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by

James B. Ramsey
and Zhifeng Zhang

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C. V. STARR CENTER
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NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

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JAMES B. RAMSEY * AND ZHIFENG ZHANG †

Abstract. This paper uses waveform dictionaries to decompose the signals contained within three foreign exchange rates using tick-by-tick observations obtained world wide. The three exchange rates examined are the Japanese Yen and the German Deutsche Mark against the U.S. dollar and the Deutsche Mark against the Yen. The data were provided by Olsen Associates.

A waveform dictionary is a class of transforms that generalizes both windowed Fourier transforms and wavelets. Each wave form is parameterized by location, frequency, and scale. Such transforms can analyze signals that have highly localized structures in either time or frequency space as well as broad band structures; that is, waveforms can, in principle, detect everything from shocks represented by Dirac Delta functions, to "chirps", short bursts of energy within a narrow band of frequencies, to the presence of frequencies that occur sporadically, and finally to the presence of frequencies that hold over the entire observed period. Waveform dictionaries are most useful in analyzing data that are not stationary and non-stationarity up to second order is well recognized in the context of foreign exchange rates.

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1. Introduction. Daily foreign exchange rates have been the subject of intense academic analysis for quite sometime. Recently, in a series of innovative papers the daily analysis has been extended to intra- daily data on postings of bid and ask quotes on foreign exchange rates. All of this literature has involved three related lines of research. The first is the question as to what distribution, or distributions, are relevant in the analysis of these data. The second concerns the degree of temporal correlation, both for means and for second moments in the form of "ARCH" analyses. The third branch deals with the question of the degree of stationarity in the data.

There is an alternative approach to these data. The sequence of price reactions to variations in information and the financial needs of the participating agents might be regarded as a sequence of signals of various types and of varying duration. We might also presume that the signals are partially obscured by noise that is generated by liquidity trades, agent errors, and isolated random shocks to the foreign exchange system. A physical example will clarify the situation.

Consider, a sequence of signals generated with background noise by someone speaking. Each syllable involves a distinct set of frequencies that lasts for a short period of time. Some syllables are of very short duration, some are much longer. For some signals, the transition is very abrupt, for others the transition is very slow. We may presume that the amplitude of the signal varies unpredictably. The background may also include a few low frequency harmonics. Statistically, these data are highly non-stationary and there would be little evidence for any frequency to hold over the entire time sequence of points.

If such a conception of the mechanics of the foreign exchange market is applicable, then the use of wavelets and their generalization to waveform dictionaries will provide a most useful approach to the analysis of these data.

The first section of this paper briefly reviews only the literature most relevant to the objectives of this paper. This is because the literature has been well reviewed quite recently by others; for one of the latest see [10] (Guillaume et al 1994). The next section discusses the idea of dictionaries of waveforms and the link between regular Fourier analysis and the scaling properties of wavelets. The following section describes the data and indicates the extent to which the sub-sampling scheme chosen facilitates the analysis. The results of the waveform analysis are discussed in the next section and the paper is completed with a brief summary.

2. Review of the Literature. The most recent literature can be usefully reviewed in terms of the three groupings indicated above. More comprehensive reviews can be found in Guillaume et al cited above, [1] (Bollerslev et al 1990), and in [14] (Muller et al 1990). At the moment, there is very little agreement about the form of the foreign exchange data and the mechanisms that might generate them. This inconclusive result is due neither to a lack of interest, nor to a lack of imagination in attempting to analyze these data.

* Department of Economics, New York University, New York, NY. Internet: ramseyj@fasecon.econ.nyu.edu

† Department of Mathematics, Stanford University, Stanford CA 94305. Internet: zhangz@georgep.stanford.edu

The truth is that the generating mechanism for these data is likely to be very complex and the foreign exchange market is known to be evolving. Consequently, attempts to provide succinct parsimonious models of the data are hampered by the extent to which the simplifying assumptions needed by such an approach are violated.

The first question is what is the relevant distribution, or rather what are the relevant classes of distributions. There is still considerable dispute about this fundamental matter as is recorded in [14] (Muller et al 1990). For example, [3] (Boothe and Glassman 1987) using daily spot foreign exchange rates, examined some alternatives to the Gaussian distribution and concluded that the Student t distribution or a mixture of normals is the preferred choice and that there was evidence that the parameters involved vary over time. More recently, and in contrast, [12] (Loretan and Phillips 1994) used the properties of the tail behavior of the Pareto-Levy form to examine the covariance stationarity of observations on the daily Standard and Poor's 500 index and daily foreign exchange spot rates. Loretan and Phillips pursued this approach in order to facilitate the examination of covariance stationarity without assuming the existence of moments. Given the assumption of a Pareto-Levy tail distribution as the inclusive class of distributions, they concluded that the relevant sub-members of that class involved distributions with finite second moments, but not finite fourth moments.

Some interesting theoretical light is shed on the issue by [16] (Tauchen and Pitts 1983) who develop a simple model of the foreign exchange market in which both the number of participating agents and the number of informed agents entering the market at any instant play a role, thereby generalizing Clark's model; [4] (Clark 1973) and that of [9] (Epps and Epps 1976). The model while highly simplified provides considerable insight into the mechanism of such markets and argues most strongly for analyzing the data in terms of mixtures of distributions. If a model of evolutionary change in foreign exchange markets were to be added to the Tauchen and Pitts model, then the rational for examining mixtures of distributions would be considerably enhanced.

The second aspect of the foreign exchange data is the degree and extent of temporal correlation. As is well known there is little evidence for correlation in the first differences of the data. However, the data are clearly not random walks, nor are they "mean reverting" processes; by this is meant that the data generating mechanism is the sum of a unit root process and a stationary process. This result was persuasively presented in [11] (Lo and MacKinlay 1988). Further, as is also well known, there is a considerable literature on the ARCH structure and its generalizations for foreign exchange data as summarized for example in [1] (Bollerslev et al 1990). That article indicates that while various generalizations of the basic ARCH structure universally provide very good fits to the data, the ability of the ARCH models to forecast volatilities is limited.

The third issue involves the stationarity of the data. Indeed, a prior consideration in the analysis of conditional heteroskedasticity is the question of the stationarity of the data. This is a very important question in that the legitimacy of the ARCH modeling relies on covariance stationarity of the unconditional variances. [12] (Loretan and Phillips 1994) confirmed the work of [15] (Pagan and Schwert 1990a). Both have demonstrated that covariance stationarity for stock market and foreign exchange data over even quite short periods of time is implausible both at monthly intervals and at daily intervals of sampling. There is also a spatial dimension to the variation of foreign exchange data. [8] (Engle et al 1990) as well as others have demonstrated that information on foreign exchange rates diffuses westward over time.

Finally, a recent series of articles, most notably, [14] (Muller et al 1990); [6] (Dacorogna et al 1993); [10] (Guillaume et al 1994) together with several others have developed new variables for the analysis of foreign exchange data, examined the conditions for sub-sampling the data, and have provided a wealth of statistical information about tick-by-tick worldwide postings of bid and ask prices for the major foreign exchange rates. This work adds to that by [2] (Bollerslev and Domowitz 1993).

In the next section, we discuss a procedure that is designed to deal directly with non- stationarity. However, in this analysis the emphasis shifts from the properties of distributions over time to the analysis of signals of various types and durations that are contaminated by noise. The distinction is an important one, even though the existence of a sequence of complex signals with varying durations can be summarized by time varying distributions. For if the sequence of observations can be resolved into a sequence of, possibly overlapping, individual signal components, a significant advance would have been made in the ability to understand and interpret the data. The alternative view taken above is that the data can only be represented by probability distributions where it is hoped that one can find equations to explain the time variation of the moments of the distributions.

3. Review of Waveform Dictionaries for Analyzing Complex Time Series. Many signals are not stationary. Indeed, some signals, such as speech, weather, foreign exchange rates, and other financial data, are highly non-stationary and the non-stationarity may involve intermixing Dirac delta function impulses with discrete shifts in frequencies. Linear expansions in terms of a single basis, whether it is a Fourier, wavelet, or any other basis, are not flexible enough. The basis that might be suitable for one part of the data, may be most unsuitable for another part of the data. A Fourier basis provides a poor representation of signals that are tightly localized in time; a wavelet basis is not well adapted to represent functions that have Fourier representations that are highly localized in frequency space. Each of these examples of an expansion basis for representing signals works well for the types of signals for which they were designed. But if one suspects, either that the signal is highly non-stationary, or that a signal is a mixture of discrete and continuous changes, then more robust, but less specific, tools of analysis are needed. Flexible decompositions are important for representing signals that are characterized by localizations in time and frequency that vary widely over the whole signal. Impulses need to be decomposed over functions that are well concentrated in time; while spectral lines that are well localized in frequency are best represented by waveforms that have a narrow frequency support. Unfortunately, one cannot obtain high resolution in both time and frequency spaces at the same location in time-frequency space. Consequently, a local choice has to be made as to whether time or frequency localization will best represent the signal. This means that a flexible method is required.

The analytical approach used in this paper generalizes both windowed Fourier transforms and wavelets. The former approach enables one to estimate frequencies locally and to separate nearby frequencies. The latter approach concentrates on the effects of scaling of the data and enables one to estimate local time effects and to separate nearby impulses. Using wave form dictionaries, we will be able to handle both difficulties in one transformation.

3.1. Definition of Time-Frequency Atoms. A general family of time-frequency atoms can be generated by scaling, translating, and modulating a single window function . We suppose that the windowing function $g(t)$ is real and centered at 0. We also impose the conditions that, $||g_\gamma|| = 1$, where $||\cdot||$ is the L^2 norm, that the integral of $g(t)$ is non-zero, and that $g(0) \neq 0$. For any scale parameter $s > 0$, frequency modulation $\xi \in \mathbf{R}$, and translation $u \in \mathbf{R}$, we define the triplet $\gamma = (s, u, \xi)$ and define the "atom", $g_\gamma(t)$ by:

$$g_\gamma(t) = \frac{1}{\sqrt{s}} g(\frac{t-u}{s}) e^{i\xi t}. \tag{1}$$

The factor normalizes the norm of $g(t)$ to 1. The function $g(t)$ is centered at the abscissa u and its energy is concentrated in a neighborhood of u, whose size is proportional to s. Its Fourier transform is centered at the frequency ξ and has an energy concentrated in a neighborhood of ξ , whose size is proportional to $1/s$.

The dictionary of time-frequency atoms $\mathcal{D}$ is defined by $\mathcal{D} = \{g_\gamma(t) : \gamma \in \Gamma\}$, where Γ is some index set, for example, $\Gamma = \mathbf{R}^+ \times \mathbf{R}^2$. The dictionary so defined is a highly redundant set of functions that includes window Fourier frames and wavelet frames, [7] (Daubechies 1991); redundancy means that the set $\mathcal{D}$ spans the signal space but is is not a basis. When the signals include time-frequency structures of very different types, that is, when non-stationarity is an important and significant aspect of the signal, one can not choose a priori a single frame that is well adapted to perform the expansion for all the constituent structures. Rather, for any given signal, we need to find a sequence of atoms from the dictionary that best match the signal structures in order to obtain a compact decomposition. In the next section we study such adaptive decompositions from redundant dictionaries.

3.2. Matching Pursuit. Let $\mathcal{H}$ represent a signal space ($L^2(\mathbf{R})$ is a good example). We choose a dictionary $\mathcal{D} \subset \mathcal{H}$. When a dictionary is redundant a signal will not have a unique representation as a sum of vector elements, or atoms. Unlike the case of restricting attention to a basis for the space $\mathcal{H}$, we have some degrees of freedom in choosing a particular representation of the signal. This freedom allows us to choose a subset of the dictionary that is tailored to the signal in question and which will provide the most compact representation. We choose that subset of the dictionary for which the signal energy is concentrated in as few terms as possible. The chosen atoms highlight the dominant signal features as measured by the energy of the signal captured by the atoms.

An optimal approximation of $f \in \mathcal{H}$ is the expansion:

$$\tilde{f}(t) = \sum_{n=1}^{N} \alpha_n g_{\gamma_n}(t), \tag{2}$$

where N is the number of terms in the expansion for a given degree of approximation. α_n and $g_{\gamma_n} \in \mathcal{D}$ are chosen in order to minimize $||f - \tilde{f}||$, where $||\cdot||$ is the L^2 norm.

Due to the high computation complexity of finding an optimal solution, we utilize a "greedy algorithm", [13] (Mallat and Zhang 1993), that computes a useful sub-optimal approximation. Let $f \in \mathcal{H}$. We want to compute a linear expansion of over a set of vectors selected from $\mathcal{D}$ in such a way as to capture best the inner structure of the signal. This is accomplished by successive approximations of $f(t)$ through orthogonal projections of the signal onto elements of $\mathcal{D}$. Let $g_{\gamma_0} \in \mathcal{D}$. The vector f can be decomposed into

$$f = < f, g_{\gamma_0} > g_{\gamma_0} + Rf, \tag{3}$$

where $< f, g_{\gamma_0} >$ is the inner product, Rf is the residual vector after approximating f in the direction of g_{γ_0}. Clearly g_{γ_0} is orthogonal to Rf, therefore,

$$||f||^2 = | < f, g_{\gamma_0} > |^2 + ||Rf||^2. \tag{4}$$

To minimize $||Rf||$, we must choose $g_{\gamma_0} \in \mathcal{D}$ such that $| < f, g_{\gamma_0} > |$ is maximum. In some cases, it is only possible to find a vector g_{γ_0} that is almost the best in the sense that

$$| < f, g_{\gamma_0} > | \geq \alpha \sup_{\gamma \in \Gamma} | < f, g_\gamma > | \tag{5}$$

where α is an optimality factor that satisfies $0 < \alpha \leq 1$.

A matching pursuit is an iterative algorithm that at successive stages decomposes the residue from a prior projection by projecting that residue onto a vector of $\mathcal{D}$, as was done for f. This procedure is repeated for each residue that is obtained from a prior projection. Let $R^0 f = f$. Suppose that the n^{th} order residue $R^n f(t)$, for some $n \geq 0$ has been computed. At the next stage we choose an element $g_{\gamma_n} \in \mathcal{D}$ which approximates the residue $R^n f$:

$$| < R^n f, g_{\gamma_n} > | \geq \alpha \sup_{\gamma \in \Gamma} | < R^n f, g_\gamma > |. \tag{6}$$

The n^{th} order residue is decomposed into:

$$R^n f = < R^n f, g_{\gamma_n} > g_{\gamma_n} + R^{n+1} f, \tag{7}$$

which defines the residue to order $n+1$. Since $R^{n+1} f$ is orthogonal to g_{γ_n}, we have

$$||R^n f||^2 = | < R^n f, g_{\gamma_n} > |^2 + ||R^{n+1} f||^2. \tag{8}$$

We can rewrite f in terms of $R^n f$ by

$$f = \sum_{n=0}^{m-1} < R^n f, g_{\gamma_n} > g_{\gamma_n} + R^m f. \tag{9}$$

Equation (8) implies

$$||f||^2 = \sum_{n=0}^{m-1} | < R^n f, g_{\gamma_n} > |^2 + ||R^m f||^2. \tag{10}$$

One can prove that the matching pursuit algorithm does converge, see [13] (Mallat and Zhang (1993)).

THEOREM 3.1. *Let $f \in \mathcal{H}$. The residue defined by (7) satisfies*

$$\lim_{m \to +\infty} ||R^m f|| = 0. \tag{11}$$

Hence

$$f = \sum_{n=0}^{+\infty} < R^n f, g_{\gamma_n} > g_{\gamma_n}, \tag{12}$$

and

$$\|f\|^2 = \sum_{n=0}^{+\infty} | < R^n f, g_{\gamma_n} > |^2. \tag{13}$$

When $\mathcal{H}$ has finite dimension, $\|R^m f\|$ decays exponentially to zero.

3.3. Matching Pursuit With Time-Frequency Dictionaries. For dictionaries of time-frequency atoms, a matching pursuit yields an adaptive time- frequency transform. It decomposes any function in the signal space into a sum of complex time-frequency atoms that best match its residues. This section studies the properties of this particular matching pursuit decomposition. We derive a new type of time-frequency energy distribution by summing the Wigner distribution of each time-frequency atom.

Since a time-frequency atom dictionary defined in §3.1 is complete in $L^2(\mathbf{R})$, matching pursuit decomposes any function $f \in L^2(\mathbf{R})$ into

$$f = \sum_{n=0}^{+\infty} < R^n f, g_{\gamma_n} > g_{\gamma_n} \tag{14}$$

where $\gamma_n = (s_n, u_n, \xi_n)$ and

$$g_{\gamma_n} = \frac{1}{\sqrt{s}} g(\frac{t-u_n}{s_n}) e^{i\xi_n t}. \tag{15}$$

These atoms are chosen one by one to approximate the sequence of residues of f. For discretely observed signals of size N, we can discretize the index γ and facilitate the computations in the following manner. We redefine γ by:

$$\gamma = (2^j, p2^{j-1}, k\pi 2^{-j}) \tag{16}$$

i.e. we choose $s = 2^j$, $u = 2^{j-1}p, \xi = k\pi 2^{-j}$, and (j,p,k) are integers, $0 < j < \log_2 N$, $0 \leq p < N2^{-(j+1)}$, $0 \leq k < 2^{j+1}$. Using these definitions, equation (15) for the n^{th} atom becomes

$$g_{\gamma_n}(t) = \frac{1}{\sqrt{2^{j_n}}} g(\frac{t - 2^{j_n - 1}p_n}{2^{j_n}}) e^{ik_n \pi 2^{-j_n} t}. \tag{17}$$

From the decomposition (14), we derive a new time-frequency energy distribution, by adding the Wigner distribution for each selected atom. The cross Wigner distribution for two functions $f(t)$ and $h(t)$ is defined by:

$$W_{fh}(t,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t+\tau/2)\overline{h(t-\tau/2)} e^{-i\omega\tau} d\tau, \tag{18}$$

where the "bar" indicates the complex conjugate. The Wigner distribution of f is $W_f(t,\omega) = W_{ff}(t,\omega)$. Since the Wigner distribution is quadratic, we derive from the atomic decomposition (14) of $f(t)$ that

$$\begin{aligned} W_f(t,\omega) &= \sum_{n=0}^{+\infty} | < R^n f, g_{\gamma_n} > |^2 W_{g_{\gamma_n}}(t,\omega) \\ &+ \sum_{n=0}^{+\infty} \sum_{m=0, m\neq n}^{+\infty} < R^n f, g_{\gamma_n} > \overline{< R^m f, g_{\gamma_m} >} W_{g_{\gamma_n} g_{\gamma_m}}(t,\omega). \end{aligned}$$

The double sum corresponds to the cross terms of the Wigner distribution. The cross terms provide missleading information on the time-frequency localizations of the atoms selected in (14). We thus keep only the first sum and define

$$Ef(t,\omega) = \sum_{n=0}^{+\infty} |< R^n f, g_{\gamma_n} >|^2 W_{g_{\gamma_n}}(t,\omega). \tag{19}$$

From the well known dilation and translation properties of the Wigner distribution and the expression (15) of a time-frequency atom, we derive for $\gamma = (s, u, \xi)$

$$W_{g_\gamma}(t,\omega) = W_g\left(\frac{t-u}{s}, s(\omega-\xi)\right), \tag{20}$$

and hence

$$Ef(t,\omega) = \sum_{n=0}^{+\infty} |< R^n f, g_{\gamma_n} >|^2 W_g\left(\frac{t-u_n}{s_n}, s_n(\omega-\xi_n)\right). \tag{21}$$

The Wigner distribution also satisfies:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_g(t,\omega)dt\ d\omega = ||g||^2 = 1, \tag{22}$$

so that the energy conservation equation (8) implies:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Ef(t,\omega)dt\ d\omega = ||f||^2. \tag{23}$$

We can interpret $Ef(t,\omega)$ as an energy density of $f(t)$ in the time-frequency plane (t,ω). Unlike the Cohen class distributions, it does not include cross product terms. It also remains positive if $W_g(t,\omega)$ is positive, which is the case when g is Gaussian. On the other hand, the energy density does not satisfy marginal properties, as opposed to certain Cohen class distributions, [5] (Cohen 1989). The importance of these marginal properties for signal processing is however not clear. If g is the Gaussian window then

$$g(t) = 2^{1/4}e^{-\pi t^2} \tag{24}$$

then

$$W_g(t,\omega) = 2e^{-2\pi(t^2+(\omega/2\pi)^2)}. \tag{25}$$

The time-frequency atoms $g_\gamma(t)$ are then called Gabor functions. The time-frequency energy distribution $Ef(t,\omega)$ is a sum of Gaussian pulses whose locations and variances along the time and frequency axes depend upon the parameters (s_n, u_n, ξ_n).

4. The Data Used in This Study. The raw tick-by-tick data on the three foreign exchange rates, U.S. dollar Deutsche Mark rate, the U.S. dollar Japanese Yen rate, and the Yen Deutsche Mark rate for the period October 1 1992 to September 30 1993 were obtained from Olsen and Associates. The official designation of the data set is "HFDF93". Included on the tape are the bid and ask quotes as recorded by the various agencies that include both the date and the time in terms of Greenwich Mean Time, (G.M.T.), so that all bids are recorded with reference to a single time frame. Each foreign exchange time series also contains codes to identify both the posting bank and the country and city from which the posting originated. There is also a code for designating gross recording errors in the data. The foreign exchange data are supplemented by data on each country's three month bill rate that is also recorded on a tick-by-tick basis. Finally, the data tape includes quotations from news services listing key events that might affect the foreign exchange markets.

The data were originally obtained from the various financial recording agencies as discussed in [10] (Guillaume et al 1994). The data represent the postings by market makers, usually banks, of their bid and ask positions at the recorded time. No data on actual transactions prices are recorded, nor are data on quantities traded.

Because the raw series are far too numerous for the processing capabilities available to most researchers and because the transient reactions to news are far too complicated to analyze as an initial step, almost all researchers use some form of sub-sampling. The transient period to news shocks has been estimated to be about ten minutes, [10] (Guillaume et al 1994), so that most scholars avoid examining the data at sampling rates that are faster than about ten to fifteen minutes in duration. [10] (Guillaume et al 1994); [14] (Muller et al 1990); [6] (Dacorogna et al 1993) discuss in some detail the problems and the suggested methods for handling these difficulties that have been tried by various researchers.

The main difficulty stems from the fact that during any twenty four hour period there is a varying number of exchanges open for business around the world. Even within a given business day there are variations in the numbers of people actively engaged in trading; for example, Asian market lunch breaks. Further, over any week there is a weekly variation in activity, partly due to the flow of information and partly due to weekends that are almost universally observed. In addition there are a variety of holidays that shut down regional exchanges at various times throughout the year. Consequently, over any year there is a considerable variation in activity levels of individual market makers in any one regional market and in the number of regional exchanges that are active at any instant. There are daily, weekly, and annual periodicities in the frequency of postings and presumably in the frequency of trades. Further, there are definite time delays between the various markets, so that the flow of information is asymmetrically affected in a west to east direction as opposed to an east to west direction.

One way to view the data set is that within the induced periodicities alluded to above the arrival of bid and ask postings represents possible Poisson arrivals of new information and that the intervals between the arrival of postings represents a local steady state. In this context, "information" is treated in a broad context so that the appearance of a posting by a market maker who is testing the market, or who is trying to steer other traders in a preferred direction, see for example, [10] (Guillaume et al 1994) represents in this context "arrival of new information". Over very short intervals of time, below about fifteen minutes, there is some evidence that transient reactions to news are most important and that "errors in the interpretation of news" will be most prevalent.

One of the difficulties with sampling such data at a fixed chronological time interval is that periods of high information flow are under sampled and that periods of low informational flow are over sampled, or sometimes are treated in a highly nonlinear fashion by either ignoring these data altogether, or folding them into the contiguous week days. One procedure designed to compensate for some of these difficulties has been termed the "θ method", see [6] (Dacorogna et al 1993). However, there is still a question whether one wishes to include in the effects to be examined the periodic patterns that are induced by purely institutional factors. Such institutional factors in the data may well mask, or at least distort, the appearance of factors that are more germane to the flow of information and traders' reactions to that flow of information.

The sub-sampling approach taken in this paper is to define an implicit "informational time scale" so that many of the difficulties mentioned above and the appearance of periodicities due solely to institutional factors can be mitigated. The choice was to observe the data at a fixed interval in terms of the sequence of posting arrivals; that is, we sampled over intervals K observations in length. K was chosen so that the average working day sampling rate in terms of chronological time would exceed the presumed lower bound of fifteen minutes, but would allow for a few samples during the weekends. One of the benefits of sampling in this way is that the sampling rate is proportional to the rate of arrival of information; the faster that the market is oscillating the higher the chronological sampling rate. Consequently, all comments in the sequel in terms of "frequencies" are formally in terms of the time varying rate of arrival of information, that is, in terms of the "flow" of postings.

Another question that needs to be addressed is to define the variable that is to be observed as "price". One suggestion is to pick either the bid or ask prices. We follow [10] (Guillaume et al 1994); [14] (Muller et al 1990) by using the "middle price" which is the mean of the logarithms of the observed bids and asks posted. One of the advantages of this approach is that the effect of the fluctuations in the bid ask spread, especially over thin markets, is mitigated. This approach is in contrast to [2] (Bollerslev and Domowitz 1993) who attempted to overcome the lack of trading prices by using a model to estimate the unobserved trading prices.

A final question is to choose an appropriate value to represent a given sampling interval, where each observation is the "middle price" as defined above. A standard choice in the past has been to use the last price to represent the interval; an alternative is to use the mean of the interval. Because some of the data have gross recording errors in addition to an unknown degree of small scale errors and because there is a high percentage of transients within the intervals, we chose to represent each interval that is K observations in length by its median. Thus, the "middle prices" are divided into contiguous segments, each containing K observations.Each segment is represented by the median over that segment. This representation potentially induces greater stability in the distribution of the "prices" in that low probability drawings from distributions of large scale price shocks will leave the distribution of the medians invariant.

Because the data are sampled in a somewhat different manner than has been used so far in the literature, our description of the data includes a variety of simple descriptive statistics that will allow the reader to confirm the extent to which the elementary properties of the data have been preserved by the sub-sampling procedure and the extent to which these data are more suitable for the waveform analysis to be discussed in the next section.

Figures 1 to 3 show the time series plots of the middle price data, that is, the variable in question is the average logarithm (to base 2) of the bid and ask prices.

Figures 4 and 5 show the spectra for the first differenced exchange rates for the Deutsche Mark-U.S. dollar and for the absolute values of those differences. With respect to the differences, the spectrum is very flat as was to be expected. The spectrum for the absolute values of the first differences demonstrates that there is some low frequency power in the spectrum and that there is very slight evidence of an oscillation in the spectrum over the remaining frequency bands. These results are very similar to those obtained for the other two exchange rates. We concentrated on examining the absolute values of differences, rather than the variances, following the results of [10] (Guillaume et al 1994) and [14] (Muller et al 1990).

Figures 6 and 7 show the autocorrelation function plots for the differenced data for the Deutsche Mark-U.S. dollar exchange rate and the corresponding plots for the absolute values of the first differences of the exchange rate. The plots for the other two exchange rates are almost identical. One notices as was to be expected that the differences have a single significant auto-correlation, the first with an observed value of about 0.18; this result is consistent with the data being characterized by an MA(1) structure at the sampling rate. This result is also true for the other two exchange rates. The auto-correlation plots for the absolute values of the first differences are interesting in that the higher degrees of correlation in the absolute values of differenced exchange rates previously observed by other researchers carries over to our sub-sampled data. Consequently, with respect to the particular sub-sampling scheme chosen, our results confirm the result that there is much more persistence in the magnitudes of the data than there is in 'direction' of the data. Indeed, if one were to calculate the auto-correlation function of the signs of the differenced data, one obtains the same result as shown in Figure 6 for the differenced data. Only for the Deutsche Mark-Yen exchange rate are these results modified in that while the qualitative result holds the degree of correlation for the absolute values for the differences is much less and the statistically significant lags do not extend as far.

These results seem to be invariant to sub-sampling sections of the data characterized either by an overall upward drift in the levels, or by an overall downward drift in the levels. Because the auto-correlations are invariant to overall drift in the levels presumably they can provide no information about changes in the drift in the levels of the exchange rates.

With all financial data, there is always the suspicion that the data are non-stationary over even fairly short intervals and, in so far as varying information arrivals affect the outcomes, the data are probably non-stationary over quite short intervals. Further, the higher the moments that are needed to describe the data, even if only implicitly, the more likely that evidence of non-stationarity will be observed. In this connection, we examined the median and the first four moments for various subsets of the data. The higher moments were standardized so that they are invariant to changes in origin and scale; the fourth standardized moment takes the value three for the Gaussian distribution.

We examined a series of sequential subsets of the data, as well as picking out segments over which the levels were drifting up and those over which the levels were drifting down. The following generalizations can be made with respect to all the data sets examined, but most especially with respect to the Deutsche Mark-U.S. dollar exchange rate. In all cases, the data examined are the first differences for the middle prices for the exchange rates. The general lack of variability of the sample moments over a wide variety of sub-samples together with the reasonableness of the values obtained, lends some plausibility to the results presented. However, these results are only preliminary and exploratory. They do indicate that a more thorough examination and exploration of the inferences drawn in this preliminary work would be well rewarded.

The median is in almost all cases zero, or occasionally very close to it. The sign of the mean tends to track the cases where the levels are drifting up or down, that is, the mean is positive when the levels are drifting upwards, and negative when the price levels are drifting down. But, the magnitude of the mean is about a twentieth of one standard deviation and often less. In short, variations in the mean are swamped by the size of the ambient variation in the data. The unconditional variance does not vary substantially from sub-sample to sub-sample, nor does there seem to be any systematic difference between periods of upward drifting in the levels and those with a downward drift.

We also examined the standardized fourth moment of various subsets of the data, notwithstanding the worrisome results of [12] (Loretan and Phillips 1994). We did this because we are at this stage engaged in exploratory analysis and because the large number of estimates obtained over different data sets and over a wide array of sub-samples exhibited a remarkable degree of stability in the estimates relative to the Gaussian asymptotic standard errors. The standardized fourth moment clearly indicates leptokurtosis with values about 9.0 to 10.0 for the Deutsche Mark-U.S. dollar rate and a value of about 6.5 for the Japanese Yen-U.S. dollar rate. There does not seem to be any systematic relationship between the size of the kurtosis and the variation in the upward or downward drift in the levels.

What is most interesting is that the standardized third moment is the one with the greatest degree of variation. While there are exceptions, the general rule is that the third moment is negative during downturns in the levels and is positive during upturns. While the results that we have obtained seem to be highly significant, the fact that skewness varies over time indicates why average calculations of skewness in previous work have found conflicting and weak evidence for skewness, see for example, [3] (Boothe and Glassman 1987).

A model that is initially implied by these results is that of time varying probability functions with the following characteristics. The time varying distributions are 'centered' about zero in that the median difference is zero and are characterized by a fourth moment that may differ between exchange rates, but seems to be reasonably stable over time. The main characteristic of the distributions that are changing over time is the third moment; the distributions are skewed to negative values in downturns and skewed positively during upturns in the levels. These properties are in contrast to those of the Standard and Poor's 500 index for example. For the Standard and Poor's 500 index, the time variation of the probability distribution functions is much more in accord with conventional thought, in that variations in the drift in the levels of the index are reflected in the changes in the mean and in the median values. There is some change in the third moments between upward drifting levels and downward drifting levels, but the case is nowhere near as clear as it is for the exchange data. We believe that these differences in statistical properties are the effect of the disparities between the operation and institutional structures of these two markets.

The key behavioral implication is that the drift in the levels of the market prices is due to the cumulative impact of low probability, but large scale, price deviations. Further, the ambient state of trading is characterized, even during quite pronounced drifts in the levels, by an equal percentage of upticks and downticks. Otherwise the time varying distributions of price deviations seem to be remarkably constant. Consequently, attention should be focused on the tails of the distribution in order to analyze the time path of the change in prices, rather than examining the mean variation of the data within the bulk of the distribution.

With respect to all the other properties of the data in so far as we have examined them, we have confirmed the general conclusions about the distributional properties of the data that were reached by other scholars in the field with one potential exception. The exception involves the existence of the fourth moment. The [12] (Loretan and Phillips 1994) result suggests that fourth moments are not defined, whereas our results indicate that the existence of finite fourth moments is plausible and worth testing more carefully. The difference is presumably due to our method of sub-sampling the data. We also expect that the various periodicities that are generated by the institutional structure of this market will not be so apparent in our data. This is to us an advantage in that we would like to isolate the effects of the market itself from the incidental effects due to institutional characteristics.

Pursuing these ideas more formally is the subject of a subsequent paper. Nevertheless, the recognition that the data may well be characterized by time varying distributions raises the question as to the extent to which these variations can be captured in terms of generalized wave transforms. One question is whether there are any low frequency components contained in the levels of the prices that may wax and wane in effect.

A second question is the extent to which the differences are generated by shocks that are best described by Dirac delta functions, or by short bursts of high frequency oscillations. And lastly, a major question is whether examining the magnitudes of the price variations will lead to greater insight into the foreign exchange market.

5. The Empirical Results for the Waveform Analysis. Figures 8 through 19 provide a graphical summary of the empirical results obtained from our analysis of the foreign exchange data. We have only presented those graphs that yield qualitatively different information; that is, those graphs that are not presented are in all cases qualitatively similar to those that are.

Figures 8 to 10 illustrate for the Deutsche Mark-U.S. dollar exchange rate the rate of decay in the relative importance of the selected atoms in the representation of the data. A complex, or a noisy, signal declines very slowly as most atoms are of approximately equal weight; whereas a signal with a few well defined structures would decline exponentially fast. For the Deutsche Mark-U.S. dollar data, the slowest rate of decay is for the raw, undifferenced, data and the fastest is for the absolute value of the differences; the atoms for the actual differences decay at an intermediate rate. The undifferenced data have the most structures in them and the data are complex. As expected, differencing reduces the observable number of structures leaving, in all cases, little more than a sequence of Dirac delta structures plus noise. The shift to the absolute values of the differences reintroduces a certain amount of structure, as will shortly be seen; this leads to the result that the absolute value of the differences has the fastest rate of decay in the relative importance of the atoms. The results for the other two frequencies are very similar.

Figures 11 through 17 summarize the time-frequency distributions obtained for the Deutsche Mark-U.S. dollar and the Japanese Yen-U.S. dollar exchange rates. Consider Figures 11 and 12 that represent approximately the two halves over the time domain of the undifferenced Deutsche Mark-U.S. dollar exchange rate. Only the first two hundred atoms are shown; this corresponds to approximately eighty percent of the total variation of the data. There are several interesting points to be made. Almost all of the energy in these data occur at frequencies less than about 0.125 cycles per sub-sampled observation. At the very lowest frequencies, there is evidence of power at the frequencies around 0.015 and 0.045 and these frequencies hold throughout the entire historical period. A careful examination of the plots indicates for all data sets that, as anticipated in the previous discussion, many frequencies "come and go"; that is, they apply for a while, die out in relative strength, and then return. The degree of variation over time is somewhat exaggerated in that comparisons of the relative strengths of atoms are always being made locally, so that a frequency may still be potentially relevant over some time interval, but if its relative importance has waned it may not be represented in the list of the most important atoms in the local interval.

What is also of interest is that a lot of the total variation in the signal is represented by a series of bursts of energy over a narrow frequency range, such objects are known as "chirps". Chirps represent intense activity that is local in time and in frequency. For these data the chirps are located over a very low frequency range. The potential implications for understanding behavior are that even over a low frequency range, price activity can be characterized by intermittent bursts of activity involving a relatively narrow range of frequencies, separated by relative quiescent periods; that is, the market moves more often in irregular short sharp oscillations, rather than in discrete jumps, and certainly does not move smoothly. The potential image is that of the arrival of small packets of information that induce brief periods of minor adjustment. A careful examination of the pattern of chirps in Figures 11 and 12 suggest the possibility of a slow oscillation in the intensity of activity over the whole time period of observation. A similar suggestion occurs for the Japanese Yen-U.S. dollar exchange rate data, see Figures 16 and 17.

This "chirp" activity is interspersed by the occasional presence of Dirac delta functions that produce a jump in price; these are indicated in the figures by the presence of heavy vertical lines over the whole frequency range. Light vertical lines that cover only a portion of the frequency range are most likely due to unfiltered noise. The "non-differenced" data do not reveal any strong or pervasive evidence for the presence of Dirac delta functions.

Figure 13 shows for the first half of the data the top one hundred atoms for the first differences of the Deutsche Mark- U.S. dollar exchange rate, see also Figure 9. As is well known the first difference regarded as a filter is a very strong low frequency stop filter in that its transfer function is very flat for a considerable range and that the transition band is broad. Consequently, one would expect to see almost all of the low frequency activity exhibited in the previous two figures to be eliminated. Such is the case. In Figure 13 the only structures left are Dirac delta functions. What is interesting here is that the Dirac delta functions are very uneven in distribution over time; that is, the Diracs are sometimes bunched together very tightly, sometimes widely separated, and they vary widely in intensity. One suspects that if Poisson arrival times were to be calculated, the analysis would most likely have to allow for the Poisson parameter to vary over time. There is the suspicion in terms of the differenced data that the major effects of the Dirac delta functions occur when they arrive in quick succession.

Figures 14 and 15 show the results of applying the waveform dictionary approach to the absolute values of the differences. The effect relative to the signed differences is to reintroduce some frequency structure into the time-frequency plots and to enhance the discrimination between the relative weights assigned to the Dirac delta functions. In addition, the few low frequencies that are reintroduced continue throughout most of the period of observation.

Finally, in order to provide a set of comparisons, Figures 16 and 17 show the results for the Japanese Yen-U.S. dollar exchange rate data. As is clear from the two figures, the general appearance is the same as 11 and 12, although the two sets of figures, Figures 11 and 12 versus Figures 16 and 17, are clearly different. There is a suggestion that the intensity of the chirps in the two sets of graphs might be correlated; but to resolve this issue requires analysis that cannot be completed for this paper.

All of the general remarks that were made with respect to the Deutsche Mark-U.S. dollar exchange rates carry over to these data and indeed to the Japanese Yen-Deutsche Mark exchange rate data as well.

Finally, Figures 18 and 19 indicate for two contiguous sections of the Deutsche Mark-U.S. dollar exchange rate the degree of fit between the raw data and the reconstructed signal using only the first 100 atoms. Figure 18 covers the first one thousand observations and Figure 19 the second thousand. From the figures it is clear that most of the intermediate term variation in the data has been captured, although there is a noticeable amount of very high frequency variation that has not been captured. These graphs do indicate even with a relatively few structures that the variation in the historical record can be tracked. However, this "success" does not imply that the data are in any way predictable.The structures that involve frequencies are predictable, in so far as the observed structure continues to be relevant to the data. However, most of the energy of the system is in the chirps and in the Dirac delta functions. Only in so far as these structures exhibit some periodicity in their occurrence will fitting the data using such structures provide forecasts of future values of the exchange rate. We observed little evidence of such periodicity in the chirps and in the Dirac delta functions.

6. Summary and Conclusions. In this paper we have examined the insights provided by the analysis of foreign exchange rates using waveform dictionaries. The specific data that we used were sampled from the tick-by-tick observations on the "middle prices" that have been collected by Olsen and Associates; the raw data were for the year October 1992 to September 1993. The three exchange rates examined were the Deutsche Mark-U.S. dollar, the Japanese Yen-U.S. dollar, and the Japanese Yen-Deutsche Mark rates. The discussion below is almost entirely restricted to the Deutsche Mark-U.S. dollar rate; this is because the results for the three rates are very similar so that the general conclusions drawn from the Deutsche Mark-U.S. dollar rate can be applied with equal force to the other rates.

We sub-sampled by picking the median "price" from contiguous intervals, K in length. By these means we were able to sample at a rate proportional to the arrival of information into the market and were able to avoid the periodicities that are induced by the institutional structure of the international market for exchange rates in the major world currencies.

The main reason for using waveform dictionaries is that the approach allows one to deal directly with the problem of non-stationarity. Further, the approach is motivated by the idea that the signal to be extracted from the data can be represented in terms of frequencies that may hold over the entire length of the data series, by instantaneous shocks to the system, and by short run bursts, or packets, of energy over a narrow range of contiguous frequencies, otherwise known as chirps.This approach is in contrast to the more conventional idea that the data are represented by a sequence of time varying probability distributions, the time varying parameters of which may be related to the internal dynamics of the system and to economic events exogenous to the foreign exchange market.

The preliminary analysis of the data indicated that the sampling scheme chosen for this study did not invalidate any of the conventional properties that have already been discovered in the data. However, we did lessen the periodicity due to the institutional structure of the market. Further, the data are best characterized in terms of probability distributions in the following manner. Over long run periods of time, the distribution of first differences remains centered at zero, the unconditional variance does not exhibit excessive variation, nor does the standardized fourth moment. The standardized fourth moment is about nine for the Deutsche Mark-U.S. dollar exchange rate, for the Japanese Yen-U.S. dollar rate the value is about six; both these numbers are relative to a Gaussian value of three. What does change and what seems to characterize these data, is the variation in the third moment. During those periods when the levels of the exchange rates are drifting down on average, the median remains centered at zero, the mean changes very little as measured by its size relative to the standard deviation of the data, the fourth moment remains constant, but the third moment is negative. When the raw data are drifting upwards, the only change from the previous case is that the third moment is now positive.

The waveform analysis of the raw, undifferenced data, indicates that there is structure only at the very lowest frequencies. There are a few very low frequencies that seem to be robust to sub-sampling and are sustained over the entire period of observation. There are a few Dirac delta functions observable, but there seems to be nothing of importance. Most of the energy of the system seems to be in the chirps. There is some weak evidence that the distribution of chirps over the frequency range seems to oscillate over the year.

Converting the data to first differences removes all the low frequency structure that was observed with the "levels" data. With these data, there is little evidence of any activity except for Dirac delta functions. These occur sporadically and in blocks; the latter would seem to designate the periods of most intense activity in the market. Transforming the data into the absolute values of the differences re-introduces a few very low frequency oscillations that last over most of the period of observation; but these frequencies do not provide much power to the total signal.

Using only the first hundred structures provides a remarkably good fit to the data at all but the highest frequencies. However, the good fit, while it indicates that the data, though noisy and complicated in appearance, can be approximated by a relatively few structures, does not provide much evidence for an improved forecasting potential. This is because the isolated structures are, except for the very lowest frequencies, themselves unpredictable; at least at this time.

This research has opened up a number of promising lines of enquiry. One that arises from the initial examination of the data is to explore the implications of the apparent fact that the distribution of the first differences of the bid and ask prices reflects non-stationarity most prominently in terms of the third moment. The intriguing implication is that the activity that moves the price level is to be found in the tails of the distribution, but neither in the mean nor in variations in the inter-quartile range.

Another line of enquiry is to relate the "grouping" of Dirac delta functions and the distribution of chirps to known factors affecting the market.The data supplied by Olsen and Associates facilitates such an investigation. Again the significance of the result is the light shed on the dynamic process of market variation. A related aspect is the concept that the arrival of information and the markets reaction to that information is better expressed in terms of "packets of information and reaction". The relevant model is not one of smooth adjustment, nor one of instantaneous jumps, but of oscillations induced by packets of information that leave the median of changes invariant at zero.The "packets" are characterized in time frequency space by short bursts of activity over narrow frequency ranges.

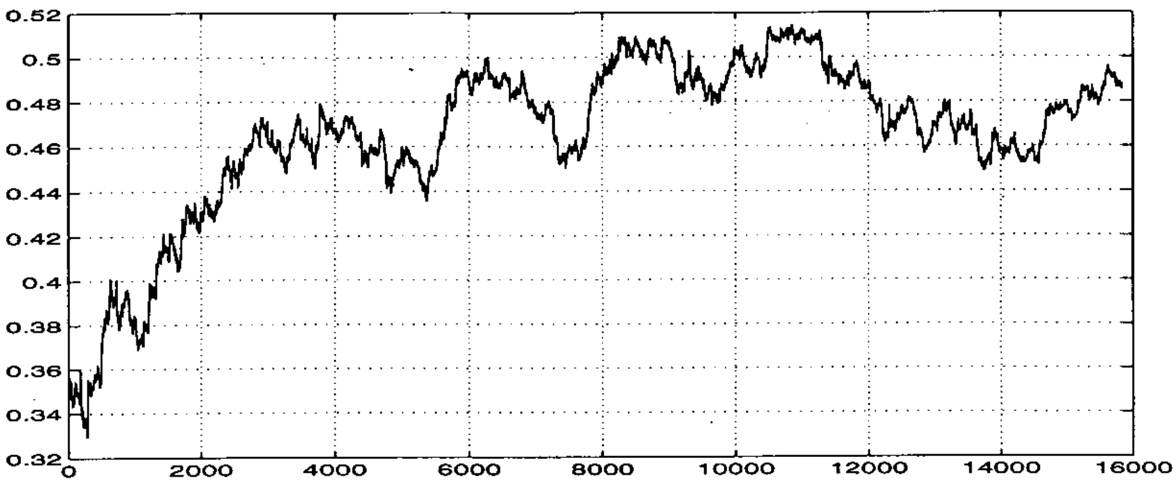

FIG. 1. The raw middle price data on the Deutsche Mark U.S dollar foreign exchange rate.

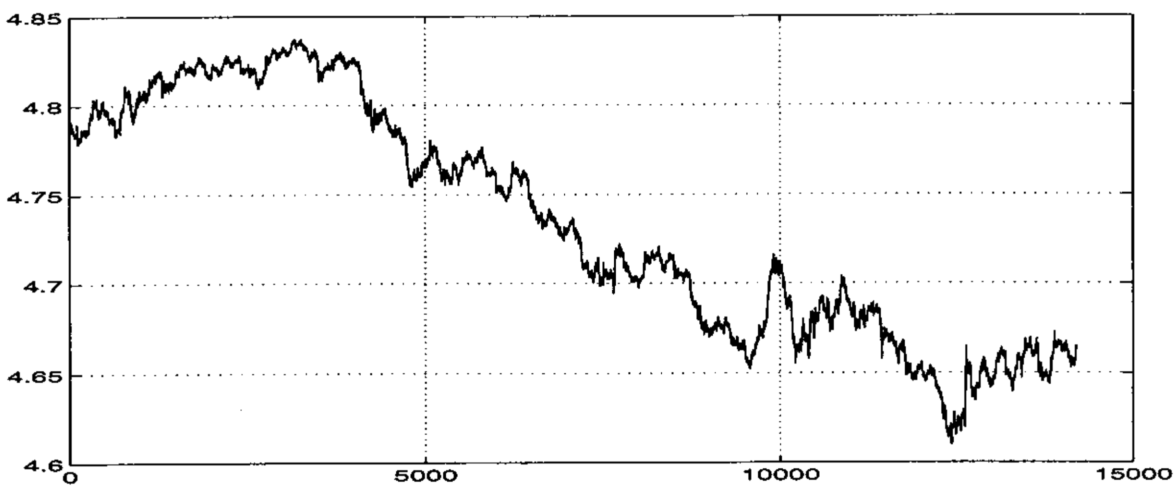

FIG. 2. The raw middle price data on the Japanese Yen U.S dollar foreign exchange rate.

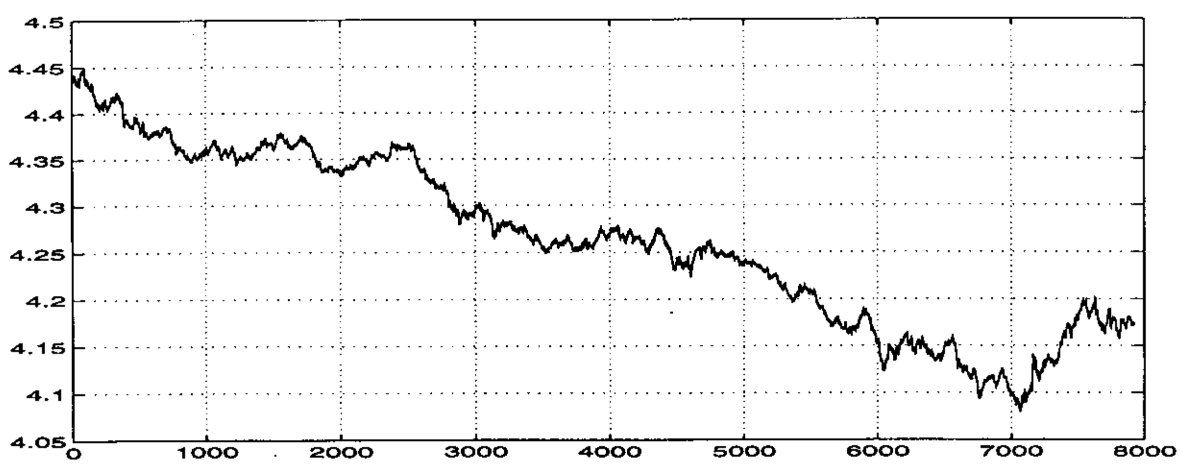

FIG. 3. The raw middle price data on the Japanese Yen Deutsche Mark foreign exchange rate.

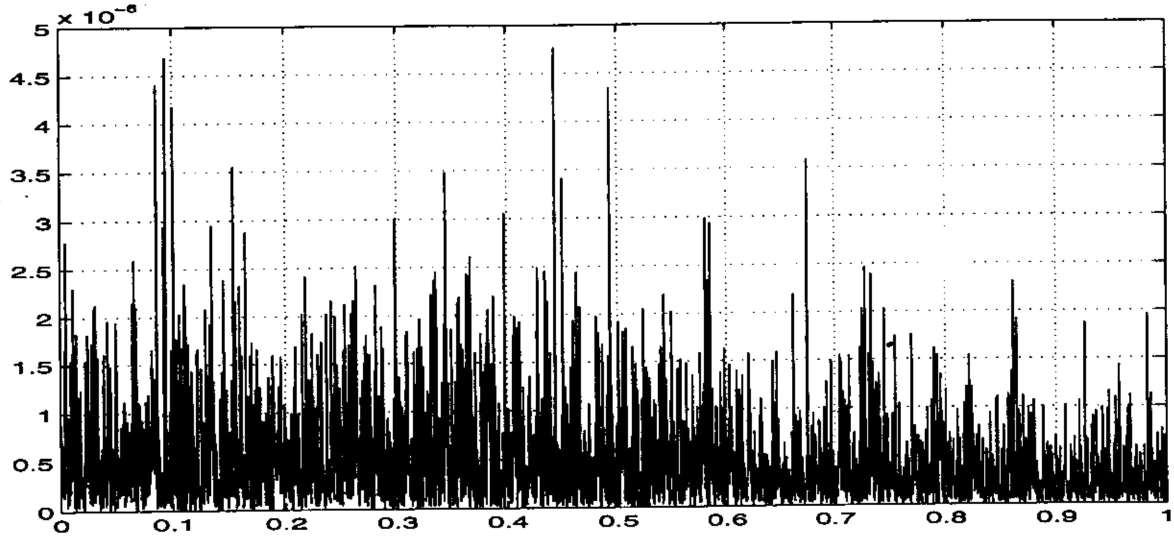

FIG. 4. The power spectral density of the difference data on the Deutsche Mark U.S dollar foreign exchange rate.

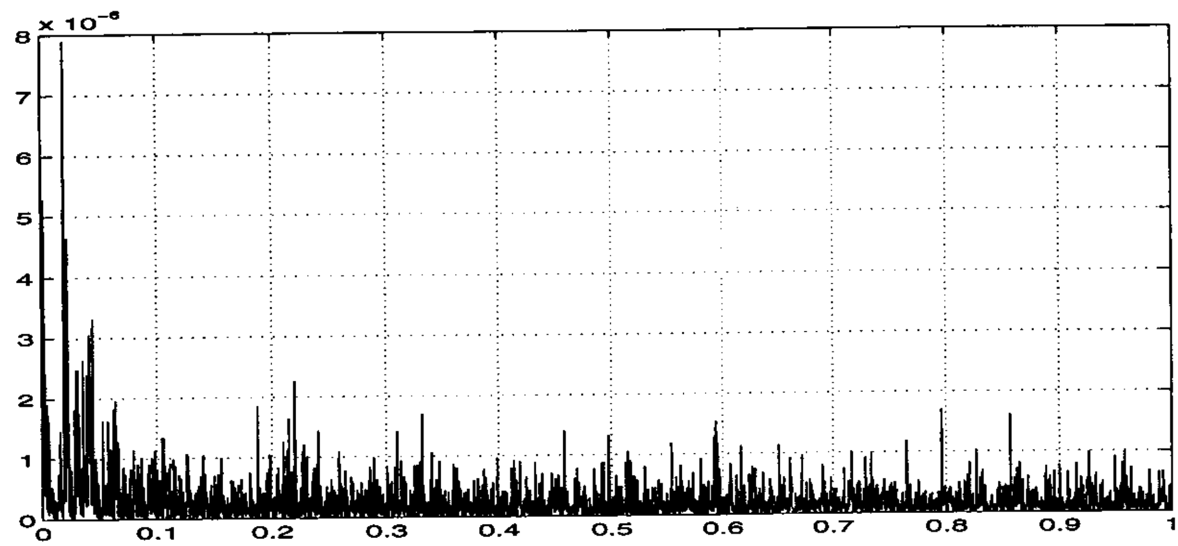

FIG. 5. The power spectral density of the absolute values of the difference data on the Deutsche Mark U.S dollar foreign exchange rate.

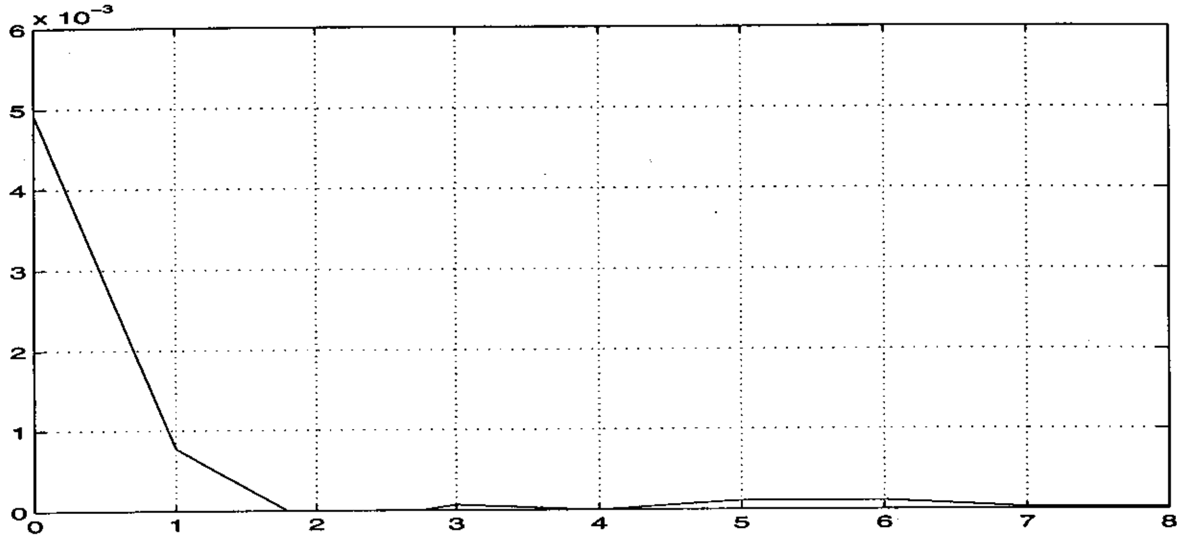

FIG. 6. The auto-correlation of the difference data on the Deutsche Mark U.S dollar foreign exchange rate.

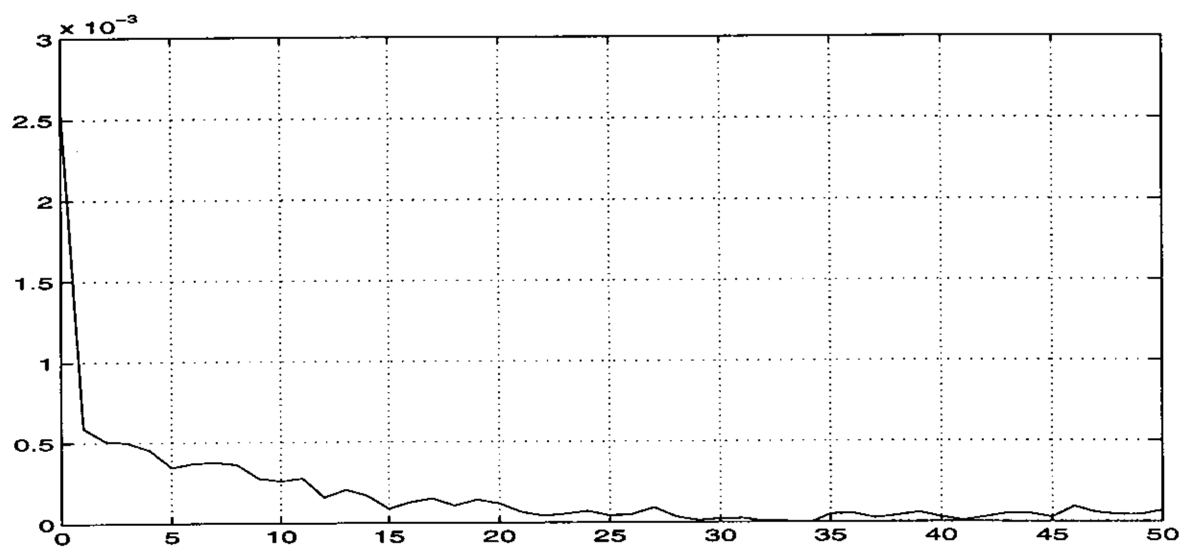

FIG. 7. The auto-correlation of the absolute values of the difference data on the Deutsche Mark U.S dollar foreign exchange rate.

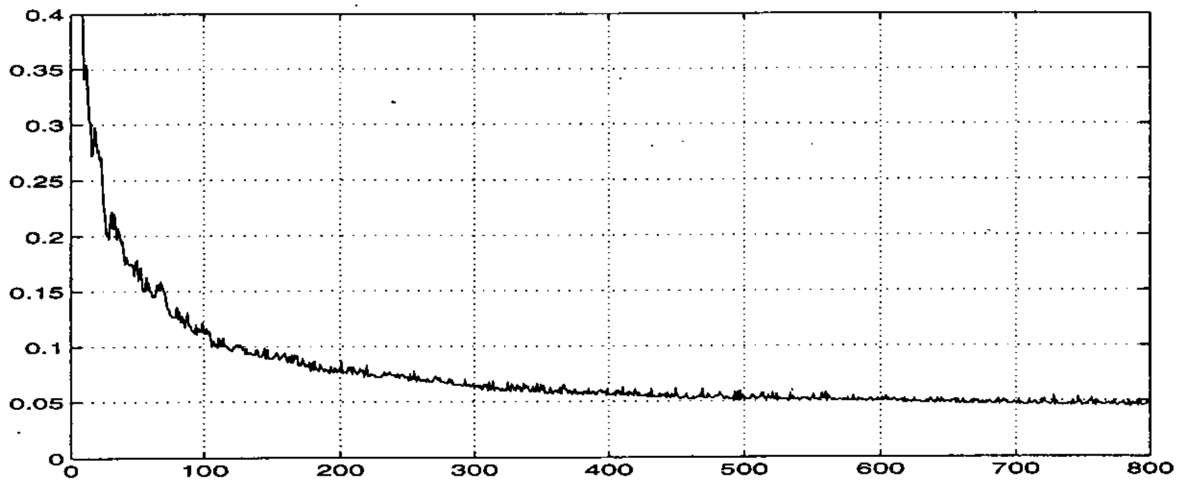

FIG. 8. The decay of the matching pursuit on the data of the Deutsche Mark U.S dollar foreign exchange rate. The x-axis is the number of iterations and the y-axis is the coefficients in the matching pursuit expansion.

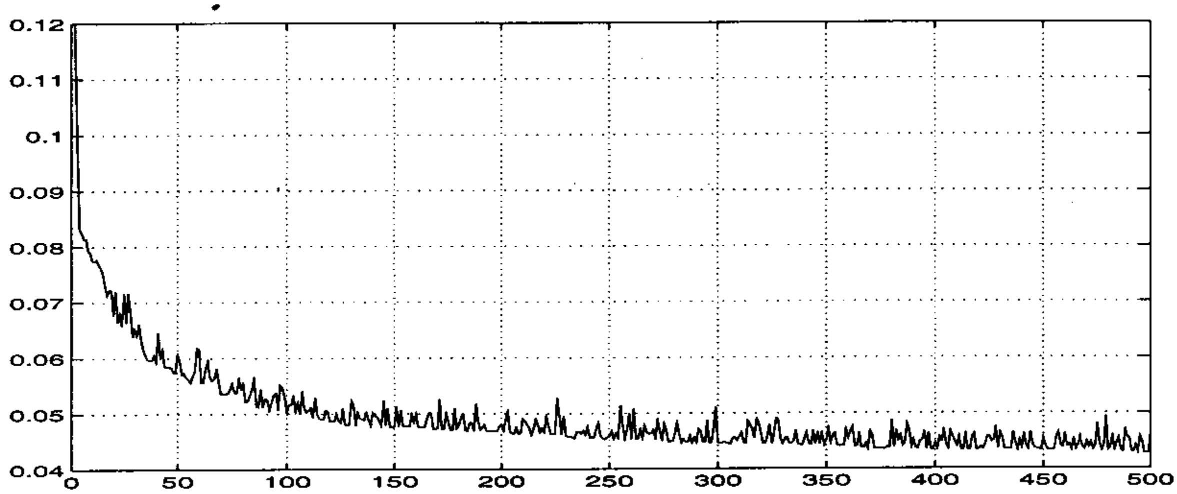

FIG. 9. The decay of the matching pursuit on the difference data of the Deutsche Mark U.S dollar foreign exchange rate. The x-axis is the number of iterations and the y-axis is the coefficients in the matching pursuit expansion.

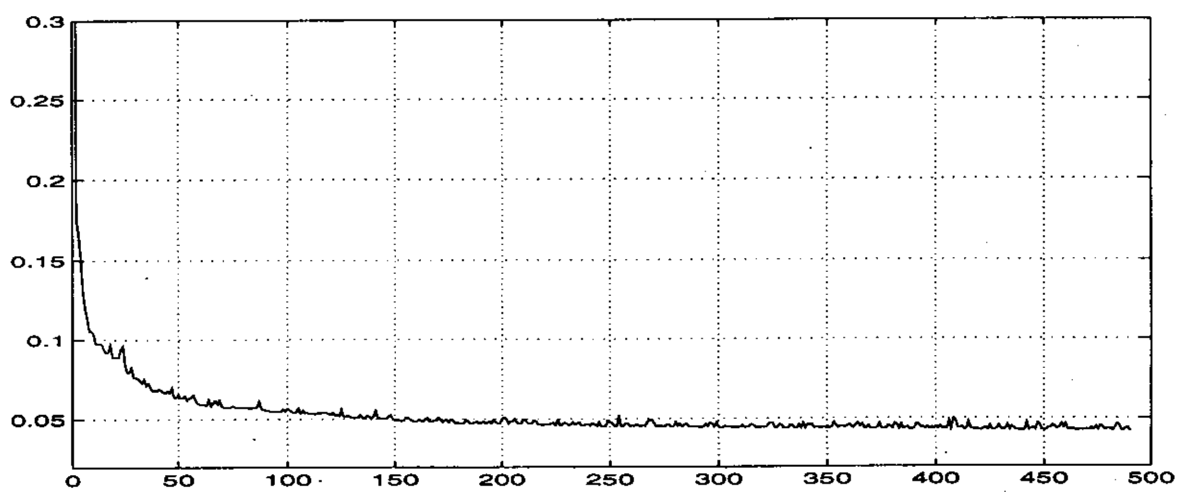

FIG. 10. The decay of the matching pursuit on the absolute values of the difference data of the Deutsche Mark U.S dollar foreign exchange rate. The x-axis is the number of iterations and the y-axis is the coefficients in the matching pursuit expansion.

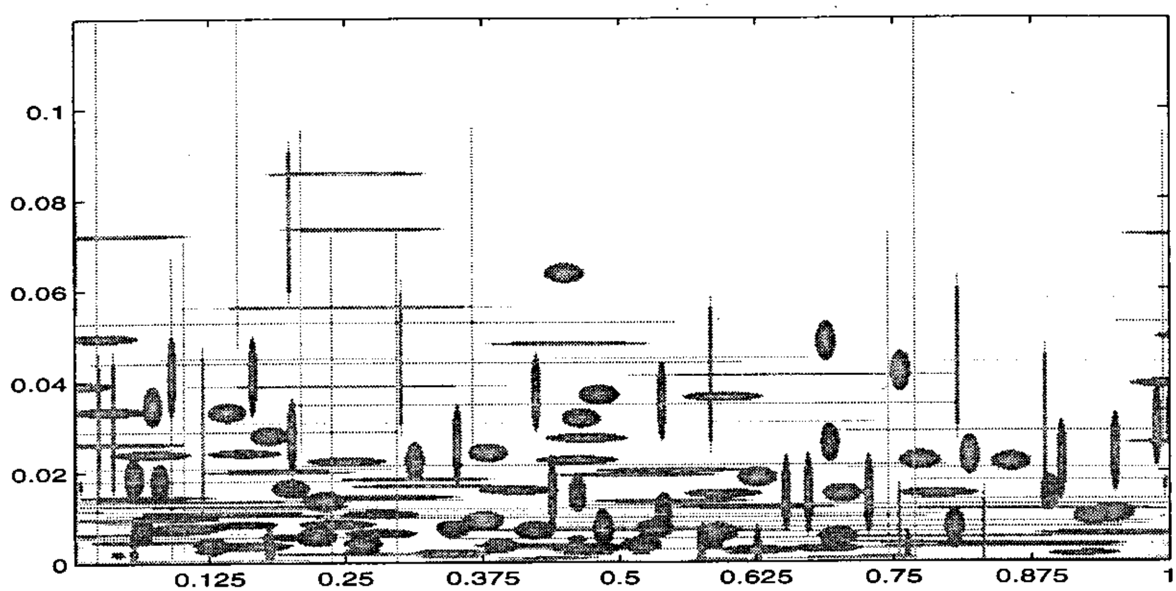

FIG. 11. The local time frequency density of the first 8192 data points on the Deutsche Mark U.S dollar foreign exchange rate. The x-axis is the time axis and the unit is $1/N$ where N is the number of observations in the sample. The y-axis is the frequency in cycles per sample.

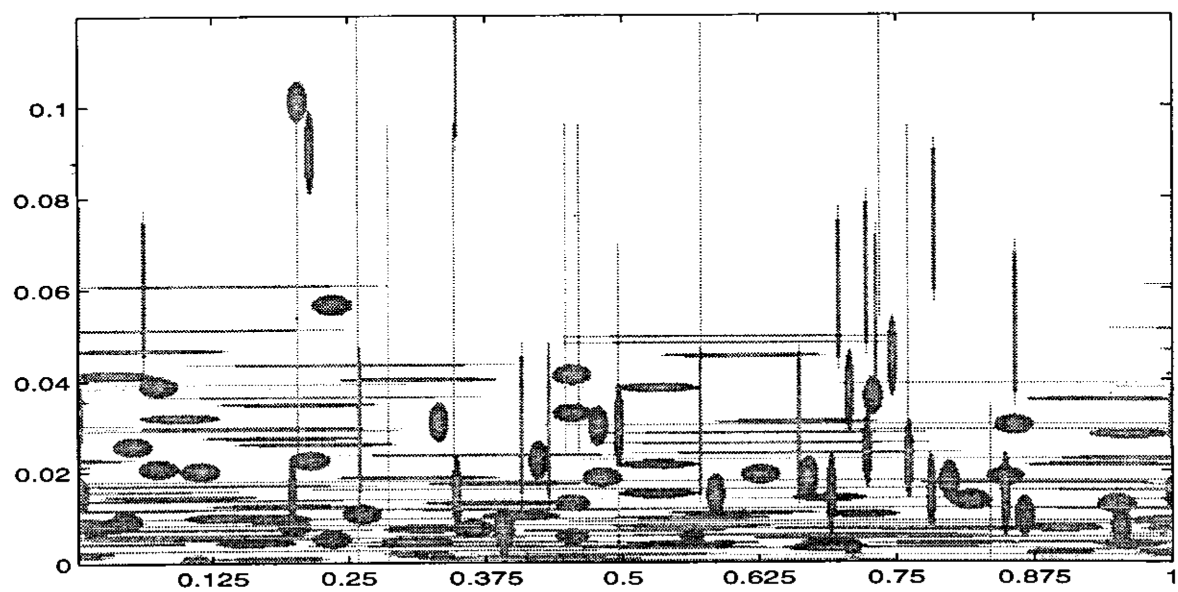

FIG. 12. The local time frequency density of the last 8192 data points on the Deutsche Mark U.S dollar foreign exchange rate. The x-axis is the time axis and the unit is $1/N$ where N is the number of observations in the sample. The y-axis is the frequency in cycles per sample.

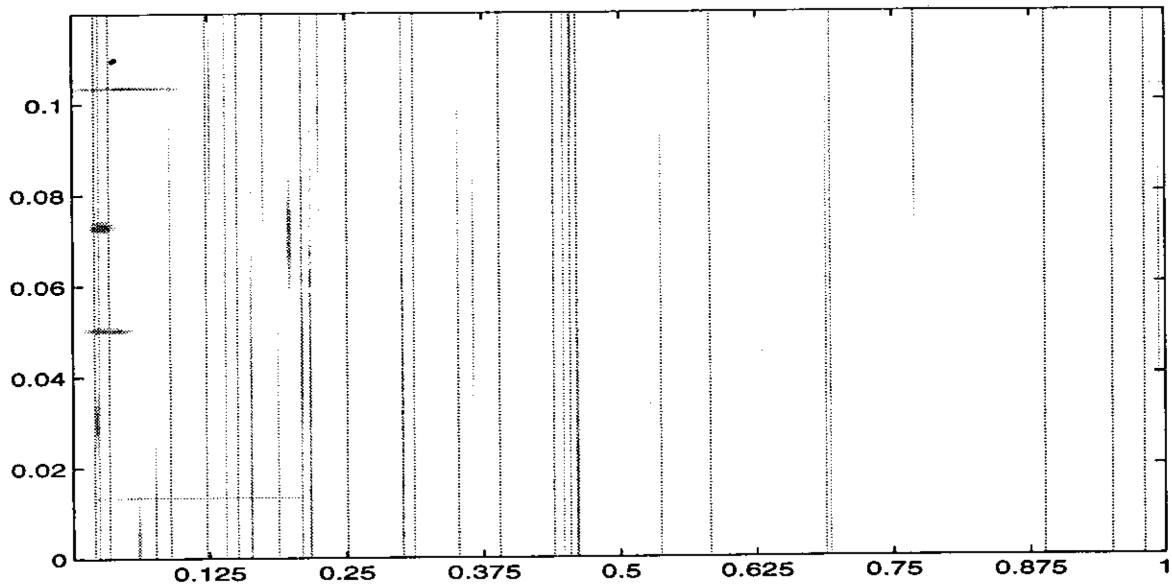

FIG. 13. The local time frequency density of the first 8192 data points of the differences on the Deutsche Mark U.S dollar foreign exchange rate. The x-axis is the time axis and the unit is $1/N$ where N is the number of observations in the sample. The y-axis is the frequency in cycles per sample.

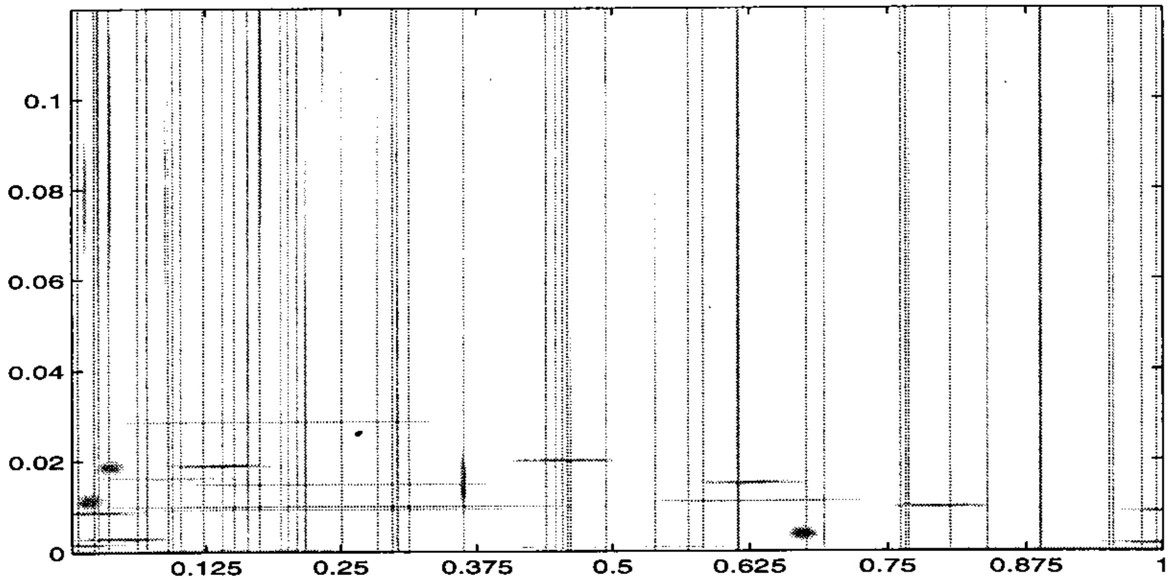

FIG. 14. The local time frequency density of the first 8192 data points of the absolute values of the differences on the Deutsche Mark U.S dollar foreign exchange rate. The x-axis is the time axis and the unit is $1/N$ where N is the number of observations in the sample. The y-axis is the frequency in cycles per sample.

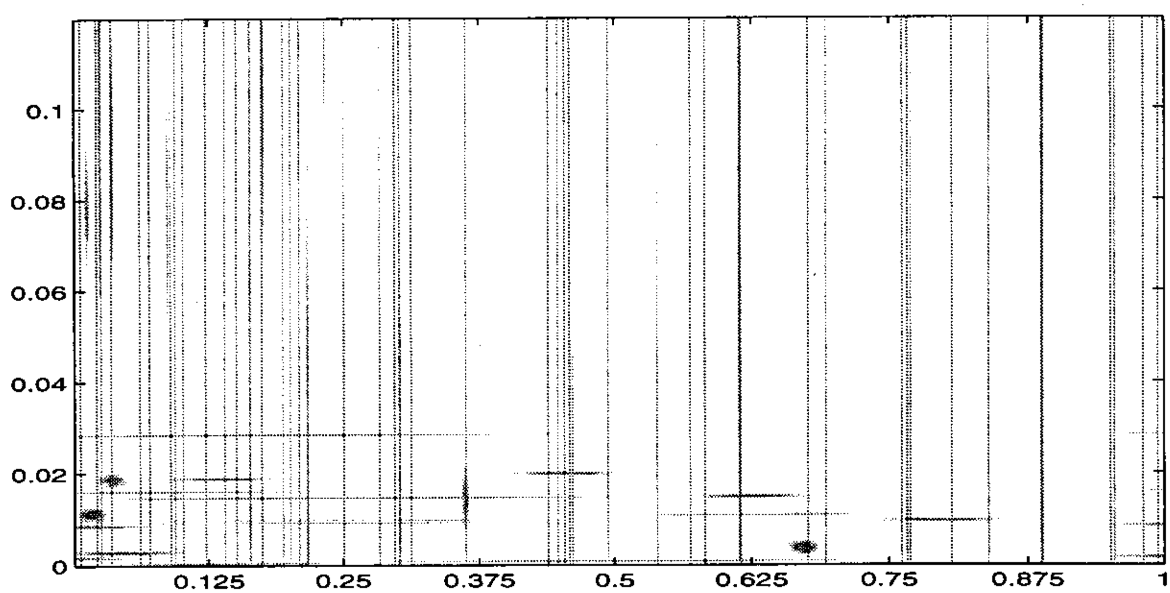

FIG. 15. The local time frequency density of the last 8192 data points of the absolute values of the differences on the Deutsche Mark U.S dollar foreign exchange rate. The x-axis is the time axis and the unit is $1/N$ where N is the number of observations in the sample. The y-axis is the frequency in cycles per sample.

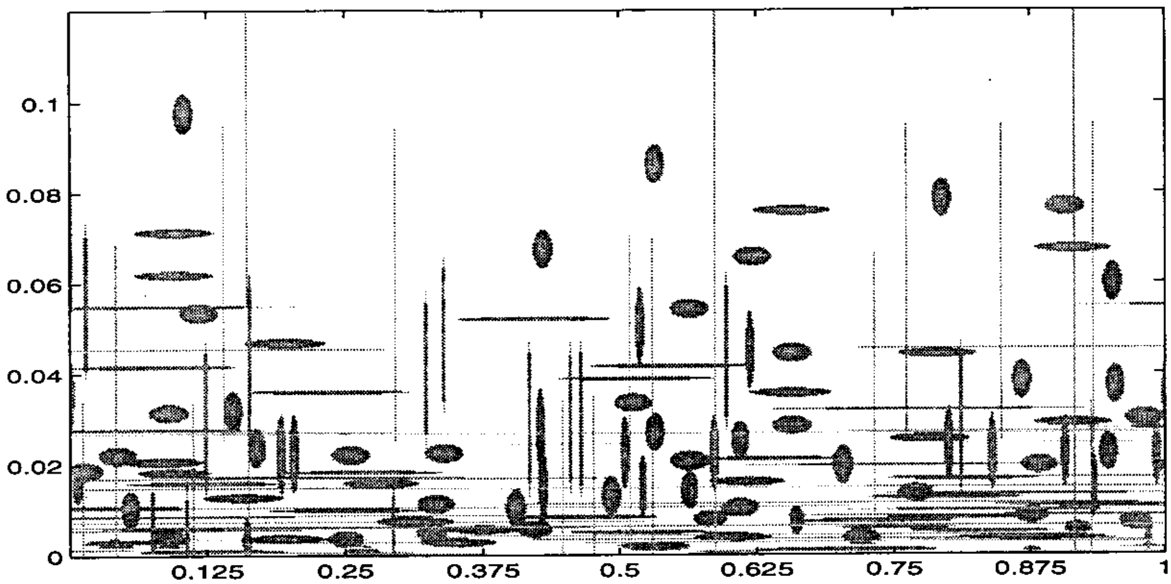

FIG. 16. The local time frequency density of the first 8192 data points on the Japanese Yen U.S dollar foreign exchange rate. The x-axis is the time axis and the unit is $1/N$ where N is the number of observations in the sample. The y-axis is the frequency in cycles per sample.

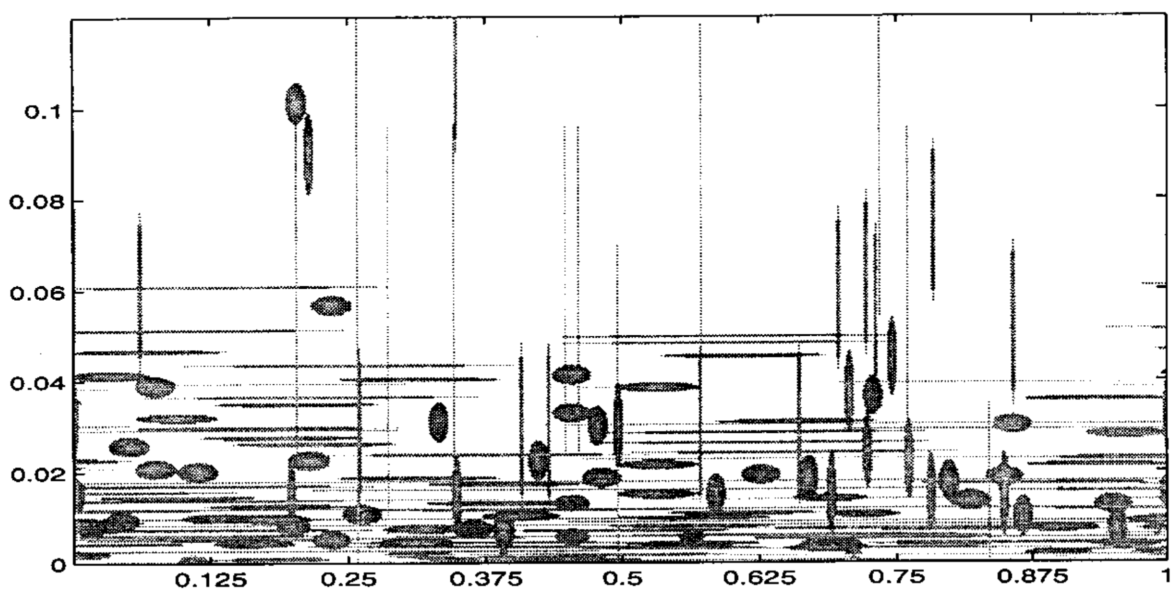

FIG. 17. The local time frequency density of the last 8192 data points on the Japanese Yen U.S dollar foreign exchange rate. The x-axis is the time axis and the unit is $1/N$ where N is the number of observations in the sample. The y-axis is the frequency in cycles per sample.

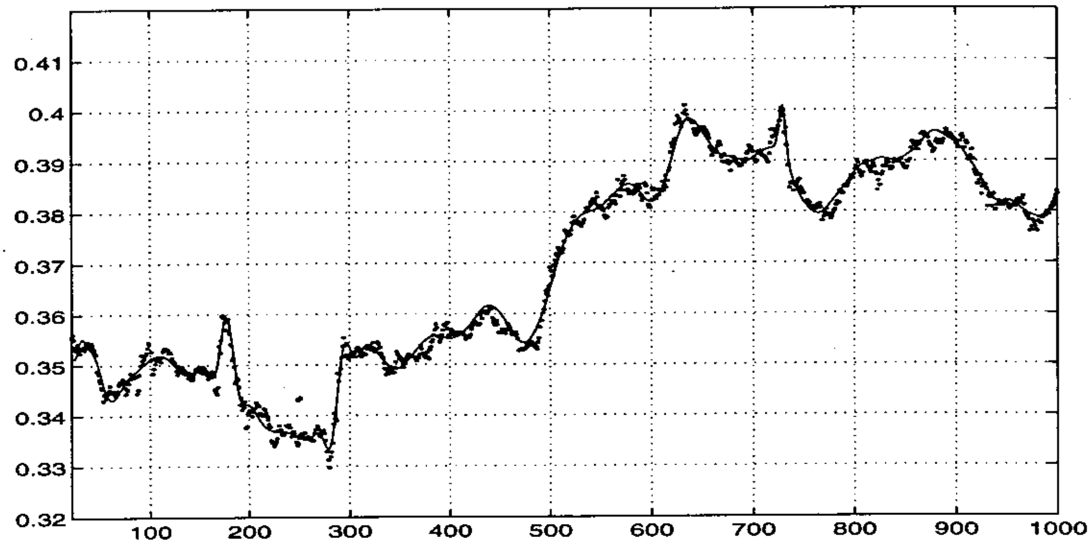

FIG. 18. The comparison of the reconstructed signal based on the first 100 iterations of the matching pursuit and the original data of the Deutsche Mark U.S dollar foreign exchange rate. The dot plot is the original signal and the solid curve is the reconstructed signal. The comparison is on data points between 20 and 1000.

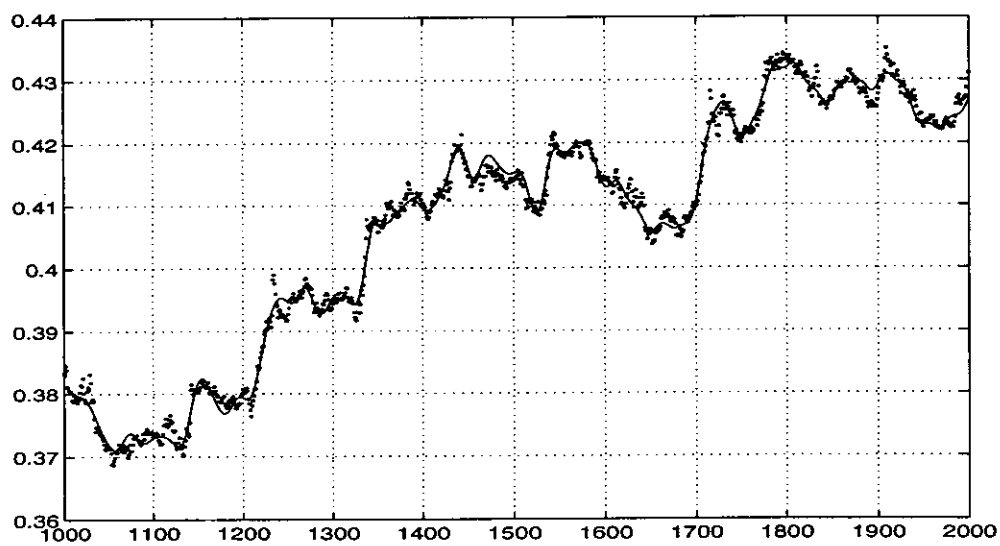

FIG. 19. The comparison of the reconstructed signal based on the first 100 iterations of the matching pursuit and the original data of the Deutsche Mark U.S dollar foreign exchange rate. The dot plot is the original signal and the solid curve is the reconstructed signal. The comparison is on data points between 1000 and 2000.

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