ON SHAPLEY VALUES AND NUCLEOLI

FOR PUBLIC GOODS ECONOMIES

by

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No. 75-53 December 1975

Introduction

The Shapley value and the nucleolus are two game theoretical solution concepts that have enjoyed wide popularity. This is so because they are unique and in their separate ways are supposed to represent "fair" outcomes for n-person games. In economies in which all preferences are convex, all goods are perfectly divisible private goods, and utility is transferable, these solution concepts are unambiguously defined. However, when the economies under study contain either pure public goods or externalities a problem arises as how best to define the characteristic function. Consequently, the resulting Shapley values and nucleoli will depend on the exact definition we use and for the same economy we may get a variety of Shapley values and nucleoli each relating to a different characteristic function.

In this paper we investigate an example of a public goods economy in which all agents have identical utility functions but differ in their initial endowment of one all purpose good.

We show that under certain interpretations of the characteristic function for these economies (namely what we shall call the Rosenthal IG_1 rule)[1] the imputations of our players associated with the Shapley value and the nucleolus may be inversely related to their initial endowments. That is, the "bigger" the agents are in these economies the less they are awarded by the Shapley value and nucleolus. Two other interpretations of the characteristic function for such games are also offered (what we shall call the Rosenthal IG_2 rule and the classical or o-type rule). For the first rule we again find some counter intuitive results, namely that seemingly asymmetric public goods economies may determine

[1]See Rosenthal (4) and Richter (3). IG is an acronym for individually and group rational behavior.

characteristic functions that are symmetric and consequently determine symmetric Shapley values and nucleoli. The second rule determines results that are more in line with our a-priori expectations. Here we notice that the Shapley value and nucleolus are directly related to the size of the initial endowment each player has.

We will proceed as follows; Section I will discuss the problem of defining a characteristic function for economies with public goods or externalities. Here we will refer to the work of Robert Rosenthal (4) and Donald Richter (3). Section II offers an example which will serve as the basis for our further discussion. Section III specifies the characteristic functions that result when we make three different assumptions about the behavior of our players, and calculates the Shapley value and nucleolus with surprising results. Several difficulties concerning the Rosenthal approach are raised here. In addition a small simulation of our example is offered in which one of the key parameters is varied in an effort to watch the impact it has on the resulting Shapley values and nucleoli. Finally, Section IV offers some interpretation of our results, and discusses some of the problems our simple example has raised that remain unsettled in the literature.

Section I: Characteristic Functions and Public Goods: What a Coalition can Achieve for Itself.

The value of a coalition S in an n-person game is unambiguously defined only in those games in which when S forms, the value that it can achieve for itself, V(S), is totally under its own control. In other words, only when the members of $\overline{S}$ (the complement of S in N - the set of all players) cannot affect the payoff of S in any way. Such games are called "orthogonal" by Shapley and Shubik (6). As an example, all market games are orthogonal games.

When games are "non-orthogonal", such as games involving externalities or public goods, the value that any coalition can guarantee itself becomes ambiguous by virtue of the fact that what it gets depends on what the counter coalition does, and their behavior cannot be satisfactorily predicted. However, it still is possible to define the characteristic function by assuming different types of behavior on their part and calculating the value of the characteristic function based on these assumptions.

Two assumptions that could be made about the behavior of the counter coalition in economies containing public goods have been specified by Robert Rosenthal (4) and studied by Donald Richter (3).
They are as follows:

1) Classical o-type behavior[1]

Under this assumption if a coalition S forms whose payoff V(S) can be affected by the actions of $\overline{S}$, it is assumed that $\overline{S}$ will organize its activities so as to make S's payoff as small as possible. This is assumed even though the resulting payoff to $\overline{S}$ may fail to be either individually rational or group rational. Consequently, from the viewpoint of coalition S, the o-type assumption is extremely pessimistic in its calculation of V(S).

2) IG-Behavior

A more appealing assumption that could be made, however, is that if coalition S forms, it can assume that the counter coalition, $\overline{S}$, will act in a fashion that is both Pareto-optimal or group rational with respect $\overline{S}$, and individually rational.

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[1]This assumption was used by Duncan Foley (1) in his work on Lindahl Equilibria and the core. It corresponds to the classical definition of the characteristic function of an n-person game by von Neumann and Morgenstern (8).

Therefore, S can rely on a certain externality to be produced by $\overline{S}$ and can coordinate its activities accordingly.

Section II: The Example

Consider a simple economy described by the following n-tuple (N,W,U,F,X), where:

N is the set of agents in the economy

W is the set of initial endowments

U is a set of utility functions one for each player

F is the production set which is available to all players and

X is the set of products that can be produced in the economy, both private and public.

To specify our particular example, assume that there are three agents in the economy each with an identical utility function $U^i = 3x_1^i + 4x_2$, where x_1^i is a private good in i's possession and x_2 is a public good. In addition, assume that there exists some all purpose good, g, of which player i possesses the amount g^i. g^i can either be transformed into a private good x_1^i or into a portion of the public godd, x_2^i, available to all, or can be arbitrarily divided between the private and public goods via the production relationship $x_1^i + x_2^i = g^i$, $x_1^i, x_2^i \geq 0$. The total amount of the public good produced is $x_2 = x_2^1 + x_2^2 + x_2^3$. g is initially distributed in the following manner to the three players: $(g^1,g^2,g^3) = (4,8,10)$. The economy is then described by the following table.

Table 1

Player	Utility Function	Initial Endowment	Production Function
1	$U^1 = 3x_1^1 + 4x_2$	$g^1 = 4$	$x_1^1 + x_2^1 = g^1$
2	$U^2 = 3x_1^2 + 4x_2$	$g^2 = 8$	$x_1^2 + x_2^2 = g^2$
3	$U^3 = 3x_1^3 + 4x_2$	$g^3 = 10$	$x_1^3 + x_2^3 = g^3$

Finally assume that a technological constraint exists which sets a maximum on the amount of x_2 that can feasibly be produced. Call this maximum x_2^* and let us set it at 20 units to begin our analysis.[1]

Section III: The Characteristic Functions

a) o-type behavior

Taking this example let us calculate the characteristic function for this game under an o-type behavioral assumption. To do this consider the value for any player acting alone (say player 1). By acting alone, this player assumes that the counter coalition (23) will produce none of the public good (even though as we said before, that behavior on their part is neither individually nor group rational). The maximum utility that 1 can then guarantee himself is achieved by transforming all of his four units of g into the public good and achieving a payoff of 16. Therefore V(1) = 16. Similarly, any coalition of two players (ij) cannot rely on any public good being built by the third and maximize their utility simply by transforming their collective g into the public good.[2] For example

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[1]Instead of making this assumption, we could have specified a concave utility function with sufficient concavity to insure an interior maximum of the type we desire. In order to use the simple linear functions that we have, however, we decided to make this assumption. There is no real loss of generality, however.

[2]If $x_2^* < g_i + g_j$ for any coalition (ij), then V(ij) is achieved by producing X_2^* and transforming the remaining g into a private good. In this example $x_2^* > g_i + g_j$ for any two $i,j \in N$. Later we will vary x_2^* and see how it affects our results.

coalition (23) can build 18 units of the public good for itself. Therefore $V(23) = 2\cdot(4(18)) = 144$. The characteristic function for this example then becomes:

$V(1) = 16$	$V(13) = 112$
$V(2) = 32$	$V(23) = 144$
$V(3) = 40$	$V(123) = 246$
$V(12) = 96$	

b) IG behavior

1) A problem

A conceptual problem arises when we try to specify a characteristic function for this game assuming IG-type behavior. Remembering that $x_2^* = 20$, what is the value for player 1 in this game? Since we are assuming that the counter coalition (23) will build the Pareto optimal amount of x_2 for itself, 1 can rely on 18 units of the public good being built. The problem is: what do the rules of the game allow player 1 to do knowing this? Do they allow him to take two of his four units of g and add them to the 18 units offered by (23), transforming the remaining two units into a private good for himself, or do they forbid this and force him to transform all of his g into the private good? Obviously the characteristic function will be affected by the rule we use.

2) The resolution

The answer as to which rule is appropriate may be found by looking at the public good to be built. If it is a good like a free public stadium or a park whose production must be coordinated and coalition S forms and threatens not to contribute to its construction, then the size of the public good x_2 will simply be that built by $\overline{S}$ and S will not be able to add to it. They will have to take x_2 as given by $\overline{S}$ and transform all of their good g into private goods for themselves. However, if the technology of the good is such that any agent or coalition can add its own resources separately and increase the amount of x_2 in the economy, (street lighting may serve as an example), then their actions may involve some public good construction.

Consequently, we will split up the IG assumption into two separate assumptions called IG_1 and IG_2. The first rule, IG_1, allows a coalition to add to the public good that it knows the counter coalition will produce if it so desires. The second, IG_2, forbids this and constrains this coalition to maximize its utility by employing strategies that don't involve public good construction.[1]

Using the two assumptions defined above, the following characteristic functions are defined for our example:

IG_1	IG_2
V(1) = 86	V(1) = 84
V(2) = 86	V(2) = 80
V(3) = 86	V(3) = 78
V(12) = 166	V(12) = 116
V(13) = 166	V(13) = 106
V(23) = 166	V(23) = 86
V(123) = 246	V(123) = 246

Computing the Shapley values and nucleoli for our three different characteristic functions, we find the following results:[2]

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[1]An ambiguity arises with the IG_2 rule. It is: What is the value of the grand coalition? Since its complement is the null set, should N also be forced to rely on the amount of the public good built by its complement, i.e. a zero amount? We will circumvent this problem in this paper by calculating V(N) in the conventional manner, but conceptually it is not resolved.

[2]It should be noted that the Shapley value and the nucleoli for the games defined by IG_1 and IG_2 are not individually rational, and therefore not imputations.

PLAYER	o-type behavior		IG_1		IG_2	
	Shapley Value	Nucleolus	Shapley Value	Nucleolus	Shapley Value	Nucleolus
1	62	59	82	82	92	85.33
2	86	85.5	82	82	80	81.33
3	98	101.5	82	82	74	79.33
Total	246	246	246	246	246	246

The results are striking. Under our o-type assumption the order of the Shapley values and nucleoli is the same as the order of the size of the initial endowment. In other words, in this economy the bigger you are the more the Shapley value and nucleolus awards you as your "fair share". Just the opposite occurs under the IG_2 rule. Here both the Shapley values and the nucleoli are inversely related to the size of a player's initial endowment. It actually hurts to be big. Finally, under IG_1 we see that the Shapley value and the nucleolus is totally symmetric and does not discriminate at all between the players. All of the values are equal.

These results however, are only defined for a value of x_2^* equal to 20. In order to investigate how they would change with different values of x_2^*, we varied x_2^* between 0 and 22 and calculated the Shapley values and nucleoli in each case for each of our assumptions.[1] The results are presented below.

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[1]One result is obvious. It is that when $x_2^* = 0$, the economy is a purely private goods economy and all three behavioral assumptions determine the same characteristic function. Obviously, for this game the payoffs for the Shapley value and nucleolus will vary directly with the initial endowment of the players, and will be identical across interpretations.

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CASE (0)

								SHAPLEY VALUE			NUCLEOLUS		
X_2^*	V1	V2	V3	V12	V13	V23	V123	X1S	X2S	X3S	X1N	X2N	X3N
0	12	24	30	36	42	54	66	12.00	24.00	30.00	12.00	24.00	30.00
1	13	25	31	41	47	59	75	15.00	27.00	33.00	15.00	27.00	33.00
2	14	26	32	46	52	64	84	18.00	30.00	36.00	18.00	30.00	36.00
3	15	27	33	51	57	69	93	21.00	33.00	39.00	21.00	33.00	39.00
4	16	28	34	56	62	74	102	24.00	36.00	42.00	24.00	36.00	42.00
5	16	29	35	61	67	79	111	26.67	39.17	45.17	27.00	39.00	45.00
6	16	30	36	66	72	84	120	29.33	42.33	48.33	30.00	42.00	48.00
7	16	31	37	71	77	89	129	32.00	45.50	51.50	33.00	45.00	51.00
8	16	32	38	76	82	94	138	34.67	48.67	54.67	36.00	48.00	54.00
9	16	32	39	81	87	99	147	37.50	51.50	58.00	39.00	51.00	57.00
10	16	32	40	86	92	104	156	40.33	54.33	61.33	42.00	54.00	60.00
11	16	32	40	91	97	109	165	43.33	57.33	64.33	45.00	57.00	63.00
12	16	32	40	96	102	114	174	46.33	60.33	67.33	48.00	60.00	66.00
13	16	32	40	96	107	119	183	48.50	62.50	72.00	49.33	61.33	72.33
14	16	32	40	96	112	124	192	50.67	64.67	76.67	50.67	62.67	78.67
15	16	32	40	96	112	129	201	52.00	68.50	80.50	50.33	67.33	83.33
16	16	32	40	96	112	134	210	53.33	72.33	84.33	50.00	72.00	88.00
17	16	32	40	96	112	139	219	54.67	76.17	88.17	49.67	76.67	92.67
18	16	32	40	96	112	144	228	56.00	80.00	92.00	50.00	81.00	97.00
19	16	32	40	96	112	144	237	59.00	83.00	95.00	54.50	83.25	99.25
20	16	32	40	96	112	144	246	62.00	86.00	98.00	59.00	85.50	101.50
21	16	32	40	96	112	144	255	65.00	89.00	101.00	63.50	87.75	103.75
22	16	32	40	96	112	144	264	68.00	92.00	104.00	68.00	92.00	104.00

CASE (IG_1)

								SHAPLEY VALUE			NUCLEOLUS		
X_2^*	V1	V2	V3	V12	V13	V23	V123	X1S	X2S	X3S	X1N	X2N	X3N
0	12	24	30	36	42	54	66	12.00	24.00	30.00	12.00	24.00	30.00
1	16	28	34	44	50	62	75	15.00	27.00	33.00	15.00	27.00	33.00*
2	20	32	38	52	58	70	84	18.00	30.00	36.00	18.00	30.00	36.00*
3	24	36	42	60	66	78	93	21.00	33.00	39.00	21.00	33.00	39.00*
4	28	40	46	68	74	86	102	24.00	36.00	42.00	24.00	36.00	42.00*
5	32	44	50	76	82	91	111	28.00	39.50	44.50	29.00	38.00	44.00*
6	36	48	54	84	90	96	120	32.00	41.00	47.00	34.00	40.00	46.00*
7	40	52	58	92	98	101	129	36.00	43.50	49.50	39.00	42.00	48.00*
8	44	56	62	100	106	106	138	40.00	46.00	52.00	44.00	44.00	50.00*
9	48	60	66	108	111	111	147	43.50	49.50	54.00	46.50	48.00	52.50*
10	52	64	70	116	116	116	156	47.00	53.00	56.00	49.00	52.00	55.00*
11	56	68	74	121	121	121	165	50.00	56.00	59.00	50.00	56.00	59.00*
12	60	72	78	126	126	126	174	53.00	59.00	62.00	51.00	60.00	63.00*
13	64	76	79	131	131	131	183	56.50	62.50	64.00	53.50	74.00	65.50*
14	68	80	80	136	136	136	192	60.00	66.00	66.00	56.00	68.00	68.00*
15	72	81	81	141	141	141	201	64.00	68.50	68.50	61.00	70.00	70.00*
16	76	82	82	146	146	146	210	68.00	71.00	71.00	66.00	72.00	72.00*
17	80	83	83	151	151	151	219	72.00	73.50	73.50	71.00	74.00	74.00*
18	84	84	84	156	156	156	228	76.00	76.00	76.00	76.00	76.00	76.00*
19	85	85	85	161	161	161	237	79.00	79.00	79.00	79.00	79.00	79.00*
20	86	86	86	166	166	166	246	82.00	82.00	82.00	82.00	82.00	82.00*
21	87	87	87	171	171	171	255	85.00	85.00	85.00	85.00	85.00	85.00*
22	88	88	88	176	176	176	264	88.00	88.00	88.00	88.00	88.00	88.00

CASE (IG_2)

								SHAPLEY VALUE			NUCLEOLUS		
X_2^*	V1	V2	V3	V12	V13	V23	V123	X1S	X2S	X3S	X1N	X2N	X3N
0	12	24	30	36	42	54	66	12.00	24.00	30.00	12.00	24.00	30.00
1	16	28	34	44	50	62	75	15.00	27.00	33.00	15.00	27.00	33.00*
2	20	32	38	52	58	70	84	18.00	30.00	36.00	18.00	30.00	36.00*
3	24	36	42	60	66	78	93	21.00	33.00	39.00	21.00	33.00	39.00*
4	28	40	46	68	74	86	102	24.00	36.00	42.00	24.00	36.00	42.00*
5	32	44	50	76	82	86	111	29.67	37.67	43.67	32.00	36.50	42.50*
6	36	48	54	84	90	86	120	35.33	39.33	45.33	36.00	39.00	45.00*
7	40	52	58	92	98	86	129	41.00	41.00	47.00	40.00	41.50	47.50*
8	44	56	62	100	106	86	138	46.67	42.67	48.67	44.00	44.00	50.00*
9	48	60	66	108	106	86	147	51.00	47.00	49.00	44.00	50.50	52.50*
10	52	64	70	116	106	86	156	55.33	51.33	49.33	44.50	56.50	55.00*
11	56	68	74	116	106	86	165	58.33	54.33	52.33	45.75	57.75	61.50*
12	60	72	78	116	106	86	174	61.33	57.33	55.33	48.00	60.00	66.00*
13	64	76	78	116	106	86	183	65.00	61.00	57.00	52.33	64.33	66.33*
14	68	80	78	116	106	86	192	68.67	64.67	58.67	56.67	68.67	66.67*
15	72	80	78	116	106	86	201	73.00	67.00	61.00	62.33	70.33	68.33*
16	76	80	78	116	106	86	210	77.33	69.33	63.33	68.00	72.00	70.00*
17	80	80	78	116	106	86	219	81.67	71.67	65.67	73.67	73.67	71.67*
18	84	80	78	116	106	86	228	86.00	74.00	68.00	79.33	75.33	73.33*
19	84	80	78	116	106	86	237	89.00	77.00	71.00	82.33	78.33	76.33*
20	84	80	78	116	106	86	246	92.00	80.00	74.00	85.33	81.33	79.33
21	84	80	78	116	106	86	255	95.00	83.00	77.00	88.33	84.33	82.33
22	84	80	78	116	106	86	264	98.00	86.00	80.00	91.33	87.33	85.33

Interpretation

The type of results which are displayed here are not new. Looking at the Shapley values and nucleoli for the IG_2 rule we notice that while the order of the values are never inversely related to the initial endowments of the players, after a point as x_2^* increases, both the Shapley value and the nucleolus become symmetric. In other words, while it can't hurt to be "big" in this game, it may not help. This exact result was found before in the computation of Shapley power indices for legislatures and committees (see Lucas (2)). Here, consider the following example of a legislature defined by the following vector, L = (51; 45,35,20), where 51 is the majority needed to win, and 45,35, and 20 are the number of votes distributed to the three players. Now in the simple majority game defined by this vector, we see that while the votes are initially distributed asymmetrically, voting power is equally distributed. This is so because in order to win it is necessary to form a coalition of two players and any two player coalition in winning. The Shapley index is $S = (1/3,1/3,1/3)$. In addition, the Shapley index is symmetric for any voting majority in the closed interval [51,55]. After that point (for required voting majorities between 56 and 80) the Shapley index is asymmetrical and varies directly with the initial allocation of votes. Here there is obviously a similarity between varying the majority voting parameter and our variation of x_2^*.

For a general interpretation of our results, however, we will have to refer to a paper by David Starrett (7) in which he discusses the question of property rights and their influence upon the proper specification of characteristic functions for games including externalities and public goods. Starrett points to a difference between the specification of characteristic functions for two models of economies with externalities or public goods, the Shapley Shubik (6) garbage game, and Foley's model of an economy with public goods. The difference between the two characteristic functions, Starrett points out, is a difference between the type of property rights assumed. Talking about external diseconomies, if we make the assumption that everyone has a right to a pure environment, - as Foley would - then what Starrett calls the "common rule" of blocking is appropriate in the specification of the characteristic function. This rule states that anything produced by the counter coalition must be totally absorbed by them - they can't costlessly dump waste on anyone. However, if we assume that when people have possession of a good that possession entails the right to dispose of it as you wish (as Shapley and Shubik do), then the game involves what Starrett calls the "possession" rule of blocking. Here the product of a coalition (counter coalition) does not have to be absorbed by them, but can be costlessly dumped.

If we were to take a closer look at our three rules, we would notice that the o-type rule is strategically equivalent to a game in which the "common" rule of blocking is in effect. Here, the countercoalition $\overline{S}$ is forbidden to dump the benefits of public good construction on the coalition S (even though of course the coalition S would like them dumped). The IG rules, on the other hand, are both strategically equivalent to variants of the possession rule.

Given this interpretation of our rules, it is not hard to see why we have achieved the results we have. In the case of the o-type behavioral assumption, since this rule is strategically equivalent to the "common" rule of Starrett, all that a coalition or player can guarantee itself is what it can produce using only its own resources. It cannot expect that any amount of the public good will be dumped on it. Therefore, the larger a player is the more of the public good it can produce, and since in our example the marginal utility of public good consumption is always greater than the marginal utility of private good consumption, "larger" players are in a better strategic position and this advantage is reflected in the Shapley value and nucleolus.[1]

With the IG rules, however the situation is different. Under IG_2, it is obvious that for larger values of x_2^* the Shapley values and nucleoli will have an inverse relationship to the initial endowment of the players. This is true because the rules of this game specify that any coalition S can guarantee itself Min $(g_{\overline{S}}, x_2^*)$ of the public good being dumped on it free of charge, and cannot add to this amount from its own resources.

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[1]For the Shapley value larger players are awarded greater imputations since their marginal contribution to each coalition is greater and the Shapley value is merely the expected value of these marginal contributions. For the Nucleolus, the potential complaint of any coalition containing a larger player is always greater than the potential complaint of that same coalition if that player were replaced by some other player who was "smaller". Therefore, those players will be given preferential treatment in the nucleolus.

Therefore, as x_2^* increases, smaller coalitions can expect greater and greater amounts of the more highly valued public good to be delivered to them and can transform their resources into private goods. Conversely, "larger" coalitions can expect to have relatively little of the public good dumped on them by the remaining "smaller" players and since the rules forbid them to build the public good for themselves, they are stuck with a non-optimal amount of the less desirable public good in their final basket of goods. Consequently, we can clearly see why in this type of game the Shapley value and nucleolus might be inversely related to the initial endowment of the players. This is obviously more likely to occur as x_2^* is increased, since it is in these cases that the value of a small coalition is increased by the potentially large amounts of free public goods it knows it can expect.

Under the IG_1 rule, this inverse relationship cannot exist. This is so because the rule specifies that any coalition S can expect Min $(g_{\overline{S}}, x_2^*)$ of the public good to be dumped on it but now it is allowed to add what it wants to this amount. Therefore it is not forced to accept what the counter coalition dumps on it as final - and it is clear that it cannot hurt to be big. To demonstrate why it may not help, consider our example with $x_2^* = 20$. Here let us calculate the value of player 1. Under IG_1 he can rely on 18 units of the public good being produced by coalition (23). Since he has 4 units of g, he simply adds two units to the 18 thereby producing the maximum 20 units, and has 2 left to transform into private goods. His value is 86. For player 2 the situation is identical. He can rely not on 18 units of the public good, but on 14, Having 8 units of g initially he can add 6 units to the 14 dumped on him, transform his remaining two into private goods and achieve a value of 86 also. The same is true for player 3 even though his initial endowment of g was 10. The same reasoning follows for any of the two player coalitions, each achieving a value of 166. The game is symmetric. Therefore, since symmetric games determine symmetric Shapley values and nucleoli, having large initial endowments in these games may not benefit you at all.

To conclude, it should be clear from what we have said that a totally appealing way to model economies with public goods or externalities game theoretically does not as yet exist. Certainly the game theoretical models are more appealing than conventional neoclassical models on behavioral grounds since no matter how confused we can get in deciding upon which blocking rule to use in our definition of the characteristic function, any of these rules are better behavioral assumptions than the parametric and "truthful" assumptions assumed by standard neoclassical - Lindahl models. In addition, despite the many conceptual problems that may exist, we feel our results are still valid. In other words, in public goods economies it does not necessarily pay to be big.[1]

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[1] For similar results for economies with syndicates, see Schotter (5).

Sources

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5) Andrew Schotter, "Disadvantageous Syndicates in Public Goods Economies: An Example", Discussion Paper no. 75-43, Center for Applied Economics, August 1975.

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