ECONOMIC RESEARCH REPORTS

Competitive Fair Division

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RR# 99-05 **March 1999**

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Prepared for Delivery at the Annual Meeting of the Public Choice Society, New Orleans, March 12-14, 1999.

Abstract

In the fair-division problem addressed here, the indivisible goods to be divided among two or more players have divisible "bads" associated with them—namely, the prices the players must pay for them. A procedure, called the Gap Procedure, is proposed whereby players bid for the goods, but the bidding competition is balanced by fairness requirements that entitle the players to certain numbers of goods.

Under the Gap Procedure, the prices the players pay for the goods reflect the bids of not only the winners but also those of players that bid less. This market-oriented approach to fair division (1) ensures non-negative prices that never exceed a player's winning bid, (2) is Pareto-optimal, precluding mutually beneficial trades, though not envy, (3) is monotonic in bids, so higher bids never hurt in obtaining a good, (4) encourages sincere bids, and (5) produces prices that are partially independent of the amounts bid (as in a Vickrey auction).

The analysis is developed in the context of the "housemates problem," in which the rent for a house (the bad) must be apportioned among several housemates, each of whom is entitled to receive one room (the good). Each housemate is assumed to have the same endowment, so the auction of rooms is a "relativized" one—only relative, not absolute, bids for each room matter. Other applications of the Gap Procedure, in which player endowments and entitlements may be different, or the procedure may be carried out in rounds, are discussed.

JEL Classification: D44, D61, D63.

Keywords: Fair division; envy-freeness; allocative efficiency; bidding; Vickrey auction.

Competitive Fair Division

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1. Introduction

Fair division involves the equitable allocation of goods among two or more players.[2] In this paper, we assume that associated with each good, which is presumed to be indivisible, is a "bad" that is divisible—namely, the price one must pay to obtain the good.

The fair-division procedure that we propose here for allocating indivisible goods and divisible bads assumes that the players make bids for the goods. As in a standard auction, the bids do double duty: they are the basis for (i) assigning the goods to the players and (ii) determining the prices they pay for them. But unlike a standard auction, the player who bids highest for a good does not necessarily receive it. Moreover, the price that a player pays for a good depends, in general, not just on its own bid but lower bids as well.[3]

We constrain the allocation of the goods by assuming

(1) *equal entitlements*: each player is entitled to receive exactly one good; and

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[1]Steven J. Brams gratefully acknowledges the support of the C. V. Starr Center for Applied Economics at New York University and the Russell Sage Foundation, and D. Marc Kilgour the support of the Social Sciences and Humanities Research Council of Canada. We thank Marco Mariotti, Hervé Moulin, Forest Simmons, and Francis Edward Su for very helpful comments.

[2]Background on fair-division procedures can be found in, among other places, Young (1994), Moulin (1995), Brams and Taylor (1996, 1999), Robertson and Webb (1998), and Raith (1998).

[3]In this respect, the procedure we will propose resembles a Vickrey (1961) auction, in which the highest bidder wins the item but pays only the second-highest bid for it. However, our procedure does not equate the prices of goods and *particular* lower bids but, rather, makes prices a function of possibly one or more lower bids.

(2) *equal endowments:* each player has the same total amount to spend, which is drawn down by the price it pays for the good it receives.

Later we relax these assumptions, but for now we suppose that there are *n* indivisible goods to be divided among *n* players. Our procedure determines which player obtains which good, and at what price.

To make this division problem more concrete, consider the *housemates problem* (Su, 1999):[4]

- there are *n* rooms in a house that *n* housemates rent;
- each housemate submits non-negative bids for each room that sum to the total rent of the house;
- each housemate is assigned, and pays rent for, exactly one room; and
- the sum of the individual rents equals the total rent for the house.

A housemate's bids are *sincere* if they mirror its relative valuations of the rooms, making it indifferent to receiving any room at its bid. In general, a housemate never pays more than its bid under the procedure we will propose, so its bids can be viewed as the maxima (possibly zero) that it would be willing to pay for each room.

The foregoing assumptions *relativize* the auction for rooms: because each housemate has the same endowment, only the relative bids matter. The equal endowments, and the requirement that each housemate be assigned one room, makes the housemates problem somewhat akin to the apportionment problem, which concerns how

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[4]This problem is different from the related problem of "house allocation," in which there are indivisible objects (houses) but no medium of exchange, as here. Different strategy-proof mechanisms that enable the players, who may or may not have property rights, iteratively to trade up to a Pareto-optimal core matching of themselves and the houses have recently been proposed in Abdulkadiroglu and Sönmez (1998a, 1998b), Svensson (1998), and Pápai (1999); earlier work on this problem is discussed in these papers.

many representatives to allocate to different-size districts in a legislature (Balinski and Young, 1982).[5]

Unfortunately, some of the most obvious solutions to this problem run amok, even in the simple case in which there are only three rooms and three housemates, whom henceforth we call *players*.

- one player might not make the highest bid for any room, which creates a problem of which room to assign that player;
- even if each player bids highest on exactly one room, these winning bids will, in general, sum to more than the required total rent;
- reducing the highest bids equally or proportionally—so they sum to the total rent—may lead to some players' (i) paying negative rent (i.e., being paid to take a room) or (ii) paying more than their bids.

To eliminate these difficulties, we propose a procedure, called the Gap Procedure, that is *market-like*:

- it assigns rooms to different players so as to maximize the *sum* of player bids;
- it makes rents for each room a function not just of the bid of the player assigned that room but also of lower bids (the more competitive these bids, the closer the winner's rent is to its bid).

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[5]The apportionment problem is often further constrained. For example, in the U.S. Senate, in which the districts are the states, all states have two senators, so size does not matter. Although size does matter in the U.S. House of Representatives, all states are entitled to at least one representative. Whereas the housemates can choose how to allocate their endowments by making different bids for different rooms, states have no such choice: they can change neither their status as states nor influence very much the size of their populations, which are the respective determinants of their apportionments in the Senate and House.

We show that the Gap Procedure assigns each player a room at a price no more, and generally less, than its bid. Moreover, there exist no trades of rooms that can make all the traders better off, rendering the allocation Pareto-optimal.

We investigate other properties of the Gap Procedure, including its envy-freeness, monotonicity, sincerity, and propensity to make prices independent of winning bids. In addition, we show how assumptions specific to the housemates problem can be relaxed to yield extensions of the Gap Procedure applicable when players' entitlements and endowments differ. We also suggest the application of the Gap Procedure to dynamic choices, in which endowments change over time as players use up their bidding points. We conclude that competition and fairness can be coherently combined: while encouraging healthy competition, the Gap Procedure preserves the rights of players to obtain certain minimum numbers of goods.

We begin with several examples that illustrate the clash between competitive bidding and fairness. We then show how the Gap Procedure ameliorates, if not resolves, these difficulties. As we proceed, we will state general results as "propositions" rather than "theorems," because most of these results do not require long proofs—sometimes only one or two sentences suffice. On the other hand, longer proofs are required for Propositions 8 and 9; we prove the latter in a special case only (a general proof would require much new notation and be too much of a diversion to include).

Proposition 8 is a general impossibility result on envy-freeness that is proved, via an extended counterexample, with virtually no mathematics, whereas Proposition 9 is a possibility result in one situation. By and large, a mathematical presentation is unnecessary, because the Gap Procedure can rigorously be studied without it. Indeed, we hope our development makes perspicuous a normative approach to fair division that is both compelling and practicable.

2. Difficulties with Equal and Proportional Reductions in the High Bids

We start with the simplest case: two players (P1 and P2) wish to share a house with two rooms (R1 and R2). Assume that the total rent for the house is 100, which are also the endowments of the two players. The rule is that each player's bids for the rooms must be non-negative and sum to 100, giving rise to three possibilities:

1. P1 bids more for R1, and P2 bids more for R2.
2. P2 bids more for R1, and P1 bids more for R2.
3. P1 and P2 bid the same amounts for R1 and R2.

For now, assume that P1 and P2 bid sincerely and so prefer the rooms on which they bid higher. Then the obvious solution in cases 1 and 2 is to assign the high bidders their preferred rooms, and to break the tie in case 3 by tossing a coin.

Even given these assignments, however, we still face the problem of determining how much P1 and P2 should pay for the rooms they get. Except in case 3—in which the sum of the bids, whichever player is assigned to each room, is 100—the sum of the high bids will be greater than 100. We begin by considering two different ways of reducing the players' bids in cases 1 and 2 so that they sum to 100, which we illustrate with Example 1.

Example 1. Bids by Two Players for Two Rooms

	R1	R2	*Total*
P1	<u>75</u>	25	100
P2	40	<u>60</u>	100

The high bids for each room (75 and 60), which are underscored, sum to 135. Consider two different ways in which the *total surplus* of 35 (above the rent of 100) might be subtracted from the two players' high bids:

1. *Equal reductions*. Subtract half the total surplus (35/2 = 17.5) from each player's high bid, so that P1 pays 57.5 for R1 and P2 pays for 42.5 for R2.

2. *Proportional reductions.* Divide the total surplus of 35 between P1 and P2 proportional to their winning bids (75/60 = 6/5), so P1's high bid is reduced by (6/11)(35) ≈ 19.1, and P2's high bid by (5/11)(35) ≈ 15.9. Thus, P1 will pay 55.9 for R1, and P2 will pay 44.1 for R2.

Clearly, the player whose winning bid is less (P2 in Example 1) will prefer equal reductions, whereas the player whose winning bid is more (P1) will prefer proportional reductions.

While arguments based on different equity considerations could be adduced for each kind of reduction, both are flawed according to other criteria: equal reductions can lead to a player's paying negative rent; and both equal and proportional reductions can lead to a player's paying more than its bid for a room. These difficulties arise when we introduce one more room and one more player into the house.

Negative Rent

Example 2. Bids by Three Players for Three Rooms

	R1	R2	R3	*Total*
P1	<u>97</u>	2	1	100
P2	1	<u>97</u>	2	100
P3	32	33	<u>35</u>	100

As in Example 1, each player most prefers a different room, again assuming the bids are sincere. The high bids for each room (underscored) sum to 229, giving a total surplus of 129.

1. If the reductions are *equal*, 129/3 = 43 is subtracted from the high bids, so P1 and P2 pay 54 each for R1 and R2, respectively, but P3 pays a negative rent of -8 for R3.

That is, P3 will have to be paid 8 to rent R3. This seems absurd, especially in light of the fact that P3 prefers R3 to the other two rooms.[6] Why should P1 and P2 compensate P3 to occupy the room it most desires?

2. If the reductions are *proportional* to the high bids of 97, 97, and 35, the portions of 129 that are subtracted from these bids are 54.6, 54.6, and 19.7, respectively. Consequently, P1 pays 42.4 for R1, P2 pays 42.4 for R2, and P3 pays 15.3 for R3.[7]

Proportional reductions, which can never produce negative rent because the amounts subtracted from the bids are always less than the bids, are not without their difficulties, as we next show.

Loss to a Player

Consider the following example:

Example 3. Bids by Three Players for Three Rooms

	R1	R2	R3	*Total*
P1	<u>50</u>	1	<u>49</u>	100
P2	29	<u>40</u>	31	100
P3	31	38	31	100

This example differs from Examples 1 and 2 in that the high bidder on each room is not always different: P1 is the high bidder on R1 and R3, with bids of 50 and 49,

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[6]This would not be absurd if the players were bidding for bads—like the chores required to maintain their house properly—in which case each would bid highest for the chores it considers least bad. In this situation, it would be reasonable for P3 to be paid 8 for R3 (if R3 were interpreted as a chore) since P1 and P2 consider it so onerous. We will return to the issue of negative rents in section 5, where we analyze the Pareto-optimality of the Gap Procedure when negative rents are both allowed and disallowed.

[7]A consequence of rounding is that the reductions sum to 128.9 (rather than 129.0), and the prices sum to 100.1 (rather than 100.0). Henceforth, we ignore the effects of rounding in this and other numerical examples.

respectively (underscored). By comparison, P2 is the high bidder on R2 (with an underscored bid of 40), and P3 is the high bidder on no room. Because the high bids sum to 139 (50 + 40 + 49), the total surplus is 39:

1. If the reductions are *equal*, 39/3 = 13 is subtracted from each of the high bids, giving rents of 37, 27, and 36 for R1, R2, and R3, respectively. Notice that R1 and R3 cannot be assigned to any player other than P1, because only P1's bids for these rooms are at least equal to their rents. Thus, whichever of R1 or R3 is assigned to P1 (say, R1), neither of the other players, which both bid 31 for R3, will be willing to pay R3's rent of 36. Hence, there is no assignment of different rooms to the three players such that all pay no more than their bids and, therefore, do not suffer a loss.

2. If the reductions are *proportional* to the high bids of 50, 40, and 49, the portions of 39 that are subtracted from these bids are 14.0, 11.2, and 13.7, giving rents of 36.0 for R1, 28.8 for R2, and 35.3 for R3. Once again, R1 and R3 cannot be assigned to any player other than P1, precluding an assignment of the three players to three different rooms such that they all pay no more than their bids.

The Gap Procedure, to be described in section 3, starts by assigning players to rooms so as to maximize the total surplus, which sometimes entails assigning rooms to players that are not the highest bidders on them.[8] But instead of making equal reductions in the total surplus, or reductions proportional to the high bids—in order to find prices that sum to 100—the Gap Procedure takes into account the lower bids for each room, setting prices competitively.

3. The Gap Procedure

We begin by describing the Gap Procedure, which proceeds in two steps, and then illustrating its application to the three preceding examples:

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[8]In this manner, it promotes the *collective* welfare of players rather than that of just the highest bidders.

Gap Procedure.

1. Maxsum Assignment. Assign the rooms to different players so as to maximize the sum of player bids. Call this sum "maxsum," which equals 100 if all bids for each room are the same; otherwise, maxsum is greater than 100.

2. Room Prices. Descend from the maxsum assignment to the next-lower bids for each room, the next-lower bids to these, and so on until the sum of the current set of bids is less than or equal to 100.[9] *Stop the descent on a room when the lowest bid on that room is reached; if necessary, continue the descent on the other rooms until the sum of the current set of bids is less than or equal to 100. If this last sum is exactly 100, then these bids are the room prices, and the procedure stops. Otherwise, go to the next-higher sum (which is necessarily greater than 100) and reduce the bids it comprises in proportion to the gaps between them and the next-lower bids so that the sum of the reduced bids equals 100. These are the room prices, and the procedure stops.*

Example 1. Maxsum is 135 (75 + 60) along the main diagonal, and the next-lower bids (40 and 25) along the off-diagonal sum to 65. Reduce the maxsum bids in proportion to the gaps—35 (75 - 40) and 35 (60 - 25), which sum to 70—between the main-diagonal and the off-diagonal bids. Because these gaps are equal, the reductions will be equal,[10] which is equivalent to making the equal reductions that we illustrated earlier for this example, to obtain prices of 57.5 and 42.5 for R1 and R2, respectively.

Example 2. Maxsum is 229 (97 + 97 + 35) along the main diagonal, and the next-lower bids (32, 33, and 2) sum to 67. Reduce the maxsum bids in proportion to the

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[9]"Next-lower bids" may be tied with those from which they descend, as Example 3 will illustrate.

[10]It is easy to show that this result holds in general for two players and two rooms. Let x be the high bid on R1 by P1, and let y be the high bid on R2 by P2. Under the Gap Procedure, x will be reduced in proportion to the gap, $x - (100 - y)$, and y will be reduced in proportion to the gap, $y - (100 - x)$; both gaps equal $x + y - 100$.

gaps—65 (97 - 32), 64 (97 - 33), and 33 (35 - 2), which sum to 162—between the maxsum and the next-lower bids Thus, the total surplus of 129 (229 - 100) is subtracted from the maxsum bids for R1, R2, and R3, respectively, in the amounts of (65/162)(129) ≈ 51.8, (64/162)(129) ≈ 51.0, and (33/162)(129) ≈ 26.3. This gives room prices of 45.2, 46.0, and 8.7 for R1, R2, and R3, respectively.[11]

Example 3. Maxsum is 121 (50 + 40 + 31) along the main diagonal. The next-lower bids for R1 and R2—and tied-for-next-lower bid in the case of R3—are 31, 38, and 31, respectively. Because these bids sum to exactly 100, no further descent is necessary—these bids *are* the room prices under the Gap Procedure. Unlike Examples 1 and 2, these prices do not depend directly on the maxsum bids but, rather, on the next-lower bids, which is a matter we will return to in section 7.

With these examples in mind, we now state two propositions. Proposition 1 demonstrates that the first difficulty identified in section 2 can never arise under the Gap Procedure:

Proposition 1. *Under the Gap Procedure, no player ever pays negative rent.*

Proof. The lowest price that a player can pay for a room is the low bid for that room, which is always greater or equal to zero. Q.E.D.

Note that the room prices can equal, or be tied for, low bid, as in Example 3 in which P3 pays 31 for R3.

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[11]These prices would change if—reflecting the competition of *all* players in the marketplace—they were based on the *average* bids below those of the winners. To illustrate, the average of the second-highest *and* third-highest bids in Example 2 is 16.5 for R1, 17.5 for R2, and 1.5 for R3. Consequently, the maxsum bids would be reduced to room prices of 43.3, 44.0, and 12.7. Compared with the Gap prices of 45.2, 46.0, and 8.7, P1 and P2 would pay somewhat less, and P3 substantially more, if the averages, rather than the second-highest bids, were used to set prices. Because it is the second-highest bids, not the averages, that *directly* compete with the bids of the maxsum bidders, however, we think they are the proper basis for pricing rooms in the marketplace.

Define P*i*'s *surplus* for R*j* to be

$s_{ij} = b_{ij} - p_j,$

or the difference between *i*'s bid for R*j*, b_{ij}, and the price, p_j, for R*j*.[12] In Example 3, the surpluses of P1, P2, and P3 for the rooms they are assigned are $s_{11} = 50 - 31 = 19$, $s_{22} = 40 - 38 = 2$, and $s_{33} = 31 - 31 = 0$. Clearly, P1 is greatly helped by the lack of competitive bids for R1, whereas P2 and P3 fare much worse—but no player receives a negative surplus. This result is general, eliminating the second difficulty identified in section 2:

Proposition 2. *Under the Gap Procedure, no player ever pays more than its bid: if Pi is assigned Rj, then* $s_{ij} \geq 0$.

Proof. Under the Gap Procedure, a player pays either (i) its bid for the room it is assigned, in which case its surplus is 0, or (ii) if its bid is not the lowest, a price less than its bid but greater than the lowest bid. Since $p_j \leq b_{ij}$, s_{ij} is non-negative. Q.E.D.

Proposition 2 renders the Gap prices *feasible*—they always give non-negative surpluses.

So far we have shown that the Gap Procedure avoids the problem of negative rent (Proposition 1), which can occur if the high bids are reduced by equal amounts (Example 2). It also ensures that no player ever suffers a negative surplus (Proposition 2), which we showed could happen if the high bids were reduced either equally or proportionally (Example 3).

Because there are *n*! possible assignments of players to different rooms, one might think that finding the maxsum assignment for large *n* will be computationally infeasible.

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[12]If players are sincere in their bidding, their bids, b_{ij}, will reflect their (true) valuations, v_{ij}, of the rooms. In this case, s_{ij} would be a plausible measure of player utilities, $u_{ij} = v_{ij} - p_j$; another plausible measure would be the proportional difference, $(v_{ij} - p_j)/v_{ij}$. We will consider later the incentives of players to be sincere under the Gap Procedure and argue that, because of the procedure's relative invulnerability to manipulation, especially when information is incomplete, s_{ij} may well be a good approximation of u_{ij}.

It turns out, however, that this problem is equivalent to the problem of assigning *n* workers to *n* jobs so as to maximize the value of their work (Bondy and Murty, 1976, pp. 86-90) and to the weighted-matching problem for bipartite graphs (Lawler, 1976, pp. 201-207). For these problems, there are algorithms computable in polynomial time that render the determination of maxsum assignments feasible in most conceivable situations (more on practical applications later).

4. Envy, Pareto-Optimality, and Trades

Consider again Example 3, wherein P3, especially, appeared to get a "raw deal" under the Gap Procedure: it had to pay exactly its bid, whereas P1 and P2 got surpluses of 19 and 2, respectively. However, P3 would *not* be better off getting R1 or R2 at the Gap prices of 38 or 31, respectively, because these prices are exactly P3's bids for these rooms as well, yielding P3 the same surplus, 0.

Can either of the other players benefit from a different assignment of rooms at the Gap prices? P1's surplus of 19 would not increase if it were assigned either R2 ($s_{12} = 1 - 38 = -37$) or R3 ($s_{13} = 49 - 31 = 18$). Neither can P2 do better than its surplus of 2 if it were assigned R1 ($s_{21} = 29 - 31 = -2$) or R3 ($s_{23} = 31 - 31 = 0$).

Call a player *envious* if its assignment to a different room at the price given by the Gap Procedure would give it a greater surplus. If this is not the case for any player, we say that the Gap assignment and prices are *envy-free*. Not only do we have envy-freeness in Example 3, but we also have it in Examples 1 and 2.

Unfortunately, this may not always be the case:

Proposition 3. *Under the Gap Procedure, one player can be envious.*

Proof. Consider the following example:

Example 4. Bids by Three Players for Three Rooms

	R1	R2	R3	*Total*
P1	57	28	15	100
P2	24	60	16	100
P3	48	47	5	100

Maxsum is 123 (48 + 60 + 15) along the off-diagonal, and the next-lower bids (24, 47, and 5) sum to 76. Under the Gap Procedure, the room prices are 36.3, 53.6, and 10.1 for R1, R2, and R3, respectively.

Now P1, which gets R3 ($s_{13} = 15 - 10.1 = 4.9$), envies P3, which gets R1; if P1 were assigned R1, its surplus would be $s_{11} = 57 - 36.3 = 20.7$, far exceeding its Gap surplus of 4.9. Consequently, maxsum assignments at the prices given by the Gap Procedure are *not* envy-free in Example 4: one player (P1) would get a greater surplus from being assigned a different room at the Gap price. Q.E.D.

In Example 4, it turns out, neither P2 nor P3 can benefit from being assigned different rooms at the Gap prices.

While P1 will desire to trade rooms with P3 in Example 4, it will not be in P3's interest to agree to such a trade: it would receive a minuscule surplus of $s_{33} = 5 - 4.9 = .1$ if it got R3 at the Gap price of 4.9, compared with its Gap surplus of $s_{31} = 11.7$. Nevertheless, this trade at Gap prices is still feasible—no player would receive a negative surplus after the trade.

We will return to the question of envy in section 5, showing that no scheme, including the Gap Procedure, can always prevent it if there are four or more players. First, however, we look at additional properties of the Gap Procedure:

Proposition 4. *The Gap Procedure maximizes the total surplus. There is no other assignment of rooms, or prices that the players pay for them, that yields individual* *surpluses Pareto-superior to the Gap surpluses, rendering the Gap surpluses Pareto-optimal.*

Proof. Because the assignment of rooms under the Gap Procedure maximizes the sum of bids, it maximizes the total surplus, which equals the bid sum minus 100. In particular, the Gap prices, which ensure that the individual surpluses of all players are non-negative, maximize the total surplus. Consequently, in order for one or more players to obtain a greater surplus from a set of different prices that sum to 100, one or more other players would have to do worse, so the Gap surpluses are Pareto-optimal. Q.E.D.

As we saw from Example 4, however, the Pareto-optimal surpluses under the Gap Procedure do not guarantee envy-freeness: a player may still prefer the room of another player, at the price the other player pays for it, to its assigned room at the Gap price. However, envy can never be two-way—if one player envies another, then the other player cannot envy the first. If this were possible, the two players could trade rooms and *both* be better off. But by Proposition 4, one player cannot be better off without hurting another—if there is no change in the surpluses of the other players—because the total surplus is maximal under the Gap Procedure.

Might a more complex set of trades be possible that benefits all the traders? For example, might P1 envy P2, P2 envy P3, and P3 envy P1, creating the possibility of a three-way trade whereby P1 would get P2's room, P2 would get P3's room, and P3 would get P1's room, which would make *all* players better off?

Proposition 5. *Under the Gap Procedure, no trade involving two or more players can improve the surpluses of all the traders without hurting non-traders.*

Proof. Assume this is not the case. Then the total surplus would be greater after the trades, which contradicts Proposition 4 that the total surplus of the Gap Procedure is maximal. Q.E.D.

In effect, Proposition 5 restates Proposition 4 in terms of trades—the Pareto-optimality of surpluses under the Gap Procedure precludes mutually beneficial trades of any kind.

Pareto-optimal surpluses are not exclusive to the Gap Procedure; they are also possible if non-maxsum assignments are used:

Proposition 6. *If the assignment of rooms is not maxsum, then there may be non-negative prices that yield the players Pareto-optimal surpluses.*

Proof. Consider the following example:

Example 5. Bids by Three Players for Three Rooms

	R1	R2	R3	*Total*
P1	40	27	33	100
P2	27	15	58	100
P3	5	20	75	100

Maxsum is 120 (40 + 15 + 75) along the main diagonal. Applying the Gap Procedure, the descent to the next-lower bids goes to 27 for R1 and and 58 for R3, but it stops at 15 for R2 since 15 is the lowest bid for R2. These three bids coincide with P2's bids in the second row of the bid matrix—and hence sum to 100—so they are the prices of the three rooms under the Gap Procedure. These gives surpluses of $s_{11} = 13$ (40 - 27), $s_{22} = 0$ (15 - 15), and $s_{33} = 22$ (75 - 58) for P1, P2, and P3, respectively.

P2 does particularly badly, obtaining a surplus of zero. Theoretically, the greatest surplus it could obtain under *any* maxsum assignment, given that negative rents are disallowed, is 15 (this is feasible, for example, if R2 is priced at 0, R1 at 35, and R3 at 65, giving surpluses of 5 and 10, respectively, to P1 and P3 for R1 and R3).

On the other hand, consider a non-maxsum assignment of R1 to P2, R2 to P1, and R3 to P3. If R1 were priced at 0 (and, say, R2 were priced at 26 and R3 at 74, giving surpluses of 1 each to P1 and P3), then P2 would obtain a surplus of 27. No pricing scheme with a maxsum assignment, including that of the Gap Procedure, can match this surplus for P2, providing rents are non-negative, so the surpluses under the aforementioned non-maxsum assignment and prices are Pareto-optimal. Q.E.D.

Although we have ruled out negative rents in the housemates problem, they would, if allowed (e.g., in the division of chores; see note 6), render maxsum assignments the *only* ones that give Pareto-optimal surpluses:

Proposition 7. *If an assignment of rooms to players is not maxsum, then there are no prices that yield the players Pareto-optimal surpluses if negative rents are allowed.*

Proof. If the assignment is not maxsum, there is a maxsum assignment in which at least two players have different rooms (since all players have one room, one player cannot be assigned a different room unless its original room is assigned to someone else). If there are exactly two players with different rooms under maxsum, then the sum of their bids for these two rooms is higher than the sum of their bids for the two rooms they get under the non-maxsum assignment. Thus, it is possible to find prices, possibly negative, under the maxsum assignment that give *both* players with different rooms greater surpluses than under the non-maxsum assignment. A similar argument can be made if the number of players that have different rooms under the maxsum assignment is greater than two: the sum of their bids under the maxsum assignment will be greater, so prices can be found that give all these players greater surpluses. Q.E.D.

While Proposition 7 establishes that non-maxsum assignments are never Pareto-optimal when negative rents are allowed, we are reluctant to admit such rents in the housemates problem to buttress the case for the Gap Procedure.[13] Nevertheless, our results for the Gap Procedure are still promising. Not only does it preclude negative rents and ensure no-loss prices, but the surpluses of players—based on their bids and the Gap

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[13]In the context of chore division, however, there is no difficulty in thinking of negative rents as side payments to the players that get assigned chores that other players abhor.

prices—are Pareto-optimal, if not exclusively so, because of maxsum. To be sure, one player may envy another for the price it pays for the room it gets, but the envy can never be two-way (or be structured so as to allow any mutually beneficial trades).

5. Envy-Freeness: An Impossible Dream

Consider Example 4, in which the maxsum assignment along the off-diagonal gave Gap prices of 36.3, 53.6, and 10.1 for R1, R2, and R3, respectively, that are not envy-free. It is not difficult to check that P3's bids of 48, 47, and 5 (third row of bid matrix) yield envy-free prices, as do other room prices close to these.[14] These prices give surpluses of $s_{13} = 0$ (48 - 48), $s_{22} = 13$ (60 - 47), and $s_{31} = 10$ (15 - 5), compared with the surpluses that the Gap prices give of 11.7, 6.4, and 4.9 for the maxsum assignments. But, as we shall next show, envy-free prices do not always exist, no matter what assignment of rooms is made:

Proposition 8. *If $n \geq 4$, there may be no feasible (i.e., no-loss) assignment of rooms at envy-free prices.*[15]

Proof. Consider the following example:

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[14]Using geometric methods, we have determined the set of *all* envy-free prices in Example 4, which can be expressed in (p_1, p_2)-space, where $p_3 = 100 - (p_1 + p_2)$. These prices are bounded by a trapezoid with vertices $(p_1, p_2) = (46\ 2/3, 48\ 2/3)$, $(47\ 1/3, 48\ 1/3)$, $(48, 47)$, and $(47\ 2/3, 46\ 2/3)$. The fact that these coordinates are nearly equal says that the envy-free price range is quite narrow. The center of the figure that encompasses the three prices is the point $(p_1, p_2, p_3) = (47.5, 47.5, 5)$. We note that envy-free prices must be based on maxsum assignments; otherwise they would admit mutually beneficial trades, which obviously are precluded if no player is envious. While the Gap prices allow for one-way envy, the fact that there exist situations in which such envy cannot be ruled out (see Proposition 8 below) means that the search for an algorithm, like the Gap Procedure, that guarantees envy-freeness is futile. On the other hand, there are other ways around the envy problem (Su, 1999) that we will discuss shortly.

[15]Remember from section 4 that envy-freeness is defined in terms of player surpluses. In section 6, we will consider the degree to which surpluses and utilities are likely to match because bidding is sincere.

Example 6. Bids by Four Players for Four Rooms

	R1	R2	R3	R4	*Total*
P1	36	34	30	0	100
P2	31	36	33	0	100
P3	34	30	36	0	100
P4	32	33	35	0	100

We begin by applying the Gap Procedure. Maxsum is 108 (36 + 36 + 36 + 0) along the main diagonal, the next-lower bids (34, 34, 35, and 0) sum to 103, and the next-lower bids to these (32, 33, 33, and 0) sum to 98.[16] Reducing the total surplus of 3 for the second set of bids in proportion to the gaps between the second set and the third set—2 (34 - 32), 1 (34 - 33), 2 (35 - 33), and 0 (0 - 0), which sum to 5—leads to reductions of 1.2, .6, 1.2, and 0 in the second-set bids, giving prices of 32.8 for R1, 33.4 for R2, 33.8 for R3, and 0 for R4. The surpluses of P1, P2, P3, and P4 for these four rooms are, respectively,

$$s_{11} = 36 - 32.8 = 3.2; \quad s_{22} = 36 - 33.4 = 2.6; \quad s_{33} = 36 - 33.8 = 2.2; \quad s_{44} = 0 - 0 = 0.$$

But P4 will envy P3 because $s_{43} = 35 - 33.8 = 1.2$, which is greater than $s_{44} = 0$.

Are there *any* feasible prices that can dispel envy for all four players? Because R4—the room nobody wants at any positive price—must be assigned to some player P*i*, these conditions imply the following:

- *Feasibility*: R4's price must be 0 to ensure no loss to P*i* ;
- *Envy-freeness*: the prices of all other rooms must be at least equal to P*i*'s bids to guarantee that P*i*'s surplus for these rooms is not more than what it gets from R4.

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[16]Since the bids for R4 are all the same, the "next-lower" bids are all 0, or, equivalently, the descent stops at 0.

Since the sum of P*i*'s bids for all the rooms is 100, the only feasible envy-free prices for P*i*, if it is assigned R4, are *exactly* its bids for the four rooms. If $i = 1$, there is no assignment of R1 to any player other than P1 that is feasible—the other players' bids are all less than 36. Similarly for $i = 2$ or 3: the assignment of R2 or R3, also at a price of 36, to any player other than P*i* is not feasible.

The only remaining possibility is $i = 4$, at prices of 32 for R1, 33 for R2, and 35 for R3. At these prices, however, P3 must be assigned to R3, because the bids of P1 and P2 are less than 35; likewise, P1 must be assigned to R1, because the bid of P2 is less than 32 (and P3 has already been assigned). This leaves P2 to be assigned to R2. But now P3's surplus from R3 ($s_{33} = 36 - 35 = 1$) is less than its surplus were it assigned R1 ($s_{31} = 34 - 32 = 2$), so P3 will envy P1. Hence, there is no feasible assignment of rooms at envy-free prices. Examples like this can readily be embedded in situations in which $n > 4$. Q.E.D.

The impossibility of envy-freeness does not depend on one room's being worthless to all players. For example, if each of the four players bid 1 for R4, any price between 0 and 1 would be feasible for the player assigned R4. Nevertheless, it is not difficult to construct an example, with no feasible envy-free assignment of rooms to players, but with no zero bids.

Perhaps surprisingly, there is no example that precludes envy-freeness if there are only two or three players.

Proposition 9. *If $n \leq 3$, there is always a feasible assignment of rooms at envy-free prices.*

Proof (Partial). If $n = 2$, either each player bids more than the other for a different room, or they bid the same for both rooms. In the former case, one can always assign the room for which each player bids more, with equal reductions in each player's bid so their prices sum to 100 and neither envies the other (because the room a player does not get gives it less surplus; see Example 1 for an illustration). In the latter case, the assignment of either room to either player, with equal reductions in the bids so that the prices sum to 100, produces a tie in surpluses and is, therefore, envy-free.

If $n = 3$, a situation analogous to Example 6, but for three players rather than four, is given by Example 7:

Example 7. Bids by Three Players for Three Rooms

	R1	R2	R3	*Total*
P1	x	$100 - x$	0	100
P2	y	$100 - y$	0	100
P3	z	$100 - z$	0	100

Without loss of generality, assume $x > y > z$. Then assigning R1 to P1 at price y, R2 to P3 at price $100 - y$, and R3 to P2 at price 0 is feasible and envy-free (note that all these prices are P2's bids in the second row of the bid matrix). Coincidentally, these are also the Gap assignments and prices.

This is a "worst-case" example for finding a feasible assignment at envy-free prices, because the zero bids for R3 by all players put the strongest possible constraint on prices for the three rooms. In fact, *whatever* P2 bids for R3 (not necessarily 0), the only envy-free prices for the aforementioned assignments are exactly P2's bids, provided $x > y > z$. We do not give the remainder of the proof here, which involves showing how envy-free prices can be constructed geometrically for three players (see note 14 for an example), because of its length. Q.E.D.

Su (1999) offers a constructive proof, using Sperner's Lemma, that if each player always prefers a free room, or one whose price is 0, to a non-free room—his "miserly tenants" condition—there is always a feasible assignment of rooms at envy-free prices.

In Example 6, it is easy to show that this condition is not satisfied by the Gap solution, which gives surpluses of

$$s_{11} = 36 - 32.8 = 3.2; \quad s_{22} = 36 - 33.4 = 2.6; \quad s_{33} = 36 - 33.8 = 2.2; \quad s_{44} = 0 - 0 = 0.$$

Because $u_{43} = 35 - 33.8 = 1.2$ is greater than $u_{44} = 0$, P4 prefers the nonfree room R3 to the free room R1. Likewise, P4 envies P3, as we showed earlier. More generally, just as there is no solution that eradicates envy in Example 6 (Proposition 8), Su's (1999) miserly tenants condition fails in this example, precluding an envy-free solution.[17]

Su's (1999) condition gives special status to a free room (priced at 0 but valued at, say, 1) compared to the most expensive room (e.g., priced at 40 but valued at, say, 50). But would a player really prefer the free room, giving it a surplus of 1, to the most expensive room, giving it a surplus of 10?

In fairness, Su (1999) admits that his miserly tenants condition—and even its relaxation, that tenants never choose the most expensive room if a free one is available—is not plausible in all situations. Not only do we concur, but we also think it better to ground preferences in surplus ($s_{ij} = b_{ij} - p_j$), wherein player valuations (v_{ij}) are traded off against prices (p_j) to give utilities ($u_{ij} = v_{ij} - p_j$), at least insofar as players are sincere.[18]

To summarize, envy-freeness cannot be guaranteed if there are four or more players, at least for the surplus function we have postulated. On the other hand, envy-freeness can be guaranteed, in a nonbidding framework, by invoking Su's (1999)

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[17]Allowing negative rents, however, will always admit envy-free prices. Thus in Example 6, prices of 34, 34, 34, and -2 for R1, R2, R3, and R4, respectively, give each of P1, P2, P3, and P4 a surplus of 2 for their maxsum assignments. But these equal surpluses do not reflect the fact that there is more competitive bidding for R3 than for R1 or R2, which is why the Gap solution makes R3 more expensive than R1 or R2.

[18]It is worth noting that Su's (1999) algorithm, which is based on successive approximations, produces one solution but does not identify all envy-free solutions, as we did for Example 4 (see note 14). Likewise, the Gap Procedure, which is not iterative, finds a unique solution, except when there is more than one maxsum assignment (in which case one could toss a coin to select one).

condition of the overriding desirability of a free room. But we find this assumption implausible, because it does not trade off value against price in the construction of utility.

While the Gap Procedure does not ensure envy-freeness, even when it is possible, its prices do take into account the competitiveness of bidding for rooms, making it a market-oriented pricing mechanism. Although envy-freeness is a desirable property, we prefer a market-like mechanism when there is a conflict between these two properties; players *should* pay more when bids are competitive, even at the sacrifice of causing envy.[19] An additional advantage of the Gap Procedure is that its prices, except in the case of maxsum ties, are specific, whereas there may be a range of envy-free prices, rendering the choice of a particular one problematic.

6. Sincerity and Independence

We consider next the potential manipulability of the Gap Procedure, first by noting the monotonicity of assignments with respect to bidding:

Proposition 10. *By raising its bid for a room under the Gap Procedure, a player never hurts, and may help, its chances of being assigned that room.*

Proof. If Pi raises its bid for Rj (and necessarily lowers one or more of its bids for other rooms), then its bid for Rj is at least as likely to be included in the maxsum assignment, holding the bids of other players constant. Hence, the probability of Pi's being assigned Rj under the Gap Procedure cannot be less and may be greater. Q.E.D.

This is not to say that Pi will necessarily receive a greater surplus after it raises its bid for Rj and, as a consequence, receives it. For example, its surplus from its old maxsum assignment of Rk ($k \neq j$)—before it raised its bid—might exceed its surplus from its new maxsum assignment of Rj.

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[19]Moulin (1995, p. 178) expresses a similar sentiment: "While no envy [i.e., envy-freeness] has a valid claim to the preeminence among other tests of justice . . ., one should not forget that alternative, conflicting tests are worthy of our attention, too."

We think it would be very hard for a player to predict, not knowing the bids of the other players, whether raising its bid for a room would increase its surplus. Consequently, players will have good reason to be sincere, making bids that reflect their valuation, v_{ij}.[20] There is still another reason for being sincere:

Proposition 11. *Sincere bids are the only bids that preclude negative utility under the Gap Procedure.*

Proof. If P*i*'s bids are not sincere, then its bid b_{ij} for some R*j* must exceed v_{ij}. In this case, P*i* may be assigned R*j* at a price greater than v_{ij}, giving it negative utility. Q.E.D.

Define the price of a room to be *independent* if it does not depend directly on the winner's bid.[21]

Proposition 12. *Under the Gap Procedure, a player's bid is independent if and only if the Gap prices are less than or equal to bids that are strictly lower than the maxsum bids.*

Proof. If the Gap price for a room does not meet the condition of the theorem, then the gap used in the determination of the price depends on the player's maxsum bid

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[20]If bid points are dollars, then a player may think that the most desirable room is worth more than $100. But because the auction is a relativized one, a player cannot bid more than 100 for a room; moreover, if a player bids the maximum of 100 for a room because this is the only room it finds acceptable, it must bid 0 for every other room. Thus, while a player may think a room is worth more, in some medium of exchange, than what it is allowed to bid, it cannot make such an overbid. Because bids express preferences only in relative terms, neither would it be motivated to do so.

[21]We distinguish here between the maxsum assignment of a room and its Gap price. A player's room assignment depends on the bids of all players, but the price it pays for its assigned room may depend only on *other* players' bids, in which case it is independent. Such independence implies that a small change in the winner's bid—small enough that its room assignment is not affected—does not change the Gap price. (In fact, only a sufficiently large decrease in a winner's bid can affect its room assignment and therefore the Gap price; no increase in its winning bid can ever change its room assignment, and hence the price it pays for this room if there is independence.)

(as well as the next-lower bid for the room). Otherwise, the price is solely a function of lower bids and, hence, is independent. Q.E.D.

Player bids are not independent in Example 1 (they can never be when there are only two players), Example 2, and Example 4, whereas they are independent in Example 3, Example 6 (four players), and Example 7. Generally speaking, the more players, or the more competitive they are in their bidding, the more likely their bids next-lower to the maxsum bids will sum to 100 or more, making prices independent of the winning player's bids (as in a Vickrey auction).

This result suggests that players can "afford" to be sincere and bid their valuations v_{ij}, because often they will have to pay only someone else's lower bid (or bids). Unlike a Vickrey auction, however, sincere bidding is not a dominant strategy, even when there is independence, because maxsum assignments do depend on player bids. Thus, we call pricing under the Gap Procedure *partially independent*.

7. Other Applications

We turn next to other possible applications of the Gap Procedure. It seems most applicable when fairness considerations dictate that players get some minimum number of items. In many business schools, for instance, MBA students bid for courses or interviews with companies recruiting on campus, but there is no assurance that these items will be equitably distributed if they go only to the highest bidders. For instance, in Examples 3, 4, and 5, one of the three bidders is not highest, or tied for highest, on any room, so under a highest-bidder-wins auction, one bidder would get two rooms, one would get one room, and one would get none.

The highest-bidder-wins auction creates incentives for players to overbid on the items they most desire, cutting out players who place middling bids on the items. By contrast, under an extension of the Gap Procedure, all players would get one item before any player gets two items, two items before any player gets three, and so on, if the procedure is repeated round by round.

To illustrate this extension of the procedure, consider the following example, in which we assume the items being allocated are MBA job interviews (I's) in four different business fields (consulting, finance, management, and marketing):

Example 8. Bids by Four Players for Four Interviews (First Round)

	Consulting (I1)	Finance (I2)	Management (I3)	Marketing (I4)	*Total*
P1	40	20	20	20	100
P2	50	31	10	9	100
P3	9	10	31	50	100
P4	20	20	20	40	100

Under the Gap Procedure, the maxsum assignment is along the main diagonal at prices of 26.5, 23.5, 23.5, and 26.5 for I1, I2, I3, and I4, respectively. Note that the highest bids of 50 by P2 for I1 and by P3 for I4 do not figure into the price calculation, because these bids are higher, not lower, than the maxsum ones and, therefore, are not used in the descent.[22]

If the four interviews were auctioned off to the high bidders in a standard highest-bidder-wins auction, then two players would get two interviews each (P2 would get I1 and I2; P3 would get I3 and I4), whereas P1 and P4—not being the highest bidders on any rooms—would win none. This seems grossly unfair if there are only the four interviews to allocate.

[22]In this example, it turns out that the unique (in integers) envy-free prices for the four rooms are 35, 15, 15, and 35. These prices reflect the high bids of 50 for I1 and I4, which are taken into account only in the second-round Gap allocations, as we will show next.

But now assume that each player is entitled to two interviews and apply the Gap Procedure again, but this time with the Gap prices paid on the first round for items won deducted from each player's second-round bids:

Example 8. Bids by Four Players for Four Interviews (Second Round)

	Consulting (I1)	Finance (I2)	Management (I3)	Marketing (I4)	*Total*
P1	13.5	20	20	20	73.5
P2	50	7.5	10	9	76.5
P3	9	10	7.5	50	76.5
P4	20	20	20	13.5	73.5

Now the totals for all players are not the same, but we can still apply the Gap Procedure, assuming the requirement that prices must sum to 100 remains in place. Maxsum will assign I1 to P2 and I4 to P3, but then there is a tie: I2 or I3 can be assigned either to P1 or P4; for definiteness, assume I2 is assigned to P1 and I3 to P4. The prices will be 30, 20, 20, and 30 for I1, I2, I3, and I4, respectively. Thus, after two rounds, each player will get the following two items at the following total prices:

P1: I1 & I2 at 56.5 P2: I1 & I2 at 43.5; P3: I3 & I4 at 43.5; P4: I3 & I4 at 56.5.

Because of the aforementioned tie, P1 and P4 would be equally happy with I1 & I3 and I2 & I4, respectively.

A glance at the original bids in Example 8 (First Round) shows that all four players get their two most-valued interviews (in the case of P1 and P4, there are ties for second place). The fact that P1 and P4 pay higher prices for their interviews than P2 and P3 pay for theirs reflects the fact that the former players value the interviews they receive more; also, their top bids are closer to each other, in both absolute and proportional terms, than the top bids of P2 and P3 for the interviews they receive.

While P2 and P3 would also win their two interviews in a standard auction, either P1 or P4 could possibly win three interviews (depending on how ties are broken), leaving the other with only one. Again, this possibility would be unfair if, in this extension of the housemates problem, we wish to ensure that each player wins the same number of items, albeit at different prices (depending on the competition in the marketplace).

If the number of items to be distributed is not a multiple of the number of players, then one cannot treat each player equally. Nevertheless, the Gap Procedure can still be applied. For example, if there are six items to be distributed in Example 8, maxsum on the second round would assign I1 to P2 and I4 to P3. Since the sum of their bids is exactly 100 (50 + 50), there would be no descent if 100 or less is the total that stops the descent. On the other hand, if, because there are only two items, the total price is set at, say, 50, then each player would have its bids reduced equally (to 25)—and, consequently, each would receive surpluses of 25 from these acquisitions.

Fair-division problems also arise at the international level. Thus, in the division of Germany into four zones after World War II, there was debate among the four Allies not only over how Germany would be divided but also about which zones each of the Allies would control.[23] Since they were considered to have more or less equal rights, the Allies might have bid for the zones, with those paying higher prices contributing more, perhaps, to the overall administration of the country.

As another example, consider the division of an estate among children, and suppose there is bidding for the indivisible goods in rounds. Then it is reasonable to suppose, as with the allocation of interviews, that the price that an heir pays after each round will be deducted from its allocation. Alternatively, an heir might be allowed to bid for as many items as it desires in a single round, but there would be a stipulation that each heir is entitled to a certain minimum number of items (not necessarily one). Similarly,

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[23]See Brams and Taylor (1996, 1999) for a discussion of this case and citations of the literature on Germany's zonal division.

members of the U.S. Congress are generally assigned to a minimum number of committees and subcommittees.[24]

The Gap Procedure could be modified to allow for different minima. For example, it could be used to assign ministerial posts in a parliamentary government to parties in the governing coalition as a function of their size.

If entitlements are equal, we presume the players who pay less for the indivisible goods they win will be entitled to receive more of the divisible goods (e.g., liquid assets) not included in the auction. Some formula, however, would probably have to be agreed to in advance to value the indivisible goods in terms of the divisible goods, which would be allocated to the players in proportion to their unspent points at the end of the auction.

We will not pursue further other applications of the Gap Procedure and its possible extensions. Clearly, it is applicable to a variety of allocation problems in which market competition needs to be tempered by considerations of fairness.

8. Conclusions

We have focused on the housemates problem because it encapsulates the trade-offs that players face when the goods they want come at a price (the bad). Fairness comes to the fore when players, either with similar or different preferences, are entitled to certain minimum numbers of goods. Even in the simple case in which two housemates like different rooms, it is important to take account in pricing them how much each is willing to pay for the room it wants.

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[24]In the U.S. House of Representatives, each member is entitled to one so-called exclusive or major commitee assignment. In the Democratic Party, freshmen members, and continuing members who seek committee transfers, submit rankings of the committees to which they would like to be assigned, but both the rankings and the selection process are conditioned by political and strategic factors (Shepsle, 1978). Allowing bidding on successive rounds under the Gap Procedure, starting with exclusive and major committees and proceeding to lesser committees and subcommittees, would presumably make the selection process more "objective," but whether this is desirable is, of course, debatable.

In a relativized auction, as we illustrated, equal reductions of high bids can lead to negative rent, and both equal and proportional reductions can lead to player losses. These problems led us to look for a procedure that takes into account not just the highest bids, but also the second-highest and possibly lower bids, in setting prices.

The Gap Procedure, starting with maxsum assignments that maximize the total surplus, lets prices descend to the second-highest bids if their sum is greater than or equal to the total rent of 100—and still lower until the sum of the bids is 100 or less. Once this level is reached, and provided the sum is not exactly 100, reductions in the next-higher bids are made in proportion to the differences, or gaps, between these bids for each room and the next-lower bids. Thereby the market helps to set prices by incorporating the most competitive lower bids into the pricing mechanism.

The Gap Procedure precludes the possibility of negative rent. In addition, it assigns rooms to players at prices that never exceed their bids, ensuring that the allocation is feasible. Because there is no other assignment of rooms, or prices that players pay for them, that yield surpluses Pareto-superior to the Gap surpluses, the Gap surpluses are Pareto-optimal, though not exclusively so (if negative rents are disallowed). While Pareto-optimality rules out mutually beneficial trades, it does not rule out envy: one player may prefer another's room at the price it pays for it. Indeed, there may be no feasible assignments at envy-free prices if there are four or more players.

While the Gap Procedure does not always give an envy-free allocation, even when this is possible (which is always the case when there are only two or three players), we believe the Gap prices take proper account of market competition. It seems only fair that prices should be higher when there is greater demand for items. Additional advantages of the Gap Procedure are that it is monotonic in bids, encourages sincere bidding, and generates prices that are partially independent of the bids.

Besides the fair division of rent among housemates, the Gap Procedure is applicable to situations in which each player is entitled to receive one item—and perhaps more than one if the procedure is applied round by round. This is the case when MBA students bid for interviews in business schools. It also might be applied to everything from estate division to chore division, including burden sharing at the international level.

Still other applications of the Gap Procedure come readily to mind, including applications in which players do not have the equal endowments and, consequently, are allocated different numbers of points at the start, as in bidding for ministerial posts in a parliamentary government. Suffice it to say that the Gap Procedure offers a promising approach to balancing competition and fairness, though neither it nor any other procedure can entirely eliminate envy.

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