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Abstract

Ordinally single-peaked preferences are distinguished from *cardinally* single-peaked preferences, in which all players have a similar perception of distances in some one-dimensional ordering. While ordinal single-peakedness can lead to disconnected coalitions that have a "hole" in the ordering, cardinal single-peakedness precludes this possibility, based on two models of coalition formation:

- *Fallback (FB)*: Players seek coalition partners by descending lower and lower in their preference rankings until a majority coalition forms.
- *Build-Up (BU)*: Similar to FB, except that when nonmajority subcoalitions form, they fuse into composite players, whose positions are defined cardinally and who are treated as single players in the convergence process.

FB better reflects the unconstrained, or nonmyopic, possibilities of coalition formation, whereas BU—because all subcoalition members must be included in any majority coalition that forms—restricts combinatorial possibilities and tends to produce less compact majority coalitions.

The "strange bedfellows" frequently observed in legislative coalitions and military alliances suggest that even when players agree on, say, a left-right ordering, their perceptions of exactly where players stand in this ordering may differ substantially. If so, a player may be acceptable to a coalition but may not find every member in it acceptable, causing that player not to join and possibly creating a disconnected coalition.

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Single-Peakedness and Disconnected Coalitions[1]

1. Introduction

That individual coherence, like transitive individual preferences, can lead to social incoherence—like voting cycles or the nonexistence of the core (Austen-Smith and Banks, 2000)—is well-known in the social-choice literature. In this paper, we show that a similar phenomenon can occur in coalition formation, whereby players with "connected" preferences may jell into "disconnected" coalitions.

This phenomenon is not surprising when strategic considerations come into play. For example, two ideologically distant players might join together if that would enable them to win, whereas either player's joining with a smaller more centrally located player would not afford this possibility. More surprising, such strange bedfellows may get together for non-strategic reasons, which we will show can occur under two models of coalition formation.

Both models assume that players have ideal positions along a one-dimensional policy space, or line. The ordering of these ideal positions—say, from left to right—is assumed to be known by all players. The goal of the players is to form simple-majority coalitions containing ideologically proximate players.

There is nothing sacrosanct about the coalition's having a simple majority of members; it could be any qualified majority, up to and including unanimity. Although players could be unequally weighted, we assume in our models that they are not to avoid

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strategic questions, such as a player's wanting to join one player (but not another) because the former (but not the latter) would make the coalition winning.

In both coalition-formation models, each player either ranks or rates *every other player* in terms of that player's desirability as a coalition partner (alternatively, players could rank or rate *policy alternatives* according to their desirability). They progressively descend in their preference rankings, or move toward the ideal positions of other players in their ratings, until there is a simple majority of members that considers every other member of that majority acceptable as a coalition partner (or every policy alternative in a set acceptable).[2]

We call coalition formation *constrained* if less-than-majority coalitions, or *subcoalitions*, which do not yet constitute a majority, fuse into composite players whose composite position restricts the positions of their component members (in a way to be made precise later). Treating these composite players as single players—whose component members can no longer be separated—in a smaller game, we repeat the movement of players toward each other until new subcoalitions form. This process continues until a majority coalition forms.

If composite players do not form, and only the preferences of individual players matter, we call coalition formation *unconstrained*. Whether constrained or unconstrained, the descent in player preferences, or movement toward the ideal positions of other players, proceeds in the manner of "fallback bargaining" (Brams and Kilgour, 2001), which we will describe and illustrate in section 4.

Players' preferences are *ordinally single-peaked* if

- all players can be ordered along a line such that each player's preferences for coalition partners declines to the left and right of its ideal position;

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[2]We will illustrate in section 6 the isomorphism between models in which players rank/rate other players and models in which they rank/rate policy alternatives.

• this ordering is the same for all players.

More stringently, preferences are *cardinally single-peaked* if

• there exists a *single* spatial representation of player positions on the line such that each player's less preferred coalition partners are farther away from it.

The existence of a single representation ensures that the players have similar perceptions of the *distances* between players' positions, as we will show later.

Disconnected coalitions may be the first majority coalitions to form under either unconstrained or constrained coalition formation if preferences are ordinally single-peaked but not if they are cardinally single-peaked. To illustrate a disconnected coalition, assume that five players can be ordered 1-2-3-4-5 along a line from left to right. Then the first majority coalition to form might be {1, 2, 4}—without player 3—creating a "hole" in an otherwise connected coalition.

This seems paradoxical, because if any individual player considers both players 2 and 4 desirable as coalition partners, ordinal single-peakedness implies that it must also consider player 3 desirable. Indeed, if preferences are ordinally single-peaked, the preferred coalition partners of each player comprise a cluster, without holes, around that player. Yet, as our models show, connected individual preferences of players can result in a disconnected majority coalition.[3]

The paper proceeds as follows. In section 2, we distinguish between ordinally single-peaked and cardinally single-peaked preferences, showing in what sense the latter

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[3]Garrett and Tsebelis (1999: 294) declare such a coalition "impossible" but offer no explanation of why this is the case, except that an "excluded member [from an otherwise connected coalition] will go along."

are consistent but the former are not. In section 3, we show that there may be no stable majority coalitions, whether preferences are ordinally single-peaked or cardinally single-peaked.

In the absence of such a stable *outcome*, we focus on *processes* of coalition formation, analyzing first what we call *fallback*, or unconstrained, coalition formation in section 4. We demonstrate that disconnected coalitions can occur with as few as five players, but a unique disconnected coalition requires at least seven players. We then analyze other properties of fallback coalitions, especially those related to its size and the "spread" of its members.

The *build-up*, or constrained, model, which requires a measure of numerical distance rather than ordinal ranks, is introduced and analyzed in section 5. While it tends to produce larger majority coalitions than the fallback model, it can lead to a unique disconnected coalition with as few as five players.

We present our conclusions in section 6, emphasizing the importance of analyzing the dynamics of coalition formation rather than just looking for stability. In addition, we comment on the applicability of the models to coalition formation in legislatures and military alliances.

2. Ordinally Single-Peaked and Cardinally Single-Peaked Preferences

Preferences are said to be *ordinally single-peaked* if there exists an ordering of players, along a single dimension, such that each player's more-preferred coalition partners are closer to it than its less-preferred coalition partners. Put another way, each

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We will show that this need not be the case. The literature on spatial models of voting is vast, but good overviews can be found in Hinich and Munger (1997) and Shepsle and Bonchek (1997).

player's preference for coalition partners declines the farther they are from the player's ideal (peak) position on this dimension.

That preferences are not always ordinally single-peaked is illustrated by a three-person example, based on the Condorcet voting paradox, wherein players rank each other as coalition partners as follows:

Example A: **1**: 2 3 **2**: 3 1 **3**: 1 2.

Thus, player 1's first choice of a coalition partner is player 2, and its second choice is player 3. While we assume that player *i* ranks itself highest—that is, it most desires that it be included in any majority coalition that forms—we indicate only its ranking of *other* players in its preference ordering.[4]

It is straightforward to check that none of the 3! = 6 orderings of the players along a single dimension, such as 1-2-3 that might be represented as,

1 ______________________ 2 __________ 3,

can be consistent with the preferences in Example A. The ordering illustrated is consistent with the rankings of players 1 and 2 in Example A, but not with that of player 3, because player 3 prefers player 1 to player 2.

Ordinal single-peakedness requires only that each player's preference be describable by the same left-right ordering of players. There is no requirement that the players have a similar perception of all players' positions, and therefore of the distances

[4] It is conceivable that a player would prefer *not* to be a member of a coalition if the position of the coalition without it as a member is closer to its ideal than the position of a coalition with it as a member. But here we are describing preferences for coalition partners, and one cannot have a "partner" unless one is a member of a coalition.

between them (e.g., that player 2 is closer to player 3 than player 1, as shown in the above representation). Indeed, this may be impossible with ordinally single-peaked preferences, as illustrated by our next example (with ordering 1-2-3-4):

Example B: **1**: 2 3 4 **2**: 3 4 1 **3**: 2 1 4 **4**: 3 2 1.

Because player 2 ranks player 1 last as a coalition partner, player 2 (in boldface below) must perceive that the distance between it and player 1 is greater than the distance between it and player 3, and even between it and player 4:

Player 2's perception: 1 ______________________ **2** 3 4.

By comparison, because player 3 ranks player 4 last as a coalition partner, player 3 must perceive that the distance between it and player 4 is greater than the distance between it and players 1 and 2:

Player 3's perception: 1 2 **3** ______________________ 4.

We say that players' preferences are *cardinally single-peaked* if it is possible to capture them in a *single* spatial representation of player positions along the real number line. If player i's position is x_i, and player j's is x_j, denote the distance between them by $d_{ij} = |x_i - x_j|$. (Note that $d_{ii} = 0$.) Then player i's preference ordering for coalition partners is given by ranking all players, j, in increasing order of d_{ij}.

To demonstrate formally that Example B is ordinally but not cardinally single-peaked, note that player 2's ranking implies $d_{34} < d_{24} < d_{12}$, whereas player 3's ranking implies $d_{12} < d_{13} < d_{34}$. This contradiction shows that Example B, while ordinally single-peaked with respect to the ordering 1-2-3-4, is not cardinally single-peaked: Different players order the distances between positions differently, so there is no single spatial representation valid for all players.

We adopt the convention that players with single-peaked preferences are named 1, 2, 3, . . ., n from left to right. At the extremes, player 1's preference ordering must be **1**: 2 3 4 . . . n, and player n's must be ***n***: n-1 n-2 . . . 1. For convenience, we assume that players are never equally preferred—that is, no two players are ever equidistant from any given player.

It is easy to see that if preferences are ordinally single-peaked, and if m is any integer satisfying $1 = m = n$, then player i's m most-preferred coalition partners, including i itself, is the subset $\{g(i), g(i) + 1, \ldots, h(i)\}$, where and $g(i) = i = h(i)$ and $h(i) = g(i) + m - 1$.[5] That is, player i's most-preferred set of coalition partners forms a cluster, without "holes," around player i. For instance, if $m = 3$ in Example B, each player's three most-preferred coalition partners are as follows:

1: {1, 2, 3} **2**: {2, 3, 4} **3**: {1, 2, 3} **4**: {2, 3, 4}.

It can be checked that when preferences are not ordinally single-peaked, any linear ordering of the players (i.e., along a line) must result in some player's set of m most-preferred coalition partners, for some m, having a hole. In Example A, for instance, when the linear ordering is 1-2-3, player 3's two most-preferred coalition partners are {1, 3}, leaving a hole because of the absence of player 2.

[5]Note that $g(i)$ and h(i) are functions of m as well as i, but we suppress m as an argument to simplify the notation.

If preferences are cardinally single-peaked and player i is to the left of player j, then the set containing player i's m most-preferred coalition partners must either be identical to player j's or start to the left of j's. More precisely,

Proposition 1. *If preferences are cardinally single-peaked, the clusters around each player satisfy the following monotonicity property: For any m, $i < j$ implies that $g(i) = g(j)$.*[6]

Proof. To prove this statement, fix m and suppose that $j > i$. If $j = h(i)$, the statement holds because $g(j) = j - m + 1 = h(i) - m + 1 = g(i)$. Otherwise, $i < j < h(i)$. Suppose that $k < g(i)$. Because $k < i < j$, $d_{kj} > d_{ki}$. Also, because $h(i)$ is among i's m most-preferred coalition partners, and k is not, it must be the case the $d_{ki} > d_{i,h(i)} > d_{j,h(i)}$. Therefore, $d_{kj} > d_{j,h(i)}$. Now assume (to obtain a contradiction) that k is among j's m most-preferred coalition partners. Then so is $h(i)$, because $d_{kj} > d_{ki} > d_{j,h(i)}$. But this is impossible, because j's m most-preferred coalition partners form an interval, $I_m(j) = \{g(j), g(j) + 1, \ldots, h(j)\}$, where $h(j) - g(j) = m - 1$. But $h(i) - k > h(i) - g(i) = m - 1$, demonstrating that both $h(i)$ and k cannot both belong to an interval containing m players. This contradiction shows that if $k < g(i)$, then $k < g(j)$, completing the proof that $g(i) = g(j)$. Q.E.D.

To illustrate Proposition 1, consider the following example, in which player preferences are cardinally single-peaked:

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[6]Demange's (1994) notion of "intermediate preferences" also satisfies this monotonicity property. Whereas Demange gives conditions under which intermediate preferences lead to stable connected coalitions in a cooperative game-theoretic model, we show in section 3 that there may be no stable coalitions, in a noncooperative sense, even when preferences are cardinally single-peaked.

Example C: **1**: 2 3 4 **2**: 1 3 4 **3**: 4 2 1 **4**: 3 2 1.

If *m* = 3, each player's most-preferred sets of coalition partners are as follows:

1: {1, 2, 3} **2**: {1, 2, 3} **3**: {2, 3, 4} **4**: {2, 3, 4}.

Notice that (i) player 1 and 2's, and player 3 and 4's, most-preferred sets of coalition partners are identical and (ii) player 2's most-preferred set starts with player 1, one position to the left of the starting player, 2, in player 3's most-preferred set. If players' preferences are cardinally single-peaked, their perceptions of distance can be described by a single spatial representation, as illustrated by Example C,

All players' perceptions: 1 2 3 4,

and their orderings are *consistent*.

By contrast, the most-preferred sets for *m* = 3 in Example B, in which preferences are ordinally but not cardinally single-peaked, do not satisfy this property: Player 2's most-preferred set starts with player 2, one position to the right of the starting player, 1, in player 3's most-preferred set.

Our results so far can be summarized as follows:

1. If preferences are *not single-peaked*, as in Example A, they cannot be described by a single linear ordering, which means that there are "holes," with respect to any linear ordering of the players, in some player's set of most-preferred coalition partners.

2. If players' preferences are *ordinally single-peaked*, as in Example B, there is such a linear ordering, and each player's most-preferred coalition partners form a cluster around its preferred position.

3. If preferences are *cardinally single-peaked*, as in Example C, player positions are describable by a single spatial model, rendering players' perceptions of distance similar and their consequent orderings consistent. Such consistency implies the following monotonicity property: If player *i*'s position is to the left of player *j*'s, then *i*'s cluster of most-preferred coalition partners may not lie to the right of *j* 's cluster.

It is worth noting that the "unfolding" technique of Coombs (1964) for determining whether stimuli and other kinds of psychological data can be represented by either unidimensional or multidimensional scales is closely related to ordinal and cardinal single-peakedness. In the unidimensional case, individual preferences (*I* scales) are ordinally single-peaked if they can be "unfolded" into a *qualitative J* ("joint") scale, and cardinally single-peaked if they can be unfolded into a *quantitative J* scale. Whereas Coombs' interest was in constructing *J* scales—qualitative or quantitative—from a set of *I* scales, ours is in determining whether a qualitative *J* scale (preferences are ordinally single-peaked) is also a quantitative *J* scale (preferences are cardinally single-peaked).

Among other things, Coombs showed that as the number of players increases, the proportion of ordinally single-peaked preferences that are cardinally single-peaked tends to zero, making disconnected coalitions more likely in our models. Before analyzing disconnected coalitions, however, we next show that neither ordinally nor cardinally single-peaked preferences ensure stable coalitions.

3. Stable Majority Coalitions: They May Not Exist

Define a majority coalition to be *stable* if no member desires to switch to another majority coalition, resulting in a so-called Tiebout equilibrium (Tiebout, 1956; Greenberg and Weber, 1985, 1986, 1993; Demange, 1994). While the cyclical majorities resulting from the Condorcet paradox in Example A obviously preclude such an equilibrium, more surprising is that the ordinal single-peakedness of Example B, and even the cardinal single-peakedness of Example C, confer no such stability on majority coalitions.

In analyzing stability, we do not define a game and analyze its equilibria. Instead, we postulate in sections 4 and 5 coalition-formation processes that seem likely to support, if not stabilize, the coalitions that form under them. In doing so, we focus on the preferences of players for each other, and on the majority coalitions they lead to, and ask if any players would prefer to be in different majority coalitions.

Proposition 2. *There may be no stable majority coalition even if preferences are cardinally or ordinally single-peaked.*

Proof. We begin by showing the instability of majority coalitions for cardinally single-peaked preferences. Consider Example C, wherein player preferences are cardinally single-peaked with respect to ordering 1-2-3-4.[7] Now consider majority coalition 123. Player 3 would prefer to be in coalition 234, because it ranks player 4 higher than player 1 (both coalitions share players 2 and 3); hence, coalition 123 is unstable. Likewise, majority coalition 234 is unstable, because player 3 has the opposite preference—it prefers player 1 to player 4 and would, therefore, prefer to be in coalition 123. Finally, in the case of the two disconnected majority coalitions, 124 and 134, it is easy to show that all three players would each prefer to be in one or the other of the connected coalitions, 123 and 234.

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[7]When there are four players, ordinal single-peakedness implies that player 2 may rank either adjacent player 1 or adjacent player 3 first—and if player 3, either player 1 or player 4 next. Player 3 may rank either adjacent player 2 or adjacent player 4 first—and if player 2, either player 1 or player 4 next. Only one of these nine possible orderings for players 2 and 3, which *is* Example B, is ordinally single-peaked without also being cardinally single-peaked.

Majority coalitions in Example B can be shown to be unstable by a similar argument. This proves that preferences that are ordinally but not cardinally single-peaked may also not yield stable majority coalitions. Q.E.D.

To be sure, the presence of one or more dissatisfied players in every majority coalition is not the only criterion of instability.[8] Stronger conditions that do ensure stability have been proposed in, among other places, Greenberg and Weber (1985, 1986, 1993), Demange and Henriet (1991), Demange (1994), Bogomolnaia and Jackson (1998), Jackson and Moselle (1998), and Burani and Zwicker (2001). In a review article, Greenberg (1994) gives several reasons why, as Greenberg and Weber (1993, p. 63) put it, "there is only a relatively small number of results that guarantee the existence of a 'stable' coalition structure."

We could follow the example of such equilibrium models and impose conditions that would render the coalitions that emerge from our models stable. However, this would detract from our main purpose of providing insight into *dynamic processes* of coalition formation that may, themselves, contribute to stability.

The approach we take next is algorithmic, rather than axiomatic, in the sense that it postulates rules for players' sequentially forming coalitions without insisting that the resulting coalitions be stable. Indeed, we have shown that one kind of stability may be impossible to achieve. This "generative" approach to deriving macroscopic behavior from microscopic assumptions is espoused in, among other places, Epstein (1999).

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[8]It is, however, similar to one proposed in a model by Milchtaich and Winter (2000, p. 10): "A partition is unstable if there is at least one individual who would like to move to a different group than the one in which he is a member." But whereas preference in their model is cardinal, based on the average distance between the individual in question and a group's members, preference in our first model is ordinal. Unlike

4. Fallback (Unconstrained) Coalition Formation

We now define and illustrate *fallback coalition formation (FB),* in which subcoalitions that form prior to the emergence of a majority coalition do not constrain the formation of such a coalition. FB proceeds as follows:

1. The most desirable coalition partner of each player is considered. If two players mutually desire each other, and this is a majority of players, then this is the majority coalition that forms. The process stops, and we call this a level 1 majority coalition.

2. If there is no level 1 majority coalition, then the next-most desirable coalition partners of all players are also considered. If there is a majority of players that mutually desire each other at this level, then this is the majority coalition (or coalitions) that forms. The process stops, and we call this a level 2 majority coalition.

3. The players descend to lower and lower levels in their rankings until a majority coalition, all of whose members mutually desire each other, forms *for the first time.* The process stops, with the resulting *largest* majority coalition(s) at this level designated an FB coalition(s).

We illustrate FB with the preceding examples:

Example A: **1**: 2 3 **2**: 3 1 **3**: 1 2.

There is no level 1 majority coalition, because no pair of members consider each other mutually desirable at this level. At level 2, however, the grand coalition (of all

either of our models, theirs always leads to stable connected ("segregating") coalitions and hence does not account for strange bedfellows.

players), 123, forms, because members of all pairs, 12, 13, and 23, become mutually desirable at this level. Thus, the FB coalition is the grand coalition.

Example B: **1**: 2 3 4 **2**: 3 4 1 **3**: 2 1 4 **4**: 3 2 1.

At level 1, coalition 23 forms; at level 2, coalitions 13 and 24 form; at level 3, coalitions 12, 14, and 34 form, as well as all 3-person coalitions and the grand coalition, 1234. Thus, the FB coalition can be the grand coalition, even when majority preferences do not cycle (as they do in Example A).

Likewise in Example C, it is not difficult to show that the FB coalition is the grand coalition. Thus, if preferences are either ordinally single-peaked (Example B) or cardinally single-peaked (Example C), the FB coalition may be the grand coalition.

We next present a five-person example in which preferences are ordinally single-peaked and there are two FB coalitions, one of which includes nonadjacent players, proving the following proposition:

Proposition 3. *If preferences are ordinally single-peaked, an FB coalition may be disconnected.*

Proof. Assume five players have the following preferences:

Example D (FB Coalition Disconnected but Not Unique)

1: 2 3 4 5 **2**: 1 3 4 5 **3**: 4 5 2 1 **4**: 3 2 1 5 **5**: 4 3 2 1

One can verify that these preferences are ordinally single-peaked, with respect to ordering 1-2-3-4-5, by checking that each player's *m* most acceptable coalition partners, for every *m*, cluster around it without holes. However, these preferences are not cardinally single-peaked, because player 3's ranking implies $d_{45} < d_{35} < d_{23}$, and player 4's ranking implies $d_{23} < d_{24} < d_{45}$, which are inconsistent.

The largest coalitions that form at each level, until two FB coalitions form at level 3, are as follows (the starred coalitions are "ordinally *m*-compact," which will be defined after the Proposition 5):

Level 1: 12*, 34* *Level 2*: 35 *Level 3*: 124, 234

Notice that we do not include coalitions 14, 23, and 24 at level 3 in our listing of coalitions because they are proper subsets of coalitions 124 or 234 at level 3. Clearly, FB coalition 124 is disconnected, with a hole due to the absence of player 3. Q.E.D.

The underlying reason that player 3 is excluded from coalition 124 is that whereas players 1 and 2 necessarily rank player 3 higher than player 4 (because of ordinal single-peakedness), player 3 ranks players 2 and 1 at the bottom of its preference order. In particular, player 3 does not consider player 1 acceptable at level 3.

That pairs of player may rank each other quite differently, even when their preferences are single-peaked, differs sharply from Axelrod's (1997, chs. 4-5) "landscape theory" of aggregation, in which the propensities of pairs of players to coalesce are assumed to be the same. Also, landscape theory predicts coalitions, not the dynamic process that leads to them.

We next show that, if the number of players is increased from five to at least seven, a disconnected coalition may be the only FB coalition:

Proposition 4. *If preferences are ordinally single-peaked, the FB coalition can be unique and disconnected. At least seven players are required for this to happen.*

Proof. Assume seven players have the following preferences:

Example E (FB Coalition Unique and Disconnected)

1: 2 3 4 5 6 7 **2**: 1 3 4 5 6 7 **3**: 2 1 4 5 6 7

4: 5 6 3 7 2 1 **5**: 4 3 2 1 6 7 **6**: 5 4 3 2 1 7 **7**: 6 5 4 3 2 1

One can verify that these preferences are ordinally single-peaked, with respect to ordering 1-2-3-4-5-6-7, by examining clusters. However, these preferences are not cardinally single-peaked, because player 4's ranking implies $d_{56} < d_{46} < d_{34}$, whereas player 5's ranking implies $d_{34} < d_{35} < d_{56}$, which are inconsistent.

The largest coalitions that form at each level until a single (disconnected) FB coalition, 1235, forms at level 4 are as follows:

Level 1: 12*, 45* *Level 2*: 46, 123* *Level 3*: 34 *Level 4*: 47, 1235*

To show that a unique disconnected FB coalition requires at least 7 players, note that any coalition of size k that forms at level k - 1 must be connected. Consequently, if n = 6, a disconnected coalition of 4 players must form by level 4; in fact, at level 5 the grand coalition forms.

Because all players must rank *either* player 1 or player 6 last at level 4, the connected coalition 2345 must form at this level. Thus, even if a disconnected coalition also forms at level 4, it will not be unique. Analogously, a unique disconnected coalition of 3 players cannot form if n = 5; if n = 4, it is easy to check that no disconnected coalition can form. Q.E.D.

Observe that the formation of disconnected coalitions does not depend on players' having preferences over subsets. In Example E, for instance, the formation of disconnected coalition 1235 is unrelated to whether player 1 prefers 125 or 134 (either preference is possible).

We next describe some properties of FB coalitions.

Proposition 5. *FB coalitions may not be minimal majority coalitions, whether preferences are ordinally single-peaked, cardinally single-peaked, or neither.*

Proof. The FB coalitions in Example B (ordinally single-peaked preferences), Example C (cardinally single-peaked preferences), and Example A (neither) are all the grand coalitions, not minimal majority coalitions of three players (Examples B and C) or two players (Example A). Q.E.D.

We next investigate properties of FB coalitions relating to their "spread." The *ordinal dispersion* (*Odisp*) of a coalition of size m is the sum, over all members of the coalition, of the minimum number of pairwise switches (in adjacent preferences) that are required to put the coalition members in position 1, 2, …, m of the player's ranking.

Example A: *Odisp* (13) = 1 + 0 = 1 (player 1 must switch 2 and 3 to induce ranking 1 3; player 3 need make no switches to induce ranking 3 1).

Example B: *Odisp* (124) = 1 + 1 + 2 = 4 (for players 1, 2, and 4, respectively).

Call a coalition of size m that minimizes *Odisp* the *ordinally m-compact* coalition. These coalitions are starred in Examples D and E. We next show that FB coalitions may or may not be ordinally m-compact if preferences are ordinally single-peaked.

Proposition 6. *If preferences are ordinally single-peaked, no FB coalition may be ordinally m-compact.*

Proof. In Example D, FB disconnected coalition 124 requires four pairwise switches, FB connected coalition 234 requires three, but non-FB 3-coalition 345 requires only two pairwise switches and is ordinally 3-compact. Q.E.D.

The fact that disconnected FB 3-coalition 124 in Example D is not ordinally 3-compact does not imply that disconnected coalitions cannot be ordinally *m*-compact:

Proposition 7. *If preferences are ordinally single-peaked, a disconnected FB coalition may be ordinally m-compact.*

Proof. Observe that disconnected FB 4-coalition 1235 in Example E, which requires six pairwise switches, is starred. Connected 4-coalitions 1234, 2345, 3456, and 4567 require, respectively, 8, 13, 8, and 10 pairwise switches; disconnected 4-coalitions, other than 1235, require even more. Hence, disconnected 4-coalition 1235 is ordinally 4-compact. Q.E.D.

Define the *ordinal diameter* (*Odiam*) of a coalition to be the maximum, over all members of the coalition, of the distance in ranks between each player (ranked first in its own ordering) and its least-preferred coalition member.

Example A: *Odiam* (13) = max {2, 1} = 2 (player 3 is ranked 3^{rd} by player 1, giving it a distance of 3 – 1 = 2 ; player 1 is ranked 2^{nd} by player 3, giving it a distance of 2 – 1 = 1).

Example B: *Odiam* (124) = max {3, 3, 3} = 3 (for players 1, 2, and 4, respectively).

Call the coalition of size *m* that minimizes *Odiam* the *ordinally m-narrow* coalition.

Proposition 8. *All FB coalitions are ordinally m-narrow.*[9]

Proof. Because the descent stops at the level at which, for the first time, a majority of players considers each other mutually acceptable, any earlier stoppage would not produce a majority coalition. Any majority coalition *not* an FB coalition must have, for some pair of players, greater rank difference. Q.E.D.

In Example D, because FB coalition 234 appears for the first time at level 3, we know there is at least one player (in this case, player 3) that ranks another player (player 2) 3rd in coalition 234; likewise for FB disconnected coalition 124. But non-FB coalition 345, which we showed earlier is ordinally 3-compact, scores worse on the *Odiam* criterion: Player 4 ranks player 5 last (i.e., its 4th choice), illustrating that an ordinally *m*-compact coalition may not be ordinally *m*-narrow. Hence, these concepts tap two different aspects of the spread of FB coalitions.

When an ordinally *m*-narrow FB coalition is not ordinally *m*-compact, as in Example D,[10] the FB coalition is probably more difficult to disrupt than the ordinally *m*-compact coalition. The reason is that the ordinally *m*-compact coalition must contain at least one player that is ranked lower by some coalition member than the FB coalition. This less-desired player would seem a more likely candidate for replacement than any member of the FB coalition, rendering the FB coalition more stable.

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[9] Unlike our earlier propositions, this proposition holds for any preferences, not just those that are ordinally or cardinally single-peaked.

[10] In Example E, by contrast, the ordinally 4-compact coalition, 1235, is also the ordinally 4-narrow FB coalition.

The property of ordinal m-narrowness of FB coalitions notwithstanding, FB coalitions that are disconnected would still seem fragile, especially if the player left out is the Condorcet player (the player preferred by a majority of players in pairwise comparisons to every other player). As a case in point, player 3 in Example D, and player 4 in Example D, are Condorcet players, but they are the players left out of the disconnected FB coalitions in each example. This can never be the case if preferences are cardinally single-peaked, which also preclude disconnected FB coalitions.

Proposition 9. *If preferences are cardinally single-peaked, then every FB coalition is connected. Moreover, if the number of players is odd, every majority coalition includes the (unique) Condorcet player.*

Proof. Recall the monotonicity property of Proposition 1: If preferences are cardinally single-peaked, then the cluster of any player i's m most-preferred coalition partners, $I_m(i) = \{g(i), g(i) + 1, \ldots, h(i)\}$, satisfies $g(i) = g(j)$ whenever $i < j$. Recall that $h(i) = g(i) + m - 1$.

Now suppose that $i < j < k$, and i and k are members of an FB coalition of size m. Then any member e of the coalition must also belong to $I_m(j)$, because $e = g(k) = g(j)$, and $e = h(i) = h(j)$. Also, j must belong to $I_m(e)$, because $j > i = g(e)$ and $j < k = h(e)$. It follows that j belongs to the coalition, so it is connected. Because the coalition has more than half the members, it must include the median player, which is the unique Condorcet player if the number of players is odd. Q.E.D.

So far we have shown that if preferences are ordinally single-peaked,

• FB can produce non-unique disconnected majority coalitions if there are at least five players, unique disconnected majority coalitions if there are at least seven players;

- FB coalitions need be neither minimal majority nor ordinally *m*-compact coalitions, but they are always ordinally *m*-narrow.

If preferences are cardinally single-peaked,

- FB coalitions will always be connected and include the Condorcet (median) player if there is an odd number of players.

Thus, if players' clusters of preferred coalition partners satisfy the monotonicity property of Proposition 1—which renders players' perceptions of distance similar and their consequent orderings consistent—disconnected coalitions are ruled out and unique Condorcet players are ruled in.

5. Build-Up (Constrained) Coalition Formation

In this section, we drop the assumption that the preferences of players are ordinal. Instead, we assume that players can indicate their *degrees* of preference for coalition partners by expressing, in quantitative terms, how much more they prefer, say, a first-choice coalition partner to a second-choice partner.

We continue to identify preference with spatial proximity, but now defined numerically. For instance, consider Example C, in which the preferences of the players are cardinally single-peaked, so the players have a common perception of the ordering of distances between player ideal positions that was illustrated in section 2. We can turn this ordinal representation into a numerical one by making the ideal positions of players real numbers on [0, 1], as illustrated underneath the line below:

Example C´

All players' perceptions: 1 2 3 4.
0 .2 .7 1

Instead of assuming, as under FB, that the players descend from their ideal points to lower and lower ranks, we assume the following in our model of *build-up coalition formation (BU)*:

1. The players increase, at a constant rate, the radii of positions, starting from their ideal positions, that they consider acceptable.[11]

2. When the players' radii touch, so they find each other mutually acceptable, they become a single composite player (or subcoalition). The *position* of the composite player is the average of the ideal positions of its members.[12]

3. Each time a composite player forms, the expansion process begins again in the new and smaller "game," comprising both individual and composite players, until a majority coalition forms for the first time. The process stops, with the resulting largest majority coalition(s) designated BU coalition(s).

We illustrate BU with Example C´:

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[11]This is analogous to a knife moving across a cake in the fair-division literature (Brams and Taylor, 1996; Robertson and Webb, 1998), except that in our coalition-formation model, two knives move in opposite directions from an ideal position. Also, no player calls "stop" to halt the knife; instead, the process stops, automatically, when a majority coalition forms for the first time.

The first subcoalition to form is 12, which becomes a composite player at .1:

All players' perceptions: 12 (.1) —— 3 (.7) —— 4 (1).

The next subcoalition to form is 34, which becomes a composite player at .85:

All players' perceptions: 12 (.1) —— 34 (.85).

Finally, the grand coalition, 1234, will form at position (.10 + .85)/2 = .475,[13] which becomes the BU coalition.

Coalition formation is constrained in the BU model, because the players in each subcoalition that forms, now fused into a single player, cannot be selectively excluded from any future majority coalition. In particular, player 4 would be left out if there were not this fusion under BU: Convergence would be to coalition 123, separated by a distance of .7, before it would be to coalition 234, separated by a distance of .8.

Thus, it is the grand coalition, 1234, that forms in Example C´ under BU. Because the build-up of coalitions has a history, whose lineage is the sequence of subcoalitions that form, the BU model is *path dependent*. The dependence in this example suggests that larger majority coalitions will tend to form under BU than under FB.

One might think that the constraints on coalition formation under BU would prevent the formation of disconnected coalitions, but this is not the case:

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[12] Note that if the BU model were ordinal, like the FB model, we could not calculate an average position, which is one reason we have based BU on numerically ideal positions.

[13] Here simple averaging of the positions of individual players, and of the pairs that combine is possible, but later we will need to do *weighted* averaging to determine the coalition position when subcoalitions of different sizes combine. The successive use of weighted averaging to determine the position of any BU coalition is equivalent to the simple averaging of the positions of *all* its individual members.

Proposition 10. *If preferences are ordinally single-peaked, a BU coalition can be the unique disconnected FB coalition if there are five, but not fewer, players.*

Proof. Assume that players 1, 2, 4, and 5 have a common perception of distance that differs from that of player 3:

Example F

Player 1, 2, 4 and 5's perceptions: 1 2 3 4 5
0 .1 .2 .3 1

Player 3's perception: 1 2 3 4 5.
0 .1 .5 .9 1

We obtain after .1 unit has been traversed subcoalition 12 (notice that subcoalitions 34 and 45 do not form, because one player in each is more than .1 units from the other):

Player 1, 2, 4, and 5's perceptions: 12 3 4 5
.05 .2 .3 1

Player 3's perception: 12 3 4 5.
.05 .5 .9 1

Because for players 1, 2, and 4 the distance separating composite player 12 and player 4 is .25 units, BU disconnected coalition 124 will form next at position [2(.05) + .3]/3 = .133 for players 1, 2, 4, and 5, and position [2(.05) + .9]/3 = .333 for player 3.

Such a disconnected coalition cannot form if there are only three or four players, because a disconnected coalition would have to include both endpoints. Consequently, the grand coalition would form at the same time as all smaller majority coalitions, so a disconnected majority coalition cannot form under the FB cardinal model with only four or fewer players. Q.E.D.

The "problem" for player 3 in Example F is that whereas players 1, 2, and 4 consider player 3 to be acceptable, player 3 does not deem them acceptable because it is too far away, on both the left (.45 units from subcoalition 12) and on the right (.40 units from player 4). To be sure, player 5 is even farther from *all* players—except as player 3 perceives the situation—so player 5 will suffer the most when disconnected coalition 124 forms at position .133.

While the BU model may *tend* to produce larger majority coalitions than the FB model, Example F demonstrates that BU may, nevertheless, produce minimal-majority coalitions, and disconnected ones at that, if preferences are ordinally single-peaked.

Grofman (1982) and Straffin and Grofman (1984) show, in a dynamic model of coalition formation that somewhat resembles our BU model, that coalitions will always be connected in one dimension but not necessarily in two or more dimensions (i.e., all players in the convex hull defined by the spatial positions of coalition members may not be in the coalition, creating "holes" in space rather than along a line).[14] Proposition 10, however, demonstrates that coalitions need not be connected, even in one dimension, if preferences are ordinally single-peaked.

For FB, we earlier defined notions of ordinal *m*-compactness and *m*-narrowness, proving that FB coalitions are always ordinally *m*-narrow but may not be ordinally *m*-compact. We can define analogous notions for BU, with a coalition's diameter (on which narrowness is based) and its dispersion (on which compactness is based) numerical distances rather than differences in ranks.

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[14] For extensions of this model, and applications to coalition-formation data in different European parliamentary democracies, see Grofman (1996) and Grofman, Straffin, and Noviello (1996). Laver and Schofield (1992), van Deemen (1997), and de Vries (1999) also analyze parliamentary coalitions.

We forego formal definitions of these notions, because the calculations we make in the example that proves Proposition 11 makes them evident.

Proposition 11. *If preferences are cardinally single-peaked, a BU coalition may be neither numerically m-compact nor numerically m-narrow.*

Proof. Consider the following 5-person example:

Example G

All players' perceptions: 1 2 3 4 5.
0 .26 .52 .72 1

The first subcoalition to form under BU is 34, which becomes a composite player at position .62:

All players' perceptions: 1 2 34 5.
0 .26 .62 1

The next subcoalition to form is 12, which becomes a composite player at position .13:

All players' perceptions: 12 34 5.
.13 . 62 1

Finally, BU coalition 345 will form at position [2(.62) + 1]/3 = .75:

All players' perceptions: 12 345.
.13 .75

It is not difficult to show that the sum of distances among all three pairs of members of BU coalition 345 is not minimal—coalition 234 minimizes this dispersion—so coalition 345 is not the (numerically) 3-compact coalition. Likewise, the maximum distance between the extreme members of BU coalition 345 is not minimal—coalition 234 minimizes this diameter—so coalition 345 is not the (numerically) 3-narrow coalition. Q.E.D.

The fact that BU coalitions may not be numerically *m*-narrow, even when preferences are cardinally single-peaked, is inconsistent with DeSwann's (1970, 1973) assumption that coalitions minimize "policy distance." Axelrod's (1970) assumption of connected coalitions may also be violated—if preferences are ordinally but not cardinally single-peaked—under both FB (Propositions 3 and 4) and BU (Proposition 10).

The assumptions of the DeSwann and Axelrod, though inconsistent with our conclusions, have proven quite accurate descriptively, at least in predicting parliamentary coalitions. While our models are intended mainly to explicate processes rather than outcomes, they do, nevertheless, pinpoint conditions under which disconnected, non-compact, or non-narrow coalitions—though perhaps exceptional—are likely to form (see section 6 for examples).

If preferences are cardinally single-peaked, BU coalitions will be connected for essentially the same reasons that FB coalitions are (see Proposition 9).

Proposition 12. *If preferences are cardinally single-peaked, BU coalitions are always connected.*

Proof. The first subcoalition to form under BU comprises the closest pair of players, which must be adjacent (otherwise, there would be a closer pair); the first subcoalition is therefore connected. At any stage, the next subcoalition to form must join the two closest players at that stage. Again, these players, which may be either individual or composite, must be adjacent. If all coalitions formed prior to this stage are connected, then the new subcoalition must also be connected. Thus, at every stage of build-up, a majority coalition that forms under BU is connected. Q.E.D.

In the final section, we compare our ordinal FB and numerical BU models when preferences are ordinally single-peaked and cardinally single-peaked. We then offer some reflections on their applicability to legislative coalitions and military alliances.

6. Conclusions and Extensions

Instability plagues many situations in which players seek to form majority coalitions. Whether the preferences of players for coalition partners are ordinally or cardinally single-peaked, at least one player in every majority coalition may prefer a different majority coalition and, therefore, have a reason to defect.

In the face of such instability, we proposed two coalition-formation models, fallback (FB) and build-up (BU), that describe how coalitions might plausibly form. The models share the assumption that players coalesce when they find each other mutually acceptable.

Alternatively, we could have assumed that players rank policy alternatives rather than each other. Thus in Example E, assume that the players rank alternatives *a*, *b*, *c*, . . . the same as they rank players 1, 2, 3, . . ., making, for instance, player 1's ranking $a > b > c > d > e > f > g$. Then it is easy to show that under FB, alternative *c* will be the first supported by a majority coalition—namely, disconnected coalition 1235 at level 3—which is the same coalition as when the players rank each other. Similarly, when the players place alternatives *a*, *b*, *c*, *d*, and *e* on a [0, 1] scale, as they do each other in Example F, the set of alternatives, {*a*, *b*, *d*}, will be the first on which a majority coalition—namely, disconnected coalition 124—converges under BU. Thus, disconnected coalitions can also form if players rank policy alternatives or place them on a continuum.

Under FB, players seek coalition partners by descending lower and lower in their preference rankings until a majority coalition emerges. Subcoalitions that form early do not restrict future choices, suggesting FB as a model of unconstrained coalition formation. Because players can abandon early subcoalition partners in order to be part of the first majority coalition to form later, they may be thought of as acting nonmyopically (though not in the sense of anticipating other players' choices in a game but rather in terms of not tying themselves down too early).

Under BU, by comparison, subcoalitions fuse into composite players that cannot be broken apart. The movement of players toward each other begins afresh each time a new composite player forms, which may constrain the build-up of coalitions that would otherwise form were only individual players the building blocks. Thus, BU is myopic in the sense that the "baggage" of coalition partners that players pick up early, when subcoalitions form that may hurt them later, cannot be detached.

We showed that both FB and BU may *not* produce minimal-majority coalitions or ordinally or numerically m-compact coalitions that minimize dispersion in m-member coalitions. The two models differ, however, on the criterion of m-narrowness: FB coalitions minimize the maximum difference in ranks of players, whereas BU m-coalitions do not necessarily ensure that the numerical distance between their most extreme members is minimal.

In legislatures, the myopia of BU is probably more common than the nonmyopia of FB in the passage of ordinary legislation. Typically, small groups of members coalesce to try to put together a larger winning coalition. Once formed, these groups rarely split apart. However, the combining of these groups can lead to "oversize" majority coalitions, as compared with the majority coalitions generated by FB from individual players.[15]

On the other hand, when a political party is asked to form a new government in a parliamentary democracy, FB may be a better mirror of the manner in which the governing coalition emerges. The party's leaders weigh simultaneously different combinations of other parties to try to find the set of coalition partners that it can best work with to advance its legislative program. Because party leaders must think beyond the next piece of legislation they want enacted, their thinking is more likely to be farsighted and strategic than that of ordinary legislators struggling to win on the next vote.

Whether it is individual legislation or parliamentary control that is sought, the old saw that "politics makes strange bedfellows" often turns out to be descriptively accurate. Strange bedfellows are also observed in international politics, wherein countries attempt to ensure their national security through "unholy alliances."

As a case in point, fascist Germany, to neutralize opposition on its eastern front just prior to its invasion of Poland that led to the outbreak of World War II, made a nonaggression pact with communist Soviet Union in August 1939 (which it violated less than two years later in June 1941). Then Germany—pledged to make Aryans the

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[15]In either case, the build-up is sequential (rather than all at once), which Downs, Rocke, and Barsoom (1998) argue contributes to the depth of cooperation among members of a multilateral organization.

controlling race—gained (and kept) two more strange bedfellows when, with Italy and Japan, it signed the Tripartite Treaty in Berlin in September 1941.[16]

More contemporary examples of what Bronner (1999) calls "crossing paths" could be given. In *Strange Bedfellows*, Grayson (1999) describes the bureaucratic battles, especially in the United States, that led to the expansion of NATO in 1998 to include three former communist states (the Czech Republic, Hungary, and Poland).

Our analysis suggests that even when there is a single left-right dimension, it may not be so much that players cross paths as have different perceptions of distance along a single path that leads them to form disconnected coalitions. True, there may be a second salient dimension. Nevertheless, we think that at least some of the strangeness or unholiness of coalition formation in the world is attributable to perceptual differences on a single dimension, which reflects ordinal single-peakedness but not cardinal single-peakedness.

[16]Of course, these allies are not so strange if the relevant dimension is democratic-totalitarian regimes, because Germany, Italy, Japan, and the Soviet Union were all, in varying degrees, totalitarian states. But because race was so central to the ideology of Nazi Germany, it seems particularly odd that the German-Japanese alliance endured until the end.

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