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Abstract

The paper investigates the relation between the equilibria of discrete and continuous time formulations of the War of Attrition. We show first that there is no analogue in continuous time for the variety of types of discrete time equilibria. Second, there is generally not a one to one correspondence between the equilibria of the continuous time game with the limiting distributions of the equilibria of discrete time games. The set of continuous time equilibria is sometimes larger and sometimes smaller. The reasons for this divergence and its relation to the results obtained by other authors are explored in some detail.

1. Introduction:

There are many problems in economics where a choice must be made between formulating a model in discrete or continuous time. Although the computations are often simpler in continuous time, the discrete time formulation frequently corresponds more closely to the actual decision problem of the agents. For single person decision problems (or problems where individual agents are insignificant), the behavioral implications of the two formulations are generally equivalent. In models of strategic problems involving two or more agents, however, the payoff functions in the continuous time formulations are frequently discontinuous. In these cases, the continuous time and discrete time formulations may have different implications for the behavior of the agents. When a choice is to be made between the two formulations, therefore, it is important to determine the circumstances under which these differences arise.[1]

In this paper, we present a systematic analysis of this comparison for a specific game of conflict known as the War of Attrition. In this game two players must decide at each instant of time whether to move (concede) or compete (wait). The game ends as soon as one of the players moves. Its essential characteristics are that the return from moving first (leading) decreases with time and that the return to following (letting the other player move first) is always greater than the return to leading.

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[1]Several authors, in their analyses of various strategic problems, have, at least implicitly, appealed to some kind of equivalence between the two formulations. For instance, Cramton (1984) analyzes a bargaining problem in continuous time, but uses the limit of the equilibria of an alternating offers game formulated in discrete time to close the model. Kreps and Wilson (1982) formulate the two sided uncertainty version of their model of the chain store paradox in discrete time, but then analyze the equilibrium of the continuous time analogue. Fudenberg and Tirole (1985) study the timing of adoption of a new technology in continuous time, but then justify the diffusion equilibrium as the limit of discrete time equilibria.

Both the discrete time and continuous time formulations have been used to analyze a number of economic problems.[2] For instance, Benoit (1985) uses a simultaneous move, discrete time model to analyze price wars while Fudenberg and Tirole (1984) use a continuous time model to analyze a similar problem. Ordover and Rubinstein (1985) use an alternating move, discrete time framework to analyze a simple bargaining problem while others, such as Bliss and Nalebuff (1984) and Osborne (1984) formulate these kinds of problems in continuous time.

Several authors have already observed that games of timing may possess more equilibria when formulated in continuous time than in discrete time. For example, Kreps and Wilson (1982) analyze a problem in which a monopolist is sequentially threatened by a single firm in N markets as a War of Attrition and note that, under the assumption of complete information, their model possess a mixed strategy equilibrium in continuous time while none exists in discrete time. Similarly, several authors have also observed that one can lose equilibria in going from discrete time to continuous time. For example, Fudenberg and Tirole (1984) note that in preemption games such as the "grab the dollar" game, the limits of discrete time equilibria are not always equilibria in the continuous time formulation of the game.

In this paper, we demonstrate that when decisions are made at discrete points, a variety of equilibrium behavior patterns are possible which have no analogue when decisions are made continuously.[3] Each of these patterns converge to the same family of distributions as the length of the period goes

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[2]In addition to applications in economics, there is an extensive literature on the War of Attrition in the field of theoretical biology. It was introduced by Maynard Smith (1974) and subsequently extended by a number of authors (e.g. Bishop and Cannings (1978), Nalebuff and Riley (1984)). A more complete literature survey is given in Hendricks and Wilson (1985).

[3]This issue is examined in greater detail in Hendricks and Wilson (1985).

to 0. However, depending upon the properties of the payoff functions, only certain local patterns may be consistent with equilibrium. This has implications for the size of the family of limiting equilibrium distributions and hence the relation between the equilibria of discrete time and continuous time models.

Specifically, we obtain the following results. First, in games with a finite horizon where the return from leading is always positive,[4] there is one mixed strategy equilibrium in discrete time but no mixed strategy equilibrium in continuous time. The reason is that the payoff function is discontinuous in a way that Dasgupta and Maskin (1982), in their study of the existence of equilibria in discontinuous games, call essential.[5] That is, the limiting distributions of the discrete time equilibria contain an atom of probability at a point in the strategy space where the payoffs to the players are discontinuous.

Second, in games with a finite horizon where the return to each player from leading is equal to 0 at the same time, there is a one parameter family of mixed strategy equilibria in continuous time, but "generically" only one mixed strategy equilibrium in discrete time. Since, when time is discrete, each player's return from leading is generally never equal to 0 at any decision point, the probability with which a player moves in any period is uniquely determined by the condition that he is either indifferent between moving in that period and waiting until the terminal period or prefers to wait. Because of the implications of this restriction on the "local"

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[4]We normalize payoffs so that both players earn a return of 0 if both wait until the terminal period T.

[5]They provide a set of sufficient conditions for the existence of a mixed strategy equilibrium even in the presence of discontinuities. The condition which is violated in the War of Attrition is that the sum of the payoffs to the players is upper semi-continuous.

structure of the equilibrium, the result is a unique mixed strategy equilibrium in discrete time games. Since there is no analogue to this local structure for continuous time strategies, games in continuouse time always possess a one parameter family of equilibria.

Finally, if there is no terminal period (i.e. the time horizon is infinite), then the possibility that a player never moves implies no restrictions on the equilibrium strategies. In these cases, there is a continuum of equilibria in both the continuous and discrete time games and the set of equilibria of the two formulations coincide in the limit.

Before discussing the organization of the paper, we should emphasize that our analysis is conducted only for games with complete information. Since much of the recent analysis has been for games with incomplete information, our results cannot be directly applied to many of these games. In some of these games, our results must be substantially modified, while for others, the analysis is essentially unchanged. Given the sensitivity of our results to exactly how incomplete information is introduced, we will not deal with such games in this paper.

The paper is organized as follows. The assumptions and description of the game are given in Section 2. In Section 3, we note that any differences in the equilibria of the discrete and continuous time formulations concern mixed strategy equilibria which we refer to as nondegenerate equilibria. In Section 4, we characterize this class of equilibria when players make decisions continuously. In Section 5, we provide a similar characterization when the players make decisions at discrete intervals. In Section 6, we derive the family of distributions which can arise as the limit of discrete time equilibria and compare it to the set of equilibria obtained in continuous time. We conclude with a discussion of the factors that need to be considered in making a choice between the two formulations.

2. The Game

Two players, a and b, must decide when to make a single move at some time t between 0 and T ($0 < T \leq \infty$). In what follows, α will refer to an arbitrary player and β to the other player. The payoffs to the players depend upon which player moves first and the time that he moves. To simplify the analysis, we will assume that all returns are discounted at a constant discount rate $\delta > 0$, that they are symmetric, and that the return from moving alone at time t is equal to the return from moving simultaneously with the other player at that time. Then, if player α moves first or simultaneously with player β at time t, he is called the leader and earns a return $Ae^{-\delta t}$. If player β moves first at time t, player α becomes the follower and earns a return $Be^{-\delta t}$. If neither player moves before time T, then player α earns a return H_α. In the analysis that follows, it will be convenient to normalize the returns so that the return to each player if neither player ever moves is 0. Given this normalization, the return to player α if he is the leader at time t becomes $L_\alpha(t) = Ae^{-\delta t} - H_\alpha$, and if he is the follower at time t, it is $W_\alpha(t) = Be^{-\delta t} - H_\alpha$.

We will study the class of games satisfying the following assumption:

A1 (i) $B > A > 0$ for $\alpha = a,b$;

(ii) $A > H_\alpha$ for $\alpha = a,b$.

Condition (i), together with the assumption that $\delta > 0$, implies that $W_\alpha(t)$ always exceeds $L_\alpha(t)$ and that $L_\alpha(t)$ decreases with time. In order to ensure that the players have an incentive to move at time 0 rather than wait until time T, we assume, in condition (ii), that $L_\alpha(0)$ exceeds 0 for both players.

In economic contexts, the interpretation of the payoffs when both players wait until time T generally depends upon whether T is finite or infinite. If T is finite, H_α may represent the equilibrium payoff to player α in some continuation game played at time T. Hence, it may be unrelated to the values of $Ae^{-\delta T}$ and $Be^{-\delta T}$. In order to examine the implications of asymmetries in the payoffs, we will not require that H_a be equal to H_b in this case. If T equals infinity, then H_α represents player α's payoff if neither player ever moves. Assuming players discount their returns over time, then it is reasonable to suppose that H_α is equal to 0. We will therefore assume

A2 If $T = \infty$, then $H_a = H_b = 0$.

We want to compare the equilibria of the game when decisions are made continuously with the equilibria of the game when decisions are made at discrete but arbitrarily small intervals. It will be necessary, therefore, to consider a sequence of discrete time games in which the partition is made successively finer. For any $\Delta > 0$, let

$$J_\Delta = \{t < T: t \in \{0,\Delta,2\Delta,3\Delta,\ldots\}\}.$$

We will denote by G_Δ a game in which all decisions are made at times in J_Δ. The returns to leading and following at any time $t \in J_\Delta$ are then given by the values of $L_\alpha(t)$ and $W_\alpha(t)$ respectively. We will denote the continuous time game as G_0 and, accordingly, define $J_0 = [0,T)$.

An (extensive) game G_1 with a finite horizon is illustrated in Figure 1. Starting at time 0, both players must decide simultaneously whether to move (M) or wait (W). If either player decides to move, then the payoff to each player is determined and the game is over. Otherwise, the game proceeds to the next period, where once again both players must decide whether to move or wait. This process continues until one of the players moves or period T is reached. A pure strategy for this game is thus a function $m: J_1 \to \{M, W\}$, where $m(t) = M$ means that the player plans to move in period t if neither player has moved prior to that period and $m(t) = W$ means that the player intends to wait in period t if this period is reached.

Figure 1 here.

In this paper we focus primarily on Nash equilibria.[6] For this purpose, the description of the strategy space given above is unnecessarily rich. For any strategy m, let $\tau(m)$ represent the earliest time at which the player plans to move. Then, since the payoff to each player depends only on who moves first and when he moves, any two strategies m_1 and m_2 such that $\tau(m_1) = \tau(m_2)$ are equivalent. Consequently, we can represent the strategy space of the strategic form of the game by a set of equivalence classes of extensive form strategies indexed by the time t at which the player plans to move first. A mixed strategy for G_Δ in the strategic form is then a probability distribution F over the set of pure strategies, J_Δ.[7]

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[6]In fact, for the class of equilibria on which we focus our attention, subgame perfection is implied. See Lemma 4.2 below.

[7]An alternative interpretation of a game with this strategy space assumes that both players commit themselves at time 0 to the time at which they will move first. In that case, the issue of subgame perfection does not arise. For most economic interpretations, however, the assumption that players commit themselves at time 0 is not consistent with our assumption that the payoff to following is independent of when the follower had planned to move.

Whenever possible we will formulate our concepts so that they apply to both the discrete and continuous time games. Thus, $F_\alpha(t)$ denotes the probability that player α plans to move on or before time t and, if F_α is differentiable at time t, $f_\alpha(t)$ denotes the probability density at time t. The probability mass at time t is denoted by $q_\alpha(t)$, and if $F_\alpha(t)-q_\alpha(t) < 1$, $r_\alpha(t) = q_\alpha(t)/[1+q_\alpha(t)-F_\alpha(t)]$ is the probability that player α plans to move at time t conditional on neither player having moved before time t.

The payoff to player α from moving at any time t less than T, given that player β is following strategy F_β, is

$$P_\alpha(t,F_\beta) = \int_{-\infty}^{t} W_\alpha(v)\,dF_\beta(v) + [L_\alpha(t)-W_\alpha(t)]q_\beta(t) + L_\alpha(t)[1-F_\beta(t)]^{8},$$

and the payoff to player α from waiting until time T is

$$P_\alpha(T,F_\beta) = \int_{-\infty}^{T} W_\alpha(v)\,dF_\beta(v).$$

For an arbitrary strategy combination (F_a,F_b), the expected payoff to player α is then

$$P_\alpha(F_a,F_b) = \int_{-\infty}^{\infty} P_\alpha(t,F_\beta)\,dF_\alpha(t).$$

A strategy combination for G_Δ, (F_a^*,F_b^*), is an <u>equilibrium</u> for G_Δ if $P_\alpha(F_\alpha^*,F_\beta^*) \geq P_\alpha(F_\alpha,F_\beta^*)$ for all G_Δ strategies F_α, $\alpha = a,b$ and $\beta \neq \alpha$.

It is useful to distinguish between two types of equilibria for these games. An equilibrium in which one of the players moves with certainty at

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[8] The expressions are complicated by the fact that the integral $\int_y^z dF(x)$ includes any probability mass at z but not at y.

time 0 we will call a <u>degenerate equilibrium</u>. All other equilibria we will call <u>nondegenerate</u>.

3. Degenerate Equilibria

Whether the game is formulated in discrete or continuous time, it is easy to show that Assumption A1 implies that, for either player α, there is a degenerate equilibrium in which he moves immediately and the other player follows.[9] If we impose the requirement of subgame perfection, one of the equilibrium outcomes may be eliminated, but there is always at least one subgame perfect degenerate equilibrium.[10] Since we wish to focus on the <u>differences</u> in the equilibria of the discrete and continuous time formulations, our attention in the remainder of the paper is on the nondegenerate equilibria.

4. Nondegenerate Equilibria in Continuous Time Games

In this section we characterize the nondegenerate equilibria of G_0. We establish that in every nondegenerate equilibrium, the strategy of each player follows an exponential distribution with possible mass points at time 0 and, if the horizon is finite, at time T. We then determine the necessary and sufficient conditions under which these equilibria exist and prove that, if these conditions are satisfied, then there is a one parameter family of such equilibria. The family is indexed by the probability with which one of the players moves immediately at time 0.

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[9]In fact, there is generally an infinity of equilibrium strategies by the other player which support the same outcome.

[10]See Hendricks and Wilson (1985) for a complete characterization of the subgame perfect equilibria for this game formulated in discrete time.

Throughout this section, (F_a, F_b) will refer to a nondegenerate equilibrium.[11]

4.1 Properties of Nondegenerate Equilibria for G_0

We begin by establishing a number of restrictions which must be satisfied by a pair of equilibrium strategies. The arguments are quite standard for games of this type. We include them here in detail in order to contrast the properties of the continuous time game with those of the corresponding discrete time game to be analyzed in Section 5.

Lemma 4.1: Suppose $F_\alpha(t-\epsilon) = F_\alpha(t) < 1$ for some $\epsilon > 0$. Then $F_\beta(t-\epsilon) = F_\beta(t)$.

Lemma 4.1 states that if player α plans to move with probability 0 in an interval $(t-\epsilon, t]$ but plans to wait until after time t with positive probability, then it cannot be optimal for player β to move in the interval $(t-\epsilon, t]$. The reasoning is as follows. By waiting beyond time $t-\epsilon$ and moving at some time $v \in (t-\epsilon, t]$, player β incurs a positive cost due to the decrease in the return to leading. There is no possibility of gain, however, since player α moves with probability 0 in this interval. Consequently, if player β plans to move prior to t, he will do so no later than at time $t-\epsilon$.

Given any strategy combination (F_a, F_b), define

$$\hat{t} = \hat{t}(F_a, F_b) \equiv \inf\{t \leq T: \max\{F_a(t), F_b(t)\} = 1\}$$

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[11]Since a more general analysis of this model has already appeared in several places (e.g. Wilson and Weiss (1984) for continuous time games and Hendricks and Wilson (1985) for discrete time games), we will not always provide complete proofs of all of our results. We will, however, provide reasonably tight heuristic arguments for important results we wish to stress.

to be the earliest time by which at least one of the players plans to move with certainty.

<u>Lemma 4.2</u>: If $\hat{t} > 0$ then $\hat{t} = T$.[12]

Lemma 4.1 implies that if the equilibrium is nondegenerate, then there is a positive probability of reaching any time t before T. The argument is as follows. Suppose t is the earliest time by which at least one of the players, say player β, plans to move with probability one. If $0 < t < T$, then Assumption A1 implies that there is an interval $[t-\epsilon,t]$ such that conditional on the game reaching any time v in that interval, the return to player α from waiting until time t and being the follower is larger than his return from being the leader at time v. But, if player α plans to move with probability 0 in the interval $(t-\epsilon,t]$, Lemma 4.1 implies that player β also plans to move with probability 0 in this interval, contradicting the hypothesis that t is the earliest time by which player β moves with probability 1.

The next step is to use Lemmata 4.1 and 4.2 to eliminate the possibility of mass points occurring at times other than 0 and T.

<u>Lemma 4.3</u>: (i) $q_\alpha(t) = 0$ for $t \in (0,T)$;

(ii) $q_a(0)q_b(0) = 0$.

The logic behind Lemma 4.3 is as follows. Suppose player β plans to move with positive probability conditional on reaching some time $t \in (0,T)$.

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[12]Note that this Lemma implies that every information set is reached with positive probability. Therefore, any nondegenerate equilibrium is subgame perfect.

Then player α strictly prefers moving just after time t to moving during some small interval prior to time t. But if player α moves with probability 0 in some interval $(t-\epsilon,t]$, Lemma 4.1 implies that player β also plans to move with probability 0 in this interval, contradicting the assumption that $q_\beta(t)$ is positive. At $t = 0$, player α does not have the option of moving earlier, so the argument implies only that at most one player moves with positive probability at time 0. At time T, player α does not have the option of moving later, so the argument does not rule out the possibility of a mass point at time T.

The next definition applies to discrete as well as continuous time games. For any pair of equilibrium strategies (F_a,F_b) of a game G_Δ, let

$$t^* = t^*(F_a,F_b) \equiv \inf\{t \in J_\Delta \cup T: F_\alpha(t)-q_\alpha(t) = 1-q_\alpha(T),\ \alpha = a,b\}$$

represent the beginning of the last interval of time during which neither player ever moves.[13] If such an interval does not exist, then t^* is defined to be equal to T.

<u>Lemma 4.4</u>: F_α is strictly increasing on $[0,t^*]$.

Lemma 4.4 states that in a nondegenerate equilibrium, the distribution representing the strategy of player α is strictly increasing from time 0 to time t^*. If $t^* < T$, then neither player moves during the interval $(t^*,T]$, in which case Lemma 4.2 implies that there is probability mass at time T.

The argument is as follows. Suppose there is an interval $(t',t'']$,

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[13]We subtract $q_\alpha(t)$ from $F_\alpha(t)$ in order make the definition consistent with its use in our analysis of discrete time games.

where t" is less than T, during which player β plans not to move. Since the payoff to leading is continuous and decreasing with time and, by Lemma 4.3, $F_\beta(t)$ is continuous at t", there is an $\epsilon > 0$ such that player α prefers to move at t′ rather than at any time in the interval (t′,t"+ϵ]. Hence, for ϵ positive and sufficiently small, $F_\alpha(t') = F_\alpha(t''+\epsilon)$. Lemma 4.1 then implies that player β also moves with probability 0 in the interval (t′,t"+ϵ]. We conclude, therefore, that if an interval exists in which player β plans not to move, then that interval must extend to time T for both players.

Our next step is to calculate the functional form of the mixed strategies on the interior of their supports, $(0,t^*)$. Since player α must be indifferent between moving at any time $t \in (0,t^*)$, we require for any $t \in (0,t^*)$ and $\epsilon \in (0,t^*-t)$ that

$$\begin{aligned} 0 &= P_\alpha(t+\epsilon,F_\beta) - P_\alpha(t,F_\beta) \\ &= \int_t^{t+\epsilon}[W_\alpha(v)-L_\alpha(t)]\, dF_\beta(v) + [1-F_\beta(t+\epsilon)][L_\alpha(t+\epsilon)-L_\alpha(t)] \\ &= e^{-\delta t}\left\{\int_t^{t+\epsilon}[Be^{-\delta(v-t)}-A]dF_\beta(v) + [1-F_\beta(t+\epsilon)]A[e^{-\delta\epsilon}-1]\right\} \\ &= e^{-\delta t}\left\{[B-A+o(\epsilon)][F_\beta(t+\epsilon)-F_\beta(t)] - [1-F_\beta(t+\epsilon)]A[\delta\epsilon+o(\epsilon^2)]\right\}, \end{aligned}$$

where the last equality uses a Taylor's approximation. Dividing by ϵ and rearranging terms then yields

$$[F_\beta(t+\epsilon)-F_\beta(t)]/\epsilon = [A(\delta+o(\epsilon))/(B-A+o(\epsilon))][1-F_\beta(t+\epsilon)].$$

Taking $\epsilon \to 0$ then yields the differential equation

$$dF_\beta(t)/dt = f_\beta(t) = [\delta A/(B-A)][1-F_\beta(t)].$$

Upon integration and fixing the initial condition $F_\beta(0) = q_\beta(0)$, we then obtain

$$(4.1) \quad F_\beta(t) = q_\beta(0) + [1-q_\beta(0)][1-e^{-[\delta A/(B-A)]t}].$$

Let

$$E(t;q,t^*) = \begin{cases} q + (1-q)[1-e^{-[\delta A/(B-A)]t}] & \text{for } t \in [0,t^*); \\ q + (1-q)[1-e^{-[\delta A/(B-A)]t^*}] & \text{for } t \in [t^*,T); \\ 1 & \text{for } t = T \end{cases}$$

Combined with Lemma 4.4, equation (4.1) implies that any nondegenerate equilibrium strategy, F_β, must equal $E(t;q,t^*)$ for some $q \in [0,1)$ and $t^* \in (0,T]$. Any differences in the equilibrium strategies must then occur either in the probability mass at time 0 or in the time t^* after which the probability density is set equal to 0.

4.2 The Terminal Conditions and a Characterization of Nondegenerate Equilibria for Continuous Time Games

Up to this point all of our restrictions on the equilibrium strategy combinations have followed from implications of the optimal behavior from time 0 to time t^*. To complete our characterization of the equilibrium strategies, we need to consider the implications for the optimal responses of the players when they take into account the possibility of waiting until period T. To facilitate a comparison with the limit of discrete time games, we distinguish between games with a finite horizon and games with an infinite horizon. We deal with the infinite horizon game first.

4.2.1. Games with an Infinite Horizon

Since Assumption A2 implies that $L_\alpha(t) > 0$ for all $t > 0$, it follows immediately that $t^* = \infty$. Otherwise, upon reaching time t^*, both players would move with probability 1, contradicting Lemma 4.2. Beyond this restriction, one may readily verify that any pair of strategies satisfying Lemma 4.3 and equation (4.1) constitute an equilibrium. Then we may state

Theorem 4.1: Suppose $T = \infty$. Then (F_a,F_b) is an equilibrium if and only if
(i) $F_\alpha = E(\cdot;q_\alpha(0),\infty)$ with $0 \leqq q_\alpha(0) < 1$ for $\alpha = a,b$, and
(ii) $q_a(0)q_b(0) = 0$.

We conclude that, when the horizon is infinite, there is a one parameter family of nondegenerate equilibria indexed by the probability with which one of the players moves at time 0. Each player moves with a constant hazard rate of $\delta A/(B-A)$ throughout the interval $(0,\infty)$.

4.2.2 Games with a Finite Horizon

Consider next games with a finite horizon. To ease notation, let $L_\alpha(T) \equiv \lim_{t\to T}L_\alpha(T)$. In contrast to the infinite horizon game, it follows from Lemma 4.2 that there is a positive probability that both players will choose neveer to move. Thus, player α must be indifferent between moving at any time before t^* and waiting until time T. But, since Lemma 4.3 implies that the probability that player β moves in the interval (t,T) approaches zero as t approaches t^*, it then follows that $L_\alpha(t)$ must approach 0 as t approaches t^*. We conclude, therefore, that

Lemma 4.5: $L_a(t^*) = L_b(t^*) = 0$.

For each player α, define

$$\tau_\alpha = \inf[\{t \in [0,T): L_\alpha(t) \leqq 0\} \cup T]$$

to be the earliest time in which the return to player α from leading is less than or equal to 0 (or T if no such time exists). Let $\tau = \inf \{\tau_a, \tau_b\}$. Then Lemmata 4.3 and 4.5 can be combined with equation (4.1) to yield the following characterization of the set of nondegenerate equilibria in finite horizon games.

Theorem 4.2: (a) If $T < \infty$, a nondegenerate equilibrium exists if and only if $\tau_a = \tau_b$ and $L_\alpha(T) \leqq 0$ for $\alpha = a,b$.
(b) If a nondegenerate equilibrium exists, then (F_a, F_b) is an equilibrium if and only if (i) $F_\alpha = E(\cdot; q_\alpha(0), \tau_\alpha)$ with $0 \leqq q_\alpha(0) < 1$, $\alpha = a,b$, and (ii) $q_a(0)q_b(0) = 0$.

We conclude that when the horizon is finite, nondegenerate equilibria exist if and only if the return from leading is equal to 0 at exactly the same time for both players. When a nondegenerate equilibrium exists, there is a one parameter family of nondegenerate equilibria, indexed by the probability with which one of the players moves at time 0. Each player moves with a constant hazard rate of $\delta A/(B-A)$ in the interval $(0,\tau)$, whereupon they wait until period T.

5. Nondegenerate Equilibria in Discrete Time Games

In this section we characterize the set of nondegenerate equilibria when the set of times at which the players can choose to move is discrete. We establish that equilibrium strategies may display three possible patterns. The pattern may be fully mixed in which case both players move with positive probability conditional on reaching any period. It may be alternating, in which case the players alternately move with positive probability in every other period. Finally, it may be a hybrid of these two patterns, in which case the pattern is initially fully mixed and then alternating for the remainder of the game. If the horizon is infinite and Assumption A2 is satisfied, all of these patterns are equilibria. If the horizon is finite, however, this is no longer the case. We derive necessary and sufficient conditions for the existence of a nondegenerate equilibrium and show that if a nondegenerate equilibrium exists, it is generally unique and fully mixed. It is only for a special (nongeneric) class of return functions that all of the patterns are equilibria in a finite horizon game.

To ensure that the discrete time version of the game captures the same tradeoffs as the continuous time version, we require that the return from following in period $t+\Delta$ always exceeds the return from leading in period t. To ease notation, let $\lambda(\Delta) = e^{-\delta\Delta}$. Then we will restrict attention to games G_Δ such that

$$\lambda(\Delta)B > A.$$

Note that since $\lambda(\Delta) \to 0$ as $\Delta \to 0$, Assumption A1 implies that this relation must be satisfied for Δ sufficiently small. In this section, we will suppose that the condition is satisfied for $\Delta = 1$ so that we may focus our analysis on G_1.

5.1 Preliminary Lemmata and the Patterns for Nondegenerate Equilibria

As in the analysis of continuous time games, we will proceed by first deriving some restrictions which must be satisfied by any pair of equilibrium strategies. We begin by establishing the discrete time analogue of Lemma 4.1.

Lemma 5.1: Suppose $F_\alpha(t+1) < 1$. Then $q_\alpha(t) = 0$ implies $q_\beta(t+1) = 0$.

The argument is similar to the argument of Lemma 4.1. If player α moves with probability 0 in period t and plans to move with positive probability in some period after period t+1, there is no advantage to player β from waiting until period t+1 to move. His return from leading is higher in period t than in period t+1 and he is no more likely to be a follower by waiting until period t+1 than by moving in period t.

Using Lemma 4.1, we may then establish the analogue of Lemma 5.2.

Lemma 5.2: If $\hat{t} > 0$, then $\hat{t} = T$.

Lemma 5.2 implies that if the equilibrium is nondegenerate, then there is a positive probability that the game reaches any finite period t less than T. The argument is similar to that of Lemma 4.2. Suppose there is a positive probability that neither player moves before period t but, conditional on reaching period t, player β plans to move with probability 1. Now consider the problem of player α upon reaching period t-1. If he moves, he gets a return of $L_\alpha(t-1)$. If he waits until period t+1, his return is at least $W_\alpha(t)$ since player β moves with certainty by period t. But $B\lambda > A$ then implies that $W_\alpha(t) > L_\alpha(t-1)$. Therefore, player α never moves in period t-1. Lemma

5.1 then implies that it cannot be optimal for player β to move in period t, contradicting our assumption that his conditional probability of moving in period t is 1.

Lemmata 5.1 and 5.2 together imply the following restriction on the support of the equilibrium strategies.

<u>Lemma 5.3</u>: For any nondegenerate equilibrium (F_a, F_b), there is a pair of integers $\tilde{t}$ and t^*, $0 \leq \tilde{t} \leq t^* \leq T$, such that

(i) $q_a(t)q_b(t) > 0$ for $t < \tilde{t}$;

(ii) $q_\alpha(t)q_\alpha(t+1) = 0$ and $(1-q_a(t))(1-q_b(t)) < 1$ for $\tilde{t} \leq t < t^*$;

(iii) $q_\alpha(T) = 1 - F_\alpha(t^*-1)$.[14]

In words, Lemma 5.3 says the following. Up to some period $\tilde{t}$, both players move with positive probability conditional on reaching any period. If $\tilde{t} < t^*$, then beginning in period $\tilde{t}$ and up to (but not including) period t^*, only one of the players moves with positive probability conditional on reaching any period and they move in alternate periods. From period t^* up to (but not including) period T, neither player ever moves. If $\tilde{t} = 0$, we will say that the equilibrium is <u>alternating</u>. If $\tilde{t} = t^*$, we will say that the equilibrium is <u>fully mixed</u>. If $0 < \tilde{t} < t^*$, we will say that the equilibrium is <u>hybrid</u>. In this case, the pattern is fully mixed up to period $\tilde{t}$, alternating from period $\tilde{t}$ up to t^*, after which neither player ever moves. Notice that Lemma 5.1 rules out the possibility that the pattern is first alternating and then fully mixed.[15]

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[14] Implicitly, we are supposing that T is an integer, More generally, if T is not a multiple of Δ for a game G_Δ, then $T-\Delta$ is to be understood to represent the largest multiple of Δ which is less than T. Note also that t^* is consistent with the definition of $t^*(F_a,F_b)$ given in Section 4.1.

[15] When the return to moving simultaneously differs from the return to leading,

Before determining the conditions under which each of these three types of equilibria exist, we first provide a more precise characterization of the three cases.

5.1.1 Fully Mixed Pattern

Consider first the fully mixed pattern. Since both players move with positive probability conditional on reaching any period between 0 and $\tilde{t}-1$, the strategy of player β must be chosen so that player α is indifferent between moving in period t and moving in period t+1 for all $t \leq \tilde{t}-2$. This implies that

$$P_\alpha(t+1,F_\beta) - P_\alpha(t,F_\beta) = \lambda^t[q_\beta(t)(B-A) + (1-F_\beta(t))(\lambda A-A)] = 0.$$

Then, using the fact that $F_\beta(t-1) = F_\beta(t)-q_\beta(t)$, we obtain

$$(5.1) \quad r_\beta(t) = q_\beta(t)/(1-F_\beta(t)) = A(1-\lambda)/(B-A\lambda) \equiv r_f, \quad t = 0,\ldots,\tilde{t}-2.$$

5.1.2 Alternating Pattern

Consider next the alternating pattern. Suppose for instance that player α plans to move with positive probability in periods t-1 and t+1 and with probability 0 in period t. Then the strategy of player β must be adjusted so that

$$P_\alpha(t+1,F_\beta) - P_\alpha(t-1,F_\beta) = \lambda^{t-1}[q_\beta(t)(\lambda B-A) + (1-F_\beta(t))(\lambda^2 A-A)] = 0.$$

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the possible patterns are much richer. The more general model is analyzed in Hendricks and Wilson (1985)

A little algebraic manipulation then yields

(5.2) $r_\beta(t) = A(1-\lambda^2)/[\lambda(B-\lambda A)] \equiv r_g, \qquad t = \tilde{t}+1,\ldots,t^*-2.$

5.1.3 <u>The Transition Probabilities</u>

For equilibria in which $\tilde{t} < t^*$ (hybrid or alternating equilibria), we need to determine the probabilities with which the players move conditional on reaching period $\tilde{t}$ and, if $\tilde{t} > 0$, period $\tilde{t}-1$. Suppose player α plans to move with probability 0 in period $\tilde{t}$. Consider first the case where $\tilde{t} = 0$ (an alternating equilibrium). In contrast to a fully mixed equilibrium, there is, in this instance, an indeterminacy at period 0. Since player α cannot move before period 0, the probability with which player β moves in period 0 can be any number large enough to ensure that player α prefers to wait until period 1 to move. Calculations similar to the fully mixed case then yield the restriction

(5.3) $q_\beta(0) \geq r_f$

Hybrid equilibria ($0 < \tilde{t} < t^*$) exhibit the same properties as the fully mixed equilibria up to period $\tilde{t}-1$. Hence, the probability with which either player moves conditional on reaching any period $t < \tilde{t}-1$ is r_f. Furthermore, since player β plans to move in period $\tilde{t}$ with positive probability, player α must move with probability r_f conditional on reaching period $\tilde{t}-1$. For player β, however, the only restriction on $r_\beta(\tilde{t}-1)$ is that it be set sufficiently small so that player α does <u>not</u> prefer to wait in period $\tilde{t}-1$ and move in period $\tilde{t}$. Therefore, for $0 < \tilde{t} < t^*$, (5.4) $\quad 0 < r_\beta(\tilde{t}-1) \leqq r_f.$

Given $r_\beta(\tilde{t}-1)$, the value of $r_\beta(\tilde{t})$ must then be adjusted so that player α is indifferent between moving in period $\tilde{t}-1$ and moving in period $\tilde{t}+1$. Thus, given any $\tilde{t}$ such that $0 < \tilde{t} < t^*-1$, appropriate calculations yield the equation

(5.5) $$r_\beta(\tilde{t}) = [A(1-\lambda^2) + r_\beta(\tilde{t}-1)(A\lambda^2-B)]/[(1-r_\beta(\tilde{t}-1))\lambda(B-\lambda A)].$$

Note that $r_\beta(\tilde{t})$ is decreasing with $r_\beta(\tilde{t}-1)$. If $r_\beta(\tilde{t}-1) = 0$, then we are already in an alternating pattern by period $\tilde{t}-1$ and $r_\beta(\tilde{t}) = r_g$. If $r_\beta(\tilde{t}-1) = r_f$, then $r_\beta(\tilde{t}) = r_f$. Conditional on reaching any period later than $\tilde{t}$ and less than t^*-1, each player alternately moves with probability r_g.

The three possible equilibrium patterns are illustrated in Figure 2 for the case $t^* < T$.

Figure 2 here

5.2 The Terminal Conditions and a Characterization of Nondegenerate Equilibria for Discrete Time Games

As in Section 4, to complete our characterization of the discrete time nondegenerate equilibria, we must consider the implications of the optimal behavior of the players at the end of the game. Looking ahead to Section 6, it will useful to state the results in this section for an arbitrary game G_Δ. Once again, we will need to distinguish between games with an infinite horizon and games with a finite horizon. We consider the infinite horizon case first.

5.2.1 Games with an Infinite Horizon

Given Assumption A2, the implications for equilibrium behavior at the end of the infinite horizon game is essentially the same as for continuous time games. Recall that t^* is the first period in which both players wait with probability 1 until period T. The analogue to Theorem 4.1 can then be stated as

Theorem 5.1: Suppose $T = \infty$ and A2 is satisfied.

(a) There is a unique fully mixed equilibrium.

(b) For each $\beta \in \{a,b\}$ and $r_\beta(0) \geqq r_f^\Delta$, there is a unique alternating equilibrium.

(c) For each $\beta \in \{a,b\}$, $\tilde{t} \in \{1,2,3,...\}$, and $r_\beta(\tilde{t}-\Delta)$ satisfying relation (5.4) there is a unique hybrid equilibrium.

(d) There are no other equilibria.

5.2.2 Games with a Finite Horizon: The Fully Mixed Equilibrium

Consider next the possibility for a fully mixed equilibrium in games with a finite horizon. Let $t^{*\Delta}$ denote the value of t^* for an equilibrium of G_Δ. If a fully mixed equilibrium exists, then both players move with positive probability conditional on reaching period $t^{*\Delta}-\Delta$. Consequently, it follows from Lemma 5.2 that both players must be indifferent between moving in period $t^{*\Delta}-\Delta$ and waiting until period T. Since neither player moves in the intervening periods, it follows that, for $\beta = a,b$, $r_\beta(t^{*\Delta}-\Delta)$ must satisfy the equation

(5.6) $r_\beta(t^{*\Delta}-\Delta) = L_\alpha(t^{*\Delta}-\Delta)/W_\alpha(t^{*\Delta}-\Delta).$

To establish the analogue of Lemma 4.5, it is useful to first define the discrete time analogue of τ_α. Let

$$\tau_\alpha^\Delta = \inf\{\{t \in J_\Delta: L_\alpha(t) \leqq 0\} \cup T\}$$

be the first period t in the game G_Δ in which $L_\alpha(t) \leqq 0$ (or T, if $L_\alpha(T) \geqq 0$). Then we may state

<u>Lemma 5.4</u>: In fully mixed equilibrium, $t^{*\Delta} = \tau_\alpha^\Delta = \tau_\beta^\Delta$.

It is sufficient to establish Lemma 5.4 for G_1. Since Lemma 5.2 implies that $q_\beta(T) > 0$, player α moves with probability 0 in period t^*-1 unless $L_\alpha(t^*-1) \geqq 0$. Our assumption that $\lambda B-A > 0$ then implies that $W_\alpha(t^*-1) > 0$ which, combined with equation (5.6) and the requirement that $r_\beta(t^*-1) > 0$, implies that $L_\alpha(t^*-1) > 0$. On the other hand, if $t^* < T$, then $L_\alpha(t^*) \leqq 0$; otherwise, upon reaching period t^*, player α would move with probability 1, contradicting the definition of t^*. Employing the definition of τ_α^Δ then yields the Lemma.

Note that the conditions of Lemma 5.4 are weaker than the corresponding conditions derived in Lemma 4.5 for the continuous time game. The analogue of Lemma 4.5 would require in addition that $L_\alpha(T) \leqq 0$. We will discuss the reasons for this difference in section 6.

We are now prepared to characterize the set of fully mixed equilibria for discrete time games with a finite horizon. As in the continuous time games, the restrictions of Section 5.1 plus Lemma 5.4 constitute not only necessary but also sufficient conditions for a fully mixed equilibrium. In this case, however, the set of equilibria reduces to a single strategy pair.

Theorem 5.2: There is unique fully mixed equilibrium for G_Δ if and only if $\tau_\alpha^\Delta = \tau_\beta^\Delta$.

5.2.3 Games with a Finite Horizon: Alternating and Hybrid Equilibria

Consider next the possibilities for alternating and hybrid equilibria in a game with a finite horizon. They differ from the fully mixed equilibria in that only one player moves with positive probability upon reaching period t^*-1. Suppose that $r_\beta(t^*-1) > 0$. Assuming $t^* \geq 2$, Lemma 5.1 implies that $r_\alpha(t^*-2) > 0$. Then if player α is to be indifferent between moving in period t^*-2 and waiting until period T, $L_\alpha(t^*-2)$ must be strictly positive. Furthermore, if he is to wait until period T upon reaching period t^*, $L_\alpha(t^*)$ must be nonpositive. We conclude that $t^*-1 \leq \tau_\alpha \leq t^*$. On the other hand, since the equilibrium is not fully mixed, $r_\alpha(t^*-1) = 0$ which means that if player β is to be indifferent between moving in period t^*-1 and waiting until period T, $L_\beta(t^*-1)$ must be equal to 0. This in turn implies that $t^*-1 = \tau_\beta$. Combining these observations then yields the analogue to Lemma 5.4 for alternating and hybrid equilibria.

Lemma 5.5: If $r_\alpha(t^{*\Delta}-\Delta) = 0$, then $L_\beta(t^{*\Delta}-\Delta) = 0$ and $\tau_\alpha^\Delta \leq \tau_\beta^\Delta+\Delta = t^{*\Delta} \leq \tau_\alpha^\Delta+\Delta$.

We demonstrate in Hendricks and Wilson (1985) that these conditions are also sufficient for the existence of a continuum of nondegenerate equilibria. For the purposes of this paper, however, we are concerned only with the equilibria of G_Δ as Δ goes to 0. Therefore, if, for some player α and some time $t < T$, $L_\alpha(t) = 0$, Lemmata 5.4 and 5.5 imply that we may confine our attention to games in which $L_a(\tau_a) = L_b(\tau_b)$.

To complete our characterization of the nondegenerate equilibria for this case, we need to determine the restrictions on the probabilities of moving in periods t^*-1 and t^*-2. Suppose that $r_\beta(t^*-1) > 0$ and hence that $r_\alpha(t^*-2) > 0$. Then if $L_\alpha(t^*-1) = 0$, player α is indifferent between moving in period t^*-2 and waiting until period T if and only if the strategy of player β is chosen to satisfy (for G_Δ)

$$(5.7)\qquad r_\beta(t^{*\Delta}-\Delta) = \frac{L_\alpha(t^{*\Delta}-2\Delta) - r_\beta(t^{*\Delta}-2\Delta)W_\alpha(t^{*\Delta}-2\Delta)}{W_\alpha(t^{*\Delta}-\Delta)(1 - r_\beta(t^{*\Delta}-2\Delta))} \leqq r_g^\Delta \text{[16]}$$

In addition, $r_\beta(t^{*\Delta}-2\Delta)$ must satisfy the analogue to relation (5.3) which is

$$(5.8)\qquad 0 \leqq r_\beta(t^{*\Delta}-2\Delta) < L_\alpha(t^{*\Delta}-2\Delta)/W_\alpha(t^{*\Delta}-2\Delta) \leqq r_g^\Delta \text{[17]}.$$

These two relations then complete the characterization of the nondegenerate equilibria.

<u>Theorem 5.3</u>: Suppose there is a $t \in J_\Delta$ such that $L_a(t) = L_b(t) = 0$. If $r_a(t^{*\Delta}-\Delta)r_b(t^{*\Delta}-\Delta) = 0$, then $t^* = t+\Delta$. The set of such nondegenerate equilibrium may be indexed as follows:

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[16]By definition, $r_g^\Delta = [L_\alpha(t^{*\Delta}-2\Delta)-L_\alpha(t^{*\Delta})]/[W_\alpha(t^{*\Delta}-\Delta)-L_\alpha(t^{*\Delta})]$. Then, since $L_\alpha(t^{*\Delta}) < 0$, it follows that $r_g^\Delta \geqq L_\alpha(t^{*\Delta}-2\Delta)/W_\alpha(t^{*\Delta}-\Delta)$, which is the upper bound on $r_\beta(t^{*\Delta}-\Delta)$.

[17]Since $W_\alpha(t^{*\Delta}-2\Delta) > W_\alpha(t^{*\Delta}-\Delta)$, it follows directly from the argument given in the previous footnote.

(a) For each value of $r_a(0)$ or $r_b(0)$ satisfying relation (5.3), there is a unique alternating equilibrium.

(b) For each $\tilde{t} \in J_\Delta$, $0 < \tilde{t} < t^{*\Delta}-\Delta$, and each value of $r_a(\tilde{t}-\Delta)$ or $r_b(\tilde{t}-\Delta)$ satisfying relation (5.4) and for $\tilde{t} = t^{*\Delta}-\Delta$ and each value of $r_a(t^{*\Delta}-2\Delta)$ or $r_b(t^{*\Delta}-2\Delta)$ satisfying relation (5.8), there is a unique hybrid equilibrium.

Note that the conditions required in Theorem 5.3 for the existence of a nondegenerate equilibrium imply that τ_α/Δ must be an integer. Generally, we should not expect this condition to be satisfied. Therefore, if $\tau_\alpha < T$ and τ_α/Δ is an integer for some player α, we will say that a game G_Δ is <u>nongeneric</u>. We will refer to all other games as generic games.

6. Discrete Versus Continuous Time

In this section we compare the pairs of nondegenerate distribution functions obtained as the limit of a sequence of equilibria of discrete time games to the equilibria of the corresponding continuous time game. We show in Section 6.1 that the limiting distributions associated with the fully mixed, alternating, and hybrid patterns are all members of the one parameter family of distributions which characterizes the set of possible equilibria of the continuous time game. In Section 6.2 we complete our characterization of the limits of nondegenerate discrete time equilibria and compare them to nondegenerate continuous time equilibria.

6.1 The Limiting Distributions

Consider any equilibrium strategy F_α^Δ for a game G_Δ. For any integer n such that $0 \leq n\Delta < T$,

$$(6.1)\quad F_\alpha^\Delta(\Delta n) = 1 - \Pi_{j=0}^{n}[1-r_\alpha(j)]$$
$$= [1-q_\alpha^\Delta(0)][1-\Pi_{j=1}^{n}[1-r_\alpha^\Delta(j)]] + q_\alpha^\Delta(0)$$

Extending the analysis of Section 5 to an arbitrary game G_Δ, we may infer from equations (5.1) to (5.5) and relation (5.8) that $r_\alpha^\Delta(j) \leqq r_g^\Delta$ for all $j \in J_\Delta$, $0 < j < t^{*\Delta}-\Delta$.[18] Then, since r_f^Δ and r_g^Δ go to zero as Δ becomes small, it follows that for any $t \in [0,\tilde{t}-\Delta)$,

$$F_\alpha^\Delta(t) = [1-q_\alpha^\Delta(0)][1 - (1-r_f^\Delta)^{t/\Delta} + o(\Delta)] + q_\alpha^\Delta(0)$$

and for any $\tilde{t}-\Delta \leqq t < t^{*\Delta}-\Delta$,

$$F_\alpha^\Delta(t) = [1-q_\alpha^\Delta(0)][1 - (1-r_f^\Delta)^{\tilde{t}/\Delta}(1-r_g^\Delta)^{(t-\tilde{t})/2\Delta} + o(\Delta)] + q_\alpha^\Delta(0).$$

Upon substituting the expressions for r_f^Δ and r_g^Δ and using a Taylor's expansion, we then obtain, for any $t < t^{*\Delta}-\Delta$,

$$(6.2)\quad F_\alpha^\Delta(t) = q_\alpha^\Delta(0) + [1-q_\alpha^\Delta(0)][1-e^{-\delta tA/(B-A)}] + o(\Delta)$$
$$= E(t;q_\alpha^\Delta(0),t^{*\Delta}) + o(\Delta)$$

If the equilibrium is fully mixed or hybrid, $q_\alpha^\Delta(0) = r_f^\Delta$ which goes to 0 as Δ becomes small. For alternating equilibria, however, if $q_\beta^\Delta(0) = 0$, then $q_\alpha^\Delta(0)$ can be any number between r_f^Δ and 1. In any case, equation (6.2) implies that, between 0 and $t^{*\Delta}-\Delta$, the equilibrium strategies are approximately the same as the solution of the differential equation which

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[18]Since $r_f^\Delta < r_g^\Delta$.

characterizes the continuous time equilibria on $(0,t^*)$. Any differences between the continuous time equilibria and the limit of discrete time equilibria, therefore, must be the result of differences in the terminal conditions.

6.2 Continuous Time Equilibria and the Limit of Discrete Time Equilibria

In this section, we complete the characterization of the distributions which arise as the limit of equilibria of discrete time games and explicitly compare these limiting distributions with the continuous time equilibrium distributions. For any two distributions F^1 and F^2, define

$$\|F^1-F^2\| = \sup_t |F^1(t)-F^2(t)|.$$

As in Sections 4 and 5, it is useful to distinguish between games with infinite and finite horizons.

6.2.1 Games with an Infinite Horizon

When the horizon is infinite, Theorems 4.1 and 5.1 guarantee that $t^{*\Delta} = \infty$ for any game G_Δ. When combined with Theorem 4.1, equation (6.2) then implies that the limit of successively finer discrete time equilibria must converge to a continuous time equilibrium. Furthermore, when combined with Theorem 5.1, equation (6.2) implies that any continuous time equilibrium is the limit of some sequence of discrete time equilibria. We may conclude, therefore, that there is a one to one relationship between the continuous time equilibria and the limiting distributions of the discrete time game.

Theorem 6.1: Suppose $T = \infty$. Then (F_a,F_b) is a nondegenerate equilibrium for G_0 if and only if for any $\epsilon > 0$ there is a $\bar{\Delta} > 0$ such that for any

$\Delta \in (0,\bar{\Delta})$, there is an equilibrium (F_a^Δ, F_b^Δ) for G_Δ such that
$\|F_\alpha - F_\alpha^\Delta\| < \epsilon$, $\alpha = a,b$.

6.2.2 Games with a Finite Horizon

When the horizon is finite, both players wait with positive probability until the end of the game in both the continuous and discrete time formulations. As a consequence, there are additional restrictions on the equilibrium strategies that do not arise when the horizon is infinite. These restrictions have different implications for discrete and continuous time games with respect to the existence and number of nondegenerate equilibria.

To complete the characterization of the distributions that can arise as the limit of discrete time equilibria in games with a finite horizon, note first that, since τ_α^Δ converges to τ_α as Δ goes to 0, it follows from Lemmata 5.5 and 5.6 that

$$t^{*\Delta} = \tau_\alpha + o(\Delta) \tag{6.3}$$

for $\alpha = a,b$. It then follows from Lemmata 5.4 and 5.5 that $\tau_a = \tau_b$ is necessary for the existence of a nondegenerate equilibrium for G_Δ as Δ goes to 0. Since Theorem 4.2 also implies that $\tau_a = \tau_b$ is necessary for the existence of an nondegenerate equilibrium in continuous time (i.e. for the game G_0), we may state

Theorem 6.2: Suppose $T < \infty$ and $\tau_a \neq \tau_b$. Then there is a $\bar{\Delta} > 0$ such that for any $\Delta \in [0,\bar{\Delta})$, there is no nondegenerate equilibrium for G_Δ.

In view of Theorem 6.2, we may confine our attention to games where $\tau_a = \tau_b$. There are two cases to consider.

Suppose first that $\tau_a = \tau_b$ but $L_\alpha(T) > 0$ for at least one player α. Then Theorem 4.2 implies that there is no nondegenerate equilibrium in the continuous time game. Now consider any discrete time game. Since $L_\alpha(T) > 0$, it follows from the definition of τ_α that $\tau_a = \tau_b = T$. It then follows from the definition of τ_β that $L_\beta(T-\Delta) > 0$ which implies that $L_\beta(t^*-\Delta) > 0$. Lemma 5.5 then implies that an alternating or hybrid equilibrium cannot exist. Theorem 5.2, on the other hand, does guarantee the existence of a fully mixed equilibrium. Using equation 6.1, we may then state the following result.

<u>Theorem 6.3</u>: Suppose $\tau_a = \tau_b$ and $L_\alpha(T) > 0$ for some α. Then

(a) There is no nondegenerate equilibrium for G_0;

(b) For any $\epsilon > 0$, there is a $\bar{\Delta} > 0$ such that for any $\Delta \in (0,\bar{\Delta})$, there is a unique nondegenerate equilibrium (F_a^Δ, F_b^Δ). For all $\Delta \in (0,\bar{\Delta})$ and $t \in [0,T-\Delta)$, $|F_\alpha^\Delta(t) - E(t;0,T)| < \epsilon$.

Consider next the case where $\tau_a = \tau_b$ and $L_\alpha(T) \leqq 0$ for $\alpha = a,b$. In this case, Theorem 4.2 implies the existence of a one parameter of equilibria $\{(F_a,F_b)\}$ where $F_\alpha = E(\cdot;q_\alpha(0),\tau_\alpha)$ and, if $q_\beta(0) = 0$, $q_\alpha(0)$ may take on any value in $[0,1)$. For a discrete time game G_Δ with Δ sufficiently small, equation (6.2) implies that F_α^Δ is approximately equal to $E(\cdot;q_\alpha^\Delta(0),t^{*\Delta})$ over the interval $[0,t^{*\Delta}-\Delta)$. To complete the characterization, therefore, we need only examine the behavior of F_α^Δ over the interval $[t^{*\Delta}-\Delta,T]$.

By definition, $r_\alpha^\Delta(\Delta n) = 0$ for any integer n with $\Delta n \in [t^{*\Delta},T)$. Then combining (6.1) with (6.2), we obtain for any $t \in [t^{*\Delta}-\Delta,T)$,

(6.4) $$F_\alpha^\Delta(t) = 1 - [1 - E(t;q_\alpha^\Delta(0),t^{*\Delta})][1-r_\alpha^\Delta(t^{*\Delta}-\Delta)] + o(\Delta).$$

If the equilibrium is not fully mixed, then either $r_\alpha^\Delta(t^{*\Delta}-\Delta) = 0$ or else $r_\beta^\Delta(t^{*\Delta}-\Delta) = 0$ in which case relation (5.7) implies that $0 < r_\alpha^\Delta(t^{*\Delta}-\Delta) \leqq r_g^\Delta$. If the equilibrium is fully mixed, then, since $L_\beta(T) \leqq 0$, Lemma 5.4 implies that $L_\beta(t^{*\Delta}) \leqq 0$. Applying the definition of r_f^Δ, it is then easy to show that equation (5.6) implies again that $r_\alpha(t^{*\Delta}-\Delta) \leqq r_f^\Delta < r_g^\Delta$. Since r_g^Δ is of order $o(\Delta)$, we may conclude that $F_\alpha^\Delta(t)$ is close to $E(t;q_\alpha^\Delta(0),\tau)$ for <u>all</u> values of t.

The relation between the limiting discrete time equilibria and the continuous time equilibria then depends on the permissible values $q_\alpha^\Delta(0)$. If G_Δ is generic, then Lemma 5.5 implies that the equilibrium must be fully mixed. Consequently, for Δ small, $q_\alpha^\Delta(0)$ is approximately 0. If G_Δ is nongeneric, however, then alternating equilibria exist. In this case, $q_\alpha^\Delta(0)$ can take on any value in the interval $[r_f^\Delta,1)$, provided $q_\beta^\Delta(0) = 0$.

We may summarize our conclusions as:

<u>Theorem 6.4</u>: Suppose $\tau_a = \tau_b = \tau$ and $L_\alpha(T) \leqq 0$, $\alpha = a,b$. Then

(a) (F_a,F_b) is a nondegenerate equilibrium for G_0 if and only if, for $\alpha = a,b$, $F_\alpha = E(\cdot;q_\alpha,\tau)$ where $q_\alpha \in [0,1)$ and $q_aq_b = 0$.

(b) For any $\epsilon > 0$, there is a $\bar{\Delta} > 0$ such that for any $\Delta \in (0,\bar{\Delta})$:

(i) If G_Δ is <u>nongeneric</u>, there is an equilibrium (F_a^Δ,F_b^Δ) such that $\|F_\alpha - F_\alpha^\Delta\| < \epsilon$, $\alpha = a,b$, if and only if (F_a,F_b) is a nondegenerate equilibrium for G_0.

(ii) If G_Δ is <u>generic</u> and (F_a^Δ,F_b^Δ) is the (unique) nondegenerate equilibrium, then $\|F_\alpha - E(\cdot;0,\tau)\| < \epsilon$ for $\alpha = \alpha,b$.

A summary of the comparison of the <u>nondegenerate</u> equilibria in the discrete and continuous time games with a finite horizon is presented in the following table.

	G_Δ generic	G_Δ nongeneric
$L_\alpha(T) > 0$	no equilibrium for G_0 one (fully mixed) equilibrium for G_Δ, $\Delta > 0$	not applicable
$L_\alpha(T) \leqq 0$	continua of equilibria for G_0 one (fully mixed) equilibrium for G_Δ, $\Delta > 0$	continua of equilibria for G_0 continua of equilibria for G_Δ, $\Delta > 0$

6.3 <u>Upper Hemicontinuity and Convergence in the Sup Norm Topology</u>

One implication of Theorem 6.3 is that the equilibrium correspondence of G_Δ is not upper hemicontinuous in Δ whenever $L_\alpha(T) > 0$. To understand this result, it is useful to clarify what we mean by saying that two strategies are close to each other. One possibility is to use the topology of weak convergence (see e.g. Loeve(1963), p.178) to say that two distributions are close to each other if they have approximately the same value <u>except near points of discontinuity</u>. Inspection of equation (6.1) reveals that, for a fixed $q_\alpha(0)$, the equilibrium strategy of a sequence of ever finer discrete time games always converges <u>weakly</u> to the distribution $E(\cdot;q_\alpha(0),\tau_\alpha)$. Consequently, if the expected payoff functions were continuous in this topology, standard limiting arguments could then be used to establish the upper hemicontinuity of the equilibrium correspondence at $\Delta = 0$. Unfortunately, in order for expected payoffs to be continuous in this topology, the payoff function must generally be <u>continuous</u> in the space of pure strategies. Since, for the games studied here, the payoff functions are not continuous on the diagonal, this notion of closeness does not imply that two distributions which are close to each other generate payoffs which are approximately equal.

A stronger notion of closeness is implied by the sup norm topology which defines two distributions to be close to each other whenever they have approximately the same value at <u>every</u> point. Under this definition of closeness, the expected payoff is continuous in the space of mixed strategies for any <u>integrable</u> payoff function. Thus, even in the War of Attrition, the expected payoffs to either player from two strategy combinations which are close to each other in the sup norm topology are approximately equal. Looked at from this perspective, the reason why upper hemicontinuity of the equilibrium correspondence may fail is that a sequence of discrete time equilibria may not converge in the sup norm topology.[19] In fact, for this particular model, Theorems 6.3 and 6.4 imply that the equilibrium correspondence is upper hemicontinuous if and only if the sequence of discrete time equilibria converge in the sup norm topology.

To see why convergence in the sup norm topology fails in those cases when a nondegenerate equilibrium in continuous time does not exist, note that equation (6.4) combined with equation (6.3) implies that in order for the discrete time equilibria to converge, the last "step" of the discrete time equilibria, $r_\alpha^\Delta(t^{*\Delta}-\Delta)$, must converge to 0 as Δ becomes small. However, when $L_\alpha(T) > 0$, equation (5.6) implies that this number must be bounded away from 0. As a result, there is a downward jump in the expected payoff to each

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[19]The space of distributions in this topology is not sequentially compact. Consequently, there may be no convergent subsequences.

player α as we reach the limiting strategy (in the topology of weak convergence). In such a case the limiting distributions do not form an equilibrium in continuous time.[20]

6.4 Approximate Equilibria

If the set of equilibria are not equivalent in the two formulations, one might hope that the set of approximate equilibria are.

Define an ϵ-equilibrium for G_Δ to be a G_Δ strategy combination (F^*_a, F^*_b) such that $P_\alpha(F^*_\alpha, F^*_\beta) \geq P_\alpha(F_\alpha, F^*_\beta) - \epsilon$ for all G_Δ strategies F_α, $\alpha = a,b$ and $\beta \neq \alpha$. Fudenberg and Levine (1983) have argued that in many instances the equilibria in the continuous time formulation can be approximated by discrete time ϵ-equilibria and conversely. They establish such an approximation theorem for degenerate (more precisely, pure strategy) equilibria in games of timing where the payoff functions are piecewise continuous.

When we consider nondegenerate equilibria, however, the approximation goes only one way: continuous time equilibria can be approximated by discrete time ϵ-equilibria, but not vice versa. The reason is that while the concept of ϵ-equilibrium may be used to guarantee lower hemicontinuity of the equilibrium correspondence, it does not help guarantee upper hemicontinuity when the exact equilibrium correspondence does not satisfy that property.

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[20] Dasgupta and Maskin (1982) have obtained similar results in a much more comprehensive study of the existence of mixed strategy equilibria for games with discontinuous payoffs. Fudenberg and Tirole (1985), in analyzing a game of timing in which the return from leading exceeds the return from following, have also demonstrated that equilibria in a game formulated in discrete time may have no analogue in the continuous time formulation. The reason is the same as in the War of Attrition. The limiting distributions contain an atom at a point of discontinuity in the payoffs (in their case it is at time 0 rather than at time T) and the discontinuity is not inessential, to use the terminolgy of Dasgupta and Maskin.

In the model analyzed here, an equilibrium strategy in continuous time is discontinuous only at 0 and T. Therefore, it is always possible to construct a discrete time strategy which is arbitrarily close to it in the sup norm topology for any Δ sufficiently small. Since the expected payoffs are continuous functions of the mixed strategies in this topology, it then follows that any equilibrium in continuous time is close to an ϵ-equilibrium in any sufficiently finely partitioned discrete time game. Formally, we may establish

Theorem 6.5: Suppose (F_a, F_b) is an equilibrium for G_0. Then for any $\epsilon > 0$, there is a $\bar{\Delta} > 0$ such that for any $\Delta \in (0, \bar{\Delta})$, there is an ϵ-equilibrium for G_Δ, (F_a^Δ, F_b^Δ), such that $\|F_\alpha^\Delta - F_\alpha\| < \epsilon$, $\alpha = a,b$.

A proof is supplied in the Appendix.

7. Discussion and Conclusion

One conclusion we draw from our analysis is that the equilibria of the continuous time and discrete time formulations of the War of Attrition are not always good approximations to each other. In some instances, there are nondegenerate equilibria in the discrete time formulation for which there is no counterpart in the continuous time formulation. For instance, if T is finite and $L_\alpha(T) > 0$ for each α, there is one nondegenerate equilibrium in the discrete time formulation but no nondegenerate equilibrium in the continuous time formulation. In other instances there are nondegenerate equilibria in the continuous time formulation which have no counterpart in the discrete time formulation. Specifically, if $\tau_a = \tau_b$ and $L_\alpha(T) \lesseqqgtr 0$ for each α, there is a continuum of nondegenerate equilibria in the continuous time game but generically only one (fully mixed) nondegenerate in the discrete time game. Corresponding to any continuous time equilibrium, there is always an "approximate" equilibrium in discrete time, however.

The question then arises, which formulation to choose? The choice between them clearly involves more than computational convenience. Thus, one needs to think carefully about the way in which the agents are likely to reason about their optimal responses in the two formulations.

One way of explaining why the continuous time formulation sometimes leads to a larger set of equilibria is to note that there is always a "last" time to move in discrete time which does not exist in continuous time. In a discrete time game, each player realizes that there is a period in which both players still have some incentive to move and after which both players plan to wait. The restrictions on the equilibrium strategies in this "last" period then eliminate all but the unique pair of fully mixed strategies. When time is continuous, however, there is no analogous restriction to eliminate any of the continuum of strategy combinations which are otherwise best responses.

If time is really discrete and agents explicitly take into account this "last" period effect, then some of the equilibria of the continuous time game are probably bad approximations to likely behavior of rational agents. On the other hand, if agents essentially ignore any discreteness in their calculations, then the discrete time formulation may eliminate some possible patterns of behavior. In this case, it may be more appropriate either to work in continuous time, or if the problem must be formulated in discrete time, to use the ϵ-equilibrium as the solution concept.

There are examples, however, where the agents might explicitly account for the discreteness of time in their calculations. Consider, for instance, a bargaining problem, organized in bouts, in which the issues are in some sense indivisible and each party hires an agent to "bargain" on its behalf. Another example is a problem where time is essentially continuous but each player observes the other player's move with a lag of Δ units of time. The presence of this observation lag implies that each player will plan to move only at times which are integer multiples of Δ.[21] Furthermore, the equilibria of this game coincide exactly with the equilibria of the corresponding discrete time game.

The "last" period effect can also explain why the discrete time formulation sometimes leads to a larger set of equilibria than the continuous time formulation. When time is discrete and the return from leading is always positive, the probability with which a player moves, conditional on reaching period $T-\Delta$, can always be adjusted to make the other player indifferent between moving and waiting to obtain a nondegenerate equilibrium. When time is continuous, however, there is no last instant prior to T at which the players can move. This means that the probability with which a player moves in an arbitrarily small interval before period T must be arbitrarily small. Consequently, neither player is willing to wait until period T, and, as a result, a nondegenerate equilibrium cannot exist.

In this case, the implications of the continuous time model are probably misleading. One can argue that either there is likely to be some uncertainty about the actual date of the terminal time T, or agents are aware that there is a last instant in which to move. If there is some uncertainty about the terminal date, then, by assuming this uncertainty is sufficiently "smooth", continuity of the payoffs can be restored and the one parameter

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[21] The observation lag implies that neither player can be a follower in the interval $[0,\Delta)$. Then since $L_\alpha(t)$ is decreasing, if player α plans to move in this interval, he must plan to move at time 0. Arguing by induction, it then follows that neither player will ever move at a time outside the set $\{0,\Delta,\ldots,T-\Delta\}$.

family of equilibria will appear. If agents are aware that there is a "last" instant to move, the discrete time formulation is probably more appropriate. In any case, we should be cautious about accepting the conclusions of a continuous time model which does not exhibit any nondegenerte equilibria in a symmetric game.

Appendix

Proof of Theorem 6.5:

Suppose (F_a, F_b) is an equilibrium for G_0. Let I be the nonnegative integers and let $\bar{n} = \sup\{n \in I: n < T/\Delta\}$. For any $\Delta > 0$, define F^{Δ}_{α} to be a right continuous Δ step function satisfying $F^{\Delta}_{\alpha}(\Delta n) = F_{\alpha}(\Delta n)$ for all integers $n \leqq \bar{n}$ and $F^{\Delta}_{\alpha}(t) = 1$ for all $t \geqq T$. Note first that

$$W_{\alpha}(0) = \sup\{P_{\alpha}(t,t'): t,t' \geqq 0\}$$

and, for all $n < \bar{n}$

$$\begin{aligned}
&\sup\{F_{\alpha}(\Delta n+s)-F^{\Delta}_{\alpha}(\Delta n+s): 0 \leqq s < \Delta\} = \sup\{F_{\alpha}(\Delta n+s)-F_{\alpha}(t): 0 \leqq s < \Delta\} \\
&= F_{\alpha}(\Delta n+\Delta)-F_{\alpha}(\Delta n) \leqq \Delta[1-F_{\alpha}(t)]\delta A/(B-A) \\
&= \Delta[\delta A/(B-A)][1-q_{\beta}(0)]\exp(-[\delta A/(B-A)]\Delta n) \\
&\leqq \Delta[\delta A/(B-A)]\exp(-[\delta A/(B-A)]\Delta n)
\end{aligned}$$

A similar argument establishes that

$$\sup\{F_{\alpha}(\Delta\bar{n}+s)-F^{\Delta}_{\alpha}(\Delta\bar{n}+s): 0 \leqq s < T-\bar{n}\Delta\} \leqq \Delta[\delta A/(B-A)]\exp(-[\delta A/(B-A)]\Delta\bar{n})$$

Therefore,

$$\int d[F_{\alpha}(t)-F^{\Delta}_{\alpha}(t)] = \sum_{n=0}^{\infty}\Delta[\delta A/(B-A)]\exp(-[\delta A/(B-A)]\Delta n) < 2\Delta[\delta A/(B-A)].$$

Choose $K > \max\{1, W_a(0), W_b(0)\}$ and choose $\bar{\Delta} > 0$ so that $2\bar{\Delta}\delta A/(B-A) < \epsilon/3K$. Then condition $\|F^{\Delta}_{\alpha}-F_{\alpha}\| < \epsilon$.

We will show that (F_a^Δ, F_b^Δ) is an ϵ-equilibrium for G_Δ. Let F_α^* be any G_Δ strategy for player α. Then for $\alpha = a,b$,

$$P_\alpha(F_\alpha^*, F_\beta^\Delta) - P_\alpha(F_\alpha^\Delta, F_\beta^\Delta) = P_\alpha(F_\alpha^*, F_\beta^\Delta) - P_\alpha(F_\alpha^*, F_\beta)$$

$$+ P_\alpha(F_\alpha^*, F_\beta) - P_\alpha(F_\alpha, F_\beta) + P_\alpha(F_\alpha, F_\beta) - P_\alpha(F_\alpha^\Delta, F_\beta)$$

$$+ P_\alpha(F_\alpha^\Delta, F_\beta) - P_\alpha(F_\alpha^\Delta, F_\beta^\Delta)$$

$$= \int P_\alpha(F_\alpha^*, t)\, d[F_\beta^\Delta(t) - F_\beta(t)] + \int P_\alpha(t, F_\beta)\, d[F_\alpha^*(t) - F_\alpha(t)]$$

$$+ \int P_\alpha(t, F_\beta)\, d[F_\alpha(t) - F_\alpha^\Delta(t)] + \int P_\alpha(F_\alpha^\Delta, t)\, d[F_\beta(t) - F_\beta^\Delta(t)]$$

$$\leqq \epsilon/3 + \epsilon/3 + 0 + \epsilon/3 = \epsilon.$$

Q.E.D.

Figure 1. The Discrete Time Game in Extensive Form

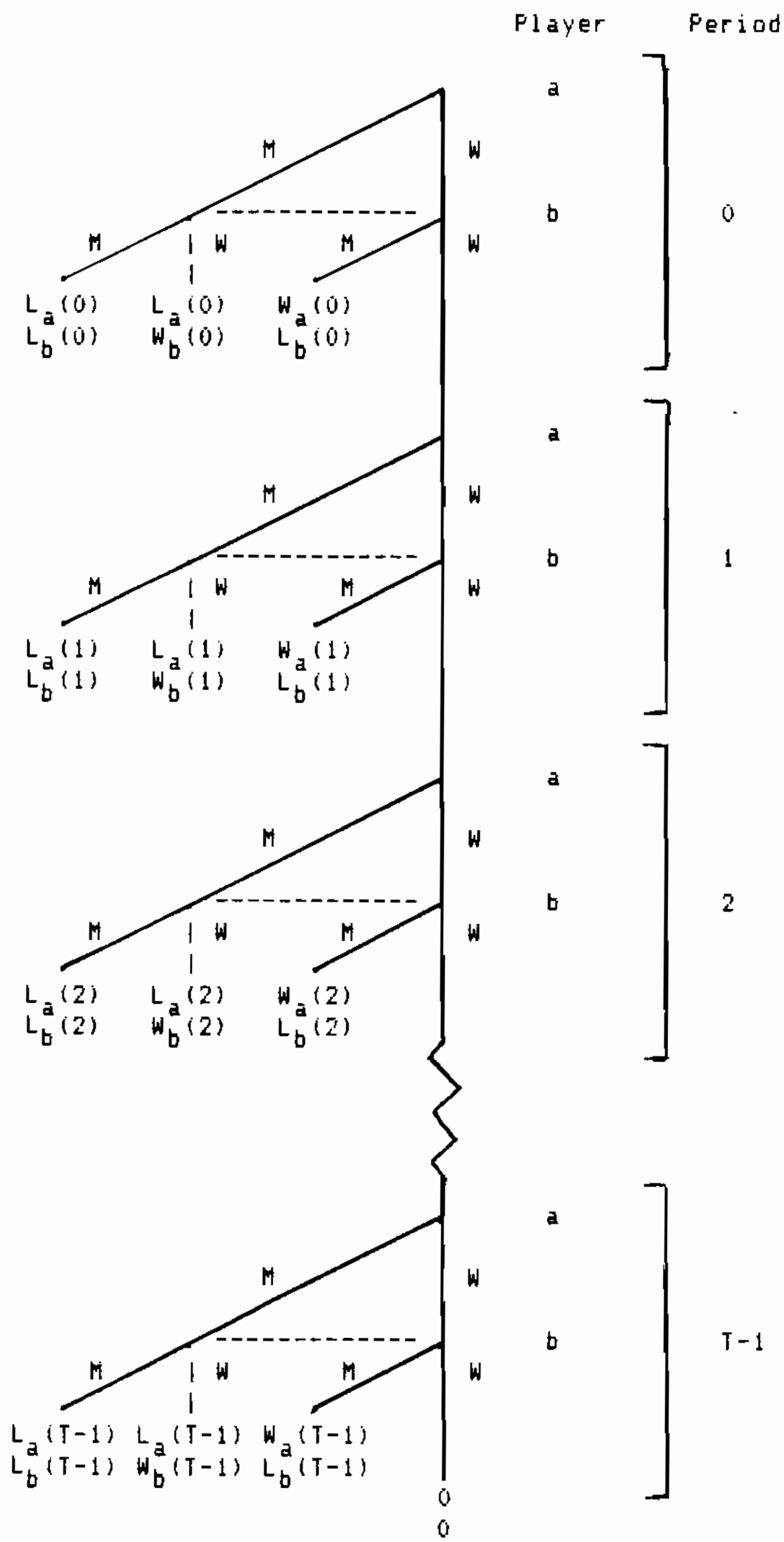

Figure 2. Possible Patterns For Nondegenerate Equilibria

Fully Mixed Equilibria

time	0	1	2	3	t^*-2	t^*-1	t^*
r_α	r_f	r_f	r_f	r_f	r_f	?	0
r_β	r_f	r_f	r_f	r_f	r_f	?	0

Alternating Equilibria

time	0	1	2	3	t^*-2	t^*-1	t^*
r_α	0	r_g	0	r_g	0	?	0
r_β	$r_\beta(0)$ [a]	0	r_g	0	r_g	?	0

Hybrid Equilibria

time	0	1	$\tilde{t}-1$	$\tilde{t}$	$\tilde{t}+1$ ...	t^*-2	t^*-1	t^*
r_α	r_f	r_f	r_f	0	r_g	0	?	0
r_β	r_f	r_f	$r_\beta(\tilde{t}-1)$ [b]	$r_\beta(\tilde{t})$ [c]	0	r_g	?	0

$r_f = A(1-\lambda)/(B-A\lambda)$

$r_g = A(1-\lambda^2)/[\lambda(B-\lambda A)]$

a $\quad r_\beta(0) \in [r_f, 1)$

b $\quad r_\beta(\tilde{t}) \in [r_f, r_g)$

c $\quad r_\beta(\tilde{t}-1) = [A(1-\lambda^2) - r_\beta(\tilde{t})\lambda(B-A\lambda)]/[(B-\lambda^2 A) - r_\beta(\tilde{t})\lambda(B-\lambda A)]$

References

Benoit, J.P. [1985], "Financially Constrained Entry in a Game with Incomplete Information," The Rand Journal of Economics, 15, pp. 490-499.

Bishop, D.T. and Cannings, C. [1978], "A Generalized War of Attrition," Journal of Theoretical Biology, 70, pp. 85-124.

Bliss, C. and Nalebuff, B. [1984], "Dragonslaying and Ballroom Dancing: The Private Supply of a Public Good," Journal of Public Economics, 25, pp. 1-12.

Cramton, P. [1984], "The Role of Time and Information in Bargaining," Research Paper No. 729, Stanford Graduate School of Business.

Dasgupta, P. and Maskin, E. [1982], "The Existence of Equilibrium in Discontinuous Games, 1: Theory," mimeo.

Fudenberg, D. and Levine, D. [1983], "Limit Games and Limit Equilibria" mimeo, (U.C. Berkeley)

Fudenberg, D. and Tirole, J. [1983], "A Theory of Exit in Oligopoly," IMSSS Working Paper No. ?.

Fudenberg, D. and Tirole, J. [1985], "Preemption and Rent Equalization in the Adoption of New Technology," Review of Economic Studies, 170, pp.383-401.

Hendricks, K. [1985], "Information Externalities and the Timing of Investments," mimeo.

Hendricks, K. and Wilson, C.A., [1985], "The War of Attrition in Discrete Time," C.V.Starr Center Working Paper No. 85-32, New York University.

Kreps, D. and Wilson, R. [1982], "Reputation and Imperfect Information," Journal of Economic Theory, 27, pp.253-279.

Loeve, Michel [1963], Probability Theory, Van Nostrand Reinhold Company, New York.

Maynard Smith, J. [1974], "The Theory of Games and the Evolution of Animal Conflicts," Journal of Theoretical Biology, 47, pp. 209-221.

Nalebuff, B. and Riley, J. [1984], "Asymmetric Equilibria in the War of Attrition," forthcoming Journal of Theoretical Biology, 1985.

Ordover, J.A. and Rubinstein, A. [1985], "A Sequential Concession Game With Asymmetric Information," mimeo, (New York University).

Osborne, M.J. [1983], "The Role of Risk Aversion in a Simple Bargaining Model," mimeo.

Riley, J. [1980], "Strong Evolutionary Equilibria and the War of Attrition," Journal of Theoretical Biology, 82, pp.383-400.

Selten, R. [1974], "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," International Journal of Game Theory, 4, pp. 25-55.

Weiss, A. and Wilson, C.A. [1984], "An Analysis of Games of Timing With Complete Information: An Application to Oil Exploration," mimeo.

Wilson, C.A. [1984], "Notes on Games of Timing with Incomplete Information," mimeo.