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Abstract

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Analyses of assembly elections often assume that voters have well-defined preferences over candidates, even though preferences over assemblies are the natural analytic starting point. This candidate-based approach is usually justified by an assumption that preferences over assemblies are separable. We show that if preferences over assemblies are themselves derived from underlying preferences over legislative or economic outcomes, then preferences over assemblies will not in general be separable. We then suggest, through discussion of a paper by Sugden, that a candidate-based analysis may be misleading even when one can legitimately assume separable preferences over assemblies.

KEYWORDS: social choice, voting scheme, sincere voting, separable preferences, median voter rule

JEL classifications: D70, D72

On Candidate-Based Analyses of Assembly Elections[1]

by
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In many elections a group of people, or assembly, is selected. These elections include the United States Senate, school boards, and city governments. Since assemblies are being elected, a natural analytical starting point is voters' preferences over these assemblies.[4] On the other hand, in these elections voters are typically asked to vote for individual candidates, not assemblies as a whole. For this reason, undoubtedly, analysts have often assumed that voters have well-defined preferences over candidates and pursued their analyses in terms of these candidate preferences. However, this starting point is problematic.

Analysts typically justify, explicitly or implicitly, a candidate-based approach with an assumption that preferences over assemblies are separable (defined below). However, little work has been done as to the reasonableness of this assumption. In this Note, we show that if preferences over assemblies are themselves derived from underlying preferences over legislative or economic outcomes, then preferences will not in general be separable. We then suggest, through discussion of a paper by Sugden, that a candidate-based analysis may be misleading even when one can legitimately assume separable preferences over assemblies.

SEPARABILITY OF PREFERENCES OVER ASSEMBLIES

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[4] See Benoit and Kornhauser [1994, 1995] for more on this.

Although assembly preferences are fundamental, there is a condition under which candidate preferences may unambiguously be derived from these assembly preferences. Consider the election of an m-sized assembly. For any two candidates a_i and a_j, let X_{ij} be a set of (m-1) candidates not including a_i or a_j. Let > and $\geq$ denote the preference relations.

Definition: Assembly preferences are separable if for all a_i, a_j, A_{ij}, and B_{ij}, $\{a_i\}UA_{ij} > \{a_j\}UA_{ij}$ implies $\{a_i\}UB_{ij} \geq \{a_j\}UB_{ij}$.

Separable assembly preferences generate a natural ranking of the candidates. Namely,

Definition: An individual ranks candidates simply if $a_i > a_j$ if and only if there exists an A_{ij} such that $\{a_i\}UA_{ij} > \{a_j\}UA_{ij}$.

Consider a voter with k votes to cast for k different candidates. *Simple voting* extends the notion of sincere voting to assembly elections in which votes are cast for candidates, rather than assemblies. A simple voter always prefers every candidate for whom she votes to every candidate for whom she does not vote.[5] Thus,

Definition: An individual votes simply if for all a_i for whom she votes and a_j for whom she does not vote, $\{a_i\}UA_{ij} \geq \{a_j\}UA_{ij}$ for all A_{ij}.

When preferences are separable, a simple ranking of candidates and simple voting are possible. An individual then votes simply by voting for her top k candidates.[6] Because simple rankings and simple voting permit a straightforward analysis of assembly elections in candidate terms, analysts often assume that preferences are separable.

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[5]The extension of sincerity outlined in the text is not the only possible extension. Cox [1990] for example considers a spatial model in which an individual votes for the candidate who is closest to the individual's ideal point in the policy space. Such a vote is "expressive" of the voter's preferences and will generally not be simple. See Benoit and Kornhauser [1995] for an extensive discussion.

[6] See Benoit and Kornhauser (1991, 1995) for more on this.

How reasonable is the assumption of separability? The assumption is clearly quite restrictive. For instance, if there are five candidates running for a two-person assembly, there are 10! possible (strict) orderings of assemblies. On the other hand, there are 5! strict rankings of candidates each of which is consistent with 12 distinct separable orderings of assemblies.[7] Thus, of the 10! possible assembly rankings, only 5! x 12 are separable. The fact that separability is restrictive, however, does not in itself indicate whether or not it is a reasonable condition. We address this question in the next section.

SEPARABILITY AND OUTCOME-BASED PREFERENCES

We have argued that the natural starting point for the analysis of assembly elections is assembly preferences. Typically, however, assemblies are not ends in themselves but rather are constituted to determine some economic or political outcomes. One could reasonably argue that preferences over these legislative outcomes should be the starting point. Note, however, that if

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[7]Suppose that some voter ranks the candidates $a_1 > a_2 > a_3 > a_4 > a_5$. Table A presents the twelve assembly rankings that are consistent with this candidate ranking:

Table A: Assembly Preferences that are consistent with the Candidate Preferences $a_1 > a_2 > a_3 > a_4 > a_5$

	1	2	3	4	5	6	7	8	9	10	11	12
1st	a_1a_2	a_1a_2	a_1a_2	a_1a_2	a_1a_2	a_1a_2	a_1a_2	a_1a_2	a_1a_2	a_1a_2	a_1a_2	a_1a_2
2nd	a_1a_3	a_1a_3	a_1a_3	a_1a_3	a_1a_3	a_1a_3	a_1a_3	a_1a_3	a_1a_3	a_1a_3	a_1a_3	a_1a_3
3rd	a_1a_4	a_1a_4	a_1a_4	a_1a_4	a_1a_4	a_1a_4	a_1a_4	a_2a_3	a_2a_3	a_2a_3	a_2a_3	a_2a_3
4th	a_1a_5	a_1a_5	a_2a_3	a_2a_3	a_2a_3	a_2a_3	a_2a_3	a_1a_4	a_1a_4	a_1a_4	a_1a_4	a_1a_4
5th	a_2a_3	a_2a_3	a_1a_5	a_2a_4	a_2a_4	a_1a_5	a_2a_4	a_1a_5	a_1a_5	a_2a_4	a_2a_4	a_2a_4
6th	a_2a_4	a_2a_4	a_2a_4	a_1a_5	a_3a_4	a_2a_4	a_1a_5	a_2a_4	a_2a_4	a_1a_5	a_1a_5	a_3a_4
7th	a_2a_5	a_3a_4	a_2a_5	a_2a_5	a_1a_5	a_3a_4	a_3a_4	a_2a_5	a_3a_4	a_2a_5	a_3a_4	a_1a_5
8th	a_3a_4	a_2a_5	a_3a_4	a_3a_4	a_2a_5	a_2a_5	a_2a_5	a_3a_4	a_2a_5	a_3a_4	a_2a_5	a_2a_5
9th	a_3a_5	a_3a_5	a_3a_5	a_3a_5	a_3a_5	a_3a_5	a_3a_5	a_3a_5	a_3a_5	a_3a_5	a_3a_5	a_3a_5
10th	a_4a_5	a_4a_5	a_4a_5	a_4a_5	a_4a_5	a_4a_5	a_4a_5	a_4a_5	a_4a_5	a_4a_5	a_4a_5	a_4a_5

the outcomes that an assembly will enact can be perfectly predicted then there is a one-to-one correspondance between assemblies and legislative outcomes, so that these two starting points are not in conflict. Nonetheless, starting from outcome preferences entails important restrictions. We now show that assembly preferences that are derived from outcome preferences will not generally be separable.

We begin with two examples. First, suppose that the legislative outcome can be described as a real number from 0 to 1. Individual candidates adopt a position on this interval. Consider two-member assemblies and suppose that the outcome of an assembly is given by the mean of the assembly members' positions. Suppose there are four candidates, two each at positions x = 0 and y = 1, vying for the two seats. Then an individual whose favorite outcome is 0.5 will favor a candidate at y to one at x, to complete an assembly whose other member is at x, but will have the reverse ranking if the other member is at y. Hence, this individual's assembly preferences are not separable.[8]

Consider now an election for a three-person assembly that will reach majority rule decisions on three separate issues, each of which can be decided 0 or 1. Consider six candidates, two each at positions x = (1,1,0), y = (1,0,1), and z = (0,1,1) and a voter whose favorite outcome is (1,1,1). The voter prefers a candidate at z to one at x to complete an assembly whose other members are at x and y, but prefers a candidate at x to one at z to complete an assembly whose other members are at y and z. Again preferences are not separable.

These two examples differ markedly both in the nature of the outcome spaces and the outcome rule of the assemblies. Nevertheless, it is no coincidence that in both cases assembly

[8]This point is not new. See, for instance, Austin-Smith and Banks (1991).

preferences are not separable.[9] We now generalize these examples.

We consider the election of an m-sized assembly. Let Y denote an <u>outcome space</u>. Voters are assumed to have well defined preferences over the elements of Y. Each candidate espouses a certain outcome. An <u>outcome rule</u> f determines the implemented outcome as a function of the outcomes espoused by those candidates who are elected. Let $a_i \in Y$, $b_i \in Y$, for $i=1,...,m$. We say that a voter's assemblies preferences are <u>derived from her outcome preferences</u> if she prefers assembly $\{a_1,...a_m\}$ to $\{b_1,...b_m\}$ if and only she prefers $f\{a_1,...a_m\}$ to $f\{b_1,...b_m\}$.

For ease of expostion we assume that for all $a_i \in Y$ and outcome rules f, $f\{a_i,a_i,...,a_i\} = a_i$. We identify each candidate with the outcome she espouses.

We say that an outcome rule is <u>non-compromising</u> if for any set of m outcomes $A = \{a_1,...a_m\}$, $f\{A\} = a_i$, for some $a_i \in A$. Otherwise, the rule is said to be <u>compromising</u>. In words, a non-compromising outcome rule always selects the outcome espoused by some assembly member. A compromising rule may result in an outcome which represents a "compromise" of the various positions.

For the following theorem, we assume that any outcome might be some voter's strictly favorite outcome. Beyond this, however, the voters' preferences over outcomes may be subject to any restriction.[10]

<u>Theorem 1</u>: Suppose that voters' preferences are derived from their outcome preferences and that the outcome rule is compromising. Then, no matter what restrictions are placed upon the form of outcome preferences, there is a group of candidate positions so that some (potential) voter's preferences are not separable, when the set of candidates running includes this group.

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[9]However these examples differ with respect to “top-separability”. See footnote 13.

[10]For instance, if the outcomes are points on an interval, one restriction would be that each voter has an ideal point and prefers A to B if A is closer to this ideal point.

Proof (reductio): Suppose that all voters always have separable preferences and let $A = \{a_1,...a_m\}$ be such that for all $a_i \in A$, $f\{A\} \neq a_i$. Let the set of candidates include m candidates at each position $a_i \in A$. Consider a voter whose (strictly) favorite outcome is $f\{A\}$. Since $f\{A\}$ is the voter's favorite outcome and $f\{a_1,a_2,...,a_m\} \neq a_m$, $\{a_1,a_2,...,a_m\} > \{a_m,a_m,...,a_m\}$. Separability then implies (iteratively) that $a_i > a_m$ for some $i \neq m$. Without loss of generality, suppose that $a_1 > a_m$. Now $\{a_1,a_2,...,a_m\} > \{a_1,a_1,...,a_1\}$ implies that $a_i > a_1$ for some $i \neq 1$ (and $\neq m$). Without loss of generality suppose that $a_2 > a_1$. Continuing in this manner we obtain a cycle among all the candidates.

Q.E.D.

Although the proof of theorem 1 is rather trivial, the theorem's consequences are far reaching. When assembly preferences are derived from outcome preferences via an outcome rule which admits of any sort of compromise among the proposed outcomes of the assembly members, then preferences cannot be presumed separable. As we shall see, even with an appropriate non-compromising outcome rule, preferences will not be separable unless severe conditions are imposed upon the way individuals can rank the outcomes.

Consider an election for a three-member assembly. From theorem 1 we know that if preferences are to be separable, then the outcome rule must be non-compromising. Thus, it must be the case that for any arbitrary proposed outcomes a_1, a_2, and a_3, $f\{a_1,a_2,a_3\} = [a_1 \text{ or } a_2 \text{ or } a_3]$. Say that $f\{a_1,a_2,a_3\} = a_1$. Say also that $f\{a_2,a_3,a_4\} = a_2$.

Consider any voter whose favorite outcome is a_2. Since $f\{a_1,a_2,a_3\} = a_1$, separability implies that this voter prefers a_4 to a_1. Hence, if assembly preferences are to be separable, the voter's outcome preferences must satisfy restrictions (which depend upon the outcome rule). A simple generaliztion of this example yields:

Theorem 2: If no a priori restriction is placed upon voters' preferences over outcomes, then there is a set of candidates and a voter whose preferences are not separable.

Proof: A simple generalization of the above example.

While theorem 2 is negative, it is not unusual to impose restrictions upon voters' preferences. For instance, in a one-dimensional model voter preferences are often assumed to be single-peaked. If the outcome rule is non-compromising and we impose suitable restrictions on outcome preferences, then assembly preferences will be separable. However, the necessary restrictions are (inordinately) severe.

To get a feel for how severe the restrictions must be, consider an election in which the outcome space Y = [0,1]. Theorem 1 tells us that if preferences are to be separable, the outcome rule must be non-compromising. One oft-discussed such rule is the median voter rule, where the oucome is given by the median assembly member's position. We first note that with a median voter rule, preferences cannot be presumed separable.

Proposition 1: Suppose the outcome space Y = [0,1] and the assembly size m is odd. Suppose the outcome rule f is the median voter rule and that voters' preferences are derived from their outcome preferences. Then, no matter what restriction is imposed upon the form of outcome preferences, there exists a group of candidates such that if the candidates running includes this group, then all voters whose (strictly) favorite outcome is in the interior of the outcome space have preferences that are not separable.

Proof: Let the set of candidates include one candidate at each voter's favorite outcome as well as (m+1)/2 candidates at a position a_l to the left of any "interior" voter and (m+1)/2 candidates at a position a_r to the right of any interior voter. Consider any voter j with favorite outcome a_j. The (sub)assembly consisting of (m-1)/2 candidates at a_l, candidate a_j ,and [(m-1)/2 - 1] candidates at a_r , will result in the outcome a_l if completed with a candidate a_l , and a_j if completed with a candidate a_r , so that j will prefer a_r to a_l here. On the other hand, similar reasoning shows that j prefers a_l to a_j to complete the (sub)assembly comsisting of [(m-1)/2 - 1] candidates at a_l, candidate a_j ,and (m-1)/2 candidates at a_r. Hence j's preferences are not separable.

Q.E.D.

Thus, the median voter rule will induce non-separable preferences. Now, for concreteness, suppose that the assembly consists of three members. With Y = [0,1], for preferences to be separable then, say, f{.3,.4,.5} = [.3, .4, or .5]. Since the median voter rule will not work, consider the rule f{A} = min $a_i \in A$. Take a voter whose favorite outcome is .3. Since f(.3,.4,.9) > f(.2,.3,.4), separability implies that the voter ranks outcome .9 above .2! This is certainly an unusual requirement, and it is in this sense that we say that the restrictions necessary to guarantee separable preferences are severe. With the outcome rule min a_i, assembly preferences will be separable if, for instance, outcome preferences are single-peaked and all outcomes greater than a voter's favorite outcome are preferred to all outcomes smaller than the favorite.

Thus, broadly speaking, the assumption of separable assembly preferences is incompatible with deriving these preferences from outcome preferences.[11] However, while separability is sufficient for the possibility of simple voting, it is not necessary. A necessary and sufficient condition for being able to vote simply for k candidates is given by k-top separability, defined below.[12]

Definition: An individual has k-top separable preferences if there exists a set of k candidates, called the individual's top candidates, such that if a_i is a top candidate and a_j is not a

[11]Of course, separable assembly preferences may still be justified on other grounds.

[12]See Benoit and Kornhauser (1991) for a proof of this propostion.

top candidate, then $\{a_i\}UA_{ij} \geq \{a_j\}UA_{ij}$ for all A_{ij}.

The set of candidates used in the proof of theorem 1 establishes the following theorem:

Theorem 3: Suppose that voters' preferences are derived from their outcome preferences and that the outcome rule is compromising. Then, no matter what restrictions are placed upon outcome preferences, there is a set of candidates for which some voter's preferences are not k-top separable for all k less than the total number of candidates.

While this negative result parallels theorem 1,[13] the weaker requirement of k-top separability permits the following positive proposition.

Proposition 2: Let the assembly size m be odd. Let the outcome space be Y = [0,1] and let the outcome rule be the median voter rule. Suppose each voter j has single-peaked preferences. Then the assembly preferences derived from these outcome preferences are 1-top separable.

Proof: Let p_j be a candidate/position that j weakly favors most among all the active candidate positions. We will show that p_j is in j's 1-top set. Let a_i be a candidate/position no better than p_j. Let A_{ij} be a set of (m-1) candidates not including p_j or a_i. We show that $p_jUA_{ij} \geq a_iUA_{ij}$. Without loss of generality, suppose that the median candidate of p_jUA_{ij} is (weakly)to the right of p_j. Then if a_i is to the left of p_j, then a_iUA_{ij} and p_jUA_{ij} yield the same outcome, or a_iUA_{ij} yields a_i and p_jUA_{ij} yields p_j. If a_i to the right of p_j then either a_iUA_{ij} and p_jUA_{ij} yield the same outcome, or the median candidate of a_iUA_{ij} is further to the right than the median candidate of p_jUA_{ij}. Since i) the median candidate of p_jUA_{ij} is to the right of p_j, ii) p_j is the position that j

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[13]Notice one important difference between theorems 1 and 3. From theorem 1, given the "right" set of candidates, adding more will not alter the non-separability of a voter's preferences. However, for some outcome functions, the same is not true for top-separability. For instance, suppose $Y = \{x \in R^n | x_i = 0 \text{ or } 1)$, m is odd, and f = majority rule on each issue. Then a voter's preferences will be 1-top separable if there is a candidate that espouses his favorite outcome.

weakly favors amongst all the candidate positions, and iii) preferences are single-peaked, $p_j UA_{ij}$ is preferred to $a_i UA_{ij}$.

Q.E.D.

CANDIDATE PREFERENCES ARE MISLEADING

In a provocative article, Sugden (1984) analyzes candidate-based assembly elections. He explicitly recognizes the primacy of assembly preferences. Thus, he begins his analysis by assuming that voters have lexicographic assembly preferences, from which candidate preferences are derived. Having done this, however, he then proceeds dubiously from a candidate-based perspective.

Using the notion of "election by free association" Sugden argues for the appropriately defined core of the candidate voting game.[14] The following example satisfies all of his assumptions. The example can also be understood without reference to Sugden's article.

There are five candidates running for three seats. The candidates are labelled a_1 through a_5 from left to right. Preferences over candidates are lexicographic and single-peaked. There are 123 voters who divide into four groups with candidate and corresponding assembly rankings given in tables 1 and 2 below:

Table 1: Candidate Preferences in Example 1

Ranking of Candidates	Groups of voters			
	1 31 Voters	2 29 Voters	3 32 Voters	4 31 Voters
1st	a_2	a_3	a_4	a_5
2nd	a_3	a_2	a_3	a_4
3rd	a_4	a_4	a_2	a_3

[14]See Sudgen (1984) for details.

4th	a_1	a_1	a_1	a_2
5th	a_5	a_5	a_5	a_1

Table 2: Assembly Preferences in Example 1

Ranking of Assemblies	Groups of voters			
	1 31 Voters	2 29 Voters	3 32 Voters	4 31 Voters
1st	$\mathbf{a_2a_3a_4}$	$\mathbf{a_2a_3a_4}$	$\mathbf{a_2a_3a_4}$	$a_3a_4a_5$
2nd	$a_2a_3a_1$	$a_1a_2a_3$	$a_1a_3a_4$	$\underline{a_2a_4a_5}$
3rd	$a_2a_3a_5$	$a_2a_3a_5$	$a_3a_4a_5$	$a_1a_4a_5$
4th	$a_1a_2a_4$	$a_1a_3a_4$	$a_1a_2a_4$	$a_2a_3a_5$
5th	$\underline{a_2a_4a_5}$	$a_3a_4a_5$	$\underline{a_2a_4a_5}$	$a_1a_3a_5$
6th	$a_1a_2a_5$	$a_1a_3a_5$	$a_1a_4a_5$	$a_1a_2a_5$
7th	$a_1a_3a_4$	$a_1a_2a_4$	$a_1a_2a_3$	$\mathbf{a_2a_3a_4}$
8th	$a_3a_4a_5$	$\underline{a_2a_4a_5}$	$a_2a_3a_5$	$a_1a_3a_4$
9th	$a_1a_3a_5$	$a_1a_2a_5$	$a_1a_3a_5$	$a_1a_2a_4$
10th	$a_1a_4a_5$	$a_1a_4a_5$	$a_1a_2a_5$	$a_1a_2a_3$

The core of the election by free association game is the assembly $a_2a_4a_5$. Looking strictly at the candidate ranking this seems like a reasonable choice. Each of the three most populous groups gets one candidate.

Looking at the assembly ranking, however, quite a different picture emerges. Assembly $a_2a_3a_4$ seems clearly superior to $a_2a_4a_5$. About 3/4 of the population ranks assembly $a_2a_3a_4$ first. On the other hand, $a_2a_4a_5$ is ranked first by no one and is ranked second by only about 1/4 of the population. While $a_2a_3a_4$ is ranked seventh by the 31 people who do not rank it first, $a_2a_4a_5$ is ranked eighth by 29 people and fifth by 63. Even if we make the bold assumption that electing one's first ranked candidate is especially important, so that a ranking in the top six positions is especially important, the fact that two more people rank $a_2a_4a_5$ in the top six than do so for $a_2a_3a_4$ does not outweigh $a_2a_3a_4$'s other advantages. Indeed, $a_2a_3a_4$ is not only a majority rule winner, but is also the Borda count winner. In any case, it is striking how much better $a_2a_4a_5$ looks with respect to the candidate preferences than with respect to the assembly preferences.[15]

More seriously, in addition to his maintained hypothesis that preferences are lexicographic, at one point (page 37) Sugden goes on to make the apparently innocuous assumption that assemblies reach decisions according to a median voter rule. However, as proposition 1 indicates, the median voter assumption is inconsistent with the assumption that preferences are separable. Since lexicographic preferences are separable, all these assumptions cannot be maintained.[16]

CONCLUSION

The assumption of separable assembly preferences is convenient in the analysis of candidate-based election procedures. However, this assumption is not warranted if assembly preferences are derived from preferences over legislative outcomes. Assembly preferences need not be derivative in this manner, and separability may still be justified on other grounds. Even if this is the case, in analyzing assembly elections it is important not to rush too quickly to a candidate-based perspective.

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[15]With lexicographic preferences (as opposed to those which are only separable) there is a one-to-one correspondence between candidate preferences and assembly preferences. Hence, the information contained in the candidate rankings alone is sufficient to make our point here. Nonetheless, as the example shows, looking only at the candidate rankings can be misleading.

[16]Sugden does not restrict the set of potential candidates.

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