maxima and R is differentiable the first order conditions of calculus imply that $R_c(c,y) = R_y(c,y) = 0$ at both these points. Therefore the sum $R_c + R_y$ must be zero at both points. However, from assumption A.3.3, $R_{cc} + R_{cy} < 0$. Thus, $R_c + R_y$ must be different at the two points (c′,y″) and (c″,y″). This is a contradiction which proves the lemma. //

LEMMA 3.6.2. *For all* $y > 0$ *sufficiently small,* $R_c(C^*(y),y) > 0$.

Proof of Lemma 3.6.2. For ease of exposition define $Y(y,r) = f(X^*(y),r)$. Suppose that $R_c(C^*(y),y) = 0$. From the Euler condition,

$R_c(C^*(y),y) =$

$$E\{f'(X^*(y),r)[R_c(C^*(Y(y,r)),Y(y,r)) + R_y(C^*(Y(y,r)),Y(y,r))]\}.$$

Using the fact that $R_y \geq 0$ by assumption and $R_c(C^*(y),y) \geq 0$ for all y (Lemma 2.6), the Euler condition implies that $R_c(C^*(Y(y,r)),Y(y,r)) + R_y(C^*(Y(y,r)),Y(y,r)) = 0$ for each r. Combining this with the fact that $R_c(C^*(y),y) \geq 0$ and $R_y \geq 0$ gives $R_c(C^*(Y(y,r)),Y(y,r)) = R_y(C^*(Y(y,r)),Y(y,r)) = 0$ for all r. The concavity of R then implies that R attains its global maximum at $(C^*(Y(y,r)),Y(y,r))$.

From Lemma 3.6.1, we know that if R(c,y) has a global maximum in $(c,y)\in\mathbb{R}_{++}$, then the global maximum is attained at a unique value of c. Let this maximum be $\bar{c}$. Then $C^*(Y(y,r)) = \bar{c}$ for all r. Under our assumptions $C^*(y) > 0$ for all y, so that $\bar{c} > 0$. However, as $y\to 0$, $Y(y,r) = f(X^*(y),r)\to 0$. This implies that $C^*(Y(y,r)) < \bar{c}$ for all $y > 0$ sufficiently small which is a contradiction. //

LEMMA 3.6.3. *Suppose that for some $\bar{r} \in \Phi$, $\gamma(\{\bar{r}\}) > 0$. Then, as a function of x, $X^*(f(x,\bar{r}))$ cannot have a sequence of positive fixed points $\{x^n\}_{n=1}^{\infty}$ with $x^n\to 0$ as $n\to\infty$.*

Proof of Lemma 3.6.3. Suppose the lemma does not hold. Then there exists a sequence $\{x^n\}_{n=1}^{\infty}$ of positive fixed points of $X^*(f(x,\bar{r}))$ such that $x^n\to 0$ as $n\to\infty$. Define $y^n = f(x^n,\bar{r})$. From the Euler condition, if $Y(y,r) = f(X^*(y,r))$ then

$$R_c(C^*(y^n),y^n) = \tag{3.6.1}$$

$$\delta\int\{f'(X^*(y^n),r)[R_c(C^*(Y(y^n,r)),Y(y^n,r))+R_y(C^*(Y(y^n,r)),Y(y^n,r))]\}\gamma(dr)$$

$$\geq \delta\gamma(\{\bar{r}\})f'(X^*(y^n),\bar{r})[R_c(C^*(Y(y^n,\bar{r})),Y(y^n,\bar{r}))+R_y(C^*(Y(y^n,\bar{r})),Y(y^n,\bar{r}))].$$

From the definition of x^n as a fixed point, $x^n = X^*(f(x^n,\bar{r})) = X^*(y^n)$. Thus, $Y(y^n,\bar{r}) = f(X^*(y^n),\bar{r}) = f(x^n,\bar{r}) = y^n$ and (3.6.1) can be rewritten as

$$R_c(C^*(y^n),y^n) \geq \delta\gamma(\{\bar{r}\})f'(x^n,\bar{r})[R_c(C^*(y^n),y^n)+R_y(C^*(y^n),y^n)]. \tag{3.6.2}$$

From Lemma 3.6.2 we may suppose $R_c(C^*(y^n),y^n) > 0$ as $n\to\infty$. Rearranging (3.6.2) then gives

$$f'(x^n,\bar{r}) \leq \frac{R_c(C^*(y^n),y^n)}{\delta\gamma(\{\bar{r}\})[R_c(C^*(y^n),y^n)+R_y(C^*(y^n),y^n)]} \leq \frac{1}{\delta\gamma(\{\bar{r}\})}. \tag{3.6.3}$$

Under assumption A.3.1(a) $\lim_{x^n\to 0} f'(x^n,\bar{r})\to\infty$. This is a contradiction since $1/\delta\gamma(\{\bar{r}\})$ is finite. //

LEMMA 3.6.4. *The function of y given by*

$$R_c(C^*(y),y) + R_y(C^*(y),y)$$

is nonincreasing in y.

Proof of Lemma 3.6.4. Consider the functional equation

$$V(y) = \max_{0 \le x \le y} R(y-x,y) + \delta EV(f(x,r)).$$

V is differentiable at y > 0 by the envelope theorem of Benveniste and Scheinkman [1979, Theorem 1] and

$$V'(y) = \frac{\partial}{\partial y} R(y-x,y) \Big|_{x = X^*(y)}.$$

$$= R_c(C^*(y),y) + R_y(C^*(y),y).$$

The lemma follows from the concavity of V (see Lemma 2.3). //

LEMMA 3.6.5. *If there exists some $\bar{r} \in \Phi$ such that $\gamma(\{\bar{r}\}) > 0$, then $X^*(f(x,\bar{r})) > x$ for all $x > 0$ sufficiently small.*

Proof of Lemma 3.6.5. From Lemma 3.6.3 the function $X^*(f(x,r))$ cannot have a sequence of fixed points arbitrarily close to 0. Hence, either $X^*(f(x,r)) > x$ for all $x > 0$ sufficiently small, or $X^*(f(x,r)) < x$ for all $x > 0$ sufficiently small (otherwise the existence of arbitrarily small fixed points would follow from the mean-value theorem). Suppose, per absurdum, that

$$X^*(f(x,\bar{r})) < x \text{ for all } x > 0 \text{ sufficiently small.} \qquad (3.6.4)$$

Using similar arguments to those preceding equation (3.6.3) we get,

$$\frac{R_c(C^*(y),y) + R_y(C^*(y),y)}{R_c(C^*(y),y)} \leq \frac{1}{\delta\gamma(\{\bar{r}\})f'(x,\bar{r})} \ . \tag{3.6.5}$$

Under A.3.1.(a), the right-hand side of (3.6.5) tends to zero as y tends to zero so for all $y > 0$ sufficiently small the left-hand side is less than one, and therefore

$$\begin{aligned} &R_c(C^*(Y(y,\bar{r})),Y(y,\bar{r})) + R_y(C^*(Y(y,\bar{r})),Y(y,\bar{r})) \\ &\quad < R_c(C^*(y),y) \leq R_c(C^*(y),y) + R_y(C^*(y),y). \end{aligned} \tag{3.6.6}$$

Lemma 3.6.4 shows that $R_c(C^*(y),y) + R_y(C^*(y),y)$ is nonincreasing in y. Using this, (3.6.6) implies $f(X^*(y),\bar{r}) = Y(y,\bar{r}) \geq y$ so that $X^*(f(X^*(y),\bar{r})) \geq X^*(y)$. As $y\to 0$, $X^*(y)\to 0$ so for arbitrarily small x (ie., for $x = X^*(y)$ in the previous inequality), $X^*(f(x,\bar{r})) \geq x$. This contradicts (3.6.4), concluding the proof. //

We now complete the proof of Lemma 3.6. The lemma follows from either of the two hypotheses in assumption A.3.1(b) since Lemma 3.6.5 states that for all r of positive probability, $X^*(f(x,r)) > x$ if $x > 0$ and sufficiently small. //

<u>PROOF OF THEOREMS 3.7, 3.8 AND 3.9.</u> These theorems are proved by adapting the results of Section 3.1. From Lemma 3.6 we know that there exists some $\underline{s} > 0$ such that $X_m(x) > x$ for all $x\in(0,\underline{s})$. Hence, if $x_0\in(0,\underline{s})$, x_t will enter the set $[\underline{s},\infty)$ in finite time and stay there forever. Further, from assumption A.2.5 there is a $\bar{y} < \infty$ such that for

all $x \geq \bar{y}$, $f(x,r) < x$. From this is follows that if $x_0 > \bar{y}$, x_t will enter the set $(0,\bar{y})$ in finite time and stay there forever. Thus, it is without loss of generality that we may suppose $x_0 \in S = [\underline{s},\bar{y}]$.

For each r define $H(\cdot,r):S \to S$ by $H(x,r) = X^*(f(x,r))$. $H(x,r)$ determines the evolution of the $(x_t)_{t=0}^{\infty}$ process. The results of Section 2 showed that $X^*(\cdot)$ is monotonic and continuous; therefore, A.H.1 and A.H.2 hold for the H defined here. Assumption A.3.2 and the fact that X is strictly increasing (Theorem 2.8) imply that condition A.H.3 holds. Hence, Theorem 3.7 follows immediately from Theorem 3.1.

Suppose that configuration A holds and $x_m \leq x_M$. Recall $S = [\underline{s},\bar{y}]$. Using the terminology of Section 3.1 it is easy to see that $b_0 = \underline{s}$, $\hat{a}_1 = x_m$, $b_1 = x_M$ and $N^* = 1$. Theorem 3.8 then results from an application of Theorem 3.4.

Now suppose that configuration B holds and $x_m > x_M$. Then it is easy to see that $b_0 = \underline{s}$, $b_1 = x_M$, $\hat{a}_1 \in (b_0,b_1)$, $N^* \geq 2$, $\hat{a}_{N^*} = x_m$ and $b_{N*} \in (\hat{a}_{N*},\bar{y})$. From $(0,\underline{s})$ (resp. $(\bar{y},\infty)$) the x_t process enters the set $[\underline{s},b_1]$ (resp. $[x_m,\bar{y}]$) in finite time. Thus, Theorem 3.9 follows directly from Theorem 3.4. ///

<u>Proof of Theorem 3.10.</u> The proof will be divided into a series of subsidiary lemmas. We will show that assuming $x_m > x_M$ leads to a contradiction.

Recall from (3.5) that y_m and y_M are defined by

$$y_m = f_m(x_m) = \min_r f(x_m,r) \quad \text{and} \quad y_M = f_M(x_M) = \max_r f(x_M,r).$$

Define

$$c_m = C^*(y_m) \quad \text{and} \quad c_M = C^*(y_M). \tag{3.10.1}$$

From the definition of X_m (see (3.3)) and the fact that x_m is a fixed point of X_m (see (3.4)), it follows that $X^*(y_m) = X^*(f_m(x_m)) = X_m(x_m) = x_m$. Hence, x_m and c_m are optimal investment and consumption from y_m. Similarly, x_M and c_M are optimal investment and consumption from y_M.

LEMMA 3.10.1. *If $x_m > x_M$, then for all r,*

$$y_M \geq f(x_M, r),\ y_m \leq f(x_m, r),\ c_M \geq C^*(f(x_M, r)),\ c_m \leq C^*(f(x_m, r)), \tag{3.10.2}$$

and

$$y_m > y_M \quad \text{and} \quad c_m \geq c_M. \tag{3.10.3}$$

<u>Proof of Lemma 3.10.1.</u> The first two inequalities in (3.10.2) follow from the definitions of y_m and y_M. The last two inequalities in (3.10.2) follow from the first two and the fact that $C^*(\cdot)$ is nondecreasing by Theorem 2.7.

By the definitions of $X_m(\cdot)$ and $X_M(\cdot)$ in (3.3), the definitions of x_m and x_M as fixed points, and the assumption $x_m > x_M$ we have

$$X^*(f_m(x_m)) = X_m(x_m) = x_m > x_M = X_M(x_M) = X^*(f_M(x_M)).$$

Since $X^*(\cdot)$ is nondecreasing by Theorem 2.8, it must be that $f_m(x_m) > f_M(x_M)$ or $y_m > y_M$. Combining this with the fact that $C^*(\cdot)$ is nondecreasing yields the second inequality in (3.10.3). //

LEMMA 3.10.2. *Under A.3.5, $R_c(c_m,y_m) > 0$ and $R_c(c_M,y_M) > 0$.*

<u>Proof of Lemma 3.10.2.</u> Suppose $R_c(c_m,y_m) = 0$. Then, from the Euler equation we have

$$0 = R_c(c_m,y_m) = \delta E\{[R_c(C^*(f(x_m,r)),f(x_m,r)) + R_y(C^*(f(x_m,r)),f(x_m,r))]f'(x_m,r)\}. \tag{3.10.4}$$

Since $f'(x_m,r) > 0$ and $R_y \geq 0$ by assumption, and $R_c(C^*(y),y) \geq 0$ by Lemma 2.6, equation (3.10.4) implies

$$R_c(C^*(f(x_m,r)),f(x_m,r)) = R_y(C^*(f(x_m,r)),f(x_m,r)) = 0 \tag{3.10.5}$$

for all r.

From Lemma 3.6.4, $R_c(C^*(y),y) + R_y(C^*(y),y)$ in nonincreasing in y. Thus, equation (3.10.5) implies that for any $y > y_m$

$$R_c(C^*(y),y) + R_y(C^*(y),y) \leq 0. \tag{3.10.6}$$

However, $R_c(C^*(y),y) \geq 0$ and $R_y \geq 0$ so (3.10.6) implies that

$$R_c(C^*(y),y) = R_y(C^*(y),y) = 0$$

for all $y > y_m$. This contradicts A.3.5 and proves the first inequality of the lemma. Proof of the second inequality follows by similar arguments. //

LEMMA 3.10.3. *If $x_m > x_M$ and A.3.5 holds, then*

$$\frac{R_y(c_M, y_M)}{R_c(c_M, y_M)} < \frac{R_y(c_m, y_m)}{R_c(c_m, y_m)} .$$

Proof of Lemma 3.10.3. The Euler equation implies

$$R_c(c_m, y_m) = \delta E\{[R_c(C^*(f(x_m,r)), f(x_m,r)) + R_y(C^*(f(x_m,r)), f(x_m,r))]f'(x_m,r)\}.$$

From Lemma 3.6.4, $R_c(C^*(y),y) + R_y(C^*(y),y)$ is nonincreasing in y. Therefore, (3.10.2) implies

$$R_c(c_m, y_m) \le \delta E\{[R_c(c_m,y_m)+R_y(c_m,y_m)]f'(x_m,r)\},$$

or

$$\frac{R_y(c_m, y_m)}{R_c(c_m, y_m)} \ge (1/\delta k) - 1, \qquad (3.10.7)$$

where $k = Ef'(x_m, r)$. Similarly, one can show that

$$\frac{R_y(c_M, y_M)}{R_c(c_M, y_M)} \le (1/\delta K) - 1, \qquad (3.10.8)$$

where $K = Ef'(x_M, r)$.

Since $x_m > x_M$ and f is strictly concave by assumption, we get that $k < K$. The lemma then follows from (3.10.7) and (3.10.8). //

LEMMA 3.10.4. *$x_m > x_M$ implies that $c_m/y_m \le c_M/y_M$.*

Proof of Lemma 3.10.4. $f_m(x)$ is concave since the minimum of an arbitrary collection of concave functions is concave. Hence, $f_m(x)/x$ is nonincreasing in x. Define $g_m(x) = (f_m(x)-x)/f_m(x) = 1 - x/f_m(x)$. Clearly, $g_m(x)$ is nonincreasing in x. As a result, $x_m > x_M$ implies

$$c_m/y_m = g_m(x_m) \le g_m(x_M). \tag{3.10.9}$$

Next, define $g_M(x) = (f_M(x)-x)/f_M(x)$. Then, $g_M(x) = 1 - x/f_M(x) \ge 1 - x/f_m(x) = g_m(x)$. This gives

$$g_m(x_M) \le g_M(x_M) = c_M/y_M. \tag{3.10.10}$$

The lemma then follows from (3.10.9) and (3.10.10). //

LEMMA 3.10.5. *If $x_m > x_M$ and A.3.6 holds, then*

$$\frac{R_y(c_M, y_M)}{R_c(c_M, y_M)} \ge \frac{R_y(c_m, y_m)}{R_c(c_m, y_m)}.$$

Proof of Lemma 3.10.5. Define $\alpha = c_M/c_m$ and $y_\alpha = \alpha y_m$. Then from (3.10.3), $0 < \alpha \le 1$ and $y_\alpha \le y_m$. Lemma 3.10.4 implies $c_M/y_M \ge c_m/y_m$ so that $y_M \le (c_M/c_m)y_m = y_\alpha$. Lemma 3.10.2 implies $R_c(c_M,y_M) > 0$. Since $R_{cy} \ge 0$ under A.2.11, $R_c(c_M,y) > 0$ for all $y \ge y_M$. Thus, for $y \ge y_M$ it follows that

$$\frac{d}{dy}\left[\frac{R_y(c_M,y)}{R_c(c_M,y)}\right] = \left.\frac{R_c R_{yy} - R_y R_{cy}}{R_c^2}\right|_{(c_M,y)} \le 0.$$

Then, $y_\alpha \ge y_M$ implies

$$\frac{R_y(c_M,y_M)}{R_c(c_M,y_M)} \ge \frac{R_y(c_M,y_\alpha)}{R_c(c_M,y_\alpha)} . \tag{3.10.11}$$

From the definitions of α and y_α, $(c_M,y_\alpha) = (\alpha c_m,\alpha y_m)$ with $0 < \alpha \le 1$. Thus, $(\lambda c_M,\lambda y_\alpha) = (c_m,y_m)$ with $\lambda \ge 1$. Using this, A.3.6 implies

$$\frac{R_y(c_M,y_\alpha)}{R_c(c_M,y_\alpha)} \ge \frac{R_y(c_m,y_m)}{R_c(c_m,y_m)} . \tag{3.10.12}$$

The lemma follows from (3.10.11) and (3.10.12). //

The proof of Theorem 3.10 follows from the fact that Lemmas 3.10.3 and 3.10.5 contradict each other. Thus, it cannot be that $x_m > x_M$. //

Proof of Theorem 4.1. The theorem is proved through a sequence of lemmas.

LEMMA 4.1.1. *Fix any production function $f(x,r)$ that satisfies A.4.1. For this production function let x_M be defined as in (3.4). If $Ef'(x_M,r) > 1/\delta$ then $R(c,y)$ attains its global maximum at a unique $\bar{c} > 0$ (see Lemma 3.6.1) and $C^*(f_m(x_M)) = C^*(f_M(x_M)) = \bar{c}$.*

Proof of Lemma 4.1.1. We first record the following facts:

Fact A. $R_c(C^*(y),y)) + R_y(C^*(y),y))$ is nonincreasing in y. (Lemma 3.6.4.)

Fact B. $R_c(C^*(y),y) \geq 0$. (Lemma 2.6).

Fact C. $R_c(C^*(y),y) + R_y(C^*(y),y)) = 0$ implies $R_c(C^*(y),y) = R_y(C^*(y),y) = 0$. (This follows from fact B and the assumption that $R_y \geq 0$.)

Define $y_M = f_M(x_M)$ and $c_M = C^*(y_M)$. Since $f(x_M,r) \leq f_M(x_M)$, the Euler condition and fact A above imply

$$R_c(c_M,y_M) \tag{4.1.1.a}$$

$$= \delta E\{f'(x_M,r)[R_c(C^*(f(x_M,r)),f(x_M,r)) + R_y(C^*(f(x_M,r)),f(x_M,r))]\}$$

$$\geq \delta E\{f'(x_M,r)[R_c(C^*(y_M),y_M) + R_y(C^*(y_M),y_M)]\}$$

$$= \delta E\{f'(x_M,r)[R_c(c_M,y_M) + R_y(c_M,y_M)]\}. \tag{4.1.1.b}$$

If $R_c(c_M,y_M) + R_y(c_M,y_M) > 0$, then (4.1.1.b) above and the hypothesis of the lemma ($Ef'(x\ ,r) > 1/\delta$) imply

$$R_c(c_M,y_M) > R_c(c_M,y_M) + R_y(c_M,y_M) \geq R_c(c_M,y_M),$$

where the second inequality follows from the assumption that $R_y \geq 0$. This is a contradiction which proves that $R_c(c_M,y_M) + R_y(c_M,y_M) = 0$.

From this equality, fact C yields $R_c(c_M,y_M) = 0$. Substituting this into the Euler condition (4.1.1.a) implies that the expression in square brackets on the right hand side of (4.1.1.a) is zero for each r (since $f' > 0$). Thus, fact C implies

$$R_c(C^*(f(x_M,r),f(x_M,r)) = R_y(C^*(f(x_M,r),f(x_M,r)) = 0$$

for all r. This, in turn gives

$$R_c(C^*(f_m(x_M)),f_m(x_M)) = R_y(C^*(f_m(x_M)),f_m(x_M)) = 0$$

and

$$R_c(C^*(f_M(x_M)),f_M(x_M)) = R_y(C^*(f_M(x_M)),f_M(x_M)) = 0.$$

Since R is concave these first order conditions imply that R attains its maximum at $(C^*(f_m(x_M)),f_m(x_M))$ and $(C^*(f_M(x_M)),f_M(x_M))$. An application of Lemma 3.6.1 then completes the proof of this lemma. //

Let $\{f^k(x,r)\}_{k=0}^{\infty}$ be a class of production functions satisfying A.4.1-A.4.4. If the model with production function f^k has more than one invariant distribution then $x_M^k \le x_m^k$ (see Theorems 3.8 and 3.9). Since $f^0(x,r)$ obeys assumption A.2.5 there exists a $\bar{y}^0$ such that $f_m^0(x) < x$ for all $x \ge \bar{y}^0$. If $x \ge \bar{y}^0$, A.4.4 implies $X^*(f_m^k(x)) \le f_m^k(x) \le f_m^0(x) < x$. Since x_m^k is a fixed point of $X^*(f_m^k(x))$ this means that

$$x_M^k \le x_m^k < \bar{y}^0. \qquad (4.1.2)$$

LEMMA 4.1.2. *Let $\{f^k(x,r)\}_{k=0}^{\infty}$ be a class of production functions obeying A.4.1-A.4.4 above. Suppose further that for all k, the model with the production function $f^k(x,r)$ has more than one non-trivial invariant distribution. Then,*

a) $E\partial f^k(x_M^k,r)/\partial x > 1/\delta$ for all k sufficiently large, where x_M^k is defined in a similar manner as x_M (see (3.4)) for the model with production function $f^k(x,r)$; and

b) $\lim_{k\to\infty} x_M^k = 0$.

<u>Proof of Lemma 4.1.2.</u> Fix any k. Since $f^k(x,r)$ is concave in x (4.1.2) implies

$$E\partial f^k(x_M^k,r)/\partial x \ge E\partial f^k(\bar{y}^0,r)/\partial x.$$

Taking limits as $k \to \infty$ and using property A.4.3 yields

$$\lim_{k\to\infty} E\partial f^k(x_M^k, r)/\partial x \geq \lim_{k\to\infty} E\partial f^k(\bar{y}^0, r)/\partial x = \infty.$$

Part (a) of the lemma follows immediately.

Part (a) of this lemma and Lemma 4.1.1 then imply that for sufficiently large k, $C^*(f_M^k(x_M^k)) = \bar{c}$. From the definition of x_M^k as a fixed point we obtain

$$x_M^k = f_M^k(x_M^k) - C^*(f_M^k(x_M^k)) = f_M^k(x_M^k) - \bar{c}.$$

If there is any sub-sequence of x_M^k that does not tend to zero as $k\to\infty$, then from condition A.4.2, along that subsequence

$$x_M^k = f_M^k(x_M^k) - \bar{c} \to \infty.$$

This is a contradiction to (4.1.2) above which proves part (b) of the lemma. //

We now complete the proof of Theorem 4.1. Let $\{f^k(x,r)\}_{k=0}^{\infty}$ be a class of production functions satisfying the hypotheses of Lemma 4.1.2. From Lemmas 4.1.1, 4.1.2(a) and A.4.4 it follows that for k sufficiently large,

$$0 < \bar{c} = C^*(f_m^k(x_M^k)) \leq f_m^k(x_M^k) \leq f_m^0(x_M^k).$$

Taking limits as $k \to \infty$ and using Lemma 4.1.2(b) then implies that $0 < \bar{c} \leq f_m^0(0) = 0$. This is a contradiction which proves Theorem 4.1. //

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Notes

1. Reed [1979] studies a class of nonlinear reward functions that can be transformed into an equivalent linear model.

2. This assumption is without loss of generality due to A.2.5.

3. This model is basically a stochastic version of that presented in Feenstra [1986, Section 3].

4. Arrow and Kurz [1970, pp. 11, 89] argue that public goods produce "joint" products that yield direct consumption benefits in addition to their influence on productivity. They cite highways and education as examples of such goods.

5. Let $\phi(y)$ be a correspondence from $Y \subset \mathbb{R}^m$ into $Z \subset \mathbb{R}^n$, where Z is assumed to be compact. Let y_0 be a point in Y. Let $\{y_j\}$ be a sequence in Y such that $y_j \to y_0$, and let $\{z_j\}$ be a sequence in Z such that $z_j \in \phi(y_j)$. If $z_j \to z_0$ implies $z_0 \in \phi(y_0)$, then ϕ is called <u>upper semicontinuous</u> (u.s.c.) at y_0. ϕ is upper semicontinuous if it is u.s.c. at each $y_0 \in Y$.

6. If $R(c,y)$ is strictly concave, then this result can be strengthened to show that $V(y)$ is also strictly concave.

7. If $R(c,y)$ is expressed as $R(y-x,y) = W(x,y)$, then $W_x = -R_c$ and $R_{cc} + R_{cy} \leq 0$ implies $W_{xy} = -R_{cc} - R_{cy} \geq 0$.

8. Define $V(x,z) = R(f(x)-z,f(x))$. Then $V_{xz} = -[R_{cc} + R_{cy}]f_x$. Benhabib and Nishimura show that if $V_{xz} < 0$ for interior optimal policies, then $x_t < x_{t+1}$ implies $x_{t+1} \geq x_{t+2}$.

9. Additional insight into A.3.6 can be obtained by considering a consumer faced with the problem of maximizing $R(c,y)$ subject to a budget constraint, $I = p_c c + p_y y$. Assume interior solutions and suppose that the sufficient second order conditions for a maximum are satisfied. Let $c(p,I)$ and $y(p,I)$ be the Marshallian demands for c and y as functions of I and $p = (p_c,p_y)$. Under these assumptions (and using the first order conditions), the implicit function theorem implies that $\partial y/\partial p_c - \partial c/\partial p_y = [R_{cc}R_y c + R_{cy}R_y y - R_{cy}R_c c - R_{yy}R_c y]/D$, where $D = \mu[2R_{cy}R_cR_y - R_{yy}R_c^2 - R_{cc}R_y^2] > 0$ and μ is the multiplier on the budget constraint ($D > 0$ follows from the second order conditions). Note that in a local neighborhood of (c,y) satisfying $R_c > 0$, A.3.6 implies

$$\frac{\partial \left[\dfrac{R_y(\lambda c,\lambda y)}{R_c(\lambda c,\lambda y)} \right]}{\partial \lambda} = \frac{(R_{yc}c + R_{yy}y)R_c - (R_{cc}c + R_{cy}y)R_y}{R_c^2} \leq 0.$$

Thus, A.3.6 is equivalent to the following cross-price restriction on the demand functions for c and y: $\partial y(p,I)/\partial p_c \geq \partial c(p,I)/\partial p_y$. For a consumer "purchasing" c and y, A.3.6 implies that the demand for resource stocks is more cross-price sensitive than the demand for consumption. Alternatively, A.3.6 implies that income expansion paths for purchases of c and y intersect balanced growth paths from below in (y,c) space.

10. While A.3.6 is somewhat restrictive it does hold for significant classes of reward functions. One such class is the class of all reward functions that are homothetic. Indeed, this may be extended to include the class of all reward functions homothetic to a point in the region $\Omega = \{(c,y) \in \mathbb{R}^2 | c \leq y, \ c \leq 0\}$. This is a subset of the class of quasihomothetic or affine-homothetic reward functions (see Blackorby, Boyce, and Russell [1978]).

11. The transversality condition is $\lim_{t\to\infty} \delta^t R_c(c^*_t, y^*_t) x^*_t = 0$. In Example 2, the transversality condition is satisfied since $R_c(c^*_t, y^*_t)$ and x^*_t are both constant and positive at the three steady states.

FIGURE 1

Illustration of fixed points a_n, $\hat{a}_n$, and b_n

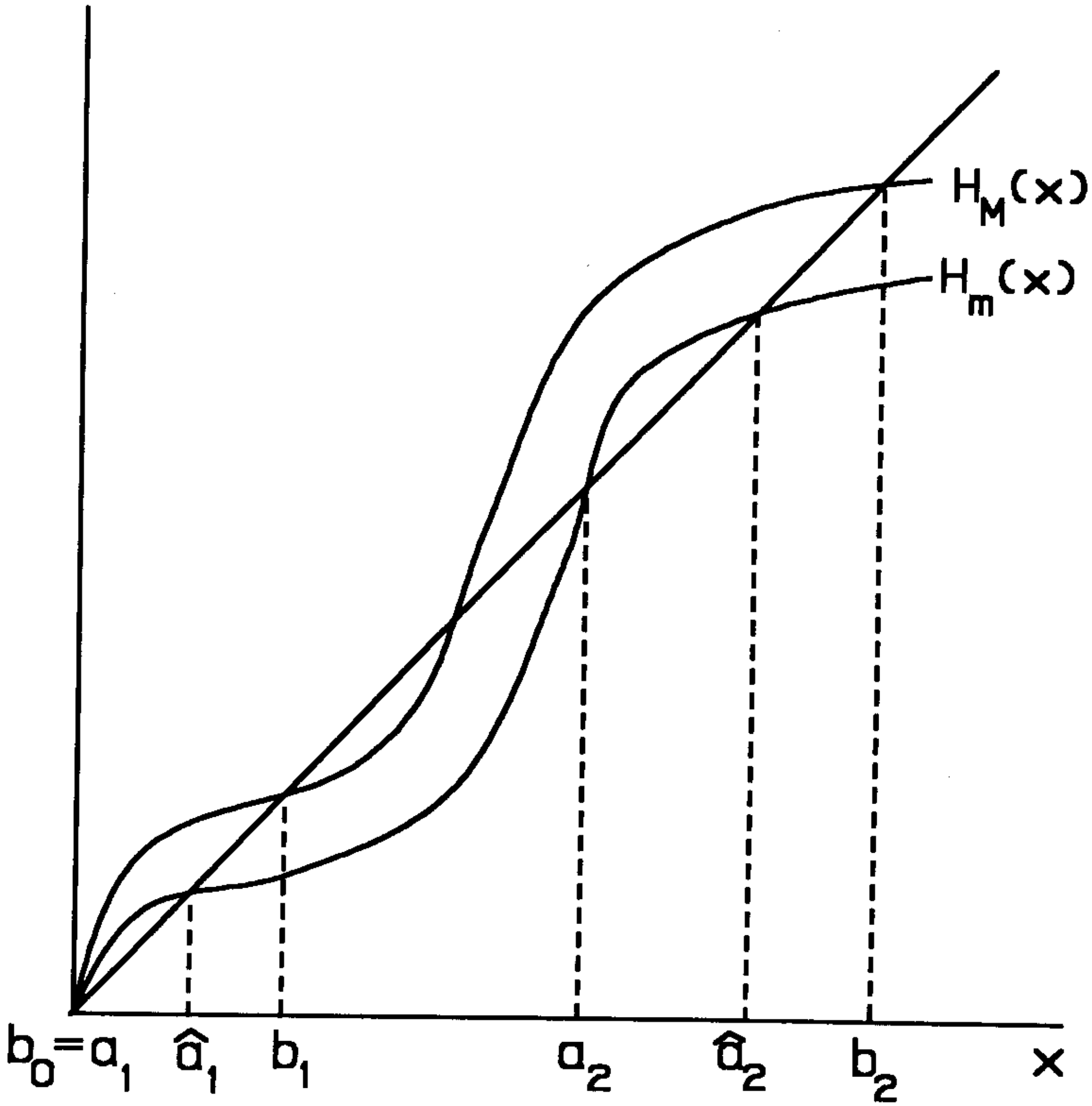

FIGURE 2

Two possible configurations for fixed points of X_m and X_M

(A) A unique limiting distribution.

(B) Nonunique limiting distributions.

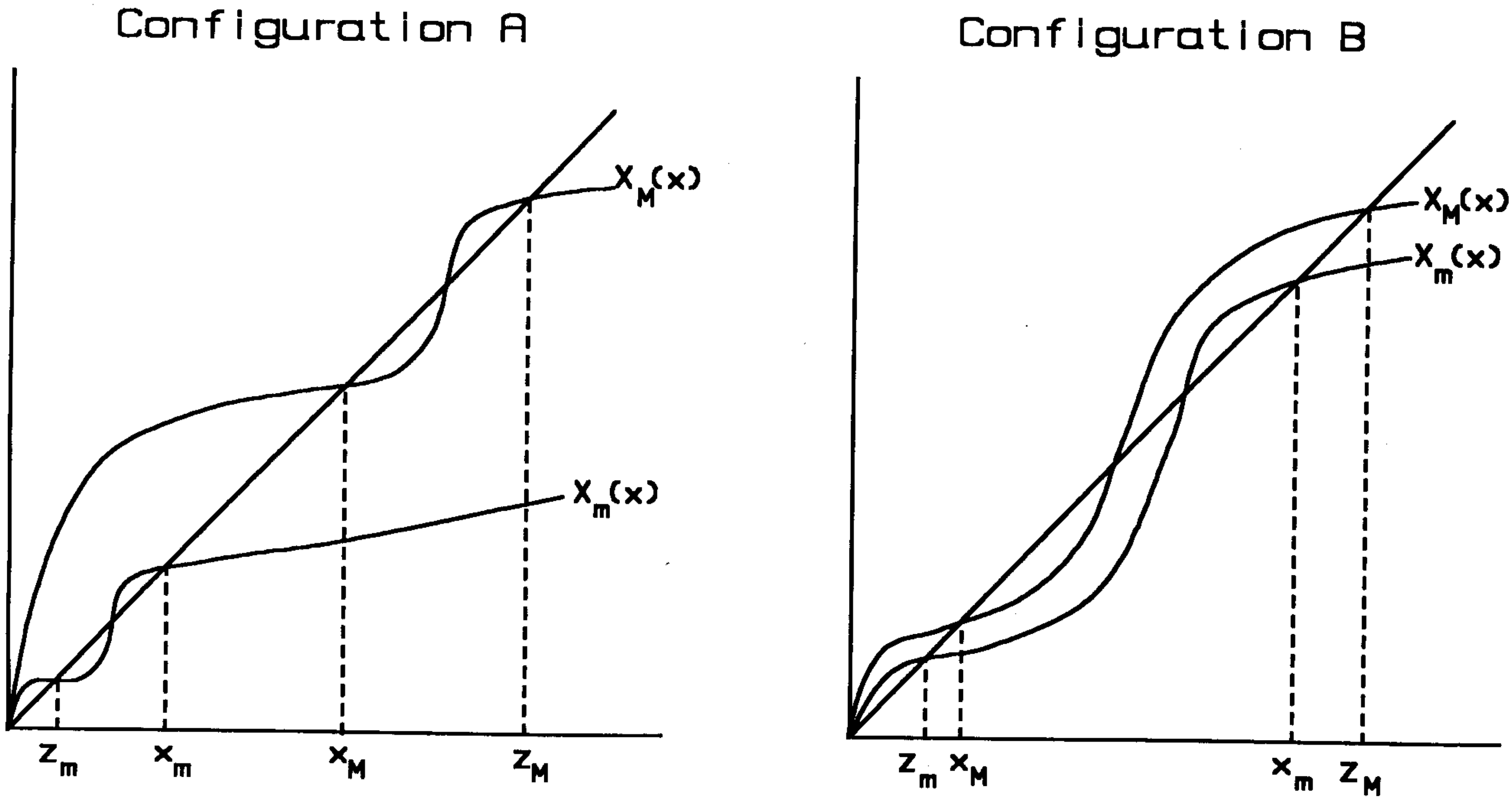

FIGURE 3

Illustration of assumption A.3.6