# LINEAR TRADE-MODEL EQUILIBRIUM REGIONS, PRODUCTIVITY, AND CONFLICTING NATIONAL INTERESTS

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# Linear Trade-Model Equilibrium Regions, Productivity, and Conflicting National Interests

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Abstract:

This paper examines the many equilibria that arise in a family of linear models in which the production parameters vary among models. The equilibria are analyzed as points in a graph. The points constitute a region of the graph with a robust and significant shape. The analysis shows that the classical linear model of international trade contains within it inherent conflict in the interests of trading partners. An analysis of various productivity levels in two trading countries shows that the set of productivity levels best for one country invariably yields poor outcomes for the other. The conflict entails rivalry over the share of the world's industries in which a given country predominates as producer.

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This paper shows that the classical linear model of international trade contains within it inherent conflict in the interests of trading partners. A systematic analysis of the possible productivity levels in two trading countries shows that the set of productivity levels in the two countries that are best for one country invariably yields poor outcomes for the other. This conflict arises even if there are no attempts, such as tariff wars, to interfere with free trade, and entails rivalry over the share of the world's industries in which a given country predominates as producer. Our results also offer systematic insights on an issue long debated among economic historians and the general public. When Germany and the U.S. surpassed British productivity did the U.K. necessarily benefit? Today, is the comparable emergence of Asia good for the Western countries?[1]

## I. Very Many Equilibria: Their Disparate Welfare Effects

As is well known, multiple equilibria arise in trade models with scale economies -- indeed it transpires that there can be very many.[2] This paper deals with a second case in which very many equilibria can arise: that of a family of *linear* models. Here, the parameters of the production functions vary from one model in the family to another, each choice of parameters producing an equilibrium.

**Families of Linear Models** In our linear model, the quantities $q_{i,j}$ produced of each good i in Country j are determined by linear production functions $e_{i,j}l_{i,j}$. Each of the two countries participating in trade has a given utility function of Cobb-Douglas form with demand parameters $d_{i,j}$. We fix the labor-force sizes $L_j$ of the two countries as well as n, the number of industries. A single model is then completely specified by the vector of productivity coefficients $\epsilon=\{e_{i,j}\}$. However, instead of dealing with just one model we will discuss the equilibrium outcomes of the *family of models* obtained by considering *all productivity coefficients* $\epsilon$ *subject only to a maximal productivity condition* $e_{i,j} \leq e^{max}_{i,j}$. This will enable us to analyze the effect of different productivities on the welfare of the two countries.

Each equilibrium of a family of linear models is represented as a point in a utility versus relative national income graph, described below. The region of that graph that contains *all* the stable equilibrium points for a family of linear models has a definite and characteristic shape. This shape has its own economic implications. Notably it shows that:

(1) The equilibrium points that yield the highest utility for one trading partner invariably yield relatively low utility to the other.

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[1]Although our approach on this issue is quite different, our results are completely consistent with the pathbreaking work of Johnson and Stafford (1993, 1995) and Hymans and Stafford (1995).

[2]Gomory (1994), and Gomory and Baumol (1994b) have shown that the most traditional of scale-economies trade models yields a set of stable equilibria whose number actually grows exponentially with the number of traded goods.

(2) A country is generally better off with a less developed trading partner than with a developed one.[3]

(3) While there are parts of the region where increased productivity in Country 2 yields improvement in welfare in both Country 2 and Country 1, there are also substantial parts of the region where generally one country can gain only at the expense of the other.

**Relation to Scale Economies**: We also show that there is a surprisingly tight connection between the scale economies and linear cases. Indeed, we will see that, in a wide variety of circumstances, the multiple equilbria from a single scale economies model are identical with a set of equilibria from a family of linear models. Thus, what can be described as "the new scale-economies slant" on trade theory, to which recent writings have contributed so much (see, e.g., Ethier (1979), (1982), Helpman and Krugman (1985), Krugman (1979), (1990), and Grossman and Helpman (1991), Gomory (1994) also casts light on the mechanism of the more traditional models. We will see that novel constructs from the scale-economies analysis, -- the *regions of equilibria* with their robust and economically significant shape, the critical role of a nation's *share* of real world output, -- are also present where production is linear.

Let us turn now to the analysis that leads to these conclusions.

## II. The Basic Graph and the Equilibrium Conditions

For any given vector of productivity parameters $\epsilon=\{e_{i,j}\}$ of our family, satisfying $e_{i,j}\leq e_{i,j}^{max}$, there is a stable equilibrium giving a national income $Y_j$ and a utility $U_j$ for each country. From the $Y_j$ we can compute *relative* national income $Z_j=Y_j/(Y_1+Y_2)$. We can then plot this equilibrium as a point $p_1(\epsilon)$ in a $(Z_1,U_1)$ diagram, which displays Country 1's utility, or as a point $p_2(\epsilon)$ in a $(Z_1,U_2)$ diagram which displays Country 2's utility.

Each $\epsilon$ gives us one point in each diagram. The 10 dots in Figure 1a represent 10 such $p_1(\epsilon)$ from one of our models. $Z_1$ is measured horizontally from 0 to 1. Utility is measured vertically with the with the scale chosen so that unity represents Country 1's utility in autarky using the maximal productivities $e^{max}_{i,1}$. In Figure 1b we have the $p_2(\epsilon)$. They have the same $Z_1$ values as the $p_1(\epsilon)$ but different utility. The unit value of utility now represents Country 2's utility in autarky using the $e^{max}_{i,2}$.

By combining the two diagrams we can see when equilibria that are good for one country are, or are not, good for the other. We do this in Figure 1c. The equilibrium of each $\epsilon$ is now

[3]The analysis that leads to this result is not trivial. However the possibility that a country can be better off when its trading partner is less developed can be shown by the following very elementary example. Consider two countries with productivities $e_{i,1}$ and $e_{i,2}$ respectively. Let us suppose that Country 1 is more productive in every industry, so that $e_{i,1}>e_{i,2}$ for all i. Nevertheless Country 1 gains by trade, so its utility in equilibrium $U_1$ is greater than its utility in autarky $U_1^A$. Next suppose that Country 2 improves its productivity so that for all industries, $e_{i,1}=e_{i,2}$. Now the productivities are the same in both countries, there is no comparative advantage, and no gain from trade. Thus the utility of Country 1 at this new equilibrium is $U^1_A$. The improvement in productivity in Country 2 has lowered Country 1's utility from $U_1$ to $U^1_A$.

represented by both $p_1(\epsilon)$ and $p_2(\epsilon)$: The black $p_1$ points represent Country 1's utility in Country 1 autarky units, and the gray $p_2$ points represent Country 2's utility in Country 2 autarky units. In the (randomly chosen) example in Figure 1c, we see that the equilibria that yield the most utility for Country 1 tend to yield a low utility for Country 2 and vice versa.

**Stable Equilibria**: Next we describe the equilibrium conditions that yield these equilibria.. For this we need some notation. $Z_j$ just defined, is Country j' s (relative) national income (Country j's *share*). We normalize analogously all our pecuniary expressions, so $p_i$ the price of good i, and $w_j$, the wage in Country j, are also divided by total income $Y_1.+ Y_2$.[4] Country j's consumption of good i is denoted by $y_{i,j}$ and its production of good i by $q_{i,j.}$ Country j's *production share* or *market share* of world output of good i is represented by $x_{i,j}=q_{i,j} / (q_{i,i}+q_{i,2})$, so that the vector $x=\{x_{i,j}\}$ describes the pattern of production. We can now describe our equilibrium conditions, noting that, henceforth, the term "equilibrium" will mean *stable* equilibrium.

First, (relative) national income of Country j must equal the total revenue from domestic and foreign sales of that country's products. Since with a Cobb-Douglas utility function Country i's expenditure on good i will be $d_{i,j}Z_j$ , this condition[5] is:

**(2.1)** $$\sum_i x_{i,j}(d_{i,1}Z_1+d_{i,2}Z_2) = Z_j.$$

Second, we have a zero-profit condition. World expenditure on Country j's output of good i all goes into the wages of the labor $l_{i,j}$ employed in that industry, so:

**(2.2)** $$w_j\ l_{i,j} = x_{i,j}(d_{i,1}Z_1+d_{i,2}Z_2).$$

Third, is the full-employment requirement for each country. This is expressed as the condition that the wage rate times the country's total labor force equals national income:[6]

**(2.3)** $$w_j\ L_j = Z_j.$$

Fourth, we have the requirement that, for each good, quantity supplied equals quantity demanded, or equivalently, that the value of the output of good i at the equilibrium price equals the amount consumers are willing to spend on it

**(2.4)** $$p_i(q_{i,1}+q_{i,2}) = d_{i,1}Z_1+d_{i,2}Z_2 \quad or \quad p_iq_{i,j} = w_j\ l_{i,j}$$

where the second form of (2.4) follows directly from the first by multiplying through by $x_{i,j}$ and using (2.2).

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[4]For example, let $Y_1=\Sigma_i p'_i q_{i,1}$ with the $p'_i$ , the prices of the goods $q_{i,1,}$ given in some arbitrary units. We divide through by $Y_1.+Y_2$ and obtain $Z_1 = \Sigma_i p_i q_{i,j}$. Both $Z_1$ and the normalized prices $p_i = p'_i /(Y_1+Y_2)$ will now be unaffected by the original units of price.

[5]There are of course two equations, one for each j value. However since $Z_1+Z_2=1$ and $x_{i,1}+x_{i,2}=1$ the two equations are dependent. If one is satisfied the other is too.

[6] This also implies that the exchange rate $w_1/w_2$ is proportional to $Z_1/Z_2$.

Finally we have the stability conditions that make entry by non-producers unprofitable. These require producers not to have higher unit costs than non-producers For example if Country 1 is the producer in industry i and Country 2 is a non-producer, we must have $e_{i,1}/w_1 \geq e_{i,2}/w_2$. More generally:

$$\text{(2.5)} \qquad \begin{array}{l} \text{if } x_{i,1}>0 \text{ and } x_{i,2}=0 \text{ then } e_{i,1}/w_1 \geq e_{i,2}/w_2 \\ \text{if } x_{i,2}>0 \text{ and } x_{i,1}=0 \text{ then } e_{i,1}/w_1 \leq e_{i,2}/w_2 \\ \text{if } x_{i,2}>0 \text{ and } x_{i,1}>0 \text{ then } e_{i,1}/w_1 = e_{i,2}/w_2. \end{array}$$

The third condition in (2.5) requires that, if both counties produce, neither can have lower unit costs than the other. The conditions (2.5) are, of course, a form of the familiar comparative-advantage criterion.

In our model then, equilibrium is determined by the relative national income relation, supply-demand equality for each good, zero profit in each industry, full employment in each country, and the stability conditions. It is easily shown that when these conditions hold trade must also be in balance.

**Property 2.1 of equilibrium conditions (2.1)-(2.4):** If any x and $Z_1$ satisfy (2.1) then together with *any* $\epsilon$ they immediately determine wages $w=(w_1,w_2)$, prices $p_i$, labor quantities $l_{i,j}$, production quantities $q_{i,j}$, and consumption $y_{i,j}$ that satisfy (2.2) (2.3) and (2.4). To see this we note that if x and $Z_1$ are chosen, they determine wages $w_j$ through (2.3), then labor quantities $l_{i,j}$ through (2.2). With the $l_{i,j}$ known the quantities produced $q_{i,j}$ are calculated from $q_{i,j}= e_{i,j}\, l_{i,j}$ and then the prices $p_i$ from (2.4). The consumption quantities $y_{i,j}$ are found by dividing the known Cobb-Douglas spending, $d_{i,j}Z_j$ by the prices $p_i$.

**Equivalent Equilibria.** To make visible the market shares and the $Z_1$ of equilibria, we adopt the notation $(x,Z_1,\epsilon)$ for the equilibrium determined by the productivity parameters $\epsilon=\{e_{i,j}\}$ and having market shares $x=\{x_{i,j}\}$, and relative national income $Z_1$ for Country 1.

Generally a large $\epsilon$ means high productivity and high utility, and therefore produces points high up in the either diagram. When the $e_{i,1}$ are large relative to the $e_{i,2}$, Country 1 is the producer in most industries so Country 1's share is large. $(x,Z_1,\epsilon)$ will have a large $Z_1$, and $p_j(\epsilon)$ is near the right edge in both diagrams. Large $e_{i,2}$ relative to the $e_{i,1}$ yields points near the left edges.

One point in a $(Z_1,U_j)$ plane can correspond to many quite different equilibria. The following definition reduces this duplication somewhat. Two equilibria $(x,Z_1,\epsilon)$ and $(x,Z,\epsilon')$ are *equivalent* if they differ only in the productivities of industries in which the country is *not* a producer. Equivalent equilibria have the same quantities of labor employed in each industry and have the same outputs, the same Z, the same x, and the same utility. They correspond to the same point in the $(Z_1,U_j)$ plane.

It is sometimes useful to have a normal form for equilibria in the same equivalence class. For this we set $e_{i,j}=0$ whenever Country j is a non-producer of good i. The normal-form equilibrium is the most stable member of the class in the sense that it most strongly fulfills the stability requirements (2.5)

**Structure of the Region.** We describe the region of equilibria by means of a curve $U_j=B^*_j(Z_1)$, which we call the *supremum curve* of the equilibrium points. The supremum curve $B^*_j(Z_1)$ traverses each diagram completely from left to right, and *every point below* $B^*_j(Z_1)$ (and no point above it) *is an equilibrium point of the family.* $B^*_j(Z_1)$ then completely defines the region of equilibria.

The first step in showing this regional structure is:

Lemma 2.1 There are equilibria for every $Z_1$, $0<Z_1<1$.

Actually there are many equilibria for each $Z_1$, but one suffices for our purposes.

Proof: To construct the desired equilibrium choose *any* x satisfying (2.1) for the given $Z_1$.[7] That is, choose any production pattern x that produces the given national income $Z_1$. If $x_{i,1}=1$ and $x_{i,2}=0$, choose any $e_{i,1}>0$ and $e_{i,2}=0$. If $x_{i,2}=1$ and $x_{i,1}=0$, choose $e_{i,2}>0$ and $e_{i,1}=0$. If both variables are positive for some industry i, choose $e_{i,1} = w_1 = Z_1/L_1$ and $e_{i,2} = w_2 = Z_2/L_2$. These choices clearly satisfy the stability condition (2.5). By Property 2.1, the wages, prices, etc. that they generate satisfy the remaining equilibrium conditions, so this is an equilibrium.

Lemma 2.2 If $(x,Z_1,\epsilon)$ is an equilibrium of the family, so is $(x,Z_1,\lambda\epsilon)$ for all positive $\lambda$ such that $\lambda e_{i,j} \leq e^{max}_{i,j}$.

Proof: Clearly the equilibrium conditions (2.1)-(2.5) remain satisfied and the parameters do not exceed $e^{max}_{i,j}$.

Thus, given an equilibrium $(x,Z_1,\epsilon)$ on $Z_1$, if we steadily decrease the $e_{i,j}$ by multiplying them by a $\lambda<1$, the new equilibria have the same x and $Z_1$, and consequently the same w and the same $l_{i,j}$. However the quantities produced, the $q_{i,j}$ will decrease because the $q_{i,j}=\lambda e_{i,j} l_{i,j}$ and $\lambda$ is decreasing. Therefore $U_1$ and $U_2$ decrease and the point in either diagram moves steadily down, tracing out a vertical line of equilibrium points with decreasing utility. This shows:

Lemma 2.3 If $(x, Z,\epsilon)$ is an equilibrium on the $Z_1$ vertical line with utility $U_j$, all the points below $(Z_1,U_j)$ on that vertical line are also equilibria.

Now we can establish our theorem:

Theorem 2.1- (Regional Structure Theorem): There is a curve $U_j=B^*_j(Z_1)$ such that every point of the $(Z_1,U_1)$ diagram under $U_j=B^*_j(Z_1)$, and no point above that curve, is an equilibrium point.

Proof: There are equilibrium points on any vertical line by Lemma 2.1. These points are bounded above because the $e_{i,j}$ and the $l_{i,j}$ are bounded, and therefore the $q_{i,j}$ and $U_j$ are also. Define $B^*_j(Z_1)$ to be the supremum of the equilibrium points on $Z_1$. Suppose there is a point p somewhere below the supremum curve that is *not* an equilibrium point. Suppose p has horizontal coordinate $Z_1$. Since $B^*_1(Z_1)$ is the supremum, there are equilibrium points on $Z_1$ between it and p. But then Lemma 2.3 asserts that p must also be an equilibrium. This contradiction proves the theorem.[8]

The supremum curves $B^*_j(Z_1)$ are usually quite jagged, even discontinuous, especially for models with a very small number of industries. Even for larger problems they can be quite jagged in detail. Fortunately, the $B^*_j(Z_1)$ can be approximated very well by an *upper frontier*

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[7]That there are (many) such x follows from $\sum (d_{i,1}Z_1+d_{i,2}Z_2)x_{i,1}=1$ when $x_{i,1}=1$ all i, and $\sum(d_{i,1}Z_1+d_{i,2}Z_2)x_{i,1}=0$ when $x_{i,1}=0$ all i.

[8]All points on the curve $B^*_j(Z_1)$ itself can also be shown to be equilibria.

*curve* $B_j(Z_1)$ that is both well behaved and easily calculated, to which we turn next.

We show first that the frontier curve is obtained by a simple linear programming calculation, and then, that it is close to the supremum curve. Then we can analyze the shape of the equilibrium region.

**Utility and Linearized Utility.** We can, in principle, obtain a particular value $U_j = B^*_j(Z_1)$ by maximizing utility over the equilibria for any given national income $Z_1$. We can obtain the entire boundary curve by repeating that process for each $Z_1$, or approximate the curve by many repetitions. We will come quite close to this. However, the task is eased considerably by use of a simplified but equivalent maximization problem that for each $Z_1$ is *linear* in the x.

This requires two steps. (1) We modify utility, the maximand, to make it *linear in x for fixed $Z_1$*, and (2) we confine ourselves to equilibria with share $Z_1$ using constraints *linear in x for fixed $Z_1$.*

Step 1. Linearizing Utility: Utility is a function of $y_{i,j}$, consumption of good i in Country j. The $y_{i,j}$ are determined at equilibrium by the equilibrium $Z_1$ and x as described in Property 2.1. Specifically, $y_{i,j}$ is obtained by multiplying world output of good i, which is $(q_{i,1}+q_{i,2})$, by the fraction of world spending on good i by Country j, $F_{i,j}(Z_1) = d_{i,1} Z_1/(d_{i,1}Z_1+d_{i,2}Z_2)$.

Consumption at equilibrium can be expressed entirely in terms of the equilibrium x and $Z_1$ by:

$$y_{i,j}=F_{i,j}(Z_1)\ \{q_{i,1}(x_{i,1},Z_1,\epsilon)+q_{i,2}(x_{i,2},Z_1,\epsilon)\}$$

(2.6a)

$$\text{where}\quad q_{i,j}(x_{i,j},Z_1,\epsilon)=e_{i,j}l_{i,j}=e_{i,j}(x_{i,j}\frac{d_{i,1}Z_1+d_{i,2}Z_2}{w_j})=e_{i,j}(x_{i,j}\frac{(d_{i,1}Z_1+d_{i,2}Z_2)L_j}{Z_j}).$$

(2.6a) enables us to express Cobb-Douglas utility for Country j, $U_j(x,Z_1,\epsilon)$, or its logarithm $u_j(x,Z_1,\epsilon)$, in terms of the equilibrium x and $Z_1$:

$$\text{(2.6b)}\qquad \ln U_j(x,Z,\varepsilon)= u_j(x,Z,\varepsilon)=\textstyle\sum_i d_{i,1}\ln\ y_{i,1}=\sum_i d_{i,1}\ln F_{i,j}(Z)\{q_{i,1}(x_{i,1},Z,\epsilon)+q_{i,2}(x_{i,2},Z,\epsilon)\}.$$

To linearize the nonlinear utility (2.6b), just as in Gomory 1994 and Gomory and Baumol 1994b, we define the *linearized utility for any triple $(x,Z_1,\epsilon)$ whether or not it is an equilibrium*, by:

$$\text{(2.7)}\qquad Lu_1(x,Z_1,\epsilon)\ =\ \textstyle\sum_i \{\ x_{i,1}\ \ (d_{i,1}\ln F_{i,1}(Z_1)q_{i.1}(1,Z_1,\epsilon))+x_{i,2}\ \ (d_{i,1}\ln F_{i,1}(Z_1)q_{i.2}(1,Z_1,\epsilon))\ \}.$$

This expression is linear in the $x_{i,j}$ for fixed Z. The $x_{i,j}$ have moved out of the $q_{i,j}(x_{i,j},Z_1,\epsilon)$ which have become the $q_{i,j}(1,Z_1,\epsilon)$, the quantities that would be produced if Country j were the sole producer. (2.7) deals with a weighted sum of outputs of sole producers rather than with the quantities actually produced. Nevertheless we have:

Theorem 2.2: The utility $u_1(x,Z_1,\epsilon)$ and the linearized utility $Lu_1(x,Z_1,\epsilon)$ are equal at every equilibrium point.

Proof: The proof is straightforward but tedious. It proceeds by comparing the ith term in each utility expression for the three cases: $x_{i,1}>0$ and $x_{i,2}=0$, $x_{i,1}=0$ and $x_{i,2}>0$, and $x_{i,1}>0$ and $x_{i,2}>0$. They are equal in all cases.

**An Upper Boundary for the Region Using Linear Programming.** The next step is:
Step 2 -Linearizing the constraint: We adopt as a linear constraint the equation (2.1) which requires total revenue to equal $Z_1$. This is satisfied by any equilibrium x and $Z_1$ as well as by many non-equilibrium x,$Z_1$ pairs. We are now ready for the maximization problem.

Let $\epsilon_{max}=\{e^{max}_{i,j.}\}$ represent the vector of maximum productivities. Consider the curve $B_1(Z_1)$ obtained by solving, for each $Z_1$, the problem of maximizing the linearized utility subject to the linear constraint:

$$\text{(2.8)} \qquad \begin{aligned} &\ln B_1(Z) = Max_x \; Lu_1(x,Z_1,\epsilon_{max}) \\ &subject\ to \;\; \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}\, x_{i,1} = Z_1 \quad and \quad x_{i,1}+x_{i,2}=1 \end{aligned}$$

The resulting curve $B_1(Z_1)$ lies *above* the supremum curve $U_1=B^*_1(Z_1)$ because: (1) all equilibria $(x,Z_1,\epsilon)$ with relative national incomes $Z_1$ satisfy the constraint (2.1) and were included in the maximization, and (2) at those equilibria $u_1(x,Z_1,\epsilon)= Lu_1(x,Z_1,\epsilon) \leq Lu_1(x,Z_1,\epsilon_{max})$. So we have proved:

Theorem 2.3: The supremum curve lies below the upper frontier curve: $B^*_1(Z_1) \leq B_1(Z_1)$.

$B_1(Z)$ is produced by solving the very simple linear program (2.8), so it is easily and rapidly computed and its shape can be analyzed. This analysis uses the fact that the optimizing x of a one-equation linear programming problem has at most one index i for which $x_{i,1}$ and $x_{i,2}$ are both positive. Therefore, the optimizing production pattern of (2.8) has at most one shared industry. We will use this property in showing that there are always equilibria near $B_1(Z_1)$, so the curves $B_1$ and $B_1$* are close.

Theorem 2.4 - (Nearby Equilibrium Theorem): The production pattern x that maximizes (2.8) for some $Z_1$ value always has a (nearby) equilibrium $e_n=(x,Z_1,\epsilon_n)$ with the same x and $Z_1$ and with $e_{i,j}=e^{max}_{i,j}$ in every *producing* industry except the one shared industry.

Proof: From Property 2.1 we know that since x satisfies (2.1) it also satisfies equations (2.2)-(2.4) for any choice of $\epsilon$. We can easily choose $\epsilon=\{e_{i,j}\}$ to make x satisfy the stability conditions (2.5). Start with $\epsilon=\epsilon_{max}$ and modify it as follows :(1) In single-producer industries, if the non-producer has lower unit cost, lower the productivity of the non-producer until its unit costs are higher than those of the producer. (2) In an industry where both countries produce, lower the productivity of the country with lower unit cost to produce equal unit costs in both countries. With this $\epsilon=\epsilon_n$ comparative advantage is satisfied and $e_n=(x,Z_1,\epsilon_n)$ is the (nearby) equilibrium required by the theorem.

In step (1) above we changed the productivity of a non-producer. Such a change has no effect on utility. Therefore *the utility change from $B_1(Z_1)$ to $e_n$ must come entirely from the one shared industry.* That change is easily estimated[9]. Since $e_n$ must lie under the supremum curve

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[9]The largest change in the utility that this can produce is, (letting k be the shared industry)

$$\frac{B_1(Z_1)}{U_1(x,Z_1,\epsilon)} \leq \left(\max\left\{\frac{e^{max}_{k,1}/w_1}{e^{max}_{k,2}/w_2}, \frac{e^{max}_{k,2}/w_2}{e^{max}_{k,1}/w_1}\right\}\right)^{d_{k,1}}.$$

$B^*_1$, $B^*_1(Z_1)$ is even nearer to $B_1(Z_1)$ than $e_n$ is. This leads to the:
Approximation Corollary: The ratio $B_1(Z_1)/ B_1{}^*(Z_1)$ is always $\leq (\max(R_1/(w_1/w_2),R_2/(w_2/w_1))^D$. Here $D=\max_i d_{i,1}$, $R_1=\max_i (e^{max}_{i,1}/e^{max}_{i,2})$, and $R_2 = \max_i(e^{max}_{i,2}/e^{max}_{i,1})$.

The Nearby Equilibrium Theorem and the Approximation Corollary suggest that in large problems, where the one shared industry is likely to be only a small slice of the total economic effort, we can expect the two curves to be very close. To prove this we need some preliminaries.
Convergence in Large Problems: Any one family of equilibria is completely characterized by the number of industries n, the labor force sizes $L_1$ and $L_2$, the demand parameters $d_{i,j}$, and the maximal productivity vector $\epsilon_{max}=\{e^{max}_{i,j}\}$. We will say that a family has *extremeness* bounded by K if

$$L_1/L_2 \leq K, \quad L_2/L_1 \leq K \quad d_{i,j} \leq K\,(\frac{1}{n}), \quad \frac{e^{max}_{i,1}}{e^{max}_{i,2}} \leq K, \quad \frac{e^{max}_{i,2}}{e^{max}_{i,1}} \leq K.$$

K restricts the extremeness of the variation in country size, productivity advantage, or in the case of the restriction on demand, the amount by which the demand for one good can exceed the average demand 1/n. If we restrict the extremeness of our models, their boundaries $B_1$ and $B^*_1$ converge as they become large:
Theorem 2.5 (Convergence Theorem): For any sequence of models with extremeness bounded by K the curves $B_1{}^*(Z_1)$ and $B_1(Z_1)$ of the nth problem approach each other as $n\to\infty$.
Proof: The formula in the Approximation Corollary shows that the nth model in the sequence has its $B_1(Z_1)/ B_1{}^*(Z_1)$ bounded by:

$$1 \leq (\max\ (R_1/(w_1/w_2),\ R_2/(w_2/w_1)\ )^D \leq (K^2 \max\ (Z_1/Z_2, Z_2/Z_1)^{K/n} \ \textit{which approaches } 1 \textit{ as } n\to\infty$$

So for models with large numbers of industries the region of equilibria becomes almost identical with the region under the frontier $B_1(Z_1)$.

## III. The Shape of the Region and Inherent Conflict

Next, using frontier $B_1(Z_1)$ as the upper boundary, we study the region of equilibria for Country 1. Figure 2a shows the characteristic regional shape. It also contains a vertical mark to show the $Z_1$ value of the equilibrium obtained when both countries attain maximal productivity ($\epsilon=\epsilon_{max}$). We call this equilibrium the "classical point" and its $Z_1$ value the "classical level" $Z_C$. Gomory and Baumol 1994a show that $B_1(Z_1)$ always starts from a zero utility level at $Z_1=0$, rises steadily to a point that is always to the *right of the classical level* $Z_C$, and then declines to the autarky level.[10] Later we will explain the economic reasons for this characteristic regional shape.

Obviously, the best outcomes for Country 1, those with greatest utility, are at or near the

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[10]Our theorems actualy assert that the $Z_1$ value of the peak $Z_p$, is $\geq Z_C$ and equality is possible. However it takes a very special choice of $\{e^{max}_{i,j}\}$ to get $Z_p$ even near $Z_C$.

peak of frontier $B_1(Z)$ and to the right of the classical level. Figure 2b shows the corresponding region for Country 2, with the best equilibria for Country 2 to the left of the classical level. Figure 2c combines Figures 2a and 2b.

Figure 2c makes it clear that, because of the position of the peaks, the outcomes best for Country 1 are always poor for Country 2 and vice versa. A country that is successful in maximizing its utility does so at the expense of its trading partner. Thus there is inherent conflict in the interests of trading partners in this classical trade model.

**Trading with a Developed Country**: We can see what happens to Country 1 if its trading partner is what we will call "fully developed", i.e., if $e_{i,2}=e^{max}_{i,2}$.

Theorem 3.1 (Fully Developed Country Theorem): If Country 1 trades with a fully developed Country 2, the resulting equilibrium will always have $Z_1 \leq Z_C$. This means that Country 1 is confined to the relatively poor outcomes shown in the left half of Figure 2a.

Proof: Suppose that there is an equilibrium e* with $Z_1 > Z_C$ and with $e_{i,2}=e^{max}_{i,2}$. At this equilibrium (by 2.3) since $w_j=Z_j/L_j$, we have a higher wage for Country 1 and a lower wage for Country 2 than at the Classical Equilibrium In every industry we have $e_{i,1}/w_1=e_{i,1}L_1/Z_1 < e^{max}_{i,1}L_1/Z_C$ so Country 1 unit costs have gone up compared to the Classical Equilibrium. Country 2 unit costs, $e_{i,2}/w_2= e^{max}_{i,2}L_2/(1-Z_2)>e^{max}_{i,2}L_2/(1-Z_C)$ have gone down. Therefore at e* Country 1 produces only in a subset of the industries in which it produced at the Classical Equilibrium. Furthermore, in each of those industries, the higher C ountry 1 wage means that the amount of labor employed, $l_{i,1}=(d_{i,1}+(Z_2/Z_1)d_{i,2})L_1$, has strictly decreased. Therefore the full employment condition can not be fulfilled at e*. This contradiction ends the proof.

If trading with a developed Country 2 yields relatively poor outcomes for Country 1, what kind of a trading partner is good for Country 1? We discuss this question next.

**The Ideal Trading Partner.** We define Country 2 to be Country 1's *ideal trading partner* when Country 2's productivity parameters $e_{i,2}$ are those that permit Country 1 to achieve its largest possible utility. We will see that this ideal trading partner produces few of the world's goods, but produces those goods very efficiently.

Finding the Ideal Trading Partner. Without loss of generality we can assume that Country 1 has attained maximal productivity, i.e., $e_{i,1}=e^{max}_{i,1}$ for all i. We calculate the regional frontier $B_1(Z_1)$ and locate the $Z_1$ value $Z_p$ of its peak. As noted, $Z_p$ always lies to the right of the classical level. Then for $Z_p$ we find the maximizing x of (2.8). We then convert x into the equilibrium $e_n$ using the Nearby Equilibrium Theorem. We assert that the $e_{i,2}$ of $\epsilon_n$ are the parameters of the ideal trading partner.

While we will not prove this here, we will try to make it plausible. Actually the result seems plausible from what we already know about the nearby equilibrium point $e_n$. For $e_n$ does give an equilibrium very near the peak, which is what the ideal trading partner is intended to do.. However there is one problem; in creating the equilibrium $e_n$ from the maximizing x and $Z_1$ we changed the $e^{max}_{i,j}$ to stabilize the equilibrium. We must make these changes in the $e_{i,2}$ to stabilize the equilibrium and define the ideal trading partner. But we can't change the $e_{i,1}$ because the $e_{i,1}=e^{max}_{i,1}$ are the given characteristics of the Country 1 whose ideal partner we are seeking.

Fortunately, *to the right of* $Z_C$, the $e^{max}_{i,1}$ do not change in constructing $e_n$.[11],[12]

<u>Characteristics of the Ideal Trading Partner</u>. Figure 3 shows the result for a 22 industry model. The equilibrium point for Country 1 is the dot at the peak. Country 2's corresponding equilibrium point is also plotted. In this case 15 of the 22 goods in the model are produced in Country 1, 6 in Country 2, and one is shared. This fits with Country 1's $Z_p$ value of 0.73.

In models run with $Z_C$ near 0.5, shares of roughly 0.7 for Country 1, and 0.3 for its ideal trading partner, are typical. For two identical countries an explicit formula can be obtained for $B_1(Z)$ (see Gomory 1994). It shows that for identical countries[13] $Z_C$ is always 0.5, and $Z_p$ is 0.76.

The productivities of its ideal trading partner allow Country 1 to make most of the world's goods. Country 2 also is at its maximum productivity in its smaller share of industries except possibly in the one shared industry (Nearby Equilibrium Theorem). A high-technology country making most things for itself but trading for a few goods with an agricultural country can be an example. We emphasize that this outcome, while very desirable for Country 1, is not a good one for Country 2.

**<u>Departure from the Ideal</u>.** *Any* departure from the ideal trading partner production parameters by Country 2 has a detrimental effect on Country 1. If Country 2's production parameters *increase*, this hurts Country 1, if Country 2's parameters *decrease*, that too hurts Country 1. For example, if the ideal trading partner becomes very highly developed so that all its parameters $e_{i,2}$ are increased to $e^{max}_{i,2}$, the result is the classical equilibrium.The dots directly above $Z_C$ in Figure 3 show this equilibrium. It clearly represents a loss of utility for Country 1 and a gain for Country 2.

Economic historians have long debated such questions as whether the U.K. lost out or benefitted from the relative rise in productivity since the 19th century in countries like Germany. Our analysis shows that the effect cannot be determined simply from the change in German productivity, but requires knowing whether Germany, for example, moved closer to or further from being an ideal partner for the U.K. given U.K. productivities at that time.

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[11]It is plausible, and also provable (see Gomory (1994), or Gomory and Baumol (1994b)), that to the right of $Z_C$, the maximizing solution x from (2.8) assigns to Country 1 all the industries in which it *does* have a cost advantage, in addition to some others in which it doesn't. In symbols, for $Z_1>Z_C$, $x_{i,1}=1$, $x_{i,2}=0$, whenever $e^{max}_{i,1}/w_1 > e^{max}_{i,2}/w_2$. In constructing the nearby equilibrium, stability only required us to change Country 1's parameters when Country 1 was the lower cost producer in an industry in which Country 2 produced, i.e., $x_{i,2}>0$. As we have just seen, Country 1 is never in this situation, so its parameters never have to be changed.

[12]To determine the ideal trading partner for Country 1 when the $e_{i,1}$ are not the $e^{max}_{i,1}$ it is sufficient to recalculate the regional boundaries treating the actual $e_{i,1}$ as if they were a new set of $e^{max}_{i,1}$, and then compute the $e_n$ near the peak of this new diagram.

[13]More precisely for identical regions, i.e., $e^{max}_{i,1}=e^{max}_{i,2}$.

**Economic Explanation of the Regional Shape.** In most of our work[14] we derive the shape of the equilibrium regions rather rigorously. Here, we describe in a more intuitive way the fundamental economics behind the regional shape.

For simplicity assume that both countries have the same Cobb-Douglas demand parameters, so that $d_{i,1}=d_{i,2}.=d_i$. Both countries then have utility $U_j = \Pi_i (y_{i,j})^{di}$ where $y_{i,j}$ is Country j's consumption share of $Q_i$, the world production of the ith good. Consumption $y_{i,j}$ in turn is determined by each country's spending $d_i Z_j$ on the ith good. With symmetric demands, consumption is $y_{i,j} = (d_iZ_j / (d_iZ_1+d_iZ_2))Q_i = Z_jQ_i$.

We first discuss *world* economic outcomes rather than outcomes for the individual countries, so we will need some measure of world economic outcome. World consumption of each good is $y_{i,1} + y_{i,2}=y_i=Q_i$ so that it is natural to measure world utility $U_w$ by $U_w =\Pi_i y_i^{di} =\Pi_iQ_i^{di}$. With this measure of world utility, Country j's utility simply equals world utility multiplied by Country j's share $U_j=\Pi_i (Z_jQ_i)^{di} = Z_jU_w$.

Next we plot for each equilibrium the world output point $(U_w,Z_1)$ and examine the resulting region of *world* outcomes. We assume that the region of world outcomes has an upper frontier that can be approximated by a reasonably smooth curve.[15] It is the shape of this curve that interests us.

At the extreme right edge of the diagram, near $Z_1=1$, we can easily see what world utility is. There Country 1 makes almost everything and Country 2 almost nothing, so the world output is very close to what Country 1 makes in autarky. Similarly, at the left end, the value of the world output is near Country 2's autarky value. This explains the two low end points of the boundary curve. Next we seek its highest point.

As a promising option we set all $e_{i,j} = e^{max}_{i,j}$ and find the resulting classical equilibrium. At this equilibrium (1) all producing industries operate at maximum possible productivity, and (2) each county that produces in a given industry is the lower cost , or, at worst, the equal cost producer. This seems hard to exceed. This intuition is roughly correct; the highest point of the boundary is usually at or near this point, so the maximum is located at or near the classical level.

Putting together what we know about the highest point and the left and right hand ends we obtain the dome shaped region of world outcomes shown in Figure 4a. The dome shaped boundary is low at either end and peaks near the classical level. This shape is intuitively appealing, when either country is held back world output is low, and when both operate at full potential we get the best *world* outcomes.

Next comes the critical step, generation of the individual-country frontiers. Since Country 1's utility is simply its share $Z_1$ times world utility $U_w$, Country 1's utility at $Z_1 =0.25$ is $0.25U_w$, at $Z_1=0.5$ it is $0.5U_w$, at $Z_1= 0.75$ it is $0.75U_w$, etc. Plotting all such points we get the familiar frontier for Country 1. This is the second curve in Figure 4a. We can repeat this for Country 2 yielding the curves in Figure 4b. We see that the shape of the countries' boundary curves, and the location of their peaks, is the unavoidable result of the dome shape of world

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[14]See for example Gomory and Baumol (1994b).

[15]This can be shown by redoing the preceding analysis using world utility instead of country utility.

output, and of taking each country's share of it.

This intuitive argument[16] relies heavily on symmetric Cobb-Douglas utility. However any world output measure that produces a dome-shaped world output region and bears the same relation to individual country outcomes will do. An important example is national income. If we were using national incomes $Y_1$ and $Y_2$ as measures of individual country output, we would use $Y=Y_1 + Y_2$ for world output. Since $Y_1 = Y(Y_1/(Y_1+Y_2))$, we have $Y_1=Z_1Y$, and similarly $Y_2 =Z_2Y$. Careful analysis of our model then shows this world outcome region to be dome-shaped. This dome-shaped world output region then yields the same familiar shape for the individual country regions.[17]

## IV. Maximal Productivity Equilibria and the Subregion of Maximal Productivity

A *maximal productivity equilibrium* is one at which the *producing* industries in each country are all at their *maximal productivities,* i.e., $x_{i,j}>0$ implies $e_{i,j}=e^{max}_{i,j}$. We will show that the maximal productivity equilibria all lie in and tend to fill up a crescent shaped subregion of the region of equilibria which we call the *subregion of maximal productivity.*

There are several reasons for special attention to this subregion of maximal productivity. (1) It is the part of the region of equilibria with most-direct conflict between trading partners, (2) its equilibria are the limit points of of learning-by-doing models, (3) it contains all the efficient equilibria of the family, and (4) it bears a special relationship to economies of scale models. All these attributes are now explained in turn below.

**A Lower Boundary for the Subregion Using Linear Programming.** If we increase parameters of a maximal productivity equilibrium $(x,Z_1,\epsilon)$ from $\epsilon$ to $\epsilon_{max}$ we do not change the (already maximal) parameters in the producing industries. Consequently the linearized utility of the new triple $(x,Z_1,\epsilon_{max})$, is the same as that of $(x,Z_1,\epsilon)$ because only productivity parameters in non-producing industries have increased. Therefore, for maximal productivity equilibria, $Lu_1(x,Z_1,\epsilon)=Lu_1(x,Z_1,\epsilon_{max})$. This enables us to prove:

Theorem 4.1 (Subregion Theorem): All maximal productivity equilibria lie within the subregion bounded above by the upper frontier $B_1(Z_1)$, and below by a lower boundary $BL_1(Z_1)$ defined by the *minimization* problem:

$$\textbf{(4.1)} \qquad \begin{array}{c} \ln\, B_1(Z_1)=Min_x\; Lu_1(x,Z_1,\epsilon_{max}) \\ subject\ to\ \ \sum_i \{d_{i,1}Z_1+d_{i,2}Z_2\}x_{i,1} = Z_1 \ \ and\ \ x_{i,1}+x_{i,2}=1. \end{array}$$

(4.1) is exactly the same as (2.8) except that it is a minimization rather than a maximization

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[16]The Appendix revisits this intuitive material and illustrates our analytic methods, showing that the world outcome boundary is quasi-concave. It therefore ascends monotonically to a peak and then descends monotonically. We conjecture that this behavior holds for a much wider class of utility maximums.

[17]This analysis is available from the authors.

problem.

Proof: If $(x,Z_1,\epsilon)$ is a maximal productivity equilibrium, its x and $Z_1$ satisfy the equation in (4.1) and was considered in the minimization problem. Therefore $Lu_1(x,Z_1,\epsilon_{max}) \geq \ln BL_1(Z_1)$. However for a maximal productivity equilibrium $Lu_1(x,Z_1,\epsilon_{max}) = Lu_1(x,Z_1,\epsilon) = \ln U_1(x,Z_1,\epsilon)$. This gives us $U_1(x,Z_1,\epsilon) \geq BL_1(Z_1)$ which proves the theorem.

The properties of the lower boundary $BL_1(Z_1)$ are similar to those of the upper frontier $B_1(Z_1)$ and can be established by similar analyses. To the left of the classical level $BL_1(Z_1)$ is monotone increasing with $Z_1$; it eventually rises above the autarky level and then descends again to the autarky level at $Z_1=1$. The region of maximal productivity for Country 1 is shown in Figure 5a, Figure 5b shows it for Country 2, and the regions for both countries are shown together in Figure 5c.

**<u>Location of the Maximal Productivity Equilibria in the Subregion</u>**: In the proof of the Nearby Equilibrium Theorem we constructed an equilibrium point p close to $B_1(Z_1)$ and having the same $Z_1$. At p all the industries except the one shared industry were at maximal productivity, and, in fact, one of the two sharing producers was also. This construction can be slightly modified to yield a true maximal productivity equilibrium p′ very near p but with a slightly different $Z_1$. This indicates that there are maximal productivity equilibria near every point of the upper boundary. This approach also applies to the lower boundary and, with further modification, to any point lying between $B_1(Z_1)$ and $BL_1(Z_1)$. This indicates that there are maximal productivity equilibria near any point of the subregion.[18]

This suggests that under the assumptions of the Convergence Theorem there will be maximal productivity equilibria arbitrarily close to any point in the region as n, the number of different industries, increases. This fill-in effect is already clear with fairly small n, such as the n=13 model of Figure 6, and then strengthens very rapidly from there.

Almost all of these maximal productivity equilibria will turn out to be perfectly specialized, with any good produced only in one country.[19] So the subregion actually fills up with specialized equilibria. Each has the form x=(1,0,0,1,1,0......,) with $e_{i,j}=e^{max}_{i,j}$ for each 1 and $e_{i,j}=0$ for each 0. There are $2^n-2$ of them.

**<u>Economic Interpretation and Consequences.</u>** In equilibria below the region of maximal productivity there are always producers who can improve their productivity. In almost all cases they can do this without changing either x, the pattern of production, or relative national income

[18]This is an intuitive sketch of the proof in Gomory (1994) that specialized equilibria "fill up" the region defined by (2.8) and (4.1). Although that proof was for economies of scale it includes the results described here.

[19] For the ith industry to be shared between producers we must have $e^{max}_{i,1}/ e^{max}_{i,2}=w_1/w_2= Z_1L_2/Z_2L_1= Z_1L_2/(1-Z_1)L_1$ which is possible for only one value of $Z_1$. We repeat this reasoning for each industry, and reach the conclusion that, outside of n special $Z_1$ values, shared industries in maximal productivity equilibria are not possible.

$Z_1$.[20] Such changes are quite benign, providing more output with fixed x and $Z_1$, benefitting both countries.

Within the region of maximal productivity such benign changes are scarcer. At maximal productivity equilibria they are unavailable. There, increases in productivity are only possible for non-producers. These increases generally have no effect at first, yielding only equivalent equilibria; but if the increase is sufficiently large a new equilibrium can emerge with the former non-producer becoming the producer. Then x and $Z_1$ must change, entailing migration of industries and change in relative national income.

These shifts can have different effects depending on circumstances. Figure 5c shows that if Country 1 can increase its share $Z_1$ anywhere within the region of maximal productivity to the left of its peak it will generally increase its utility and do so at the expense of its trading partner. On the other hand, in the region on the right of Country 1's peak, increases in Country 1's share generally *decrease* utility for *both* countries since the regions of both countries are descending. Similarly, decreases in Country 1's share through loss of industries to Country 2 generally benefit both countries.

The subregion of maximal productivity, then, separates the region of equilibria into two parts: the part below the subregion, where increases in productivity generally benefit both countries, and the subregion itself where increases in productivity cause shifts in industries, utility and relative national income. Within the subregion itself we have a region of conflict between the two peaks, where gain in one country usually comes at the expense of the other, and a region of potential cooperation outside the two peaks, where changes either benefit both countries or harm them both.

**Motion in the Region of Equilibria.** It is natural to consider productivity parameters that vary over time and therefore produce motion through a series of equilibrium points in the region of equilibria (see Gomory and Baumol 1994a). A natural model entails *learning by doing* in which the parameters of the producing country or countries in each industry steadily rise toward $e^{max}_{i,j}$ while the non-producing country improves less or not at all. Almost any model of this type yields a trajectory of equilibria tending toward a maximum productivity point as the producers approach $e^{max}_{i,j}$.[21] Even from an initial point below the region of maximal productivity the economy will move into it.

In economic terms this describes the evolution of industries from an early stage, where

[20]The lone exception is an equilibrium where production in all single-producer industries is at maximal productivity, and in the shared industry one country is at its maximum and the other isn't. Then increased productivity of that one non-maximal producer will change both x and $Z_1$.

[21] In such a model we may want the maximal productivities themselves to evolve. In the Cobb-Douglas model this can be done very simply. It is only necessary to introduce for each industry an underlying maximal productivity $e^{max}_i(t)$ that evolves over time. We can then reinterpret the $e_{i,j}$ and the $e^{max}_{i,j}$ as fractions of that productivity. The results of the present model are changed only via multiplication of utilities by a single factor that grows steadily.

competition may well hinge on the rapid evolution of capabilities, to a later stage where, with productivity approaching its limits, competition is more a matter of wage levels and of that ultimate manufacturing capability.

**Efficient and Inefficient Equilibria.** With so many maximal productivity equilibria one may wonder which are efficient and which are not. We call an equilibrium *efficient* if no other assignment of labor to industries, using the productivities $e^{max}_{i,j}$, can outproduce that equilibrium.

Certainly all efficient equilibria are maximal productivity equilibria. At any other equilibrium there is some producing industry in which productivity can be increased, so the same labor assignment produces more. Efficient equilibria, therefore, all lie in the subregion of maximal productivity. However, maximal productivity equilibria need not all be efficient. Many entail an apparently "wrong" assignment in terms of efficiency, for example, assigning Country 1 an industry in which Country 2's maximal productivity is larger.

Which of the maximal productivity equilibria *are* efficient? The complete answer emerges from Baumol and Gomory (1996). Even though that paper focuses on of economies of scale, its results apply to linear production functions in a particularly simple form. Applying these results[22] to the linear model we obtain:

Theorem 4.2 (Efficiency Theorem): The efficient equilibria form a single string of 2n+1 equilibria stretching all the way across the region of maximal productivity, and always includes the classical equilibrium[23] (Figure 7).

This means that efficiency is not confined to the region near the classical equilibrium where the resources of both countries are used in balanced fashion. There are efficient equilibria in which Country 1 produces most goods, while Country 2 produces large quantities of a few, as well as the other way around. There are also efficient equilibria that are poor for both countries from a utility standpoint, as well as the classical equilibrium which is always efficient and provides substantial but not maximal utility for each country.

**Relation to Scale-Economies Models.** This paper has repeatedly referred to studies of scale economies. One direct connection is that equations (2.6) and (4.1), delimiting the region of maximal productivity, are special cases of the equations that delimit the region of all *specialized* equilibria under scale-economies in Gomory (1994), and Gomory and Baumol (1994b). This suggests a fundamental connection between the region of maximal productivity and economies of scale models. We now define that connection.

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[22]Especially Theorem 4.2 and the subsection on Productivity Ratios That Do Not Depend On the Qi.

[23]The efficient equilibria are obtained by transferring industries in turn from Country 2 to Country 1 in the order of Country 1's comparative advantage. In the case of the classical equilibrium a partial transfer is also possible.

## V. The Correspondence Principle

The Correspondence Principle, indicating that the same equilibrium can arise in both a linear and a scale-economies model, suggests itself in the following way. A given *specialized* equilibrium can be stable for two very different reasons. In a model with linear production functions, it can be stable because the $e_{i,j}$ satisfy the stability conditions (2.5), with the producing industries the low-cost producers. Alternatively, an equilibrium can be stable because its production functions have scale-economies. These stabilize the specialized equilibrium by preventing new producers from entering industries on a small scale, where there is an established large-scale producer.

This suggests that the same specialized equilibrium with the same assignments of industries and perhaps even the same output of goods can be obtained from a linear model and from a model with scale economies. To show this we must define our scale-economies model and its equilibria.

**The Scale-Economies Model and its Equilibrium Conditions.** We say that a *scale-economies model $M(f_{i,j})$ corresponds to a linear family model* if it has the same labor-force sizes $L_1$ and $L_2$ and the same country demand values $d_{i,j}$. However, instead of linear production functions $e_{i,j}l_{i,j}$, the model $M(f_{i,j})$ has production functions $f_{i,j}(l)$ with economies of scale, defined as non-decreasing average productivity, $f_{i,j}(l)/l$. We assume that there is a well defined derivative $df_{i,j}(l)/dl$ at $l$=0, and that $f_{i,j}(L_j)/L_j$, which is the largest productivity value that $f_{i,j}(l)/l$ can attain in the model, is $e^{max}_{i,j}$.

We adapt the equilibrium requirements (2.1)-(2.5) for this model. The conditions (2.1)-(2.4) can be retained unchanged; we need only remember that $q_{i,j}$, the quantity produced, now equals $f_{i,j}(l_{i,j})$ not $e_{i,j}l_{i,j}$. The conditions (2.5) that stabilize the equilibria also translate easily. If there are two producers of good i, we require them to have the same average cost, and we do not allow a non-producer to have an average cost on entering lower than the current producer. The average productivity for small scale entry is $df_{i,j}(0)/dl$, so (2.5) becomes:

$$\text{(5.1)}\quad \begin{aligned} &\textit{if } x_{i,1}>0 \textit{ and } x_{i,2}=0 \textit{ then } \frac{f_{i,1}(l_{i,1})}{l_{i,1}w_1} \geq \frac{df_{i,2}(0)/dl_{i,2}}{w_2} \\ &\textit{if } x_{i,2}>0 \textit{ and } x_{i,1}=0 \textit{ then } \frac{df_{i,1}(0)/dl_{i,1}}{w_1} \leq \frac{f_{i,2}(l_{i,2})}{l_{i,2}w_2} \\ &\textit{if } x_{i,2}>0 \textit{ and } x_{i,1}>0 \textit{ then } \frac{f_{i,1}(l_{i,1})}{l_{i,1}w_1} = \frac{f_{i,2}(l_{i,2})}{l_{i,2}w_2}. \end{aligned}$$

Conditions (2.1)-(2.4) and (5.1) are equilibrium conditions for a stable zero-profit equilibrium. The equilibria $(x,Z_1)$ of such an economies of scale model $M(f_{i,j})$ can be very numerous.

An important special case occurs when all the production functions $f_{i,j}(l)$ entail startup costs. With output zero for small $l$ values, the production functions have $df_{i,j}(0)/dl$=0. Using this in (5.1) we see that any *specialized* x automatically satisfies the conditions (5.1). In economic terms, startup costs stabilize *any* specialized production pattern.

Now we relate the many equilibria that arise in $M(f_{i,j})$ to the linear equilibria.

**Corresponding Equilibria.** We say that a *specialized* equilibrium point from the linear family and a *specialized* equilibrium of a corresponding scale-economies model are *corresponding equilibria* if the $Z_1$, the market share variables $x_{i,j,}$ the wages, the quantities of labor $l_{i,j}$ employed in each industry and the prices and the quantities produced are the same in both equilibria. Clearly, any two corresponding equilibria are represented by the same point in our graph. We assert that for each equilibrium of the economies model there is a corresponding equilibrium of the linear family:

Theorem 5.1 (Correspondence Theorem): From any specialized equilibrium $(x,Z_1)$ of the scale-economies model we can construct a corresponding equilibrium $(x,Z_1,\epsilon)$ of the linear family having the same x and $Z_1$ and an $\epsilon$ given by: (1) the $e_{i,j}$ for producers is average productivity at the economies equilibrium, so $e_{i,j}=f_{i,j}(l_{i,j})/l_{i,j}$, and (2) the $e_{i,j}$ for non-producers is average productivity at output zero, so $e_{i,j}=df_{i,j}(0)/dl_{i,j}$ .

Proof: We can verify directly that the x and $Z_1$ with this $\epsilon$ satisfy the equilibrium conditions (2.1)-(2.5) so $(x,Z_1,\epsilon)$ is a linear equilibrium. Since $e_{i,j}=f_{i,j}(l_{i,j})/l_{i,j}\leq f(L_j)/L_j = e^{max}_{i,j}$ this is one of the equilibria of the linear family. Since the x and $Z_1$ are the same in both equilibria they yield the same labor quantities through (2.2) and then, because of the choice of the $e_{i,j}$, the same quantities are produced at both equilibria. Since the demands are the same, so are the prices. Therefore $(x,Z_1)$ and $(x,Z_1,\epsilon)$ are corresponding equilibria.

Many Corresponding Equilibria: If the economies model has many equilibria, each will clearly correspond to a different equilibrium $(x,Z_1,\epsilon)$ of the linear model. One economies model is therefore a way of looking at a large sample of the equilibria of a family of linear models. Figure 8 shows the equilibria corresponding to one rather small economies model.

The location of the equilibria corresponding to $M(f_{i,j})$ in the region of equilibria of the linear family depends on the nature of the scale economies. If the production functions $f_{i,j}(l)$ have productivities $f_{i,j}(l)/l$ that go on increasing until $l=L_j$ , the corresponding equilibria tend to be low in the region of equilibria of the linear model. This is because equilibrium labor quantities $l_{i,j}$ are generally small compared to the entire work force $L_j$. Therefore the $e_{i,j}=f_{i,j}(l_{i,j})/l_{i,j}$ they produce in the corresponding equilibria will tend to be small compared with $e^{max}_{i,j}= f_{i,j}(L_{i,j})/L_{i,j}$. This results in equilibria with relatively low productivity and low utility. On the other hand, if the production functions have already reached full economies of scale when each country is supplying its own needs in autarky, the corresponding equilibria are high up in the region. In fact they are all maximal productivity equilibria, because $e_{i,j}=f_{i,j}(l_{i,j})/l_{i,j} = f_{i,j}(L_{i,j})/L_{i,j}=e^{max}_{i,j}$. Figure 8 is a case with mild scale economies.

Next we look at the correspondence in the other direction, given a set of equilibria of the linear model, when do these *all* correspond to equilibria from *one* economies model?

Any one linear equilibrium $(x,Z_1,\epsilon)$ has well determined labor inputs $l_{i,j}$ and the resulting outputs $q_{i,j}=e_{i,j}l_{i,j}$. There is therefore a well determined input-output pair $(l_{i,j},q_{i,j})$ for each industry in each country. Any m linear equilibria provide m such pairs. We refer to each collection of m pairs as the set $S_{i,j}$.

Given an $S_{i,j}$ with kth element $(l^k,q^k)$ we say that $S_{i,j}$ is an *economies set* if there is a scale-economies production function f($l$), such that $f(l^k)=q^k$ for all m of the $(l^k,q^k)$ pairs. From Figure 9a it is clear that the points $(l^k,q^k)$ can have a single economies curve passing through all of them, (and therefore have such a production function), if and only if the slopes from the origin (average productivities) of successive points, when they are arranged in order of increasing $l$, are non-decreasing.

Theorem 5.2 (Multi-Equilibrium Correspondence Theorem): m specialized equilibria of a family of linear models will correspond to m equilibria of a single economies of scale model $M(f_{i,j})$ if and only if each $S_{i,j}$ is an economies set.

Proof: If there are m corresponding equilibria in some economies model $M(f_{i,j})$, each one has the same input and output as its corresponding linear equilibrium. Therefore, together these equilibria generate the same set $S_{i,j}$. However, they produce each $S_{i,j}$ by assigning the various input quantities $l^k$ of the m equilibria of $M(f_{i,j})$ to a *single* production function $f_{i,j}(l^k)$ and obtaining the corresponding outputs $q^k$. This is only possible if $S_{i,j}$ is an economies set. So the condition is clearly necessary.

To show it is also sufficient, we will construct an $M(f_{i,j})$ that satisfies Theorem 5.2. To do this we add (see Figure 9b) to each of the given $S_{i,j}$ the pair (0,0), (if it not already included), and also a pair $(l^*_{i,j}, 0)$ which lies on the $l$ axis halfway between the origin and the first pair that is not (0,0). Then we will add the pair $(L_j, e^{max}_{i,j}L_j)$ which lies further to the right than any existing pair. The pairs we have added on the left have zero slopes and are to the left of any successive pairs with positive slopes. The new pair on the right has a larger $l$ than any of the other pairs and also a larger slope. This augmented set of points has increasing slopes with increasing $l$ and is therefore an economies set. Any production function $f_{i,j}(l)$ that passes through this augmented set not only has economies of scale but also zero derivative at $l$=0, and $f_{i,j}(L_j)/L_j=e^{max}_{i,j}$. We will use these $f_{i,j}$ in our economies model $M(f_{i,j})$.

Now take $(x,Z_1,\epsilon)$, one of the set of m linear equilibria, and use its x and $Z_1$ as a candidate equilibrium for the economies model $M(f_{i,j})$. With $(x,Z_1)$ in the economies model we will get the same demand and hence the same labor inputs as at $(x,Z_1,\epsilon)$. Because of the construction of the $f_{i,j}$ we will have the same outputs from those inputs, and hence the same prices. Furthermore, the candidate equilibrium is stable. This is because it is specialized, and the $f_{i,j}$ have been constructed with setup costs. So we have $df_{i,j}(0)/dl= 0$ and (5.1) is satisfied. This shows that $(x,Z_1)$ is a stable equilibrium and that it corresponds to $(x,Z_1,\epsilon)$. This ends the proof.

If we apply this theorem to the maximal productivity equilibria we get:

Theorem 5.3 (Maximal Productivity Correspondence Theorem) The $2^n$-2 specialized maximal productivity equilibria always correspond to the equilibria of a single economies model.

Proof: Each $S_{i,j}$ cotain points of the form $(l,e^{max}_{i,j}l)$ when Country j produces in industry i, and also the point (0,0) when Country j is a non-producer. These points give us a constant slope $e^{max}i,j$ for every $l$, so $S_{i,j}$ is an economies set. Theorem 5.2 then gives the result.

This theorem shows the tight connection between families of linear models and economies models with startup costs. The region of maximal productivity and its equilibria are virtually identical with the equilibria of such an economies model. This tends to explain why we obtain such similar economic results, such as conflict in the interests of trading partners, in both settings.

## VI. Summary and Conclusions

We have shown that the equilibria of a family of linear models form a well defined region with a robust shape. One implication of the shape is that the best possible equilibrium for one country is a poor one for its trading partner, so that a policy that succeeds in attaining or retaining such a position inherently involves conflict with the interests of the other country. This conflict, which occurs under unhampered free trade, is driven by each country's share of industries and income.

We have introduced the notion of the ideal trading partner for Country 1, and have shown how the productivity parameter values of a country's ideal trading partner can be determined. Any departure from these values, whether through increases or decreases in productivity of Country 2, the partner, will harm Country 1. Thus the welfare of a country is sometimes enhanced and sometimes reduced by a rise in productivity of its trading partner, but these outcomes follow a systematic pattern that is easily understood.

Within the region of equilibria there lies a maximal-productivity subregion. Below this subregion improvements in productivity of the producing nation tend to benefit both countries. Within the region the interests of the two countries are in conflict over a wide range, but there is also a smaller range where there are opportunities for mutually-beneficial change. The maximal-productivity region also contains all the efficient equilibria, and we have been able to find which of the many maximal productivity equilibria are efficient.

We have shown the close ties between these regions of maximal productivity, their defining equations, their regional shapes, their equilibria, and the equations, regional shapes and equilibria obtained from scale-economies models. So the patterns of multiple equilibria and their regional shape that emerge in the presence of scale economies are not peculiar to that state of affairs, but have direct counterparts in the linear models that characterize the classical theory of international trade.

This correspondence has direct implications for both theory and policy. It implies, for example, that the welfare effect of the acquisition of an industry by a country in a world of scale economies, whether through natural evolution or government intervention, can be similar to the welfare effects of a rise in that country's productivity in that industry in a linear world.

### Appendix. The Shape of the World Upper Utility Boundary.

To derive the shape of the world's upper utility frontier assume for simplicity $d_{i,1}=d_{i,2}=d_i$ which gives us a world utility $U_w = \Pi_i (Q_i)^{di} = \Pi_i (q_{i,1}+q_{i,2})^{di}$. The analogue of the linearized utility $Lu_1$ of (2.7) is linearized world utility $Lu_w(x,Z_1,\epsilon_{max})$ which is:

**(A.1)** $$Lu_w(x,Z_1,\epsilon_{max})=Max_x \sum_i \{ x_{i,1}(d_i\ln q_{i,1})+x_{i,2}(d_i\ln q_{i,2}) \}.$$

To obtain the world boundary $B_w(Z_1)$ we have the analogue of (2.8). Since the equation in (2.8) is simplified by $d_{i,1}=d_{i,2}=d_i$ the analogue of (2.8) is:

**(A.2)** $$lnB_w(Z_1) = Max_x \sum_i \{x_{i,1}(d_i\ln q_{i,1})+x_{i,2}(d_i\ln q_{i,2})\}$$
$$subject\ to\ \sum_i d_i x_{i,1} = Z_1\ and\ x_{i,1}+x_{i,2}=1.$$

If $x^*(Z_1)$ is the maximizing solution to (A.2) we have

**(A.3)** $$lnB_w(Z_1)=\sum_i d_i\{x^*_{i,1}\ln\, q_{i,1}+x^*_{i,2}\ln\, q_{i,2}\}=\sum_i d_i\{x^*_{i,1}\ln\, \frac{e_{i,1}d_iL_1}{Z_1}+x^*_{i,2}\ln\, \frac{e_{i,2}d_iL_2}{Z_2}\}.$$

Differentiating (A.3) we have the slope of the world utility boundary. In this differentiation we are helped by the fact that the maximizing solution $x^*(Z_1)$ always has $x^*_{i,j}=1$ or 0 except for the one shared industry where $x_{k,1}$ and $x_{k,2}$ can both be positive. When $Z_1$ changes, only $x_{k,1}$ and $x_{k,2}$ change, except at a finite number of exceptional points.[24] Therefore:

**(A.4)** $$\frac{d}{dZ_1}\,(Lu_w(Z_1)) = d_k(\frac{dx^*_{k,1}}{dZ_1}\ln\frac{e_{k,1}d_kL_1}{Z_1} + \frac{dx^*_{k,2}}{dZ_1}\ln\frac{e_{k,2}d_kL_2}{Z_2}) + \sum_i \{\frac{-d_ix^*_{i,1}}{Z_1}+\frac{d_ix^*_{i,2}}{Z_2}\}.$$

From (A.2) we have $\Sigma_i\,(d_ix^*_{i,1})/Z_1=1$. Similarly $\Sigma_i\,(d_ix^*_{i,2})/Z_2=1$ so the sum term in (A.4) is $-1+1=0$. Differentiating $\Sigma_i\,d_ix^*_{i,1}=Z_1$ and $\Sigma_i\,d_ix^*_{i,2}=Z_2$ we have:

**(A.5)** $$d_k\frac{dx^*_{k,1}}{dZ_1}=1 \quad and \quad d_k\frac{dx^*_{k,1}}{dZ_1}=-1$$

so (A.4) becomes

**(A.6)** $$\frac{d}{dZ_1}\,(Lu_w(Z_1)) = \ln\frac{e_{k,1}d_kL_1}{Z_1} - \ln\frac{e_{k,2}d_kL_2}{Z_2}.$$

We differentiate this to obtain the second derivative:

**(A.7)** $$\frac{d^2}{dZ_1^{\,2}}\,(Lu_w(Z_1)) = -\frac{1}{Z_1}-\frac{1}{Z_2}.$$

(A.7) shows that the second derivative of log world utility is negative. This means that log world utility is concave. It follows that world utility itself is quasi-concave and therefore rises monotonically to its highest point and then descends monotonically.

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[24]Analysis of these points shows that they do not affect the conclusions reached here.

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Figure 1a 10 Equilibria Showing Country 1 Utility

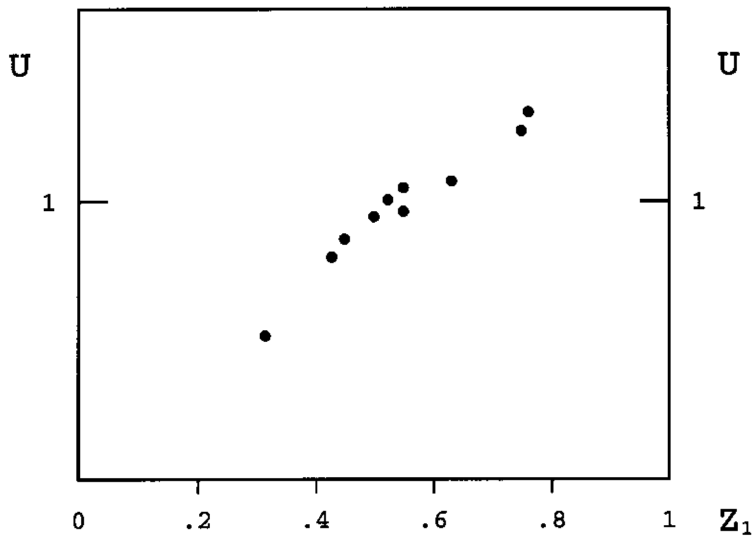


Figure 1b 10 Equilibria Showing Country 2 Utility

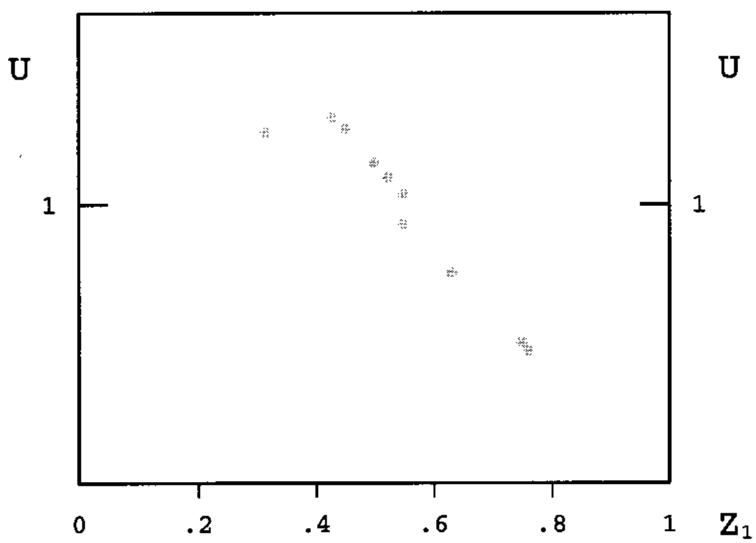

Figure 1c Combined Plot: 10 Equilibria Showing Both Utilities

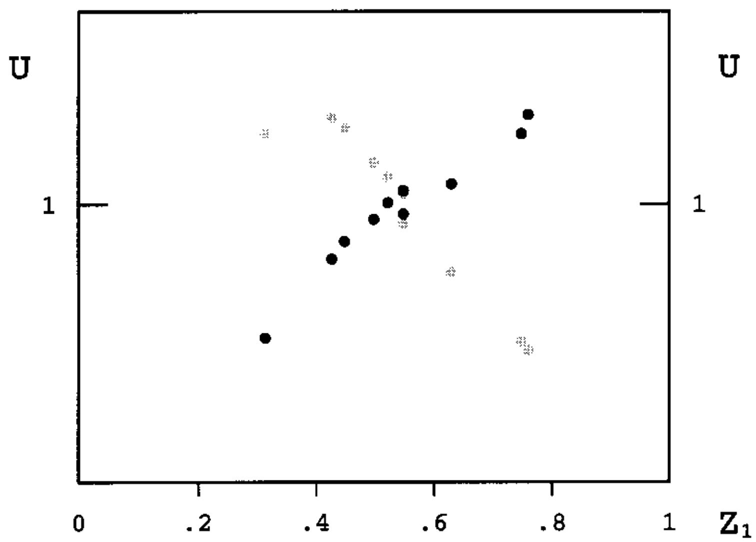

Figure 2a Boundary of Country 1

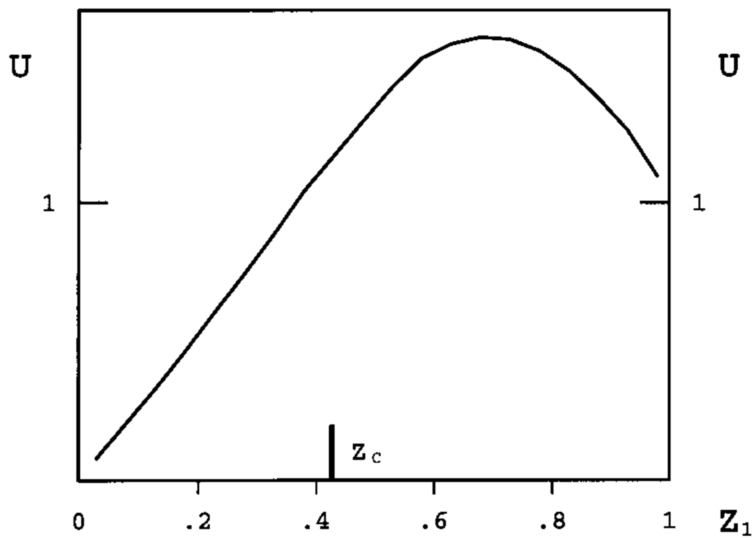


Figure 2b Boundary of Country 2

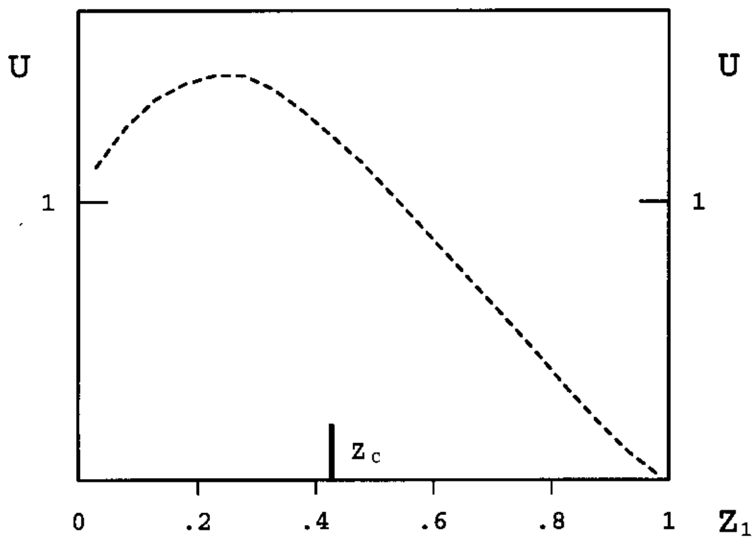

Figure 2c Boundaries of Both Countries

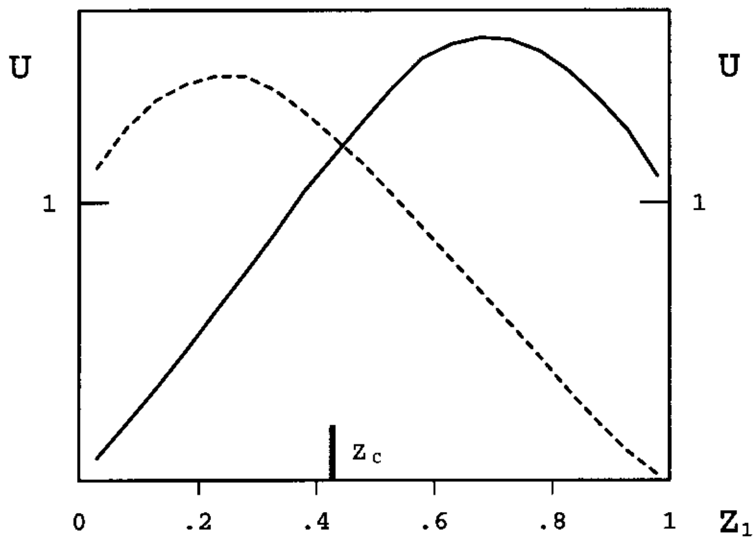


Figure 3 The Ideal Trading Partner
Black Dots - Country 1 Utility
Gray Dots - Country 2 Utility

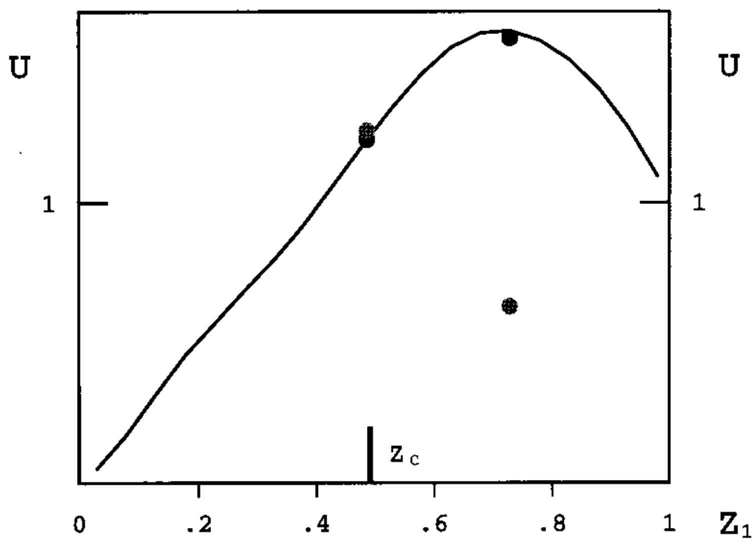

Figure 4a World Utility and its Relation to Couintry 1 Utility

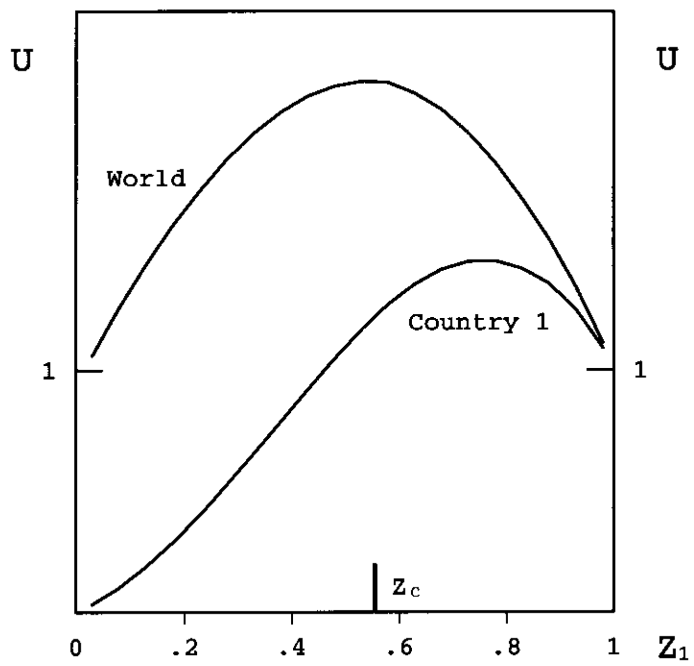


Figure 4b World Utility, Country 1 Utility, and Country 2 Utility

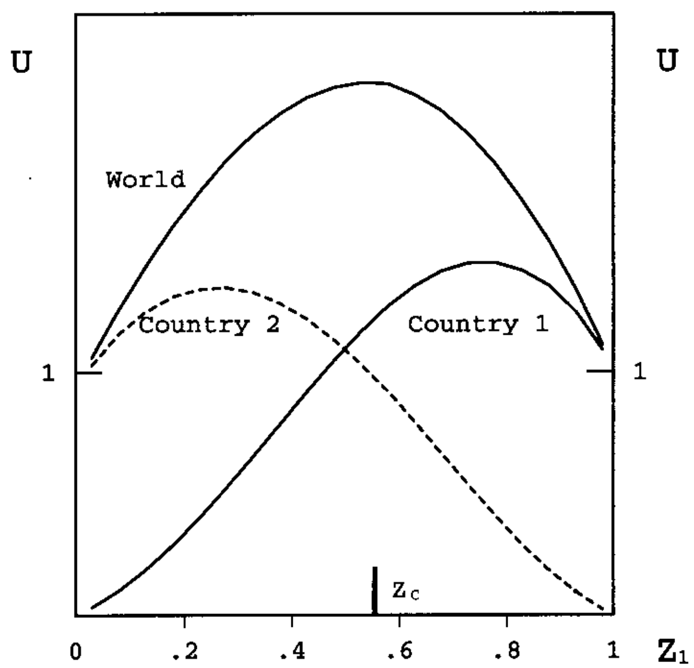

Figure 5a Region of Maximal Productivity, Country 1

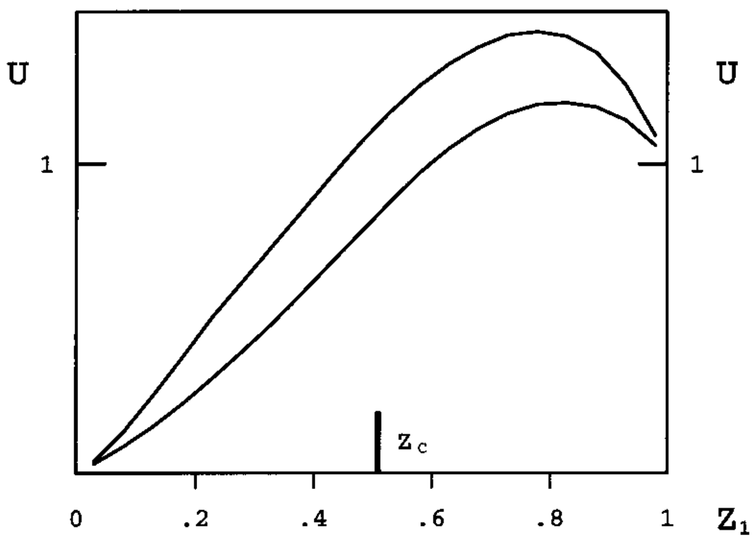


Figure 5b Region of Maximal Productivity, Country 2

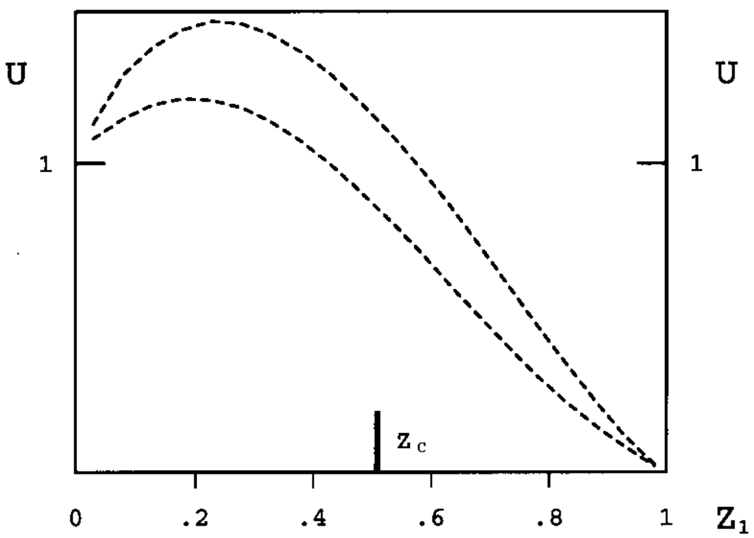

Figure 5c Region of Maximal Productivity, Combined Plot

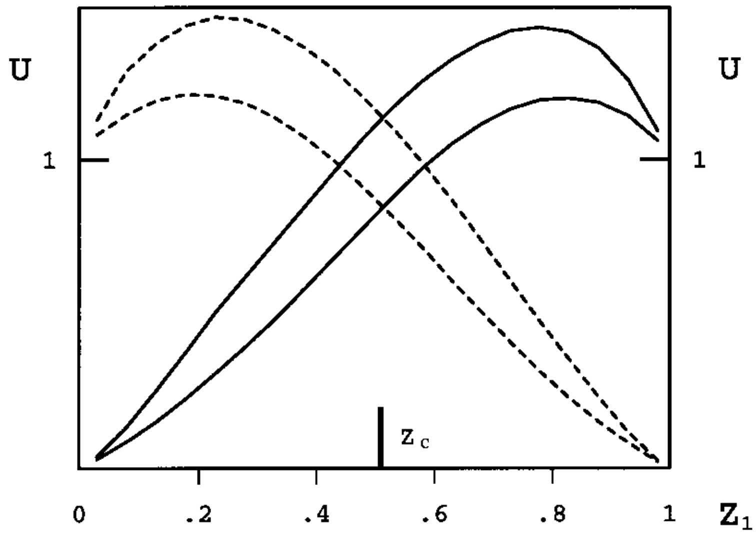


Figure 6 Filling in with Maximal Productivity Equilibria

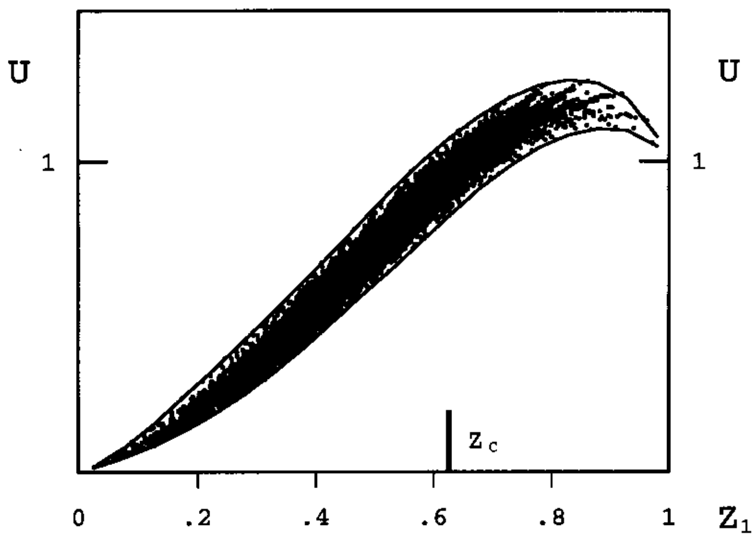

Figure 7 21 Efficient Points and 1 Classical Point

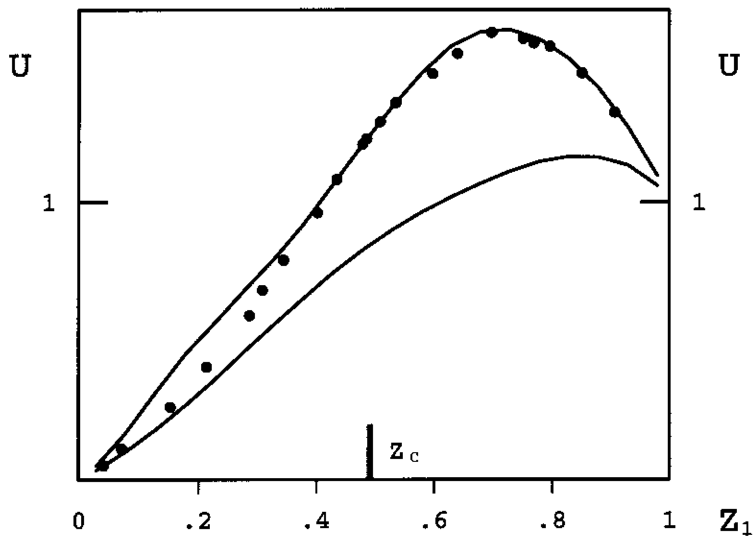


Figure 8 A Linear Region and Some Corresponding Points

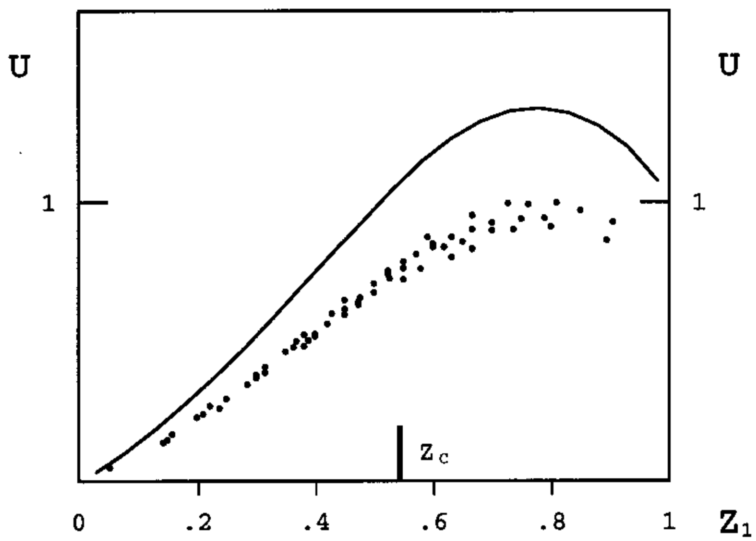

Figure 9a The (l,q) Diagram

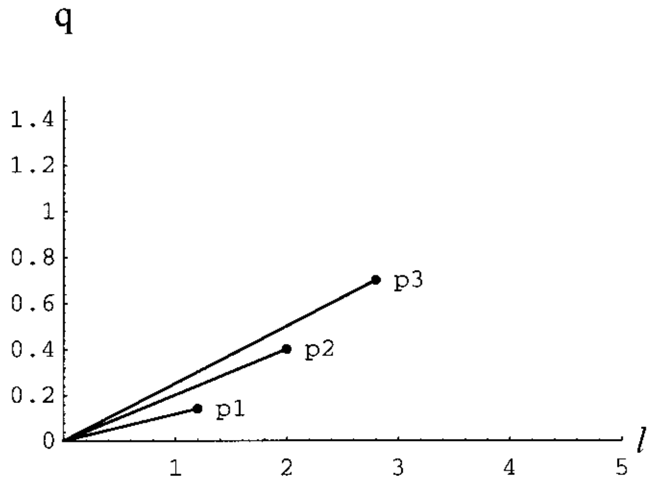


Figure 9b The Extended {l,q) Diagram