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# Reputation and Patience in the "War of Attrition"

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## Abstract

The paper presents an approach to selecting among the many subgame perfect equilibria which exist in a standard concession game with complete information. We extend the description of a game to include a specific "irrational" (mixed) strategy for each player. Depending on the irrational strategies chosen, we demonstrate that this approach may select a unique equilibrium in which the weaker player concedes immediately. A player is weaker if either he is more impatient or his irrational strategy is to wait in any period with the higher probability.

## 1. The War of Attrition

The following game is a variant of the "War of Attrition" which is now a standard paradigm in economic theory. (See Hendricks and Wilson (1985) for a survey of the literature) Two players, 1 and 2, are involved in a dispute. Time is discrete and the players alternately have the option to concede. If player 1 concedes in period t, the outcome is (A,t). If player 2 concedes in period t, the outcome is (B,t). If neither ever concedes the outcome is $(C,\infty)$.

The game form is illustrated in Figure 1. To enforce the alternation of moves, we restrict player 1 to move only in the even periods and player 2 to move only in the odd periods. A strategy for player 1 is then a sequence $\alpha_1 = (\alpha_1(t))_{t=0,2,4,\ldots}$, where $[1-\alpha_1(t)] \in [0,1]$ is the probability that player 1 concedes in period t conditional on neither player conceding before period t. Similarly, a strategy for player 2 is a sequence $\alpha_2 = (\alpha_2(t))_{t=1,3,5,\ldots}$.

We suppose that the preferences of the players can be represented by VNM utilities $v_i$ satisfying $v_1(C,\infty) = v_2(C,\infty) = 0$ and, for $t < \infty$, $v_i(x,t) = u_i(x)\delta_i^t$, where $u_1(A) = u_2(B) = L$ (the low payoff) is the return to conceding and $u_1(B) = u_2(A) = H$ (the high payoff) is a player's return if the other player concedes. We assume that $0 < \delta_i < 1$ and $H\delta_i > L > 0$ for $i = 1,2$. Thus, if a player is sure his opponent will concede in the next period, it is optimal for him not to concede, but, if he is to be the first to concede, he prefers to do it sooner rather than later.

There are two asymmetries in the model. One is due to the order of the moves in the game. For our purposes, this asymmetry is not important since our results below do not depend on the order in which the players move.[1]

The second potential asymmetry is in the time preferences of the players ($\delta_1$ may be different than $\delta_2$). It is on this asymmetry that we will focus our attention.

Regardless of the relative size of the discount factors, there is an infinity of subgame perfect equilibrium outcomes. One of these outcomes is for player 1 to concede immediately. Another is for player 1 to wait and for player 2 to concede immediately. Our own intuition, however, suggests that the weaker player, the one with the lower discount factor, should concede immediately. One of the aims of the paper is to develop a criterion for selecting this particular equilibrium outcome.

## 2. What Is Missing in the Model

We take the position that a game of complete information can generally be thought of as an approximation to a multi–person decision problem in which each player is reasonably certain about the objectives of the other players but does entertain the possibility that one or more of the other players will act irrationally. There are, of course, many ways to model an "irrational" player. For the purposes of this paper, we will identify an irrational player with a particular mixed strategy. Our criterion for selecting an equilibrium will then require that it be close to an equilibrium of the corresponding perturbed game obtained by introducing some irrational player with arbitrary small probability.

Formally, let $\Gamma$ be an n player game in extensive form and $\sigma^* = (\sigma_1^*,\dots,\sigma_n^*)$ be any (behavior) strategy combination for $\Gamma$. For any $\epsilon \in (0,1)$, let $(\Gamma,\sigma^*,\epsilon)$ be a game with the same extensive form and payoffs as $\Gamma$ with the property that any strategy combination $\sigma$ for $(\Gamma,\sigma^*,\epsilon)$ is equivalent to the strategy combination $\epsilon\sigma+(1-\epsilon)\sigma^*$ for $\Gamma$.[2] Then, $\sigma$ is a (sequential) $\underline{\sigma}^*$–

equilibrium for $\Gamma$ if it is the limit of a sequence of strategy combinations $\{\sigma^k\}$ each of which is a (sequential) equilibrium for $(\Gamma,\sigma^*,\epsilon^k)$ for some sequence $\{\epsilon^k\}$ such that $\epsilon^k \to 0$.[3] We interpret a $\sigma^*$–equilibrium to be the outcome of a game $\Gamma$ when all of the participants believe that there is some (small) chance that the strategy of player i will deviate from his rational play and adopt strategy $\sigma_i^*$.

One may view this approach as the combination of two ideas that are already well established in the literature. In a series of papers, Kreps, Milgrom, Roberts, and Wilson (1982), Kreps and Wilson (1982), and Milgrom and Roberts (1982) have included the players' doubts explicitly in the model. In determining the equilibrium outcomes in the chain store paradox and in the prisoner's dilemma, they suppose that the players assign a small probability that their opponents use a certain strategy specified exogenously by the modeller (the "tough" chain store strategy and the "tit–for–tat" accordingly). Under such an assumption they are able to obtain sequential equilibria which are not equilibria of the original game.

Our objective differs from theirs in two ways. First, we want to determine how the equilibrium changes as we vary the fixed strategy $\sigma^*$ and/or the payoffs of the rational players. Second, we are interested in selecting an equilibrium from the set of equilibria of the original game of complete information rather than trying to justify a new equilibrium outcome. Thus, although we modify the game by adding an irrational player, we are only interested in the equilibria as the probability of the irrational player goes to zero. This leads us to the second idea to which our idea is related.

Selten (1975), Myerson (1978), Kohlberg and Mertens (1986), and others have suggested equilibrium concepts based on the limit of the equilibria of sequences of perturbed games. The most widely used of these ideas is

Selten's concept of a "trembling hand" perfect equilibrium. An equilibrium is trembling hand perfect if it is the limit of the equilibria of some sequence of games in which the behavior strategy at each information set is perturbed with increasingly small probability. Thus not only is the specific perturbation unspecified in advance but any errors across information sets are uncorrelated. Evidently, mistakes are to be interpreted as errors of execution rather than errors of rationality.

The primary motivation behind Selten's approach was to extend the intuition of subgame perfection to games with incomplete information. The motivation behind our approach is to test an equilibrium against a prespecified possibility of irrational behavior. Thus, the ideas behind the $\sigma^*$–equilibrium differs from the trembling hand perfect equilibrium in two ways. First, as in the work on the chain store paradox cited above, our perturbations are in mixed (or behavior) strategies rather than local strategies (or mixed strategies in the agent–normal form). This leads to the possibility that mistakes are correlated across information sets.[4] Second, it implicitly incorporates a precise form of irrational behavior (and hence specific perturbations of the strategies) in the concept of the equilibrium. As we emphasized above, our main objective is to study how changes in the form of irrational behavior affects the outcome of the game.

Finally, we should note that the techniques used to establish our results are also well established in the literature. A number of authors (e.g. Kreps and Wilson (1982), Fudenberg and Tirole (1986) and Wilson (1982)) have demonstrated that introducing certain kinds of irrational players in the war of attrition will lead to unique (or locally unique) equilibria. The form of the irrationality, however, is generally quite simple. The irrational player never concedes. That is, he represents the most extreme version of a "tough" player.

Because we wish to parameterize the "toughness" of a player and examine its implications for the equilibrium of the game, however, we are forced to require even the irrational players to mix in each period. As a consequence, the details of the analysis become considerably more complicated.

We turn now to the concession game described in Section 1.

## 3. The Main Result

The theme of our results is illustrated in the following example. Suppose that each player is irrational with probability $\epsilon > 0$ where for player 2 irrationality means never to concede and for player 1 it means to concede always. It is easy to check that, regardless of the values of $\delta_1$ and $\delta_2$, the only equilibrium outcome is for player 1 to concede immediately. This observation fits nicely our intuition that the asymmetry in the content of "out of rationality" behavior is critical in determining the outcome of the game. Player 2 can build up a reputation of playing tough whereas player 1 does not have the tools to do that. In this section, we parameterize the ability of players to build up their reputations and investigate its implications for the equilibrium of the game.[5]

In general, there are irrational opponents against whom it is optimal to concede immediately but to wait if the game reaches some later stage without a concession. Consequently, if the influence of an irrational player is to be independent of time, we must impose some stationarity in the strategies of the irrational opponents. We will therefore restrict attention to irrational players whose strategies are of the form $(\gamma,\gamma,\gamma,...)$. That is, the irrational player plans to concede with the same probability $(1-\gamma)$, conditional upon reaching any period in which he is permitted to move. In what follows, let i refer to an arbitrary player and j to the other player and assume any integer t is odd or even as the definitions require.

To state our main result, we define $p_j$ to be the solution of

$$\delta_i[(1-p_j)H + p_j\delta_i L] = L.$$

Suppose that, conditional on reaching period t, player j concedes with probability $(1-p_j)$. Then player i is indifferent between conceding in period t-1 and waiting until period t+1 to concede. If, upon reaching period t, player j plans to concede with a probability smaller than $[1-p_j]$, then player i prefers to concede in period t-1 rather in period t+1. If, upon reaching any period t, player j plans to concede with a probability greater than $[1-p_j]$, then player i prefers to wait until period t+1 rather than concede in period t-1. Since we suppose that $\delta_i H > L$, it follows that $0 < p_j < 1$. Furthermore, $\delta_i > \delta_j$ implies $p_j > p_i$.

Let $\Gamma(\delta_1,\delta_2)$ be the concession game defined by the discount factors $\delta_1$ and $\delta_2$, and let $(\gamma_1,\gamma_2) \in [0,1]^2$ represent a pair of stationary strategies. Then, given the definition of $p_i$, we may state our main result as follows.

<u>Theorem 1</u>: Suppose $\gamma_2 > p_2$ and $\gamma_2/p_2 > \gamma_1/p_1$. Then the unique sequential $(\gamma_1,\gamma_2)$-equilibrium outcome for $\Gamma(\delta_1,\delta_2)$ is for player 1 to move immediately with probability 1 (i.e. $\alpha_1(0) = 0$).

Recall that $[1-\gamma_i]$ is the probability that, upon reaching any period, the irrational player i concedes, and $[1-p_i]$ is the probability of moving in any period that induces indifference for player j between immediate concession and waiting to concede at his next turn. If $\gamma_2$ is greater than $p_2$, then, faced with his irrational opponent, player 1 would concede immediately. If in addition, the ratio $\gamma_2/p_2$ is greater than $\gamma_1/p_1$, then Theorem 1 implies that player 1 concedes immediately in any sequential $(\gamma_1/\gamma_2)$–equilibrium. In particular, if the players have identical time preferences ($\delta_1 = \delta_2$), the player with the better facility for building a reputation for toughness (the highest $\gamma_i$) will win, while if the players have the same facilities for establishing a reputation ($\gamma_1 = \gamma_2$), then the more impatient player concedes immediately.

Theorem 1 is a statement about the equilibrium outcomes. For almost all parameter values, the sequential $(\gamma_1,\gamma_2)$–equilibrium is itself unique.

<u>Theorem 2</u>: (a) Suppose $\gamma_2/p_2 > \gamma_1/p_1 > 1$. Then $\alpha_1 = (0,p_1,p_1,p_1,...)$ and $\alpha_2 = (p_2,p_2,p_2,...)$ is the unique sequential $(\gamma_1,\gamma_2)$–equilibrium for $\Gamma(\delta_1,\delta_2)$.

(b) Suppose $\gamma_2/p_2 > 1 > \gamma_1/p_1$. Then $\alpha_1 = (0,0,0,...)$ and $\alpha_2 = (1,1,1,...)$ is the unique sequential $(\gamma_1,\gamma_2)$–equilibrium for $\Gamma(\delta_1,\delta_2)$.

Given the conditions of Theorem 1, Theorem 2 reveals a kind of second order benefit to player 1 if his irrational counterpart (who plays $\gamma_1$) is sufficiently tough. When $\gamma_1/p_1 < 1$, player 2 always waits and player 1 always concedes, regardless of the history of the game. However, when $\gamma_1/p_1 > 1$, each player i concedes with probability $(1-p_i)$ upon reaching any later period. Thus, if player 1 makes a "mistake" in the first period and waits, there is positive probability that player 2 will eventually concede.

Theorem 1 depends upon the satisfaction of two conditions. First, at least one of the players must have the ability to build a reputation for toughness. Second, one of the players must have an advantage over his opponent in building his reputation. If either of these conditions are violated, we obtain a different set of sequential $(\gamma_1,\gamma_2)$–equilibrium outcomes. If $\gamma_1/p_1$

$= \gamma_2/p_2 > 1$, then both players have an equal facility for building a reputation for toughness. In this case, player 1 concedes immediately with a probability between $1-p_1$ and $1-\gamma_1$. Thereafter, the probability with which player 1 concedes depends only on the impatience of the other player.[6] On the other hand, if neither irrational player is sufficiently tough to induce a rational opponent to concede, then it is a sequential $(\gamma_1,\gamma_2)$–equilibrium outcome for either player to concede immediately.[7]

If we reverse the order of $\gamma_1/p_1$ and $\gamma_2/p_2$, the statement of the theorems must be modified, but the results are essentially the same.

## 4. Proofs

In the section, we provide a complete proof of Theorem 2(a). The proof of Theorem 2(b) (and Theorem 1 for the case where $\gamma_1 = p_1$) is provided in an earlier version of the paper.

Suppose player 1 is playing strategy $\alpha_1$. We will use the following notation. For any odd period t, let $\mu_1(t)$ be the probability player 2 assigns to the possibility that player 1 is an irrational player, conditional on the game reaching period t. Let $1-\beta_1(t+1)$ be the probability player 2 assigns to the possibility that player 1 plans to concede in period t+1 conditional on reaching period t+1. Then, letting $\mu_1(-1) = \epsilon$, we may define, for any odd period t $> 0$,

$$\mu_1(t+2) = \gamma_1\mu_1(t)/\beta_1(t+1),$$

and

$$\beta_1(t+1) = [1-\mu_1(t)]\alpha_1(t+1) + \mu_1(t)\gamma_1.$$

We begin by characterizing the equilibria of the perturbed game

$(\Gamma(\delta_1,\delta_2),(\gamma_1,\gamma_2),\epsilon)$. The argument is roughly as follows. We first establish that as long as the rational player i has not yet moved with probability 1, he must adjust his strategy so that (after the initial period) his opponent is indifferent between conceding immediately and waiting another period. Since $\gamma_i > p_i$, this implies that, in any period t, the rational player j concedes with a higher probability than his irrational counterpart. Consequently, we eventually reach a period $\hat{t}_i$ by which he has conceded with probability 1. At this point, faced with a relatively "tough" irrational opponent, it is optimal for the rational player j to concede immediately if he has not already done so. This argument leads to the conclusions of Lemma 2 below.

In Lemmata 1 to 4, assume $\gamma_2/p_2 > \gamma_1/p_1 > 1$ and take $\beta_i(t)$, $\mu_i(t)$, etc. to be equilibrium values for player i in the game $(\Gamma(\delta_1,\delta_2),(\gamma_1,\gamma_2),\epsilon)$.

Lemma 1: Suppose $t > 0$. Then, for $i = 1,2$,

(a) $\beta_i(t+1) \geq p_i$, and

(b) $\beta_i(t) > p_i$ implies $\alpha_i(t) = 0$ (and hence $\mu_i(t+1) = 1$).

Proof: We will establish part (b) first. Suppose $\beta_i(t) > p_i$ for some $t > 0$. Then, if it is optimal for the rational player j to wait in period $t-1$, it cannot be optimal for him to move in period $t+1$. Therefore, the rational player j either moves with certainty upon reaching period $t-1$ or waits with certainty upon reaching period $t+1$. In the first instance, player j is irrational with probability 1 in period $t+1$ so that $\beta_j(t+1) = \gamma_j$. In the second instance, $\beta_j(t+1) = [1-\mu_j(t)] + \mu_j(t)\gamma_j \geq \gamma_j$. In either instance, $\beta_j(t+1) \geq \gamma_j > p_j$.

Proceeding by induction, it follows that $\beta_j(t+k) > p_j$ for all odd $k > 0$. Consequently, for every nonnegative even k, upon reaching period t+k, the rational player j strictly prefers conceding immediately to waiting and conceding in period t+k+2. But since the game is continuous at infinity,[8] it follows that, upon reaching period t, the rational player j strictly prefers conceding in period t to waiting until any later period (including $\infty$). Therefore, $\alpha_i(t) = 0$ and hence $\mu_i(t+1) = 1$.

To establish part (a), suppose $\beta_i(t+1) < p_i$ for some $t > 0$. Then, upon reaching period t, the rational player j will choose to wait, i.e $\alpha_j(t) = 1$. Therefore, $\beta_j(t) = [1-\mu_j(t-1)] + \mu_j(t-1)\gamma_j \geq \gamma_j > p_j$, contradicting part (b). Q.E.D.

For $i = 1,2$, define $\hat{t}_i \equiv \sup\{t: \mu_i(t) < 1\}$ to be the last period in which player j moves for which there is still a positive probability that player i is rational (or $\infty$).[9] Then we may establish

<u>Lemma 2</u>: (a) $\beta_i(t) = p_i$ for $2 \leq t < \hat{t}_i$, $i = 1,2$;

(b) $\hat{t}_i < \infty$ for $i = 1,2$; and

(c) $|\hat{t}_2 - \hat{t}_1| = 1$.

<u>Proof</u>: Part (a) follows immediately from Lemma 1 and the definition of $\hat{t}_i$. To establish (b), suppose that $\hat{t} = \infty$. Then it follows from part (a) that, for all $t > 1$, $\beta_i(t) = p_i$ and hence that $\mu_i(t) = (\gamma_i/p_i)\mu_i(t-2)$. But then $\gamma_i/p_i > 1$ implies that $\mu_i(t) > 1$ for t sufficiently large. A contradiction.

To establish (c), suppose that $\mu_i(t) = 1$. Then, for all odd $k > 0$, $\beta_i(k+t) = \gamma_i > p_i$. The lemmma then follows from part (a) of Lemma 1. Q.E.D.

The next step is to note that $\gamma_2/p_2 > \gamma_1/p_1$ implies that the rate at which rational player 1 must concede in order to make his opponent indifferent between conceding and waiting is smaller than for player 2. Consequently, to ensure that the rational players first concede with probability 1 in adjacent periods, player 1 must concede with a relatively high probability at the outset of the game. Furthermore, the smaller is the initial probability that the players are irrational, the longer the rational players must wait before conceding with probability 1, and consequently, the larger is the probability that player 1 must concede at the outset in order to ensure that both players first concede with certainty in adjacent periods. Theorem 2(a) is then proved by establishing that, in the limit, this relationship requires player 1 to concede immediately with probability 1.

Lemma 3: (a) $\hat{t}_1 = \hat{t}_2 - 1$ implies

$$\mu_2(2)(\gamma_2/p_2)^{(\hat{t}_2-2)/2} < 1 = \mu_1(1)(\gamma_1/p_1)^{\hat{t}_2/2} \le \mu_2(2)(\gamma_2/p_2)^{\hat{t}_2/2},$$

(b) $\hat{t}_1 = \hat{t}_2 + 1$ implies

$$\mu_1(1)(\gamma_1/p_1)^{\hat{t}_2/2} < 1 = \mu_2(2)(\gamma_2/p_2)^{\hat{t}_2/2} \le \mu_1(1)(\gamma_1/p_1)^{(\hat{t}_2+2)/2}.$$

Proof: We establish first that $\hat{t}_j = \hat{t}_i + 1$ implies $\beta_i(\hat{t}_i+1) = p_i$. Suppose $\hat{t}_j = \hat{t}_i + 1$ but $\beta_i(\hat{t}_i+1) \neq p_i$. Then Lemma 1(a) implies that $\beta_i(\hat{t}_i+1) > p_i$. Consequently, upon reaching period $\hat{t}_i$, the rational player j strictly prefers conceding immediately to waiting and moving in period $\hat{t}_i+2$. Therefore, either $\alpha_j(\hat{t}_i) = 0$ or $\alpha_j(\hat{t}_i+2) = 1$. In the first case, $\mu_j(\hat{t}_i) = 1$, violating the definition of $\hat{t}_i$, and in the second case, $\mu_j(\hat{t}_j+2) < \mu_j(\hat{t}_j) < 1$, violating the definition of $\hat{t}_j$.

Now suppose $\hat{t}_1 = \hat{t}_2 - 1$. Then the previous paragraph implies that $\beta_i(\hat{t}_i+1) = p_i$ and hence that

$$\mu_2(2)(\gamma_2/p_2)^{(\hat{t}_2-2)/2} = \mu_2(\hat{t}_2) < 1 = \mu_1(\hat{t}_2+2) = \mu_1(1)(\gamma_1/p_1)^{\hat{t}_2/2}.$$

By assumption, $\hat{t}_2 > 0$ and, therefore, Lemma 1 implies that $\beta_2(\hat{t}_2+1) \geq p_2$. Consequently,

$$1 = \mu_2(\hat{t}_2+2) = \mu_2(2)(\gamma_2/p_2)^{(\hat{t}_2-2)/2}(\gamma_2/\beta_2(\hat{t}_2+1)) \leq \mu_2(2)(\gamma_2/p_2)^{\hat{t}_2/2}.$$

These two relations establish part (a). A similar argument establishes part (b). Q.E.D.

Using Lemma 3, we may establish some properties of the sequential equilibria of $(\Gamma(\delta_1,\delta_2),(\gamma_1,\gamma_2),\epsilon)$ as $\epsilon$ becomes small.

Lemma 4: For any $t > 0$ and any $\epsilon_1 > 0$, there is a $\psi > 0$ such that, for all $\epsilon < \psi$: (a) $\beta_2(1) = p_2$, (b) $\hat{t}_i > t$, and (c) $|\alpha_i(t) - p_i| < \epsilon_1$.

Proof: We establish first that $\beta_2(1) = p_2$ for $\epsilon > 0$ sufficiently small. Suppose $\beta_2(1) < p_2$. Then $\mu_2(2) > \mu_2(0)\gamma_2/p_2 = \epsilon\gamma_2/p_2$. Furthermore, player 1 never concedes in period 0, and, therefore, is more likely to rational in period 1 than at the outset of the game. That is, $\mu_1(1) < \mu_1(-1) = \epsilon$. Then using both parts (a) and (b) of Lemma 3, we may show that

$$\epsilon(\gamma_2/p_2)^{\hat{t}_2/2} < \mu_2(2)(\gamma_2/p_2)^{(\hat{t}_2-2)/2} \leq \mu_1(1)(\gamma_1/p_1)^{\hat{t}_2/2} < \epsilon(\gamma_1/p_1)^{\hat{t}_2/2},$$

contradicting the assumption that $\gamma_1/p_1 \le \gamma_2/p_2$. We conclude that $\beta_2(1) \ge p_2$. This implies in turn that $\mu_2(2) \le \epsilon\gamma_2/p_2 < 1$ for $\epsilon$ sufficiently small. Part (a) then follows from Lemma 1(b).

Since $\beta_2(1) = p_2$, Lemma 3 implies that

$$\epsilon(\gamma_2/p_2)^{(\hat{t}_2+2)/2} = \mu_2(2)(\gamma_2/p_2)^{\hat{t}_2/2} \ge 1.$$

Fix $t > 1$. Then there is an $\psi > 0$ such that $\epsilon < \psi$ implies $t < \hat{t}_2$. Part (b) then follows from Lemma 2(c). Furthermore, it follows from Lemma 2(a) that $\beta_1(k) = p_1$ for even k, $0 < k \le t$. Lemma 3 then yields

$$\mu_1(t) = \mu_1(1)(\gamma_1/p_1)^{(t-1)/2} = \mu_1(1)(\gamma_1/p_1)^{(\hat{t}_1-1)/2}(p_1/\gamma_1)^{(\hat{t}_1-t)/2} < (p_1/\gamma_1)^{(\hat{t}_1-t)/2}.$$

Since $\hat{t}_1 \to \infty$ as $\epsilon \to 0$, it follows that $\mu_1(t) \to 0$ as $\epsilon \to 0$. Then since $\beta_1(t) = p_1$, it follows from the definition of $\beta_1(t)$ that $\alpha_1(t) \to p_1$ as $\epsilon \to 0$. A similar argument establishes that, for any odd $t > 0$, $\alpha_2(t) \to p_2$ as $\epsilon \to 0$. Q.E.D.

<u>Proof of Theorem 2(a)</u>: For each $\epsilon > 0$, it is straightforward to construct a sequential equilibrium satisfying the conditions of Lemmata 3 and 4. (Alternatively, see Fudenberg and Levine (1986)).

Using the definitions of $\mu_1(1)$ and $\beta_1(0)$ and both parts of Lemma 3, we obtain

$$[\epsilon\gamma_1+(1-\epsilon)\alpha_1(0)]/\gamma_1 = \beta_1(0)/\gamma_1 = \epsilon/\mu_1(1) = (p_2/\gamma_2)\mu_2(2)/\mu_1(1) \le [\gamma_1 p_2/\gamma_2 p_1]^{(\hat{t}_2+2)/2}.$$

Then since $\gamma_2/p_2 > \gamma_1/p_1$ and $\hat{t}_2 \to \infty$ as $\epsilon \to 0$ (by Lemma 4(b)), it follows that $\alpha_1(0) \to 0$ as $\epsilon \to 0$. Lemma 4 also implies that $\alpha_i(t) \to p_i$ for $t > 0$, $i = 1,2$. By definition, this strategy pair forms a sequential $(\gamma_1,\gamma_2)$–equilibrium. Q.E.D.

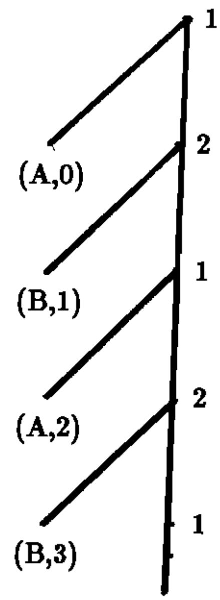
The precise proof of part (b) of Theorem 2 is a bit more cumbersome. Roughly, the idea is to show that, even in the perturbed game, the rational player 1 must move with probability 1 at the outset of the game. Suppose not. Then, since the irrational player 1 concedes with a relatively high probability in any period, the rational player 1 must concede with a lower probability in order to make the rational player 2 indifferent to conceding and waiting. This implies that the rational player 1 never concedes with probability 1. However, to make the rational player 1 indifferent between moving and waiting in each period, an argument similar to that given above implies that the rational player 2 must concede with probability 1 in some period. At this point, faced with the relatively tough irrational player 2, it is optimal for the rational player 1 to also concede immediately. This contradiction establishes the result. A complete proof is given in the unpublished version of this paper.

## 5. Concluding Remarks

In this paper, we have illustrated an approach which, in some cases, allow us to select a specific outcome in games with multiple equilibria. Essentially, we are suggesting that the specification of the equilibrium be extended to reflect the details about the kind of response a player expects to face if his opponent deviates from rational behavior. For the particular concession game we have examined above, we have parameterized how the differences in the tendency of players to play excessively tough (or weak) affects the interaction of rational players.

As we have seen, for some parameter values (i.e. when the irrational behavior of both players tends to be excessively weak), this approach yields no additional restrictions on the equilibrium outcomes. For other parameter values, however, our concept leads to a unique equilibrium outcome.[10] Philosophically, this approach is very different from the approach of many other writers, including Kohlberg–Mertens who seek a single criterion which all games must satisfy. Although our approach may seem less satisfying than using more rigid criteria, we believe it is preferable to make explicit the presumptions we have in certain situations rather than obscure them behind artificial criteria the motivation of which is somewhat vague.

Figure 1. The Extensive Game



## Footnotes

[1] This asymmetry could be eliminated by supposing that the players move simultaneously in each period. Our general results remain unchanged, but the analysis of the equilibrium becomes more complicated.

[2] If $\Pi_i(\underline{\sigma})$ is the payoff to player i from strategy combination $\underline{\sigma}$ for the game $\Gamma$, then $\Pi_i^*(\underline{\sigma}) = \Pi_i(\epsilon\underline{\sigma} + (1-\epsilon)\underline{\sigma}^*)$ is the payoff to player i in the game $(\Gamma,\underline{\sigma}^*,\epsilon)$ from strategy combination $\underline{\sigma}$.

[3] We use the product topology induced by the Euclidean norm on the space of local strategies (see Fudenberg and Levine (1983)). Given any strategy combination $\sigma^*$, the existence of a (sequential) $\sigma^*$-equilibrium for games with a finite strategy space follows from standard arguments. The results of Fudenberg and Levine can also be used to guarantee the existence of a (sequential) $\sigma^*$-equilibrium for an important class of infinite horizon games, including the one we study here. The equilibrium concept could obviously be used with other refinements of the equilibrium concept besides subgame perfection.

[4] See Binmore (1985) for a discussion about the relation between correlated trembles and irrational behavior.

[5] For a discussion of this use of the concept of reputation, see Wilson (1985).

[6] This result is established in the unpublished version of the paper.

[7] It is enough to verify that (i) $\alpha_1 = (0,0,0,...)$ and $\alpha_2 = (1,1,1,...)$ and (ii) $\alpha_1 = (1,1,1,...)$ and $\alpha_2 = (0,0,0,...)$ are both sequential $(\gamma_1,\gamma_2)$-equilibria when $\gamma_2/p_2, \gamma_1/p_1 \leq 1$. In fact, we can show that, under this assumption, the set of sequential $(\gamma_1,\gamma_2)$-equilibria for $(\Gamma(\delta_1,\delta_2),(\gamma_1,\gamma_2))$ is equal to the set of subgame perfect equilibria for $\Gamma(\delta_1,\delta_2)$.

[8] See Fudenberg and Levine (1986) for a precise definition.

[9] If $\hat{t}_i < \infty$, then $\mu_i(\hat{t}_i+2) = 1$.

[10] The possibility that the choice of irrational perturbations may affect the equilibrium outcome is illustrated most dramatically in a recent paper by Fudenberg and Maskin (1986). In the context of a repeated game, they show that every individually rational payoff can be approximated as an equilibrium payoff of a game with the proper choice of irrational behavior.

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