ECONOMIC RESEARCH REPORTS

INVESTMENT, DEPRECIATION, AND CAPITAL UTILIZATION: THEORETICAL AND SIMULATION RESULTS

BY

Jeffrey I. Bernstein
and
M. Ishaq Nadiri

RR # 91-21 **April, 1991**

C. V. STARR CENTER FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

INVESTMENT, DEPRECIATION, AND CAPITAL UTILIZATION: THEORETICAL AND SIMULATION RESULTS

Jeffrey I. Bernstein*

M. Ishaq Nadiri**

Revised
February 1991

* Carleton University and NBER
** New York University and NBER

ABSTRACT

A model of investment and capital utilization is developed. The problem of capital utilization is considered in a context of joint products of current output and capital output. Capital utilization decisions are forward-looking and represent a trade-off between current and future output production.

In the paper the dynamic adjustment paths of investment and utilization are determined. The path of capital utilization is a flexible accelerator and depends on changes in product price, wage rate and interest rate. In addition, model simulations are undertaken through a parameterization of the production, adjustment cost, and duration cost functions to show how sensitive capital utilization is to changes in the coefficients determining capital accumulation.

JEL classification: 641, 022

Address correspondence to:

Jeffrey I. Bernstein
Department of Economics
Carleton University
Ottawa, Ontario K1S 5B6
CANADA

M. Ishaq Nadiri
Department of Economics
New York University
269 Mercer Street, 7th fl.
New York, NY 10003
U.S.A.

1. Introduction*

The analysis of investment has generally ignored the interdependence between capital depreciation, utilization, and accumulation. Utilization is usually assumed to be costless (and hence there is no incentive for a firm to retain idle capacity), while depreciation is often assumed to be constant. Recently, investment theory has been extended to account for costly utilization. A. Abel (1981) and J. Bernstein (1983), following Betancourt and Clague (1978), Lucas (1970), and Winston and McCoy (1974) have characterized the determination of utilization by the trade-off between output expansion and a higher wage bill. In this framework, the wage rate varies with the utilization rate, but decisions on capital utilization were not forward looking or did not involve an intertemporal dimension because depreciation was assumed to be constant. In other words, the lifetime of capital was unaffected by the rate at which the factor was utilized.

The first purpose of this paper is to analyze the determinants of capital depreciation and utilization and their interdependence with investment decisions. Research on depreciation and utilization by K. Smith (1970), P. Taubman and M. Wilkinson (1970), W. Oi (1981), and M. Kim (1988) has emphasized the dependence of depreciation on the utilization rate. This rate was determined by balancing the increase in current output against the increase in depreciation costs. However, decisions affecting capital depreciation influence not only current but also future production through their effect on investment demand.

In this paper we incorporate into the theory of investment a general view of capital utilization, first developed by J. Hicks (1946), E. Malinvaud (1953), and later by C. Bliss (1975) and E. Diewert (1980). This approach characterizes the flow of undepreciated capital as a current output to be used

as an input in the future. At each date, capital and noncapital inputs are combined to produce current output and the capital inputs to be used for future production. Thus capital accumulation occurs in a joint product context, as two kinds of output are produced: one type for current sale and one type for future production. Epstein and Denny (1980) estimated a short-run model incorporating undepreciated capital as an output. Their interest was in the estimation of short-run factor demand and output supply functions and not with the dynamics and comparative dynamics associated with choices relating to capital depreciation, utilization, and accumulation.

The second purpose of this paper is to develop a model that captures the stylized facts obtained by M. Foss (1981) and the estimation results due to M.I. Nadiri and S. Rosen (1969). First, Foss found that, as the wage rate increased, the rate of capital utilization increased while the growth rates of capital and labor declined. Second, as product demand grew the growth rates of capital and labor increased along with the rate of utilization. Third, Nadiri and Rosen estimated that the capital utilization rate exhibited a dynamic adjustment process. They found that the utilization rate not only interacts with the rates of capital and labor accumulation but can indeed be characterized by a flexible accelerator adjustment process.

Our third purpose in this paper is to simulate the model in order to investigate how sensitive depreciation, capital stock and output are to specifications in production technology, adjustment, and duration costs. The magnitudes of depreciation, capital stock and output are determined from the dynamic adjustment path for various parameter values of the production, adjustment, and duration cost functions. The sensitivity of the variables are based on a comparison to the measured quantities from 1986 data for the manufacturing sector.

In Section 2 of this paper the model is developed and the nature of the short-run equilibrium is established. The dynamic properties and the steady state are analyzed in section 3. In part 4 the comparative steady state and comparative dynamic results are obtained. Section 5 contains the model simulations and lastly, we summarize and conclude the paper.

2. The Model and Short-Run Equilibrium

A production process is represented by

$$(1) \qquad y(t) = F(K_N(t),\ L(t),\ K_O(t))$$

where $y(t)$ is the output quantity, $K_N(t)$ and $L(t)$ are the quantities of the capital and labor inputs respectively, $K_O(t)$ is the quantity of the capital output, F is the twice continuously differentiable production function which is homogeneous of degree one, with positive and diminishing marginal products in the two inputs, while increases in the capital output decrease output at a decreasing rate. Thus $F_N > 0$, $F_L > 0$, $F_O < 0$, $F_{NN} < 0$, $F_{LL} < 0$, $F_{OO} < 0$.

The inputs $K_N(t)$ and $L(t)$ are combined to produce the joint products $y(t)$ and $K_O(t)$. The former output is produced for current sale and the latter is to be used for future production. The variable $y(t)$ can be referred to as the final product or output in the current period, while $K_O(t)$ represents an intermediate product which is used in production in the next period. The endogeneity of capital utilization is captured through the selection of the capital output. The choice regarding capital available for future production reflects decisions on the utilization of the capital input in current production (see Bliss (1975) and Diewert (1980)).[1]

There are two ways in which capital becomes available for future production: internal investment through nonutilization and external investment through acquisition. This implies that capital accumulates according to

$$(2) \qquad \dot{K}_N(t) = I_N(t) - K_O(t) - K_N(t), \quad K_N(0) = K_N^o > 0$$

where $I_N(T)$ is gross investment in capital. Equation (2) generalizes the standard formulation of capital accumulation. This can be seen by noting that the depreciation rate is defined by $(K_N(t) - K_O(t))/K_N(t) = \delta(t)$. Thus, equation (2) can be rewritten as $\dot{K}_N(t) = I_N(t) - \delta(t)K_N(t)$. If $\delta(t)$ is time invariant and exogenous, then depreciation occurs in the usual manner. The depreciation rate represents the outcome regarding the decision on capital utilization. In this model it is assumed that capital output is nonnegative and does not exceed the capital input. Hence, $0 \leq \delta(t) \leq 1$.

The definition of $\delta(t)$ enables the production function to be written as $y(t) = F[K_N(t), L(t), (1-\delta(t))K_N(t)]$, where $\delta(t)K_N(t)$ is the depreciated or utilized capital. The depreciation rate is thereby considered synonymous with the utilization rate. In this model, following Nadiri and Rosen (1969), and Taubman and Wilkinson (1970), it is assumed that the marginal product of capital input ($F_N > 0$) is not necessarily equal to the marginal product of utilized capital ($-F_O > 0$).[2]

As emphasized by Oi (1962), Nadiri and Rosen (1969), Abel (1981) and Bernstein (1983), labor is also treated as a quasi-fixed factor in this model. However, since the focus is on capital utilization and depreciation, we assume that

$$(3) \qquad \dot{L}(t) = I_L(t) - \mu L(t), \qquad L(0) = L^\circ > 0$$

$I_L(t)$ is gross investment in labor, and $0 \leq \mu \leq 1$ is the fixed rate of labor departure, reflecting in a simple way quits, retirements, firings, and layoffs.[3]

The distinction between capital stock and flow decisions can be noted from equations (1) and (2). At any time, the capital stock to be used in current production is predetermined. This means that there exists a given bundle of capital services which is embedded in the stock of capital. The flow of services from the capital stock actually used or capital utilization is selected and combined with labor services to produce current output. The choice on utilization is captured indirectly through the decision on the capital output or the flow of capital services available for production in the next period. The additions to the stock of capital consist of newly acquired capital (or gross investment) and the difference between the stock of capital available for future production and the amount that was available for current production.

There are duration costs associated with capital utilization. As labor works longer hours, remuneration rises. Following Whinston and McCoy (1974), Abel (1981), Bernstein (1983), and Betancourt, Clague and Panagariya (1988), the wage rate depends on capital utilization. In this model $w = W(K_0/L)$ with W the twice continuously differentiable wage function, $W' < 0$ and $W'' > 0$. An increase in capital output implies that capital utilization decreases and therefore the wage rate falls. Given the definition of the depreciation rate, the wage function can also be written as $W((1-\delta)K_N/L)$. An increase in capital utilization means that δ rises and so the wage rate changes by $-W'K_N/L > 0$.[4]

There are adjustment costs associated with the quasi-fixed factors, which are internal to and separable from the production process (see R. Lucas (1967), J. Gould (1968), A. Treadway (1971), D. Mortenson (1973) and L. Epstein (1982)). These costs affect the flow of funds which can be represented as

$$(4)\quad V(t) = p(t)y(t) - W(K_0(t)/L(t))L(t) - C(I_N(t)/K_N(t))I_N(t) - D(I_L(t)/L(t))I_L(t)$$

where $V(t)$ is the flow of funds, $p(t) > 0$ is the exogenous product price, C is the twice-continuously differentiable unit capital adjustment cost function with $C(0) = p_N(t)$, $C' > 0$ for $I_N(t) > 0$ and $p_N(t) > 0$ is the exogeneous purchase price of capital. In addition, total capital adjustment costs are strictly convex in $I_N(t)$. The unit labor adjustment cost function, D, has the same properties as the unit capital adjustment cost function except $D(0) = 0$. Adjustment costs for capital and labor are internal but separable from the production process and arise from the installation of capital and labor into the production process.[5]

The objective is to maximize the present value of the flow of funds, which is discounted by the interest rate r, subject to equations (1)-(3). Capital output and gross investment in capital and labor are selected in order to carry out this program.[6] The Hamiltonian is

$$(5)\quad H = pLf(k,(1-\delta)k) - W((1-\delta)k)L - C(I_N/K_N)I_N - D(I_L/L)I_L + q_1(I_N-\delta K_N) + q_2(I_L-\mu L)$$

where $y = Lf(k,(1-\delta)k)$ is derived from equation (1) using the homogeneity condition on the technology, $k = K_N/L$ is the capital intensity and f_i $i = 1,2$ are the derivatives of the production function defined in intensive form. In addition, q_1 is the capital investment shadow or demand price and q_2 is the labor investment shadow price. These variables also represent the price of installed or unutilized capital and the price of installed labor.

The first order and canonical conditions are

$$(6.1) \quad \partial H/\partial K_O = pf_2 - W' + q_1 = 0$$

$$(6.2) \quad \partial H/\partial I_N = -C'I_N/K_N - C(I_N/K_N) + q_1 = 0$$

$$(6.3) \quad \partial H/\partial I_L = -D'I_L/L - D(I_L/L) + q_2 = 0$$

$$(6.4) \quad \dot{k} = k(I_N/K_N - \delta - I_L/L + \mu)$$

$$(6.5) \quad \dot{q}_1 = (r+\delta)q_1 - p(f_1 + (1-\delta)f_2) + W'(1-\delta) - C'(I_N/K_N)^2$$

$$(6.6) \quad \dot{q}_2 = (r+\mu)q_2 - p(f(k,(1-\delta)k) - kf_1 - (1-\delta)kf_2) + W((1-\delta)k) - W'(1-\delta)k - D'(I_L/L)^2.$$

To understand the implications of the equilibrium conditions, consider the short run equilibrium.[7] This equilibrium is defined for given k, q_1 and q_2, by equations (6.1)-(6.3). The first equation (6.1) shows the determination of the allocation of the given stocks of capital and labor between current output and capital output. This is illustrated in (K_O, y) space in Figure 1. The slope of the product transformation curve is $f_2 < 0$

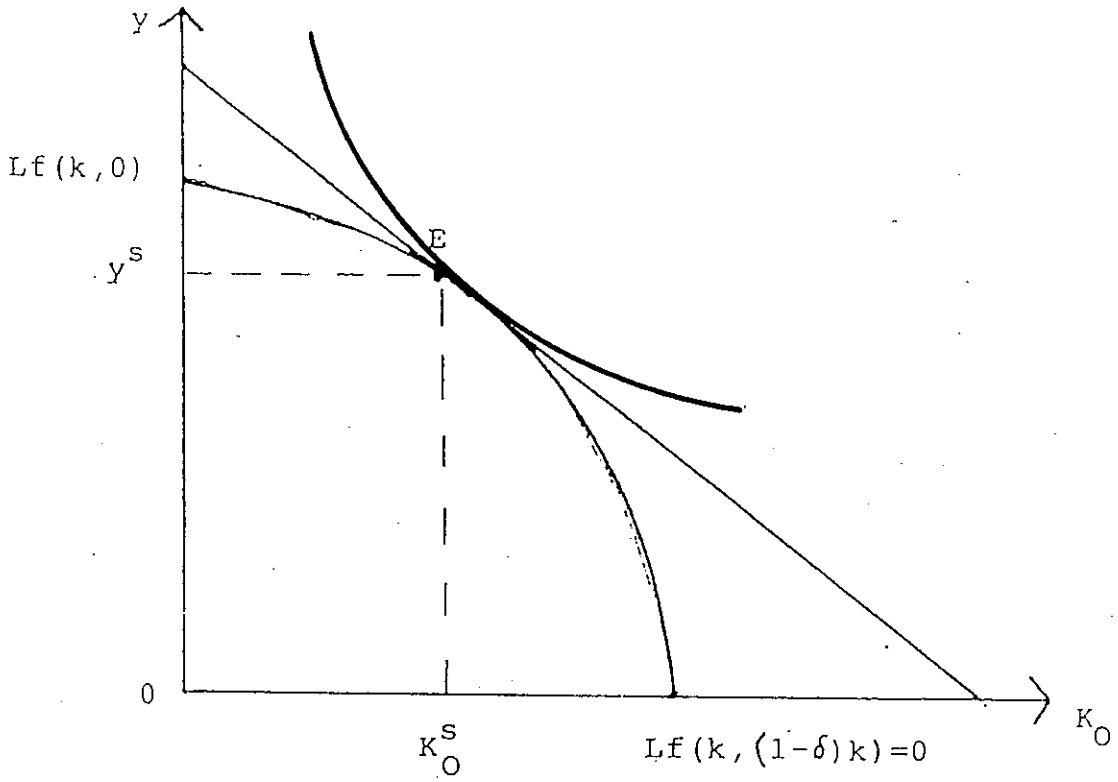

Figure 1. Short Run Equilibrium for Current Output and Capital Output

and since f is strictly concave, the curve is also strictly concave. The slope of the isorevenue curve in Figure 1 is $(W'-q_1)/p$, and since $W'' > 0$, the curve is strictly convex. Figure 1 shows that the trade-off is between current output and future output, manifested by the stock of unused capital available for future production.[8] In equilibrium the relative marginal cost $(-f_2)$ is equal to the relative marginal revenue $(-W' + q_1)/p)$. There are two components of marginal revenue. The first part is the reduction in duration costs as capital is utilized less and thereby more of it is available for

future production. The second component is the shadow price q_1. Since the equilibrium magnitudes depend on the shadow price, the allocation decision is forward looking, as this price equals the present value of the marginal benefits from installed capital or capital available for future production. This means that the utilization of capital embodied in the selection of K_0 is an investment decision.

Alternatively, equation (6.1) can be viewed as the short run solution for the depreciation or the utilization rate, δ, which depends on the capital intensity, the product price and the price of installed or unutilized capital. For an increase in q_1, the marginal value of unutilized capital rises, and as a consequence, the utilization rate falls. This, of course, implies that current output decreases. The converse occurs for an increase in the product price p.[9] Lastly, an increase in the capital intensity generates the following effect on δ, from equation (6.1),

$$(7) \qquad \partial\delta/\partial k = (p(f_{21} + f_{22}(1-\delta)) - W''(1-\delta))/(pf_{22} - W'')k.$$

The sign of the right side of (7) depends on $f_{22}(1-\delta) + f_{21}$, since $f_{22} < 0$ and $W'' > 0$. It is assumed that $f_{22}(1+\delta) + f_{21} < 0$. The reasonableness of this assumption can be noted from $-f_1/f_2 > 0$ which is the marginal product of capital in the production of capital for future use. Generally it is assumed that marginal products diminish. Therefore assuming that as the capital input (K_N) increases, the marginal product of capital decreases in the production of the capital output (K_O), implies that $f_{22}(1+\delta) + f_{21} < 0$.[10]

Summarizing the results from (6.1),

(8) $$\delta = \Gamma(k, q_1, p), \quad \Gamma_1 > 0, \quad \Gamma_2 < 0, \quad \Gamma_3 > 0.$$

The gross investment decisions for both capital and labor are forward looking. From (6.2) an increase in the marginal value of capital investment raises the rate of capital investment,

(9) $$I_N/K_N = X_k(q_1), \quad X_k' = 1/(C''(I_N/K_N) + 2C') > 0.$$

Similarly for labor investment,

(10) $$I_L/L = X_\ell(q_2), \quad X_\ell' = 1/(D''(I_L/L) + 2D') > 0.$$

Capital is utilized and investment occurs until the marginal cost of producing capital for future use net of the reduction in duration cost equals the marginal cost of purchasing and installing capital (see equations (6.1) and (6.2)). The equality between these marginal costs points out that there are indeed two forms of capital investment in this model. One type of investment can be considered internal through utilization decisions and the other can be considered external through acquisition decisions.

3. The Dynamics and the Steady State

The purpose of this section is to characterize the dynamic adjustment paths of the rates of capital and labor investment along with the path of the capital utilization rate. First the capital intensity growth rate is determined by substituting equations (8)-(10) into (6.4) so that,

$$\dot{k}/k = X_k(q_1) - \Gamma(k,q_1,p) - X_\ell(q_2) + \mu. \tag{11}$$

The capital intensity growth rate depends on the investment shadow prices. Unlike the situation with exogenous capital depreciation, the capital intensity growth rate also depends on the capital intensity itself. The growth rate is a decreasing function of the capital intensity because as the latter increases, diminishing marginal productivities of labor and capital cause there to be less of a need for further increases in the capital intensity. Thus

$$\partial(\dot{k}/k)/\partial k = -\Gamma < 0. \tag{12.1}$$

An increase in the marginal value of capital leads to more capital investment (both internal and external), thereby causing an increase in the capital intensity growth rate by

$$\partial(\dot{k}/k)/\partial q_1 = X_k' - \Gamma_2 > 0. \tag{12.2}$$

Lastly, since an increase in the marginal value of labor investment increases the labor growth rate, then the capital intensity growth rate decreases by

$$\partial(\dot{k}/k)/\partial q_2 = -X_\ell' < 0. \tag{12.3}$$

Because the capital intensity is changing through time, the marginal values of capital and labor investment exhibit intertemporal movement. Substituting equations (8) and (9) into (6.5), the dynamic path of the price of installed or unutilized capital is given by

$$(13) \quad \dot{q}_1 = (r + \Gamma(k,q_1,p))q_1 - p(f_1(k,(1-\Gamma(k,q_1,p))k) + (1 - \Gamma(k,q_1,p))f_2(k,(1-\Gamma(k,q_1,p))k) + W'(1-\Gamma(k,q_1,p)k)(1-\Gamma(k,q_1,p)) - C'(X_k(q_1))(X_k(q_1))^2.$$

From (13) it can be seen that the capital stock is chosen such that the opportunity cost of funds or the interest rate equals the rate of return on capital. The latter consists of four elements: The value of the marginal product of capital net of depreciation, the decline in installation costs arising from having a larger stock of capital, the decline in duration costs from having more capital, and capital gains associated with the installed capital.

When the price of installed or unutilized capital increases, a capital gain must occur in order to keep the rate of return on capital equal to the interest rate. Indeed

$$(14.1) \quad \partial\dot{q}_1/\partial q_1 = r + \delta + k\Gamma_2(pf_{12} + p(1-\delta)f_{22} - W''(1-\delta)) - I_N/K_N > 0.$$

The right side of (14.1) is positive because $r+\delta > I_N/K_N$ in order for the flow of funds to be finite.

An increase in the capital intensity causes a decrease in the value of the marginal product of capital. To retain the equality between the interest rate and the rate of return, a capital gain must occur. Thus differentiating (13) with respect to the capital intensity and making use of equation (7) yields

$$(14.2) \qquad \partial \dot{q}_1/\partial k = -p[p(f_{11}f_{22} - f_{12}^2) - f_{11}W"]/(pf_{22} - W") > 0.$$

Next, consider the path of the price of installed labor in the production process.

$$(15) \qquad \dot{q}_2 = (r+\mu)q_2 - p[f(k,(1-\Gamma(k,q_1,p))k) - f_1(k,1-\Gamma(k,q_1,p))k)k - f_2(k,1-\Gamma(k,q_1,p))k)(1-\Gamma(k,q_1))k] + W((1-\Gamma(k,q_1,p))k) - W'((1-\Gamma(k,q_1,p))k)(1-\Gamma(k,q_1,p))k - D'(X_\ell(q_2))(X_\ell(q_2))^2.$$

The interest rate is equated to the rate of return on labor. The latter consists of four elements: the value of the marginal product of labor net of departures, the reduction in adjustment costs due to a larger labor force, the wage rate including the duration premium paid to the workers, and the capital gains from training labor.

In order to retain equilibrium condition (15) when the price of installed labor increases, a capital gain must accrue and when the capital intensity increases, a capital loss must occur. Thus

$$(16.1) \qquad \partial \dot{q}_2/\partial q_2 = r + \mu - I_L/L > 0$$

and

$$(16.2) \qquad \partial \dot{q}_2/\partial k = pk[p(f_{11}f_{22} - f_{12}^2) - f_{11}W"]/(pf_{22} - W") < 0.$$

Moreover, the price of installed labor depends on the price of installed or unutilized capital since the marginal product of labor is affected by capital

utilization. Differentiating (15) with respect to the price of unutilized capital yields

$$(16.3) \qquad \partial \dot{q}_2/\partial q_1 = -k^2\Gamma_2[p(f_{12} + f_{22}(1-\delta)) - W''(1-\delta)] < 0.$$

An increase in the price of unutilized capital lowers the depreciation rate and thereby increases the marginal product of labor. In order for the interest rate to remain equal to the rate of return on labor, a capital loss must occur.

The properties of the time paths of the capital intensity and the prices of installed capital and labor have been analyzed. Hence the dynamic path and long-run equilibrium can now be characterized. The long-run equilibrium or steady state, defined for $\dot{k} = \dot{q}_1 = \dot{q}_2 = 0$ can be illustrated in a four quadrant diagram. Figure 2 shows the steady state in the following manner. First, since the $\dot{k} = 0$ locus in (q_1, k) space depends on q_2, in this quadrant the locus must be defined for the steady state value of q_2 which is q_2^e. Similarly, $\dot{k} = 0$ in the (q_2, k) quadrant and $\dot{q}_2 = 0$ in (q_2, k) must be drawn for the steady state value, q_1^e. In $(q_2, q_1,)$ space, the $\dot{q}_2 = 0$ curve is consistent with the steady state capital intensity, k^e. Second, the curves must be drawn such that their intersections form a rectangle. The two properties together, one relating to the position of each locus, and the other to the position of the intersections, permit the illustration of the steady state.

Not only does the steady state exist (from the properties of the production, wage, and adjustment cost functions), but it is unique. Uniqueness can be demonstrated from Figure 3. Suppose point A represents

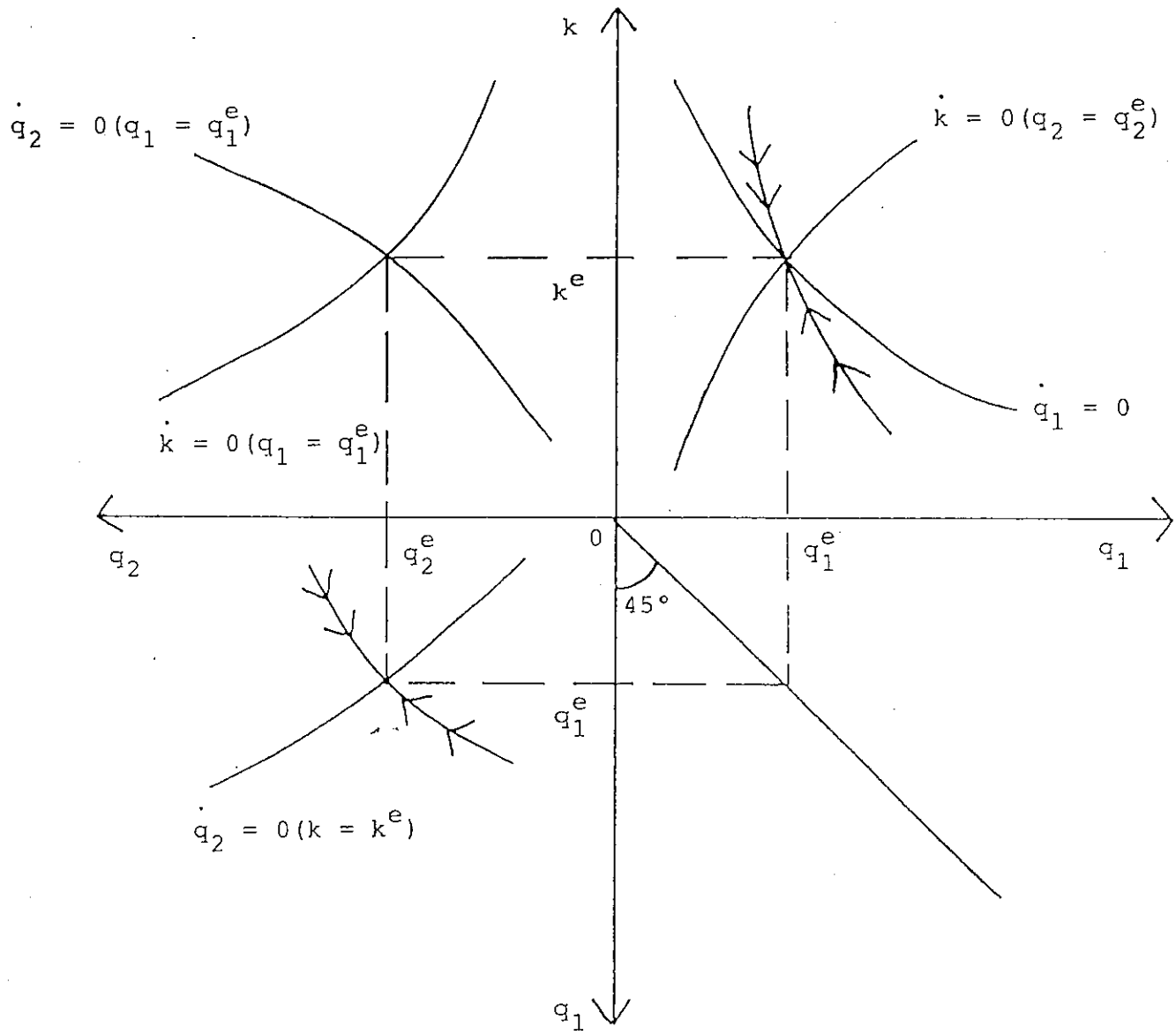

Figure 2. Steady State and Dynamic Paths

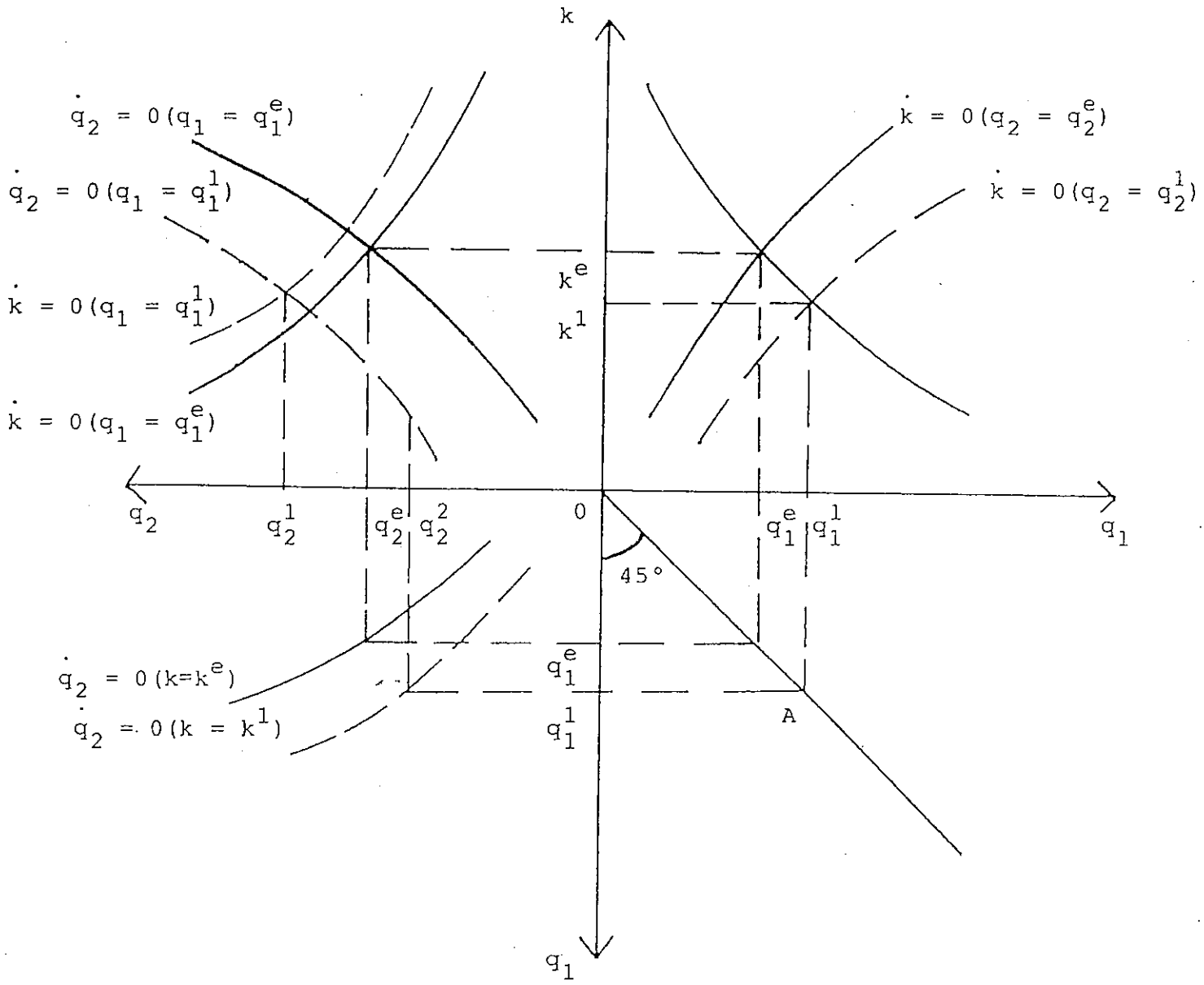

Figure 3. Uniqueness of the Steady State

another steady state value of q_1, q_1^1. By construction, $q_1^1 > q_1^e$. The higher price means that in (q_2, k) space the $\dot{q}_2 = 0$ locus shifts down and to the left (by equation (16.3)) and the $\dot{k} = 0$ locus shifts up and to the left (by equation (12.2)). The new curves intersect such that $q_2 = q_2^1 > q_2^e$. The higher q_2 causes the $\dot{k} = 0$ curve in (q_1, k) space to shift down and to the right (by equation (12.3)) so that capital intensity decreases to $k^1 < k^e$. But the decrease in capital intensity shifts the $\dot{q}_2 = 0$ locus down and to the right in (q_2, q_1) space (by equation (16.2)). Hence with q_1^1 and k^1 the price of installed labor in the production process is q_2^2 and not q_2^1. This means that there is only a single rectangle consistent with the various curves and intersections and therefore there is a unique steady state.

The stability of the steady state and the characteristics of the path can be determined from the linearization of equations (11), (13) and (15) around the steady state (k^e, q_1^e, q_2^e). The system can be written as

$$(17)\qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} k^a \\ q_1^a \\ q_2^a \end{bmatrix} = \begin{bmatrix} \dot{k} \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

where

$$a_{11} = -\Gamma_1,\ a_{12} = k(X_k' - \Gamma_2),\ a_{13} = kX_\ell'$$

$$a_{21} = -p[p(f_{11}f_{22}-f_{12}^2)-f_{11}W'']/(pf_{22}-W'')$$

$$a_{22} = r+\delta+k\Gamma_2[p(f_{12}+(1-\delta)f_{22})-W''(1-\delta)]-I_N/K_N$$

$$a_{23} = 0$$

$$a_{31} = pk[p(f_{11}f_{22}-f_{12}^2)-f_{11}W'']/(pf_{22}-W'')$$

$$a_{32} = -k^2\Gamma_2[p(f_{12}+(1-\delta)f_{22})-W''(1-\delta)]$$

$$a_{33} = r+\mu-I_L/L$$

$k^a = k-k^e$, $q_1^a = q_1-q_1^e$, and $q_2^a = q_2-q_2^e$.

There are three characteristic roots or eigenvalues which solve equation (17). Using $(a_{21}a_{32} - a_{31}a_{22}) = a_{33}a_{31}$, then the first root is $\lambda_1 = a_{33} > 0$ and with $b = a_{11} + a_{22} = r + \delta - I_N/K_N > 0$ and $c = a_{11}a_{22} - a_{21}a_{12} - a_{13}a_{31} < 0$, the second and third roots are $\lambda_2, \lambda_3 = [b \pm (b^2 - 4c)^{1/2}]/2$. Since $c < 0$, then the second and third roots are real and because $(b^2 - 4c)^{1/2} > b > 0$, one root is positive and the other is negative. This means that the steady state is a saddle point. In addition, because the roots are real, the path to the steady state does not involve any cycles. The unstable roots are positive and the stable root is negative.

Using the stable solution to equation set (17) yields

$$\dot{k} = \lambda(k - k^e) \tag{18.1}$$

$$\dot{q}_1 = [\lambda a_{21}/(\lambda - a_{22})](k - k^e) \tag{18.2}$$

$$\dot{q}_2 = [\lambda a_{31}/(\lambda - a_{22})](k - k^e), \tag{18.3}$$

where $\lambda = \lambda_2 < 0$. The shape of the adjustment paths of the capital intensity and the prices of unutilized or installed capital and labor are given by equation set (18) and illustrated in Figure 2. From equation (18.1) $k^o > k^e$ then $\dot{k} < 0$ and k decreases along the path. Simultaneously, from (18.2) since $a_{21} > 0$, $a_{22} > 0$ then $\dot{q}_1 > 0$ and q_1 increases. Thus there is an inverse relationship between k and q_1 along the path. This movement is shown in the (q_1, k) quadrant in Figure 2. Next, from (18.2) with $a_{31} < 0$, and $a_{22} > 0$,

then $\dot{q}_2 < 0$ and q_2 decreases. Thus there is a direct relationship between k and q_2 along the path and an inverse relationship between q_1 and q_2. This latter movement is illustrated in the (q_2, q_1) quadrant of Figure 2.

To understand the intuition behind these results, consider an initial situation with insufficient installed labor relative to installed (or unutilized) capital ($k^o > k^e$). The marginal value of the labor force in the production process must exceed the long-run magnitude ($q_2^o > q_2^e$), in order for the labor force to increase. Simultaneously, the marginal value of capital is below the steady state value ($q_1^o < q_1^e$) so there is less incentive to accumulate capital either through acquisition or nonutilization.

The results on the prices for installed capital and labor imply (from equations (9) and (10)) that $(I_N/K_N)^o < (I_N/K_N)^e$ and $(I_L/L)^e > (I_L/L)^e$. Since the capital intensity decreases to the steady state, the rate of labor investment must exceed the steady state rate, while the converse must occur for the rate of capital investment.

The behavior of the depreciation or the utilization rate is governed by the intertemporal movement of the capital intensity and the price of unutilized capital (from equation (8)). By time differentiating equation (8) and using equations (18.1) and (18.2), the adjustment path of the depreciation or utilization rate is

$$\dot{\delta} = [(\Gamma_1(\lambda - a_{22}) + \Gamma_2 a_{21})\lambda/(\lambda - a_{22})](k - k^e). \tag{19}$$

Thus the adjustment path for the capital utilization rate is a flexible accelerator. This result rigorously establishes the empirical finding obtained by Nadiri and Rosen (1969) that the adjustment path of the capital

utilization rate is similar to the paths of capital and labor growth rates, which are themselves governed by flexible accelerators. Along the dynamic path, as the capital intensity decreases the utilization rate declines for two reasons. First, there is the direct effect of the capital intensity on the utilization rate. A decrease in the capital intensity leads to a reallocation of resources towards capital output which decreases the rate. Second, there is the indirect effect, which arises because the decrease in capital intensity causes the price of unutilized capital to increase. Since the marginal value of unutilized capital increases, the utilization rate falls. Thus $k^o > k^e$ implies that $\delta^o > \delta^e$. These results mean that along the dynamic path the capital utilization rate and the rate of capital investment are inversely related, while the capital utilization rate and the labor investment rate are directly correlated.

4. Comparative Steady State and Dynamics

This section is concerned with the analysis of the effects of unanticipated changes in input supply and product demand conditions on the steady state and dynamic adjustment path. The stable adjustment path (from equation set (18)) can be written as

$$\dot{k} = \lambda(k - \Lambda(s,r,\mu,p)) \tag{20.1}$$

$$q_i = Q^i(s,r,\mu,p) + \lambda\phi_i(k - \Lambda(s,r,\mu,p)), \quad i=1,2, \tag{20.2}$$

where $\lambda < 0$, $\phi_1 = a_{21}/(\lambda - a_{22}) < 0$, $\phi_2 = a_{31}/(\lambda - a_{22}) > 0$, $k^e = \Lambda(s,r,\mu,p)$ and $q_i^e = Q^i(s,r,\mu,p)$. Clearly, in order to determine the effects of unanticipated

shocks to the dynamic adjustment path, the effects on the steady state must be derived. These results are presented in table 1 and they will be discussed as we consider each shock to the dynamic adjustment path. The expressions in Table 1 were determined from equations (11), (13) and (15) with $\dot{k} = 0 = \dot{q}_1 = \dot{q}_2$ so that $k = k^e$, $q_1 = q_1^e$ and $q_2 = q_2^e$ and then differentiating the three equations with respect to the scale factor or rate (s, see footnote 4), interest rate (r), departure rate (μ) and product price (p).

To begin the analysis, from equation (20.1) it is clear that unanticipated changes in the scale rate, interest rate, departure rate, and product price cause the capital intensity to change along the path in a direct and proportional manner to the steady state capital intensity.[11]

In order to determine the effects of unanticipated product demand and factor supply shocks on the rates of capital and labor investment and on the rate of capital utilization, the results presented in table 1 and equation (20.2) must be combined. First, for an increase in the scale rate using the first row of table 1,

$$(21.1) \qquad \partial q_1^e/\partial s < (\partial q_1^e/\partial s)\lambda/(\lambda - a_{22}) = \partial q_1/\partial s < 0$$

$$(21.2) \qquad \partial q_2/\partial s = \partial q_2^e/\partial s - \phi_2 \partial k^e/\partial s < \partial q_2^e/\partial s < 0.$$

The results from (21.1) and (21.2) together with the investment demand functions defined by equations (9) and (10) imply that the rate of capital investment declines along the dynamic path in response to an increase in the scale rate but not by as much as the steady state capital investment rate decreases. The labor investment rate also decreases along the adjustment

TABLE 1

Comparative Steady State Results

Exog. Var.	Endogenous Variable		
	k^e	q_1^e	q_2^e
s	$a_{22}a_{13}/A^* > 0$	$-a_{21}a_{13}/A > 0$	$-(a_{11}a_{22}-a_{21}a_{12})/A < 0$
r	$[q_1(a_{33}a_{12}-a_{13}a_{32}) + q_2a_{13}a_{22}]/A$	$[a_{13}(q_1a_{31}-q_2a_{21}) -q_1a_{11}a_{33}]/A < 0$	$[q_1(a_{32}a_{11}-a_{31}a_{12}) -q_2(a_{11}a_{22}-a_{12}a_{21})]/A < 0$
μ	$(a_{22}a_{33}+q_2a_{22}a_{13})/A > 0$	$a_{21}(a_{33}-q_2a_{13})/A < 0$	$[a_{31}a_{22}-a_{21}a_{32}) -q_2(a_{11}a_{32}-a_{21}a_{12})]/A$
p	$[(\partial\dot{q}_1/\partial p)(a_{12}a_{33}-a_{32}a_{13}) +a_{22}(a_{13}\partial\dot{q}_2/\partial p) -a_{33}\partial\dot{k}/\partial p)]/A$	$[a_{21}(a_{33}\partial\dot{k}/\partial p-a_{13}\partial\dot{q}_2/\partial p) -(\partial\dot{q}_1/\partial p)(a_{11}a_{33} -a_{13}a_{31})]/A > 0$	$[a_{11}a_{32}\partial\dot{q}_1/\partial p + (\partial\dot{q}_2/\partial p)(a_{21}a_{12}-a_{11}a_{22})- a_{21}a_{32}\partial\dot{k}/\partial p + a_{31} (a_{22}\partial\dot{k}/\partial p- a_{12}\partial\dot{q}_1/\partial p)]/A > 0$

* $A < 0$ is the determinant of the matrix in (17).

path, but the decrease exceeds the steady state decline in the rate of labor investment as the scale rate increases.

The dynamic path of the capital utilization rate is also affected by the scale rate. Using equation (8), the results in table 1 and (21.1),

$$(21.3) \qquad 0 < \partial\delta/\partial s = \Gamma_2(\partial q_1^e/\partial s)\lambda/(\lambda - a_{22}) < \Gamma_2\partial q_1^e/\partial s < \partial\delta^e/\partial s.$$

Thus an increase in the scale rate increases the capital utilization rate but the increase in the rate is not as great as the increase in the steady state rate. These results establish that along the adjustment path increases in the scale rate cause the rates of capital and labor investment to move in the opposite direction to the utilization rate.

Next, for an increase in the interest rate, using the second row of table 1 and equation (20.2),[12]

$$(22.1) \quad \partial q_1/\partial r = -q_1[a_{11}a_{33} + \phi_1 a_{33}a_{12}]/A - q_2 a_{13}\lambda\phi_1/A + q_1\phi_1 a_{13}a_{31}[\lambda - a_{33}]/Aa_{21} < 0,$$

$$(22.2) \quad \partial q_2/\partial r = q_1[a_{32}a_{11} + \phi_2 a_{13}a_{32}]/A - q_2[a_{11}a_{22} - a_{21}a_{12} + \phi_2 a_{13}a_{22}]/A$$
$$- q_1\phi_2 a_{12}[\lambda - a_{22} + a_{33}]/A < 0.$$

The results from (22.1) and (22.2) imply that the rates of capital and labor investment decline in response to an unanticipated change in the interest rate along the dynamic path. In addition, from (22.1) and equation (8) an increase in the interest rate causes the capital utilization rate to increase since the price of unutilized capital falls along the adjustment path. Thus, as for the scale rate, an unanticipated increase in the interest rate causes the counter

movement along the adjustment path between the capital utilization rate and the rates of capital and labor investment.

If there is an unanticipated increase in the rate of labor departures, then from equation (20.2), the third row of table 1, the price of installed capital decreases along the adjustment path but not by as much as in the steady state. However, there is an ambiguous effect on the price of installed labor along the adjustment path. The ambiguity arises because an increase in the departure rate decreases the rate of return on labor but simultaneously decreases the capital intensity and the price of unutilized capital. The latter two effects serve to increase the rate of return on labor. Therefore, although the rate of capital investment decreases along the path in response to an increase in the departure rate, and in addition, this decrease is less than that found in the steady state, it is not possible to unambiguously determine the effect along the path of the rate of labor investment.

Increases in the labor departure rate cause the capital utilization rate to increase. Thus, there is a direct relationship between the two rates. Moreover, the movement in the capital utilization rate in the steady state is more pronounced than that found along the adjustment path. Indeed,

$$(23) \qquad 0 < \partial\delta/\partial\mu = \Gamma_2 \partial q_1^e/\partial\mu(\lambda/(\lambda - a_{22})) < \Gamma_2 \partial q_1^e/\partial\mu < \partial\delta^e/\partial\mu.$$

Turning to the product demand shocks, suppose that there is an unanticipated increase in product demand along the dynamic adjustment path, so that there is an increase in the product price. The increase in the product price generates an increase in the price of installed capital such that from (20.2) and the fourth row of table 1,[13]

$$\partial q_1/\partial p = -(\phi_1 a_{13}\lambda\partial\dot{q}_2/\partial p)/A + \phi_1 a_{33}(a_{22}\partial\dot{k}/\partial p - a_{12}\partial\dot{q}_1/\partial p)/A$$

(24.1)

$$+ a_{33}(a_{21}\partial\dot{k}/\partial p - a_{11}\partial\dot{q}_1/\partial p)/A + (a_{13}(a_{31}+\phi_1 a_{32})\partial\dot{q}_1/\partial p)/A > 0.$$

In addition, the price of installed labor increases along the path for an increase in the product price,

$$\partial q_2/\partial p = (\partial\dot{q}_1/\partial p)(a_{11}a_{32} + \phi_2 a_{32}a_{13})/A + (\partial\dot{q}_2/\partial p)$$

(24.2)

$$(a_{21}a_{12} - a_{11}a_{22} - \phi_2 a_{22}a_{13})/A - (\partial\dot{k}/\partial p)a_{21}a_{32})/A$$

$$+ a_{31}(\lambda - a_{22} + a_{33})[-(\partial\dot{q}_1/\partial p)a_{12} + (\partial\dot{k}/\partial p)a_{22}]/A(\lambda - a_{22}) > 0.$$

These results imply that the rates of investment in capital and labor increase and move in the same direction.

There is also a tendency for the capital utilization rate to increase in response to changing product demand conditions. The effect on the capital utilization rate (using (8) and (24,1)) can be seen from

(24.3) $$\partial\delta/\partial p = \delta\eta_q(\xi_p - \xi_q)/p\xi_q,$$

where $\eta_q < 0$ is the elasticity of the utilization rate with respect to the price of unutilized capital (or the purchase price of capital), $\xi_p > 0$ is the elasticity of the rate of capital investment with respect to the product price and $\xi_q > 0$ is the elasticity of the rate of capital investment with respect to the price of unutilized capital. An unanticipated increase in the product price will increase capital utilization if the rate of capital investment is

relatively more inelastic with respect to the product price compared to the price of unutilized capital. In this situation, the increase in product demand will cause relatively more resources to be devoted to capital utilization and thereby current output will rise compared to capital investment and future output.

To summarize the results, the present model is able to capture the stylized facts of Foss (1981). Unanticipated changes in factor supply conditions generate movements in the rates of capital and labor investment in the same direction. These rates generally decrease. The capital utilization rate increases in response to changes in the supply side conditions and thereby moves in the opposite direction to the rates of investment. Unanticipated changes in product demand conditions, however, cause both rates of investment to increase and there is also the possibility for the capital utilization rate to increase. Thus, unlike changes in the supply side conditions, changes in product demand conditions can generate comovement in all three variables.

5. Model Simulation

In this section of the paper the production function, wage function and adjustment cost functions are parameterized in order to determine the sensitivity of the capital stock, output and capital utilization to changes in the parameter values. The parameterized discrete time equations are

$$(25.1) \qquad F(K_N(t-1),\ K_O(t)) = \beta_v + \beta_n K_N(t-1) + \beta_o K_O(t) + 0.5(\beta_{nn} K_N(t-1)^2 + \beta_{oo} K_O(t)^2) + \beta_{no} K_N(t-1) K_O(t)$$

(25.2) $W(K_O(t)) = 0.5\alpha_{oo}K_O(t)^2$

(25.3) $C(K_N(t) - K_N(t-1)) = 0.5\beta_{ii}(K_N(t) - K_N(t-1))^2$.

Due to constant returns to scale, for simplicity the labor input is normalized to unity. In addition, labor adjustment costs can be ignored so capital adjustment costs depend on investment.

The optimality conditions based on equation set (25) for the depreciation rate ($\delta(t) = [K_N(t-1) - K_O(t)]/K_N(t-1)$ and capital stock are

(26.1) $\delta(t) = [K_N(t-1) - (\alpha + \beta K_N(t-1))]/K_N(t-1)$

(26.2) $K_N(t) = mK_N^e(t) + (1-m)K_N(t-1)$,

where $\alpha = -(p_N + p\beta_o)/(p\beta_{oo})/(p\beta_{oo} - w_s\alpha_{oo})$, $\beta = -p\beta_{no}/(p\beta_{oo} - w_s\alpha_{oo})$, $m = -r/2 + p(\beta_{nn} + \beta_{no}\beta)/2\beta_{ii} + [(p(\beta_{nn} + \beta_{no}\beta)/2\beta_{ii} - r/2)^2 - p(\beta_{nn} + \beta_{no}\beta)]^{1/2}$ and $K_N^e = [p_N(1+r) - p(\beta_{no}\alpha + \beta_n)]/p(\beta_{nn} + \beta_{no}\beta)$, ($w_s$ is the scale wage rate, see footnote 4).

The simulation of equation set (26) proceeds in the following way. First, values of the exogenous variables are obtained. The product price, p, and the capital purchase price, p_N, are the 1986 indices relating to the manufacturing sector, the scale wage rate, w_s, is normalized to unity and the interest rate, r, is set at 0.10. The endogenous variables are the depreciation (or utilization) rate, the capital stock and output. The parameter values are obtained by successive iterations until the model solution for the capital stock, depreciation rate, and output are within 1% of the actual 1986 values of these variables for the manufacturing sector. This

solution is considered the base case. In addition, the parameter values must satisfy $\beta_n > 0$, $\beta_o < 0$, $\beta_{nn} < 0$, $\beta_{oo} < 0$, $\beta_{nn}\beta_{oo}-\beta_{no}^2 > 0$, $\alpha_{oo} > 0$, $\beta_{ii} > 0$. The base case parameter values are found to be $\beta_v = -1375$, $\beta_o = -0.02$, $\beta_n = 6.80$, $\beta_{oo} = -0.0743$, $\beta_{nn} = -0.0720$, $\beta_{no} = 0.0690$, $\alpha_{oo} = 0.00045$, $\beta_{ii} = 0.20$.

Model simulations are undertaken around the parameters that directly affect capital depreciation. The parameters are β_o, β_{oo}, β_{no}, and α_{oo}. The β_o parameter relates to the marginal cost of capital output, which is the capital available for future production (β_o also relates to the marginal product of capital utilization). However, simultaneously, the rate of marginal cost expansion, is held constant. In other words the concavity of the production function is not altered when β_o changes. Table 2 shows the results of the simulations for β_o. The middle row in the table is the base case simulation.

TABLE 2

Simulation around β_o
(Solution Values and Percentage Change from Actual)

Values of β_o	Capital Stock		Depreciation Rate		Output	
-0.002	970.50	0.20	0.0625	0.80	1493.61	-0.13
-0.011	970.50	0.20	0.0625	0.80	1493.61	-0.13
-0.020	970.36	0.21	0.0626	1.02	1485.74	-0.60
-0.029	970.22	0.20	0.0628	1.20	1477.87	-1.20
-0.038	970.08	0.18	0.0629	1.40	1470.01	-1.70

In addition, simulations are undertaken for both positive and negative increments around the base case. Table 2 shows that changes in the marginal cost of capital output, while holding constant the rate of cost increases,

exert virtually no effects on the depreciation rate, capital stock and output. This result can be seen from equation set (26). The parameter β_o only affects the depreciation rate through α, the intercept coefficient. The adjustment coefficient, m, is not affected. Thus, there are no effects on depreciation through the capital adjustment process. Indeed from table 2, changes in the value of β_o by as much as 1800% do not affect the model solution.

The situation is quite different for changes in β_{oo}. This paremeter affects the marginal cost of capital output and the rate of marginal cost expansion as capital output rises. From equation set (26), β_{oo} affects the intercept and slope coefficients of the depreciation rate function. This means that (since β_{oo} affects β which affects m) the depreciation rate is affected by the capital adjustment process. Table 3 relates to the simulations for changes in β_{oo}. The middle row is the base case simulation.

TABLE 3

Simulation around β_{oo}
(Solution Values and Percentage Change from Actual)

Values of β_{oo}	Capital Stock		Depreciation Rate		Output	
-0.0763	944.11	-2.50	0.0867	39.78	779.27	-47.90
-0.0753	956.51	-1.22	0.0748	20.65	1118.97	-25.18
-0.0743	970.36	0.21	0.0626	1.02	1485.74	-0.66
-0.0733	985.99	1.83	0.0501	-19.09	1884.40	25.90
-0.0723	1003.92	3.67	0.0373	-39.72	2221.41	55.21

The results show that for a 3% change in the value of β_{oo}, the depreciation rate changes by 40%. However, the percentage change in β_{oo} generates

approximately an equiproportional change in the capital stock. Hence relatively large variations in the depreciation rate are consistent with small changes in the capital stock. Simultaneously, through the production function, changes in β_{oo} directly exert strong effects on output supply. A 3% change in the parameter causes output to change by 55%.

The β_{no} parameter also affects the marginal cost of capital output and the rate of marginal cost expansion as the capital stock rises. The results from table 4 show that this parameter exerts a strong influence on the endogenous variables. Indeed, a 3% change in β_{no}, from the base case given in the middle row of table 4, causes the depreciation rate to change by 40%, the capital stock to change by 8% and output to change by 125%.

TABLE 4

Simulation around β_{no}
(Solution Values and Percentage Change from Actual)

Values of β_{no}	Capital Stock		Depreciation Rate		Output	
0.067	917.73	-5.22	0.0886	42.82	25.40	-98.30
0.068	941.76	-2.74	0.0756	21.88	713.52	-52.29
0.069	970.36	0.21	0.0626	1.02	1485.74	-0.66
0.070	1005.69	3.86	0.0498	-19.71	2371.15	58.53
0.071	1051.98	8.64	0.0370	-40.29	3422.67	128.84

The last parameter considered in the simulations relates to the wage function. Changes in α_{oo} affect the duration costs associated with capital output. Table 5 shows the results, with the middle row representing the base case. This parameter has no direct effect on the production function and so

there is virtually no effect on the capital stock and output. In addition, an 85% change in α_{oo} causes an 8% change in the depreciation rate. However, the depreciation rate is sensitive to the parameterized value of the wage function in the sense that very small values of α_{oo} do affect capital depreciation.

TABLE 5

Simulation around α_{oo}
(Solution Values and Percentage Change from Actual)

Values of α_{oo}	Capital Stock		Depreciation Rate		Output	
0.00005	976.44	0.83	0.0576	-7.04	1476.96	-1.25
0.00025	973.36	0.52	0.0601	-3.00	1481.65	-0.93
0.00045	970.36	0.21	0.0626	1.02	1485.74	-0.66
0.00065	967.42	-0.09	0.0651	5.04	1489.27	-0.04
0.00085	964.57	-0.39	0.0676	9.02	1492.24	-0.22

The first general conclusion from the simulations is that the depreciation rate, capital stock and output are affected by the parameterization of capital utilization through the production function. Second, the parameters that affect the rate of marginal cost expansion of capital output (or the rate of marginal product expansion of capital utilization) exert the greatest influence on the depreciation rate, capital stock and output.

6. Conclusion

In this paper it was established that the path of the capital utilization rate can be characterized as a flexible accelerator and is similar in nature to the paths for the rates of capital and labor investment. Along the adjustment path the rates of investment are inversely related to each other while the capital utilization rate is directly related to the rate of labor investment and inversely related to the capital investment rate.

The model also captures the stylized facts obtained by Foss (1981). Unanticipated changes in factor supply conditions (as represented by changes in the scale rate, interest rate and labor departure rate) cause comovement in the rates of capital and labor investment, while the capital utilization rate is inversely related to the rates of investment. In addition, unanticipated product demand changes generate comovement in both rates of investment and the capital utilization rate along the dynamic adjustment path.

Simulation results show that the rate of capital utilization or depreciation is very senstitive to the parameterization, through the production function, of changes in the marginal product of capital utilization. Indeed, it was found that a 3% change in the parameters that affect the rate of increase in the marginal product of capital utilization alter the depreciation rate by 40%.

There are many areas open to further research on capital utilization in a dynamic context. First, the model can be extended to a general equilibrium framework to investigate the role of capital utilization as a cause and consequence of aggregate growth. Second, the dynamic effects of corporate tax policy can be investigated when there is a trade-off between capital investment and capital utilization.

NOTES

* The authors would like to thank Ernst Berndt, Michael Denny, Erwin Diewert, and Zvi Griliches for their comments on a previous version of this paper. They also acknowledge support from the C.V. Starr Center for Applied Economics at New York University. Thanks are also extended to Catherine Labio for her help in preparing this manuscript.

[1] There is no distinct maintenance decision. Maintenance costs are assumed to be captured within the production function and are reflected in the costs associated with capital output.

[2] If as a special case $F_N = -F_O$, so the marginal product of the capital input equals the marginal product of utilized capital, then $y(t) = F(\delta(t)K_N(t), L(t))$. This special case is the way capital utilization is often introduced into the production function. Here it is clearly seen that $\delta(t)$ is also the utilization rate.

[3] The results from this model can be generalized to a situation where there are two labor inputs, with one treated as a variable factor of production and the other as a quasi-fixed factor. Also the results apply to the special case where labor is only a variable input.

[4] The wage function is derived in the standard manner from $w = w_s(s + Z(K_O/L) = W(K_O/L)$, where $w_s > 0$ is the exogenous scale wage rate, $s>0$ is the scale factor (s can be normalized to unity), Z is the twice continuously wage premium function, with $Z(K_O/L) = Z((1-\delta)K_N/L)$, at $\delta = 0$ $Z(K_N/L) = 0$ (i.e., $K_O = K_N$), and $Z' < 0$, $Z'' > 0$ for $K_O/L \geq 0$.

[5] The unit of adjustment cost function $C(I_N(t)/K_N(t))$ is composed of the

purchase price and the internal cost of installing capital. Thus $C(I_N(t)/K_N(t)) = p_N(t) + A(I_N(t)/K_N(t))$. Now with $A(0) = 0$ then $C(0) = p_N(0)$. Also $A' > 0$ for $I_N(t) > 0$ so $C' > 0$. Finally, we assume that the total capital adjustment cost, $A(I_N(t)/K_N(t))I_N(t)$, is strictly convex in $I_N(t)$. We do not assume that unit adjustment costs are strictly convex. This implies that $C"I_N(t)K_N(t) + 2C' > 0$. Since $C' > 0$ then $C"$ can be negative but not too negative.

[6] We drop the notation (t) for simplicity. In addition, K_0, I_N and I_L are piecewise continuous functions of time, while K_N and L are continuous functions with piecewise continuous first derivatives.

[7] The transversality conditions are $\lim_{t\to\infty} q_i \geq 0$, $i = 1,2$, $\lim_{t\to\infty} q_1K_N = \lim_{t\to\infty} q_2L = 0$. The Legendre-Clebsch conditions imply that the matrix of second order derivatives of the control variables (K_0, I_N, and I_L) is negative definite.

[8] In equilibrium short-run net revenue is maximized. Consider the problem,

$\max_{(y,K_0)}$ $py - W(K_0/L)L + q_1K_0$ subject to $y = Lf(k,(1-\delta)k)$ given p,q_1,L,K_N,

and recall $(1-\delta) = K_0/K_N$. The first order conditions are $p-\lambda = 0$, $-W' + q_1 + \lambda f_2 = 0$. Thus $f_2 = (W'-q_1)/p$, which is equation (6.1). Equation (6.1) also shows that the isorevenue curve would be a straight line if there were no duration costs associated with capital output. Short-run equilibrium would still be unique.

[9] An increase in q_1 leads to an increase in K_0 and a decrease in y. This can be seen from Figure 1 where the isorevenue curve becomes more negatively sloped. The opposite occurs for an increase in p.

[10] A sufficient condition for the marginal product to diminish is that

$f_{21} \leq 0$. It is also possible to have f_{21} positive but small and still have $\partial K_0/\partial K_N = -f_1/f_2$ decrease as K_N decreases.

[11] The results of this section are derived by assuming that the dynamic path is close to the steady state so that the derivatives are evaluated at $k = k^e$. Alternatively, it can be assumed that the elements of the A matrix in (17) are constants. This is the usual assumption to obtain local comparative equilibrium results.

[12] Since the comparative dynamics are local results, it is true that $-a_{22} + a_{33} < 0$. This result enables us to establish that the right side of (22.2) is negative.

[13] To establish that $\partial q_1/\partial p > 0$ we use the fact that $a_{22}\partial \dot{k}/\partial p - a_{12}\partial \dot{q}_1/\partial p > 0$ which is derived from equation (6.1), (6.2) and (6.5) for $\dot{q}_1 = 0$. In addition, we use the results that $\partial \dot{q}_1/\partial p < 0$, $\partial \dot{q}_2/\partial p < 0$ and $\partial \dot{k}/\partial p < 0$.

REFERENCES

Abel, A., "A Dynamic Model of Investment and Capacity Utilization," Quarterly Journal of Economics, 96 (1981), 379-403.

Bernstein, J.I., "Investment, Labour Skills, and Variable Factor Utilization in the Theory of the Firm," Canadian Journal of Economics, 16 (1983), 463-479.

Betancourt, R. and C. Clague, "An Econometric Analysis of Capital Utilization," International Economic Review, 19 (1978), 211-227.

Betancourt, R., C. Clague and A. Panagiriya, "Capital Utilization and Factor Specificity," The Review of Economic Studies, 52 (1985), 311-329.

Bliss, C.J., Capital Theory and the Distribution of Income, New York: American Elsevier Publishing Co., 1975.

Diewert, W.E., "Aggregation Problems in the Measurement of Capital," in D. Usher (ed.), The Measurement of Capital, Chicago: University of Chicago Press, 1980.

Epstein, L., "Comparative Dynamics in the Adjustment Cost Model of the Firm," Journal of Economic Theory, 27 (1982), 77-100.

Epstein, L. and M. Denny, "Endogenous Capital Utilization in a Short-Run Production Model: Theory and Empirical Application," Journal of Econometrics, 12 (1980), 189-207.

Foss, M., "Long-run Changes in the Workweek of Fixed Capital," American Economic Review Papers and Proceedings, 71 (1981), 58-63.

Gould, J.P., "Adjustment Costs in the Theory of Investment of the Firm," Review of Economic Studies, 35 (1968), 41-55.

Hicks, J.R., Value and Capital, 2nd edition, Oxford: Oxford University Press, 1946.

Kim, M. "The Structure of Technology with Endogenous Capital Utilization." International Economic Review, 29 (1988), 111-130.

Lucas, R.E., "Optimal Investment Policy and the Flexible Accelerator," International Economic Review, 8 (1967), 78-85.

———————, "Capacity, Overtime and Empirical Production Functions," American Economic Review Papers and Proceedings, 60 (1970), 23-27.

Malinvaud, E., "Capital Accumulation and the Efficient Allocation of Resources," Econometrica, 21 (1953), 233-268.

Mortensen, D., "Generalized Costs of Adjustment and Dynamic Factor Demand Theory," Econometrica, 41 (1973), 657-666.

Nadiri, M.I. and S. Rosen, "Interrelated Factor Demand Functions," American Economic Review, 59 (1969), 457-471.

Oi, W., "Labor as a Quasi-Fixed Factor," Journal of Polictical Economy, 70 (1962), 538-555.

———, "Slack Capacity: Productive or Wasteful," American Economic Review, Papers and Proceedings, 71 (1981), 64-69.

Smith, K.R., "Risk and the Optimal Utilization of Capital," Review of Economic Studies, 37 (1970), 253-59.

Taubman, P. and M. Wilkinson, "User Cost, Capital utilization and Investment Theory," International Economic Review, 11 (1970), 209-215.

Treadway, A. B., "On the Multivariate Flexible Accelerator," Econometrica, 39 (1971), 845-855.

Winston, G.C. and T. McCoy, "Investment and the Optimal Idleness of Capital," Review of Economic Studies, 41 (1974), 419-428.