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THE "TYPES" OF A BAYESIAN EQUILIBRIUM

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Abstract

In this paper I formally define and compare the various notions of a "***type***" and the associated concept of a Bayesian Nash equilibrium. I discuss re-parametrizations of the basic model and indicate which of the concepts of a type become equivalent under various re-parametrizations of the model. This paper will use the framework developed in Nyarko (1993b) which is itself a generalization of the papers of Ambruster and Boge (1979), Boge and Eisele (1979), Mertens and Zamir (1985), and others. The framework will be a model where agents have imperfect information over both the underlying fundamentals of the economy (or game) ***and*** the strategies being used by the other agents. Agents will also have imperfect information about the beliefs of others, about the beliefs about other agents beliefs, etc.

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1. Introduction

1.1. Consider a multi-agent model where agents have incomplete information over both the underlying fundamentals of the economy (or game) and the strategies being used by the other agents. The decision-theoretic or "Savage-Bayesian" framework for this incomplete information problem would model each agent as being characterized by her utility parameter and her belief over the ***both*** the fundamentals ***and*** the strategies of other agents. The agent is also characterized by her belief about other agents' beliefs; and her belief about their beliefs about other agents beliefs, etc. We will define a Savage-Bayesian type of an agent to be a specification of her own utility parameter and her hierarchy of beliefs over the fundamentals and the strategies of others. This is the concept of a type that would be used in decision theory (of the kind in Savage (1954) for example).

A game-theoretic formulation of the incomplete information problem would model agents as being characterized by some arbitrarily specified "type." An equilibrium would then specify how each agent would choose strategies as a function of her type. For example, in the Harsanyi (1968) Bayesian equilibrium concept an agent's type specifies her utility parameter and her hierarchy of beliefs over the utility parameters of others (and in particular does not specify beliefs about the strategies of others). Most of the work in applied game theory defines an agent's type to be her utility parameter. Aumann (1987) studies the correlated equilibrium concept and implicitly defines a type to be a Savage-Bayesian type.

In this paper I formally define and compare the various notions of a type and the associated concept of a Bayesian Nash equilibrium. I discuss re-parametrizations of the basic model and indicate which of the concepts of a type become equivalent under various re-parametrizations of the model. This paper will use the framework developed in Nyarko (1993b) which is itself a generalization of the papers of Ambruster and Boge (1979), Boge and Eisele (1979), Mertens and Zamir (1985), Tan and Werlang (1988), Brandenberger and Dekel (1993) and Heifetz (1990).

1.2. A Motivating Example. The following model of competitive firms facing an unknown demand curve is studied in much greater detail in Nyarko (1991): Suppose that there is a set of agents indexed by the unit interval $I=[0,1]$ and uniformly distributed along that interval. (For technical reasons suppose also that there are finitely many classes of agents within that interval with all agents of the same class identical in all respects.) Fix any date n. At that date agent i must choose an output level y_{in}. The aggregate output is then $y_n \equiv \int_0^1 y_{in} di$. The price of that output is determined via a linear demand curve $p_n = \alpha - \beta y_n + \epsilon_n$, where α and β are fixed parameters, "the fundamentals," and ϵ_n is the date n shock to the demand curve - a zero mean unobserved random variable. We suppose that the parameter β of the demand curve is "common knowledge" among the agents. However there is imperfect information over the parameter α. The cost to firm i of choosing the output y_{in} is $c(y_{in})=0.5y_{in}^2$. The profit of firm i is then $p_n y_{in}-0.5y_{in}^2$. Let E_{in} denote the date n "expectations operator" of agent i. The profit maximizing output of firm i is then $y_{in}=E_{in}p_n=E_{in}\alpha-\beta E_{in}y_n$. Notice that to choose an optimal action agent i must form a belief over both the fundamentals, α, and the (aggregate) actions of other agents, y_n.

Given any "random variable" x let $G_n x$ denote the "average opinion" of x, i.e., the average of the date n expectations of agents over x, $G_n x \equiv \int_0^1 (E_{in}x)di$. If agents do not know the beliefs of others then in general there will be uncertainty over expressions like $E_{in}G_n x$, agent i's expectation of the average opinion of x, and $G_n^2 x$, the average opinion of the average opinion of x. Inductively, we may defined $G_n^r x$ to be the r-times average opinion of the average opinion ... of x. If there is ***maximizing behavior*** of firms (which we write as ***(MB)***) or 1-level knowledge of (MB) (i.e., if agents know that other agents engage in (MB)); 2-level knowledge of (MB) (i.e., if agents know that other agents know that other agents engage in (MB)) or R-levels of knowledge of (MB) or common- or ∞-level knowledge (the latter with the added assumption that $0<\beta<1$) then we obtain:

(MB) $y_{in}=E_{in}\alpha-\beta E_{in}y_n$ so by integration over i, $y_n=G_n\alpha-\beta G_n y_n$.

(1-level knowledge of (MB)): $y_{in}=E_{in}\alpha-\beta E_{in}G_n\alpha+\beta^2 E_{in}G_n y_n$.

(2-level knowledge of (MB)): $y_{in}=E_{in}\alpha-\beta E_{in}G_n\alpha+\beta^2 E_{in}G_n^2\alpha-\beta^3 E_{in}G_n^2 y_n$.

(R-level knowledge of (MB)): $y_{in}=\Sigma_{r=1}^{R+1}(-\beta)^{r-1}E_{in}(G_n^{r-1}\alpha)+(-\beta)^{R+1}E_{in}G_n^R y_n$(where $G^0\alpha=\alpha$).

(common knowledge of (MB) and $0<\beta<1$): $y_{in}=\Sigma^{\infty}_{r=1}(-\beta)^{r-1}E_{in}(G_n^{r-1}\alpha)$.

In Nyarko (1993b) a framework was provided to represent the model above. In this paper we will discuss various definitions of a type and the resulting definition of a Bayesian Nash equilibrium ***(BNE)***. For example, a Savage-Bayesian type will specify an agent's preferences and that agent's belief hierarchy over both actions and unknown parameters. If we assume (MB) then each agent will choose actions as a function of their Savage-Bayesian type. If we assume knowledge of (MB) by all agents then each agent can compute the other agents' actions as a function of their Savage-Bayesian type. We will then be in a Savage-Bayesian BNE. Another example of the concept of a type that will be studied is that of a Harsanyi type. An agent's Harsanyi type will be defined to be that agent's belief hierarchy over the "fundamentals", which in the example above will be the parameter α. Expressions like $E_{in}(G_n^{r-1}\alpha)$ are determined by agent i's Harsanyi type. If we assume common knowledge of (MB) and $0 < \beta < 1$, then we saw above that each agent's action is a function only of that agent's Harsanyi type. Further, each agent will be able to compute the actions of others as a function of their Harsanyi types. We will then be in what will be defined as a Harsanyi BNE.

2. Some Terminology and Mathematical Preliminaries

2.1. I is the ***finite*** set of economic agents. Nature is agent 0, and is not a member of I. Given any collection of sets $\{X_i\}_{i\epsilon I}$, we define $X \equiv \Pi_{i\epsilon I} X_i$ and $X_{-i} \equiv \Pi_{j \neq i} X_j$ unless otherwise stated; (given X_0 and $\{X_i\}_{i\epsilon I}$, we shall sometimes state that $X_{-i} \equiv X_0 x \Pi_{j \neq i} X_j$). Given any collection of functions $f_i : X_i \to Y_i$ for $i \epsilon I$, $f_{-i} : X_{-i} \to Y_{-i}$ is defined by $f_{-i}(x_{-i}) \equiv \Pi_{j \neq i} f_j(x_j)$. The cartesian product of metric spaces will always be endowed with the product topology. Let X be any metric space. $\mathcal{P}(X)$ denotes the set of probability measures on X (with X endowed with its Borel σ-algebra, generated by the open sets of X). The set $\mathcal{P}(X)$ will be endowed with the weak topology of measures; (see Billingsley (1968) for more on this). The following fact will be used repeatedly: If X is a complete and separable metric space then so is $\mathcal{P}(X)$. (See, e.g., Parthasarathy Theorems II.6.2 and II.6.5.) For ease of exposition, wherever the intent is obvious we shall assume, ***without mentioning this***, that generic sets and functions are Borel-measurable and generic conditional probabilities are fixed regular versions. $\mathbf{R}$ denotes the real line.

3. The Basic Model

3.1. In this section I summarize the basic model. All of the details appear in Nyarko (1993b).

3.2. Belief Hierarchies. Recall that I is the set of agents and nature is referred to as agent 0 (not a member of I). Suppose we are given a collection of complete and separable metric spaces

Y_0 and $\{Y_i\}_{i\epsilon I}$. We shall consider Y_i to be the set pertaining to agent i; this will have the meaning that i "knows" her own value of $y_i \epsilon Y_i$. We consider Y_0 to be the parameters of "nature." We proceed to construct the space of hierarchies of beliefs over the space $Y = Y_0 x \Pi_{i\epsilon I} Y_i$. Construct the sets $\{B_i^r\}^{\infty}_{r=1}$ inductively as follows:

$$B_i^1 \equiv \mathcal{P}(Y_{-i}) \qquad \text{where } Y_{-i} \equiv Y_0 x \Pi_{j \neq i} Y_j; \tag{3.3}$$

and given $\{B_j^r\}_{j\epsilon I}$ for some $r \geq 1$, define

$$B_i^{r+1} \equiv \mathcal{P}(B^r_{-i} x Y_{-i}). \tag{3.4}$$

An element $b_i^1 \epsilon B_i^1$ represents agent i's beliefs about $y_{-i} \epsilon Y_{-i}$ and shall be referred to as agent i's first order belief. An element $b_i^2 \epsilon B_i^2$ specifies agent i's belief about the first order beliefs of others and shall be referred to as agent i's second order belief. An element $b_i^r \epsilon B_i^r$ is i's r-th order belief and it specifies agent i's belief about the (r-1)-th order beliefs of other agents. The set of hierarchies of beliefs of agent i is a subset B_i of the Cartesian product $\Pi^{\infty}_{r=1} B_i^r$ obeying a certain "***probabilistic coherence condition***," requiring that lower order beliefs be a "projection" of higher order beliefs; e.g., the first order belief of an agent should be equal to the marginal on Y_{-i} of that agent's second order belief.

There is a mapping P_i: $B_i \rightarrow \mathcal{P}(B_{-i} x Y_{-i})$ which associates with each hierarchy of beliefs $b_i = (b_i^1, b_i^2, \ldots) \epsilon B_i$ an associated probability $P_i(b_i) \epsilon \mathcal{P}(B_{-i} x Y_{-i})$ with the property that for each integer r, the marginal of $P_i(b_i)$ on $B_{-i}^r x Y_{-i}$ is equal to b_i^{r+1}. In particular, $P_i(b_i)$ is the belief that agent i with hierarchy of beliefs b_i has about the belief hierarchies of other agents. The mapping P_i is a homomorphism between B_i and $\mathcal{P}(B_{-i} x Y_{-i})$.

3.5. The Economic Model. Time is discrete and has dates $n=1,2,3,\ldots$. At each date n agent i chooses an action a_{in} in an action space A_i. Let $z_{in} \in Z_i$ denote the vector of all observations of agent i during the course of date n. We assume that agents observe their own actions, so z_{in} is a vector which includes a specification of a_{in}. Just before choosing the date n action $a_{in} \in A_i$, agent i would have information on the date n partial history $z_i^{N-1} = \{z_{i1},\ldots,z_{iN-1}\} \in Z_i^{N-1}$. ($z_i^0$ is the null or empty history.) We suppose Z_i and A_i are complete and separable metric spaces for all $i \in I$. Define

$$F_{iN} \equiv \{f_{iN}:Z_i^{N-1} \rightarrow A_i \text{ with } f_{iN} \text{ Borel-measurable}\}, \quad F_i \equiv \Pi^{\infty}_{N=1} F_{iN} \text{ and } F \equiv \Pi_{i \in I} F_i. \tag{3.6}$$

A ***behavior strategy*** for agent i is any $f_i \in F_i$. We assume that for all i and N, F_{iN} is endowed with a metric which makes it a complete and separable metric space.

We let $\Theta \equiv \Theta_0 x \Pi_{i \in I} \Theta_i$ denote the space of ***"fundamentals"*** or ***attribute vectors*** of agents. $\theta_0 \in \Theta_0$ will denote nature's attribute vector; this parameter will determine any underlying randomness of the economy. $\theta_i \in \Theta_i$ denotes the utility parameter or attribute vector of agent i. Agent i's utility or payoff function is some function $u_i:\Theta_0 x \Theta_i x F_i x F_{-i} \rightarrow \mathbf{R}$ which depends upon nature's attribute vector, θ_0, agent i's attribute vector, θ_i, agent i's strategy vector, f_i, and the strategy vector of the other agents, f_{-i}. It will be assumed that the functional forms of the utility functions are common knowledge; however agents will have imperfect information over the attribute vectors and the behavior strategies of other agents. Θ_0 and Θ_i for each $i \in I$ are assumed to be complete and separable metric spaces.

We may think of agents actions or behavior strategies as resulting in certain physical outcomes which in turn result in utility to the agents. Following Harsanyi (1968 pt. I p. 167) we shall think of the utility attribute vectors of agents, $\theta \in \Theta$, as being broadly defined and in particular as specifying (i) the physical outcome function produced by any given tuple of behavior strategies; (ii) the utilities of agents following any physical outcome; (ii) the strategy spaces available to the agent. Harsanyi (1967, pt. I) also argued that any private information signal received by an agent may also be included in a specification of the utility attribute vector. Incomplete information on any of the above three items may therefore be represented as incomplete information over the agents' attribute vectors. We shall refer the this loosely as the ***Harsanyi Principle.*** We shall from time to time consider the implications of very broad interpretations of the "Harsanyi Principle."

We shall suppose that sequence of observations and actions of the economy, $\{z_n\}^{\infty}_{n=1} \in Z^{\infty}$ has a probability distribution P_{γ} which depends upon the true date 0 vector $\gamma=(\theta_0,\{f_i\}_{i\in I})$ of nature's attribute vector and the behavior strategies of agents. We may without loss of generality suppose that this probability distribution ***as a function of*** γ is "common knowledge" among the agents in the economy. We shall assume that P_{γ} is a ***regular conditional probability*** on Z^{∞}. (Of course, by assuming that θ_0 includes a specification of the utility parameters of ***all*** agents, we may model the situation where agent i's utility function and P_{γ} are functions of agent j's attribute vector (for some or all $j\in I$), as in some formulations of adverse selection models in economics).

3.7. The Savage-Bayesian Types of Agents. At date 0 there is imperfect information over space of attribute vectors $\Theta=\Theta_0 x \Pi_{i\in I}\Theta_i$, ***and*** over the space of behavior strategies, $F=\Pi_{j\in I}F_j$. Let Q_i^{∞} be agent i's space of belief hierarchies over ΘxF defined and constructed as in section 3.1. (In that construction set $Y_0=\Theta_0$ and $Y_j=\Theta_j xF_j$ for all $j\in I$; what we refer to here as Q_i^{∞} is the same as what was referred to in that construction as B_i.) Any $q_i^{\infty}=(q_i^1,q_i^2,\ldots)\in Q_i^{\infty}$ is a possible belief hierarchy for agent i over ΘxF. At date 0 each agent i will be characterized by some attribute vector, $\theta_i\in\Theta_i$, and some belief hierarchy $q_i^{\infty}\in Q_i^{\infty}$. We refer to the tuple $q_i=(\theta_i,q_i^{\infty})$ as agent i's

Savage-Bayesian type and we define $Q_i \equiv \Theta_i x Q_i^\infty$ and $Q \equiv \Pi_{i\epsilon I} Q_i$. An agent's Savage-Bayesian type, $q_i = (\theta_i, q_i^\infty)$, contains all the information for that agent to engage in decision-making: preferences are specified by θ_i and beliefs specified by q_i^∞.

3.8. Behavior Strategy Choice Rules and Expected Utility Maximization.

We define a ***behavior strategy choice rule*** to be any (measurable) function $\mu_i''' : Q_i \rightarrow \mathcal{P}(F_i)$ which determines agent i's (possibly randomized) behavior strategy as a function of that agent's Savage-Bayesian type, q_i. Define $U_i(q_i, f_i)$ to be the expected utility function of agent i of Savage-Bayesian type $q_i = (\theta_i, q_i^1, q_i^2, q_i^3, \ldots)$ obtained by integrating out the coordinates Θ_0 and F_{-i} from the utility function u_i with respect to the measure q_i^1:

$$U_i(q_i, f_i) \equiv \int u_i(\theta_0, \theta_i, f_i, f_{-i}) dq_i^1. \tag{3.9}$$

Conditional on any q_i an expected utility maximizer will choose a behavior strategy to maximize the expression in (3.9). If there is more than one solution to this maximization problem the agent could in general randomize over the set of maximizers. Expected utility maximization will therefore under fairly general conditions result in a behavior strategy choice rule. We suppose that each agent i has a behavior strategy choice rule which determines how that agent will choose behavior strategies as a function of her Savage-Bayesian type.

3.10. The State Space Ω.

We define the state space to be the set $\Omega \equiv Q x \Theta_0 x F x Z^\infty$. Any $\omega = (\{q_i\}_{i\epsilon I}, \theta_0, \{f_i\}_{i\epsilon I}, z^\infty) \epsilon \Omega$ specifies the Savage-Bayesian types of agents, $\{q_i\}_{i\epsilon I}$, nature's attribute vector, θ_0, the vector of agents' behavior strategies, $\{f_i\}_{i\epsilon I}$, and the sample path of actions and observations, $z^\infty \epsilon Z^\infty$.

3.11. The Ex Post Beliefs of Agents over Ω. In Nyarko (1993b, (2.2)) the following "product" operation $\otimes$ was introduced: Let X and W be two complete and separable metric spaces. Suppose we are given a ("marginal") distribution, Ψ', over X; i.e., $\Psi' \epsilon \mathcal{P}(X)$. Let $G:X \rightarrow \mathcal{P}(W)$ be any function mapping X into the set of probability measures on W. Let G(.;x) denote the value of G at x (so $G(.;x) \epsilon \mathcal{P}(W)$). Then each x defines a probability G(.;x) on W ("conditional" on x). We may therefore "integrate" the conditionals with respect to the marginal to obtain a joint distribution, Ψ, over XxW. This joint probability, Ψ, will have a marginal over X equal to Ψ' and a conditional over W given x equal to G(.;x). We shall use the notation $\Psi' \otimes G$ or $\Psi' \otimes G(.;x)$ to denote this joint probability and refer to it as the "***product*** of Ψ' and G(.;x)". (For this "product" operation we will require the measurability of G(.;x) in x; in Nyarko (1993b) this is shown to be equivalent to the requirement that G(.;x) be a regular conditional probability.)

Each Savage-Bayesian type q_i of agent i induces a unique measure $\mu_i(.;q_i)$ over Ω representing that agent's belief over Ω under the following conditions: (i) agent i knows her own Savage-Bayesian type q_i; (ii) agent i knows her behavior strategy choice rule $\mu_i''':Q_i \rightarrow \mathcal{P}(F_i)$; (iii) agent i's belief over the space $Q_{-i} x \Theta_0 x F_{-i}$ of Savage-Bayesian types of others, nature's attribute vector and the behavior strategies of others is obtained via the measure $P_i(q_i^\infty)$ defined in (3.2) where $q_i = (\theta_i, q_i^\infty)$; and (iv) agent i's belief over the action and observation space Z^∞ is computed via the measure P_γ of section (3.5). Let $1_i(q_i)$ be the probability over Q_i which assigns probability one to the given q_i. Then using the notation just introduced we may write the measure $\mu_i(.;q_i)$ just defined as follows:

$$\mu_i(.;q_i) \equiv [[1_i(q_i) \otimes \mu_i'''] \otimes P_i(.)] \otimes P_\gamma. \tag{3.12}$$

3.13. Ex Ante Subjective Beliefs. We assume that each agent i has an ex ante subjective belief μ_i over the state space Ω. The interpretation is that at date 0 agent i is "born" and realizes her Savage-Bayesian type q_i. The ex post belief of that agent over the state space is then represented by the conditional probability $\mu_i(. \mid q_i)$ of μ_i given q_i, and must agree with the measure $\mu_i(.;q_i)$ of (3.11) for μ_i almost every $q_i \in Q_i$. A probability $\mu \in \mathcal{P}(\Omega)$ is a ***common prior*** for the agents if for all $i \in I$, μ is an ex ante subjective belief for agent i.

3.14. Condition (GH). We shall say that the collection of subjective ex ante beliefs of agents, $\{\mu_i\}_{i \in I}$, obey ***condition (GH)*** if μ_i and μ_j are mutually absolutely continuous $\forall i,j \in I$; (i.e., for all measurable $D \subseteq \Omega$ and $\forall i,j \in I$, $\mu_i(D) > 0$ if and only $\mu_j(D) > 0$). Condition (GH) requires that agents agree ex ante about the events which have zero probability. Condition (GH) does ***not*** require the ex post probabilities, $\mu_i(. \mid q_i)$ and $\mu_j(. \mid q_j)$, to be mutually absolutely continuous. It should be clear that if $\mu_i = \mu$ for all i so that μ is a common prior then condition (GH) holds. Condition (GH) is therefore weaker than the common prior assumption. We therefore name this "condition (GH)" for "Generalized Harsanyi" common prior condition. To avoid unnecessary "probability zero" complications we shall impose condition (GH) in the remainder of this paper.

4. A T-based Pseudo Bayesian-Nash-Equilibrium

4.1. We shall first present the "intuitive" definition of a Bayesian Nash equilibrium. We will take as given a collection of (complete and separable) metric spaces, $\{T_i^{\#}\}_{i \in I}$. Define for each i, $T_i \equiv \Theta_i x T_i^{\#}$. Then any $\tau_i = (\theta_i, \tau_i^{\#}) \in T_i$ specifies agent i's attribute vector θ_i and possibly some other parameter $\tau_i^{\#} \in T_i^{\#}$. We will refer to a T_i as a space of agent i's "***characteristics.***" Define $T \equiv \Theta_0 x \Pi_{j \in I} T_i$, the cartesian product of the space of nature's attribute vectors and the T_i's. We will take as given a collection of probability measures $\{\pi_i\}_{i\epsilon I}$ where for each $i\epsilon I$ π_i is a probability measure over the space of characteristics. We may interpret π_i as an ex ante belief for agent i over the characteristic space T; the conditional probability $\pi_i(. \mid \tau_i)$ will then be the ex post belief of agent i with "characteristics" $\tau_i \epsilon T_i$.

A ***T_i-based decision function*** for agent i is any (measurable) mapping $D_i:T_i \rightarrow \mathcal{P}(F_i)$; it represents a (possibly randomized) decision rule for agent i, used in choosing a behavior strategy as a function of agent i's characteristics, $\tau_i \epsilon T_i$. Suppose that each agent i knows that agent j uses some T_j-based decision function D_j. Let $D_{-i}(\tau_{-i}) \equiv \Pi_{j \neq i} D_j(\tau_j)$ be the measure over F_{-i} equal to product of the measures $\{D_j(\tau_j)\}_{j \neq i}$. Recall that the utility function of agent i is the function $u_i:\Theta_0 x \Theta_i x F_i x F_{-i} \rightarrow \mathbf{R}$. We may denote, with obvious abuse of notation, the following (which is equal to the utility of agent i when agents use the decision functions $\{D_j(\tau_j)\}_{j\epsilon I}$):

$$u_i(\theta_0,\theta_i,D_i(\tau_i),D_{-i}(\tau_{-i})) \equiv \int_{F_i} \int_{F_{-i}} u_i(\theta_0,\theta_i,f_i,f_{-i}) dD_i(\tau_i) dD_{-i}(\tau_{-i}), \tag{4.2}$$

where $dD_i(\tau_i)$ (resp. $dD_{-i}(\tau_{-i})$) denotes integration over F_i (resp. F_{-i}) with respect to the measure $D_i(\tau_i)$ (resp. $D_{-i}(\tau_{-i})$). The ex ante expected utility of the agent is then obtained by integration of (4.2) over $\tau \epsilon T$ with respect to agent i's ex ante belief π_i. We denote this by

$$W_i(D_i,D_{-i}) \equiv \int u_i(\theta_0,\theta_i,D_i(\tau_i),D_{-i}(\tau_{-i})) d\pi_i, \tag{4.3}$$

where the integral is with respect to π_i over the variables θ_0,θ_i,τ_i and τ_{-i}. A collection of decision functions $\{D_i^*\}_{i\epsilon I}$ is said to be a ***T-based pseudo Bayesian-Nash-Equilibrium (BNE)*** for $\{\pi_i\}_{i\epsilon I}$, if for each $i\epsilon I$ and for all T_i-based decision functions D_i for agent i,

$$W_i(D_i^*,D_{-i}^*)) \geq W_i(D_i,D_{-i}^*)). \tag{4.4}$$

The above should really be referred to as a BNE without common priors. If the ex ante beliefs $\{\pi_i\}_{i\epsilon I}$ are common (i.e., $\pi_i=\pi_j$ $\forall i,j\epsilon I$) then we have a ***T-based pseudo BNE with common priors.***

4.5. Some Problems with the Definition of a Pseudo BNE. Since we wish to interpret the state space as consisting of all relevant variables it seems natural to require characteristics to be random variables on the state space. This also helps resolve some problems. To begin with, it is possible that a characteristic has an "intrinsic definition." Our definition of a pseudo BNE may not respect any such "intrinsic definition" of a characteristic. For example suppose that one of the agents, Agent A say, may have one of two possible characteristics, "LEFT" or "RIGHT." Suppose also that agent A has two actions "left" and "right." It is easy to design a pseudo BNE where the agent of with characteristic LEFT chooses action right, and the agent of characteristic RIGHT chooses action left. Of course by relabelling we may get around this problem - and indeed this is essentially the "revelation principle." However, we may want our equilibrium to have such consistency embedded into it. The situation where an agent's characteristic is her "suggested" action or behavior strategy occurs in the study of correlated equilibria.

The above mentioned problem occurs in a slightly different context in our model with hierarchies of beliefs. In the definition of a pseudo BNE we make no restrictions on agent i's ex ante belief π_i. Suppose for example that the space of characteristics is the space of Savage-Bayesian types. There are two ways of computing the beliefs of agent i of Savage-Bayesian type q_i over the types of others. First, we may use the conditional of π_i given q_i, $\pi_i(. \mid q_i)$; alternatively, each $q_i=(\theta_i,q_i^\infty)$ defines a probability over the types of others given by $P_i(q_i^\infty)$ as described in section 3.2.

Since the prior belief π_i is arbitrary there is no a priori reason why the conditional, $\pi_i(. \mid q_i)$ should agree with $P_i(q_i^{\infty})$ over the space of Savage-Bayesian types of others.

Another related issue is the following: In principle an agent's characteristic may reveal information about the future which has not yet occurred. For example, we may have an agent's characteristic equal to the outcome of the toss of a coin which the agent is yet to flip. The potential for self-referencing problems in this case should be obvious.

To handle the above issues in the next section we shall require the characteristics to be random variables on the state space Ω and we shall impose some measurability conditions. We will in that case refer to a characteristic as a type. Bayesian Nash equilibria will be defined to be measures on the state space obeying properties analogous to those in the definition of a pseudo BNE - in which case we drop the qualifier "pseudo."

5. The Types of Agents

5.1. Recall that q_i denotes agent i's Savage-Bayesian type and it may be considered a random variable on the state space Ω. Let $\mathfrak{F}_{i0} \equiv \sigma(\{q_i\})$ denote the σ-algebra generated by q_i. This represents agent i's information at date 0. We define a type ***space*** for agent i to be the same as a space of characteristics for agent i: i.e., any $T_i \equiv \Theta_i x T_i^{\#}$ where $T_i^{\#}$ is a complete and separable metric space. We refer to $T \equiv \Theta_0 x \Pi_{j \in I} T_j$ as a type ***space*** (where, recall, Θ_0 is the space of nature's attributes). We shall define a ***type for agent i*** to be any $\mathfrak{F}_{i0}$-measurable random variable on Ω of the form $\tau_i = (\theta_i, \tau_i^{\#})$ taking values in T_i. Hence $\tau_i: \Omega \rightarrow T_i$. By an abuse of terminology we shall also refer to any realization of this random variable as agent i's type. The rationale for the measurability conditions on the definition of a type should be straightforward: At date zero the only information that agent i has is encoded in the vector $q_i=(\theta_i,q_i^\infty)$.

5.2. Savage-Bayesian Types. The belief of agent i is completely specified by that agent's hierarchy of beliefs, q_i^∞, and the utility function is completely specified by the attribute vector θ_i. Hence the decision problem facing the agent i is completely specified by the vector $q_i=(\theta_i,q_i^\infty)$. For example the expected utility maximization problem of (3.9) is completely characterized by q_i. It is for this reason that we refer to q_i as agent i's ***Savage-Bayesian type***. By construction q_i may be considered a random variable on Ω. It also is also trivially $\sigma(\{q_i\})$ measurable and hence satisfies all the requirements to refer to it as a type.

By assumption (see (3.8)) agents use behavior strategy rules $\mu_i'''(df_i \mid q_i)$ which are, conditional on q_i, independent of the Savage-Bayesian types or realized behavior strategies of other agents. This implies that the notion of a Savage-Bayesian type is "rich" enough so that conditional on their types agents choose behavior strategies independently of each other. Hence implicitly, and following the Harsanyi Principle, we have modelled the attribute vector as specifying any correlation signals or private information the agent recent receives.

5.3. Harsanyi Types. Nyarko (1993b, section 9) showed how to construct a date n hierarchy of beliefs over any random variable. We may consider θ, the attribute vector, to be a random variable on Ω; indeed, θ is the projection from $\Omega\equiv[\Pi_{i\in I}\Theta_i x Q_i^\infty]x\Theta_0 x F x Z^\infty$ onto $\Theta\equiv\Theta_0 x\Pi_{i\in I}\Theta_i$. Let h_i^∞ denote agent i's hierarchy of beliefs over Θ at date 0 (i.e., conditional on $\mathfrak{I}_{i0}=\sigma(\{q_i\})$). The vector $h_i\equiv(\theta_i,h_i^\infty)$ is a random variable on Ω which is $\mathfrak{I}_{i0}$-measurable, so is a type for agent i. We refer to $h_i\equiv(\theta_i,h_i^\infty)$ as agent i's ***Harsanyi type***. An agent's Harsanyi type specifies that agent's belief about θ; that agent's belief about other agents' beliefs about θ; that agent's beliefs about other agents' beliefs about other agents' beliefs about θ; etc. Define H_i^∞ to be the ***space*** of hierarchies of beliefs over the space of attribute vectors, Θ. The space $H_i \equiv \Theta_i x H_i^\infty$ is the Harsanyi type space of agent i and $H = \Theta_0 x \Pi_{i \in I} H_i$ is the Harsanyi type space.

In Harsanyi (1967, Pt. I) a vector c_i was identified. This vector specifies in the Harsanyi framework both the utility parameters of agent i and the subjective belief $R_i(. \mid c_i)$ of agent i over the vectors c_{-i} of other agents. On p. 170 Harsanyi writes "R_i ... is a function whose mathematical form ... is known to all n players." Harsanyi continues on p. 171 "... the rules of the game as such allow any given player to belong to any one of a number of possible "types," corresponding to alternative values his ... vector c_i could take and so representing alternative payoff functions U_i and the alternate subjective probability distributions ... that player i could have in the game. Each player is always assumed to know his own actual type but to be in general ignorant about the other players' actual types."

In the Harsanyi setup c_i does ***not*** specify beliefs about the strategy vectors of others. We therefore interpret the vector c_i of Harsanyi (1967) to be what we have referred to above as the Harsanyi type $h_i = (\theta_i, h_i^\infty)$. This vector of course specifies agent i's utility parameter, θ_i, and agent i's beliefs about the vectors of other agents, h_{-i}, which recall is specified by the measure $P_i(h_i^\infty)$ of section (3.2). Hence we feel justified in referring to h_i as a "Harsanyi type." (The vector c_i was also referred to by Harsanyi (1967) as an "attribute vector;" this should not be confused with our use of the term in referring to θ_i, which should really be referred to as a ***utility*** attribute vector).

5.4. Attribute Types. Agent i's attribute type will be defined to be the same as agent i's attribute vector. It should be obvious that θ_i satisfies the definition of a type.

One may also be looking for the notion of a "sunspot" type. One may argue thus: suppose that the utility parameters are common knowledge. Suppose however that each agent observes some private but payoff irrelevant variable, "sunspots." Should we not model this as a type different from the attribute vector? The answer is no! All opportunities for correlations and private information via the receipt of exogenous information signals should be considered a part of the attribute vector. This is a possible interpretation of what we referred to as the Harsanyi Principle in (3.5).

5.5. The Relationship Between the different notions of Types.

We have a partial ordering on the "informativeness" of the various notions of a type. The most "informative" of course is a Savage-Bayesian type and the least "informative" is the attribute type. If τ_i is a type for agent i and $\sigma(\{\tau_i\})$ denotes the σ-algebra generated by τ_i then $\sigma(\{\theta_i\}) \subseteq \sigma(\{\tau_i\}) \subseteq \sigma(\{q_i\})$ in the sense that $\sigma(\{q_i\})$ is the finest σ-algebra and $\sigma(\{\theta_i\})$ is the coarsest with everything else lying in between.

6. T-Based Bayesian Nash Equilibria (BNE)

6.1. Some Terminology.

We shall use the notion of a pseudo Bayesian Nash equilibrium to motivate our definition here of a Bayesian Nash equilibrium. There will be two differences: Where previously we had "characteristics" we now have "types" defined on the state space; and second our definition will be in terms of ex ante beliefs over the state space (as opposed to decision functions and prior beliefs over types). So fix a collection of types $\{\tau_i\}_{i\in I}$ as in (5.1) taking values in a collection of type spaces for the agents $\{T_i\}_{i\in I}$ and set $T \equiv \Theta_0 x \Pi_{i\in I} T_i$. The definition below will use the following terminology: given any random variable x on Ω taking values in a set X we let $\mu_i(. \mid x)$ denote (any fixed regular version of) the probability μ_i conditional on x. Also, given any probability ν on Ω we let $\nu(dx)$ denote the induced distribution of x by ν. Further, if for all j in some finite index set J ν_j is a probability measure on some metric space X_j we let $\Pi_{j\epsilon J}\nu_j$ denote the product of the measures ν_j over $\Pi_{j\epsilon J}X_j$.

6.2. Definition (BNE). The collection of ex ante subjective beliefs of agents over Ω, $\{\mu_i\}_{i\epsilon I}$, will be referred to as a T-based Bayesian Nash equilibrium (BNE) if there exists a subset Ω' of Ω such that $\mu_i(\Omega')=1$ for all $i\epsilon I$ and such that at each $\omega=(q,\theta_0,f,z^\infty)\epsilon\Omega'$ and for each $i,j\epsilon I$,

i. $f_i \epsilon$ Argmax $U_i(q_i,.)$ (agent i is maximizing, see (3.8));

ii. $\mu_i(df_i \mid q_i,\tau_i)=\mu_i(df_i \mid \tau_i)$ (agent i uses a T_i-based decision function);

iii. $\mu_i(df_j \mid q_i,\tau_j)=\mu_j(df_j \mid \tau_j)$ (i knows j's true decision function); and

iv. $\mu_i(df_{-i} \mid q_i,\{\tau_j\}_{j\neq i})=\Pi_{j\neq i}\,\mu_i(df_j \mid q_i,\tau_j)$ (i believes that each agent's behavior strategy choice is independent of the types of others).

6.3. A BNE with Common Priors. The definition in (6.2) should really be referred to as a T-based BNE ***without*** common priors. If $\mu_i=\mu_j$ for all $i,j\epsilon I$ in (6.2) then we have a T-based BNE ***with*** common priors.

6.4. From a BNE to a Pseudo BNE. Fix a collection of ex ante subjective beliefs over Ω, $\{\mu_i\}_{i\epsilon I}$. Recall that a type is a random variable on the state space Ω. Define π_i to be the probability distribution over the type space T induced by μ_i. Then π_i is agent i's ex ante belief over the type space T. For each $i\epsilon I$, define $D_i^*(\tau_i)$ to be the marginal distribution on F_i of $\mu_i(. \mid \tau_i)$, i's ex ante belief conditional on i's type τ_i. We verify in appendix A that if $\{\mu_i\}_{i\epsilon I}$ obey the conditions in (6.2) then $\{D_i^*\}_{i\epsilon I}$ is a T-based pseudo BNE for $\{\pi_i\}_{i\epsilon I}$. To avoid tedious and uninteresting problems with the appropriate choice of versions of the conditional probabilities, we impose condition (GH) of (3.14) in this verification. Hence we see that the definition in (6.2) yields the "intuitive" definition of a Pseudo BNE of section 4.

6.5. From a Pseudo BNE to a BNE. Fix a characteristic space T. Suppose we are given ex ante beliefs over the characteristic space, $\{\pi_i\}_{i\epsilon I}$, and suppose that $\{D_i\}_{i\epsilon I}$ is a T-based pseudo BNE for $\{\pi_i\}_{i\epsilon I}$. In appendix B we will construct an associated ex ante belief, $\bar{\mu}_i$ for each $i\epsilon I$, over the cartesian product of the characteristic space and the state space, $T\text{x}\Omega$. The measure $\bar{\mu}_i$ will have the property that agent i's beliefs about the items in the state space Ω under the pseudo BNE agrees with the value of the conditional, $\bar{\mu}_i(. \mid \tau_i)$, over Ω. Hence we may indeed interpret $\bar{\mu}_i$ to be the ex ante belief of agent i in the pseudo BNE. We will also verify that the marginal of $\bar{\mu}_i$ over Ω, which we denote by μ_i, is an ex ante subjective belief for agent i in the sense of (3.13). Implicit in the construction will be an assumption that the beliefs $\{\pi_i\}_{i\epsilon I}$ and the decision rules $\{D_i\}_{i\epsilon I}$ are "common knowledge;" further, to avoid irrelevant questions about versions of conditional probabilities we assume that the beliefs $\{\pi_i\}_{i\epsilon I}$ are mutually absolutely continuous.

In section 4.5 we mentioned that a pseudo BNE does not necessarily respect the "intrinsic" definition of a characteristic. The above construction illustrates this problem. Suppose that the characteristic has an intrinsic definition. Then there exists a (measurable) function $m:\Omega\rightarrow T$ which specifies the "intrinsic meaning," $m(\omega)$, of the characteristic at each $\omega\epsilon\Omega$. The measure $\bar{\mu}_i$ will not necessarily respect this "intrinsic meaning" and in particular it will not necessarily be the case that $\bar{\mu}_i(\{(\tau,\omega)\epsilon T\text{x}\Omega\text{: } m(\omega)=\tau\})=1$. Therefore, if μ_i is the marginal of $\bar{\mu}_i$ over Ω, in general the measures $\{\mu_i\}_{i\epsilon I}$ over Ω do not constitute a T-based BNE (where here we are referring to a BNE with types given by $m(\omega)$ at each $\omega\epsilon\Omega$). On the other hand if we suppose that $\bar{\mu}_i(\{(\tau,\omega)\epsilon T x\Omega: m(\omega)=\tau\})=1$ so that the pseudo BNE respects the definition of a characteristic, then μ_i is a T-based BNE. (This is verified in the appendix B.) In summary "a T-based Pseudo BNE which respects the intrinsic meaning or definition of a characteristic induces a T-based BNE."

7. Some Special Cases of a BNE

7.1. A Savage-Bayesian BNE. Consider each agent i's type space to be Q_i, her space of Savage-Bayesian types. A Savage-Bayesian BNE is a T-based BNE with $T=\Theta_0 x \Pi_{i\epsilon I} Q_i$. One may ask: What does the assumption of Savage-Bayesian BNE provide us with that we do not already have by construction of the ex ante subjective beliefs? Well, suppose that the type is indeed the Savage Bayesian type so that $\tau_i=q_i$. Since by assumption agents choose actions conditional on q_i 6.2(ii) will hold for any ex ante subjective probability μ_i. Condition 6.2(i) holds if i is maximizing expected utility; conditions 6.2(iii)-(iv) will hold when i knows that j is maximizing expected utility and i can solve j's maximization problem to determine j's behavior ***as a function of*** j's Savage Bayesian type $q_j\equiv(\theta_j,q_j^\infty)$). Hence the assumption that a collection of ex ante subjective beliefs constitutes a Savage-Bayesian BNE equilibrium is equivalent to (a) "maximizing behavior (b) "knowledge of maximizing behavior" and (c) in the event of a player having a non-unique optimal behavior strategy from a given Savage-Bayesian type, all agents know the selection rule she uses. The concept of a Savage-Bayesian BNE otherwise provides no restrictions on behavior. (As regards condition (c) above, one may want to assume that the selection rule for choice among optima is encoded in the attribute vector. This could be an interpretation of the Harsanyi Principle of (3.5). Alternatively, a re-parametrization of the model which achieves this effect is studied in (8.2).

Under such an assumption (or model re-parametrization) maximizing behavior (MB) and knowledge of (MB) is equivalent to the assumption of a Savage-Bayesian BNE.)

7.2. A Harsanyi BNE. Recall from (5.3) that $\{H_i\}_{i\epsilon I}$ is the space of Harsanyi types for the agents and $H \equiv \Theta_0 x \Pi_{i\epsilon I} H_i$ is the Harsanyi type space. A ***Harsanyi***-Bayesian Nash equilibrium is any H-based BNE. The Harsanyi BNE is the version of a BNE that Harsanyi (1967) modelled.

7.3. An Attribute BNE. This is nothing other than an Attribute-based Bayesian Nash Equilibrium; i.e., a T-Based BNE where the type space T is equal to the space of attribute vectors, Θ. Due to its simplicity the notion of an Attribute BNE (with common priors) is the BNE most used in the applications in the game theory and economics literature.

7.4. A Partial Ordering of BNE's. The partial ordering over types discussed in (5.5) implies an analogous partial ordering over equilibria. For example, suppose we are given an attribute BNE. Then each agent's behavior strategy choice is a function of that agent's attribute vector. This defines a Harsanyi BNE for example by requiring each agent i of Harsanyi type $h_i \equiv (\theta_i, h_i^\infty)$ to choose a behavior strategy equal to that which would be chosen in the Attribute BNE when agent i's attribute vector is θ_i. More formally, note from the definition in (6.2) that a T-based BNE is requires each agent to be maximizing utility and requires that "everything" (an in particular all decision-making) can be stated in terms of the T-types. If decision-making can be stated in terms of θ_i, then decision-making can also be stated in terms of $\tau_i \equiv (\theta_i, \tau_i^\#)$ by making the parameter $\tau_i^\#$ redundant. This argument shows that if $\{\mu_i\}_{i\epsilon I}$ is an attribute-based BNE then it is also a T-based BNE for all type spaces T. A similar argument implies that if $\{\mu_i\}_{i\epsilon I}$ is a T-based BNE, for any type space T, then it is also necessarily a Savage-Bayesian-Nash-equilibrium.

More generally, suppose that for each agent $i\epsilon I$ we have two types τ_i and $\hat{\tau}_i$ taking values in the spaces T_i and $\hat{T}_i$ respectively. (In particular, $\tau_i:\Omega\to T_i$ and $\hat{\tau}_i:\hat{T}\to\Omega$.) Suppose further that τ_i contains more information than $\hat{\tau}_i$ in the sense that the σ-algebra generated by τ_i is finer than that generated by $\hat{\tau}_i$ (i.e., $\sigma(\{\hat{\tau}_i\})\subseteq\sigma(\{\tau_i\})$). Define $T\equiv\Theta_0 x \Pi_{i\epsilon I} T_i$ and $\hat{T}\equiv\Theta_0 x\Pi_{i\epsilon I}\hat{T}_i$. Then it may easily be shown that if the collection of ex ante beliefs $\{\mu_i\}_{i\epsilon I}$ is a $\hat{T}$-based BNE then it is also a T-based BNE. The partial ordering of types in (5.5) therefore induces an analogous ranking of T-based BNE's.

7.3. A (complete information) Nash equilibrium.

In defining a Nash equilibrium we take as given the attribute vectors of agents, $\bar{\theta}=(\bar{\theta}_0,\{\bar{\theta}_i\}_{i\epsilon I})\epsilon\Theta$, say. A Nash equilibrium for the given vector of attribute vectors is then typically defined to be a collection of (possibly randomized) behavior strategies with respect to which each agent i with attribute vector $\bar{\theta}_i$ is best-responding to the behavior strategies of others. Now suppose that $\{\mu_i\}_{i\epsilon I}$ is an Attribute-based BNE such that each μ_i assigns probability one to the vector $\bar{\theta}$. Then it should be clear that the (possibly randomized) behavior strategies, $\{D_i(\bar{\theta}_i)\}_{i\epsilon I}$, defined by the decision functions at $\bar{\theta}$ constitute a Nash equilibrium for the game with attribute vector profile $\bar{\theta}$. In particular, we may define a Nash equilibrium for $\bar{\theta}$ to be an attributed Based BNE, $\{\mu_i\}_{i\epsilon I}$, where $\mu_i(\{\bar{\theta}\})=1$ for all $i\epsilon I$.

8. Is there any Real Difference Between the Notions of a Type?

8.1. Re-Parametrizing the Original Model.

In our analysis the "primitives" are made up of two parts: the economic primitives $<I,A,\Theta,Z^\infty,P_\gamma>$ from which the space of hierarchies of beliefs, Q, and the "state" space $\Omega \equiv Q x \Theta_0 x F x Z^\infty$ are constructed, and the ex ante subjective beliefs of the agents $\{\mu_i\}_{i\epsilon I}$. Fix such a set of primitives and call this the ***original model.*** We now study the consequence of re-parameterizing the model. In particular, we construct from the original model a ***new model*** (which we index by a hat, ^). In the new model the definition of agent i's space of attribute vectors is expanded from the space Θ_i in the original model to some augmented space $\hat{\Theta}_i$ which specifies some additional variables. The space of behavior strategies and nature's space of attribute vectors remain the same. This will define in a natural way a new "state" space $\hat{\Omega} \equiv \hat{Q} x \Theta_0 x F x Z^\infty$ where $\hat{Q} = \Pi_{i\epsilon I} \hat{Q}_i$ and where $\hat{Q}_i$ specifies agent i's hierarchy of beliefs over the augmented space of attribute vectors and space of behavior strategies of others $\hat{\Theta}_{-i} x F_{-i}$. The subjective ex ante beliefs of the old model, $\{\mu_i\}_{i\epsilon I}$, will then generate in the obvious manner a collection of ex ante subjective beliefs $\{\hat{\mu}_i\}_{i\epsilon I}$ over the "state space" of the new model, $\hat{\Omega}$. The original model and the new model will be different parametrizations of each other will not result in any change in the decision-making of agents or the "economics" of the problem.

8.2. The Re-Parametrization with $\hat{\Theta}_i = \Theta_i x F_i$.

For our first re-parametrization let us define the new model by indexing each agent by both her attribute vector of the original model ***and*** the behavior strategy she chooses. In particular suppose that we set $\hat{\Theta}_i \equiv \Theta_i x F_i$. Let us focus our attention on the ***new*** model. Any attribute vector $\hat{\theta}_j = (\theta_j, f_j)$ of agent j specifies a behavior strategy for that agent. Agent $i \neq j$ should therefore be able to "read" off the attribute vector $\hat{\theta}_j = (\theta_j, f_j)$ to infer the behavior strategy of agent j, f_j. Let us refer to agent i as being ***literate*** if, when forming a belief over the set $\hat{\Theta}_j x F_j$ agent i does indeed "read"; i.e., if for all $j \epsilon I$ agent i's (first order) belief assigns probability one to the set $\{(\hat{\theta}_j, \hat{f}_j) \equiv ((\theta_j, f_j), \hat{f}_j) \epsilon \hat{\Theta}_j x \hat{F}_j: \ f_j = \hat{f}_j\}$. Suppose that this literacy condition is common knowledge. Then it should be clear that any belief hierarchy $\hat{q}_i$ for agent i over

$\hat{\Theta}x\hat{F}$ under which the literacy condition is common knowledge will induce a unique belief hierarchy over $\hat{\Theta}$ (and vice versa). In particular, when agent i is forming a belief (hierarchy) over $\hat{\Theta}xF$, her belief (hierarchy) over the second coordinate, $\hat{F}$, is redundant since it is encoded in the attribute vector space $\hat{\Theta}$. Hence in the new model, belief hierarchies over $\hat{\Theta}xF$ (and in particular ***Savage-Bayesian*** types) and belief hierarchies over $\hat{\Theta}$ (and in particular ***Harsanyi*** types) are essentially one and the same thing. By re-parametrizing the model in this way we are able to eliminate any distinction between Harsanyi and Savage-Bayesian types! (In appendix C this argument is made formally. In particular we construct there the new model. The common knowledge of literacy condition will be implicit in the construction. We then show that there is a homomorphism between the space of Savage-Bayesian types and Harsanyi types.)

Notice however that an attribute type $\hat{\theta}_i=(\theta_i,f_i)$ in the new model merely specifies the original utility parameter θ_i and the behavior strategy f_i; it does not specify any beliefs. Hence an attribute type need not be the same as a Harsanyi or Savage-Bayesian type in this re-parametrization. As regards BNE's the above argument shows that a Savage-Bayesian BNE is the same as a Harsanyi BNE; that is, if $\{\hat{\mu}_i\}_{i\epsilon I}$ is a Savage-Bayesian BNE then it is also a Harsanyi BNE. (The vice-versa is of course always true - see (7.4).)

Even more is true! From our common knowledge of literacy condition each agent can "read" the behavior strategy from the attribute vector. Notice that this is the principal requirement for an attribute BNE. Indeed, suppose that $\{\hat{\mu}_i\}_{i\epsilon I}$ is a Savage-Bayesian BNE in the re-parameterized model. Then there is "maximizing behavior" and agents know other agents' behavior strategy choices as a function of their attribute vectors (by "reading" it off the attribute vector). Hence if the collection $\{\hat{\mu}_i\}_{i\epsilon I}$ is a Savage-Bayesian BNE then it is necessarily an attribute BNE. In (7.4) we argued that all BNE's "lie between" a Savage-Bayesian BNE on the one hand and an attribute BNE on the other. We may therefore conclude that under the above re-parametrization all notions of a BNE are equivalent; that is, in the re-parametrized model, if $\{\hat{\mu}_i\}_{i\epsilon I}$ is a T-based BNE then it is also a T′-based BNE for all type spaces T and T′.

8.3. The Re-Parametrization with $\hat{\Theta}_i = \Theta_i x Q_i^\infty$.

Now let us suppose instead that in the new model we index each agent by both her attribute vector of the original model ***and*** her hierarchy of beliefs, q_i^∞. Hence agent i's attribute vector in the new model is the same as her Savage-Bayesian type in the original model. In particular, we set $\hat{\Theta}_i \equiv \Theta_i x Q_i^\infty \equiv Q_i$. Let the space $\hat{Q}_i^\infty$ denote the space specifying agent i's hierarchy of beliefs over $\hat{\Theta}xF$. Now, the space Q_i^∞ is a space of hierarchies of beliefs over ΘxF; and the space $\hat{Q}_i^\infty$ is a space of hierarchies of beliefs over the space $\hat{\Theta}xF \equiv Qx\Theta_0 xF$ which itself involves a space Q of hierarchies of beliefs. It is well-known that there is a homomorphism between a space of hierarchies of beliefs and a space of hierarchies of beliefs over hierarchies of beliefs. In particular, there is a homomorphism $\Psi_i:Q_i^\infty \rightarrow \hat{Q}_i^\infty$ between the spaces Q_i^∞ and $\hat{Q}_i^\infty$, where $\hat{q}_i^\infty = \Psi_i(q_i^\infty)$ may be interpreted as the hierarchy of beliefs over $\hat{\Theta}xF$ of the agent i with hierarchy of beliefs q_i^∞ over ΘxF. (See e.g., Brandenberger and Dekel (1993).) The space of Savage-Bayesian types of agent i in the new model is by definition $\hat{Q}_i \equiv \hat{\Theta}_i x \hat{Q}_i^\infty = \Theta_i x Q_i^\infty x \hat{Q}_i^\infty$. However, the only Savage-Bayesian types we should be worried about are those which respect the homomorphism above; i.e., those in the set $\hat{Q}_i^* \equiv \{(\theta_i, q_i^\infty, \hat{q}_i^\infty)\epsilon \hat{Q}_i \mid \hat{q}_i^\infty = \Psi_i(q_i^\infty)\}$. This requirement is analogous to the "common knowledge of literacy" condition used in (8.2). It should be clear that the spaces $\hat{\Theta}_i \equiv \Theta_i x Q_i^\infty$ and $\hat{Q}_i^*$ are homomorphic. In particular, in the new model knowledge of the attribute vector $\hat{\theta}_i$ is equivalent to knowledge of the Savage-Bayesian type $\hat{q}_i$. In the new, re-parametrized model an attribute type is the same as a Savage-Bayesian type!

From (5.5) we argued that a type must contain at least as much information as the attribute vector and no more information than the Savage-Bayesian type. Hence we may conclude that in this re-parametrization of the model ***all*** notions of a type are equivalent. This in turn implies that in this new model an attribute BNE is the same as a Savage-Bayesian BNE and indeed that all notions of a T-based BNE are the same, regardless of the notion of a type used in its definition.

8.5. It all depends upon the definition of an attribute! Given the results of the previous sub-section one may now ask why so much fuss was made in section 5 about the distinction between Savage-Bayesian types on the one hand and the Harsanyi, the attribute and other types on the other hand. After all, can we not re-parametrize away any distinction between the various notions of a type and the notions of a type-based BNE? The answer of course should be straightforward to see. If the type space is modelled to be a very "simple" space then the requirement of a BNE imposes a lot of restrictions. When the type space is a very "complex" space then the requirement of a BNE imposes few restrictions on the model. Hence in some sense the "less complex" is the definition of a type the "better" is the resulting concept of a BNE.

A Savage-Bayesian type is that which completely specifies beliefs and hence the decision-making problem of the agent. It is the "most complex" notion of a type. In section (7.1) we argued that a BNE based on Savage-Bayesian types imposes no restrictions other than the requirement that each agent should be maximizing utility and should know that others are maximizing utility. This very complex notion of a type results in a "bad" equilibrium concept since it results in few restrictions on behavior.

Consider next the notion of a Harsanyi type. Harsanyi types, recall, are belief hierarchies over the attribute vector Θ. The Harsanyi BNE concept is one which results in a relationship between Harsanyi types and the behavior strategies chosen by the agents. The "less complex" is the definition of the attribute vectors, the "less complex" will be the notion of a Harsanyi type. The greater then is the restriction imposed by the equilibrium concept. When we define, as in (8.2), the attribute vectors of agents' to include a specification of their chosen behavior strategies then we have in some sense a very "complex" definition of attribute vectors; this then results in a very "complex" definition of a Harsanyi type and therefore a "bad" equilibrium concept. Indeed in this case we argued earlier that a Harsanyi type then becomes equivalent to a Savage-Bayesian type and so is the "most complex" possible. The resulting concept of a Harsanyi BNE therefore results in very few restrictions.

Finally consider the restrictions of an attribute BNE. When the attribute vector is "simple" this equilibrium notion will provide a lot of restrictions. Suppose however we consider, as we did in the previous section, a re-parametrization of the model where an agent's attribute vector in the new model also specifies that agent's Savage-Bayesian type in the original model. We argued in this case that in the new, re-parametrized model, an agent's attribute vector is essentially the same as the agent's Savage-Bayesian type. This re-parametrization therefore results in the "most complex" definition of an attribute vector, and therefore a very "bad" equilibrium concept.

In summary we may conclude thus: "the less you put into the definition of a type the more you get out of the definition of a Bayesian-Nash-Equilibrium."

9. BNE's and Correlated Equilibrium

9.1. Correlation. One may ask whether or not our definition of a BNE allows for the correlations in the standard definitions of a correlated equilibrium of Aumann (1974). (See also Forges (1993).) One may argue as follows: First, we modelled each agent $i \in I$ as choosing behavior strategies, $\mu_i'''(df_i \mid q_i)$, as a function of q_i, agent i's Savage-Bayesian type. Therefore agent i's behavior strategy choice is independent of the behavior strategies that will be chosen by others. So, one may conclude, no correlation is allowed in the behavior strategies of agents.

Well, this conclusion argument is incorrect! We are free to broadly interpret the meaning of the attribute vector, θ_i, which, recall, is specified in the Savage-Bayesian type $q_i=(\theta_i,q_i^\infty)$. It is through this parameter that correlations may be introduced. In particular, an agent's attribute vector may specify not only that agent's utility parameters but may also specify some extraneous (i.e., payoff irrelevant) parameters that are used by agents to coordinate their behavior strategies at date 0. (Correlations over time may be modelled through the observation process $\{z_n\}^\infty_{n=1}$.) Indeed, from the previous section we know that we may re-parametrize the model so that agent i's attribute vector also specifies a "suggested" behavior strategy which may used in correlating agents' actions. Hence our definition of a BNE encompasses the standard definitions of correlated equilibrium.

9.2. Independence. We now ask the following question: What is the nature of the independence assumptions we need to ***exclude*** correlations in a BNE. Well, let $\{\mu_i\}_{i\epsilon I}$ be a T-based BNE and let π_i denote the distribution over T induced by μ_i. I claim that the following assumption on the ex ante subjective beliefs of agents over the type space, $\{\pi_i\}_{i\epsilon I}$, suffices:

(9.3) for each $i\epsilon I$, π_i is a product measure over $\Theta_0 x \Pi_{j\epsilon I} T_j$.

To see this notice that the conclusion in (6.4) was that in a T-based BNE agents' behavior is the same as in a T-based pseudo BNE and in particular agents choose behavior strategies via decision functions which are a function of their own type and is, conditional on own type, independent of the types and behavior strategies of others. If those types are themselves "independent," and in particular if (9.3) holds then the agents' behavior strategies must be independent of the types and behavior strategies of other agents. In particular when (9.3) holds the induced distribution over the types and behavior strategies TxF is a product measure over the spaces $\Theta_0, \{T_j x F_j\}_{j\epsilon I}$. The independence assumption (9.3) (or its absence) is important in the model of a BNE of Milgrom and Weber (1985) and also of Jordan (1991a,b) and Nyarko (1992 and 1993a).

9.4. With Independence Types are Attributes. Suppose now that we have a T-based BNE, $\{\mu_i\}_{i\epsilon I}$, which obeys the independence assumption (9.3). It turns out this implies that $\{\mu_i\}_{i\epsilon I}$ is then necessarily an attribute based BNE. In particular, under the independence assumption types are "essentially" attribute vectors. The argument for this is as follows: Agent i's type is of the form $\tau_i \equiv (\theta_i, \tau_i^{\#})$. Her Savage-Bayesian type is of the form $q_i = (\theta_i, q_i^{\infty})$. Since agent i's type τ_i is by definition a $\sigma(\{q_i\})$-measurable random variable there exists a Borel-measurable function $g_i: Q_i \rightarrow T_i$ such that $\tau_i = g_i(\theta_i, q_i^{\infty})$. However, q_i^{∞} determines agent i's belief over the behavior strategies and Savage-Bayesian types of other agents via the measure $P_i(q_i^{\infty})$ of (3.2). Under the independence assumption (9.3) agent i's beliefs about the other agents' Savage-Bayesian types must be independent of her own τ_i type. So $P_i(q_i^{\infty})$ must be independent of q_i^{∞} with μ_i probability one. In particular with μ_i-probability one $P_i(q_i^{\infty})$ is equal to some (non-random) measure, $\bar{p}_i^{\infty}$ say, (over $Q_{-i} x \Theta_0 x F_{-i}$). From (3.2) we know that the function P_i is one-to-one. Hence there must be some (non-random) hierarchy of beliefs $\bar{q}_i^{\infty}$ such that μ_i assigns probability one to the event that $q_i^{\infty} = \bar{q}_i^{\infty}$. This in turn implies that with μ_i-probability one $\tau_i = g_i(\theta_i, \bar{q}_i^{\infty})$. This is true for all $i \epsilon I$ with μ_i-probability one (and hence under condition (GH) with μ_j probability one). In particular for any agent i, conditioning with respect to τ_i or τ_j is the same as conditioning with respect to θ_i or θ_j respectively. We may therefore replace τ_i with θ_i and τ_j with θ_j in each of the conditions of (6.2). This implies that if $\{\mu_i\}_{i\epsilon I}$ is a T-based BNE and the independence condition (9.3) holds then $\{\mu_i\}_{i\epsilon I}$ is also an attribute-based BNE.

9.5. On Aumann (1987). The main theorem of Aumann (1987) states that "if there is Bayes' rationality (i.e., maximizing behavior) at every state of the world then the distribution of actions is a correlated equilibrium." Well, as argued in (9.1), under our broad definition of an attribute vector the difference between Nash and correlated Nash equilibrium disappears. Next, if maximizing behavior occurs at every state of the world then there will necessarily be "common knowledge" of maximizing behavior since there are no states where the event "non-maximizing behavior" occurs. We argued in (7.1) that under maximizing behavior and knowledge of maximizing behavior any collection of ex ante beliefs constitutes a Savage-Bayesian BNE. Hence we obtain the implication of the main theorem of Aumann (1987) if we interpret the types as Savage-Bayesian types.

10. Conclusion

We have provided a discussion of the various notions of a type. The concept of a Savage-Bayesian type on the one hand is that required for decision theory. The concepts of a Harsanyi type, an attribute type and other notions of a type are used in game-theory. Our formal definition and comparison of these notions of type will hopefully provide some insights into the similarities and differences between the decision-theoretic and game-theoretic approaches to modelling multi-agent interactions where agents have incomplete information over both the fundamentals and the behavior strategies used by other agents.

11. Technical Appendices

11.1. Appendix A: From a BNE to a Pseudo BNE.

We now show that the conditions in (6.2) define a T-based pseudo BNE as asserted in (6.4). Let $\{\mu_i\}_{i\epsilon I}$, $\{\pi_i\}_{i\epsilon I}$ and $\{D_i^*\}_{i\epsilon I}$ be as in (6.4). Fix any $i\epsilon I$ and let $D_i:T_i \rightarrow \mathcal{P}(F_i)$ be any alternative decision function for agent i. We seek to show that (4.4) is true. We shall use the notation of (6.1): given any probability ν on Ω and any random variable x on Ω, $\nu(dx)$ denotes the induced distribution of x by ν and $\int .\nu(dx)$ denotes integration with respect to $\nu(dx)$; further, given any finite collection of Borel measures $\{\nu_j\}_j$ with ν_j a probability over a metric space X_j, $\Pi_j\nu_j$ denotes the product of the measures over the cartesian product $\Pi_j X_j$.

The following statements are true for μ_i-almost every ***fixed*** $q_i\epsilon Q_i$ and $\tau_i\epsilon T_i$: The measure $\mu_i(df_i \mid q_i)$ is agent i's behavior strategy choice rule as a function of q_i. Denote by $D_i(\tau_i)(df_i)$, the measure over F_i induced by the alternative decision function D_i when agent i's type is τ_i. From 6.2(i) agent i is maximizing utility so, recalling the definition of U_i of (3.9) and the fact that τ_i is the type of the agent with Savage-Bayesian type q_i, we conclude that

$$\int U_i(q_i,f_i)\mu_i(df_i \mid q_i) \geq \int U_i(q_i,f_i)D_i(\tau_i)(df_i). \tag{11.2}$$

We make the following three observations: first, we may re-write the utility function U_i of (3.9) of agent i as $U_i(q_i,f_i)= \int \int u_i(\theta_0,\theta_i,f_i,f_{-i})\mu_i(df_{-i} \mid q_i,\tau_{-i})\mu_i(d\tau_{-i} \mid \tau_i)$ where $\mu_i(d\tau_{-i} \mid \tau_i)$ integrates over $\theta_0\epsilon\Theta_0$ which recall is a coordinate of $T_{-i}\equiv\Theta_0 x\Pi_{j\neq i}T_j$; second, from (6.2iv) $\mu_i(df_{-i} \mid q_i,\tau_{-i})=\Pi_{j\neq i}\mu_j(df_j \mid q_i,\tau_j)$ (which from 6.2iii) $= \Pi_{j\neq i}\mu_j(df_j \mid \tau_j)= \Pi_{j\neq i}D_j^*(\tau_j)(df_j)\equiv D_{-i}^*(\tau_{-i})(df_{-i})$; and third, $\mu_i(d\tau_{-i} \mid \tau_i)= \pi_i(d\tau_i \mid \tau_i)$. These three observations imply that

$$U_i(q_i,f_i)= \int \int u_i(\theta_0,\theta_i,f_i,f_{-i})D_{-i}^*(\tau_{-i})(df_{-i})\pi_i(d\tau_{-i} \mid \tau_i). \tag{11.3}$$

Since τ_i is measurable with respect to the σ-algebra generated by q_i, conditioning μ_i on (q_i,τ_i) is the same as conditioning on only q_i. Using this fact (for the first equality below) and condition 6.2(ii) (for the second) we conclude that

$$\mu_i(df_i \mid q_i)=\mu_i(df_i \mid q_i,\tau_i)=\mu_i(df_i \mid \tau_i)\equiv D_i^*(\tau_i). \tag{11.4}$$

Recalling the definition of W_i in (4.3) we see that by putting (11.3) and (11.4) into (11.2) and integrating over τ_i with respect to π_i, we obtain (4.4).//

11.5. Appendix B: From a Pseudo BNE to a BNE.

Let T, $\{\pi_i\}_{i\epsilon I}$ and $\{D_i\}_{i\epsilon I}$ be as in (6.5). We proceed to perform the following exercises mentioned in (6.5): (i) construct the measure $\bar{\mu}_i$ over ΩxT; (ii) show that the marginal, μ_i, of $\bar{\mu}_i$ over Ω is a subjective ex ante belief for agent i over Ω; and (iii) show that if the pseudo BNE respects the "intrinsic" definition of a characteristic then $\{\mu_i\}_{i\epsilon I}$ is a T-based BNE.

__(i) Construction of the measure $\bar{\mu}_i$:__ Given any $\tau\epsilon T$, define $D(\tau)$ to be the product measure over $\Pi_{i\epsilon I}F_i$ induced by the decision functions $\{D_i(\tau_i)\}_{i\epsilon I}$. Define η_i to be the measure over $\Omega_0\equiv TxF$ whose marginal on T is π_i and whose conditional on F given τ is $D(\tau)$; (using the notation of (3.11) this is the measure $\pi_i\otimes D(\tau)$). Now, the vector of attributes and behavior strategies, $(\theta,f)\epsilon\Theta xF$, may be considered to be a random variable over Ω_0, equal to the projection of $\Omega_0\equiv TxF=\Theta xT^{\#}xF$ onto the ΘxF coordinate. We denote this by $(\theta(\omega_0),f(\omega_0))$. We construct at $\omega_0\epsilon\Omega_0$ agent i's belief hierarchy $q_i^\infty(\omega_0)=(q_i^1(\omega_0),q_i^2(\omega_0),\ldots)\epsilon Q_i^\infty$ over the random variable (θ,f) by induction as follows: the first order belief at ω_0, $q_i^1(\omega_0)$, is defined by setting for $M \subseteq \Theta_{-i} x F_{-i}$,

$$q_i^1(\omega_0)(M) \equiv \eta_i(\{(\theta_{-i}, f_{-i}) \epsilon M\} \mid \tau_i)(\omega_0); \tag{11.6}$$

and given r-th order beliefs for each $j \epsilon I$ at each $\omega_0 \epsilon \Omega_0$, (r+1)-th order beliefs are defined by setting for each $M \subseteq Q_{-i}^r x \Theta_{-i} x F_{-i}$,

$$q_i^{r+1}(\omega_0)(M) = \eta_i(M' \mid \tau_i)(\omega_0) \quad \text{where } M' = \{\omega_0' \epsilon \Omega_0 \mid (q_{-i}^r(\omega_0'), \theta_{-i}(\omega_0'), f_{-i}(\omega_0')) \epsilon M\}, \tag{11.7}$$

and where $\eta_i(. \mid \tau_i)$ denotes a fixed regular version of the conditional of η_i given τ_i. Define $q_i(\omega_0) \equiv (\theta_i(\omega_0), q_i^\infty(\omega_0))$. It can be shown that $q_i^\infty(\omega_0)$ is measurable with respect to the σ-aglebra generated by τ_i on the Borel measure space of Ω_0. Hence there is a Borel-measurable function $L_i: T_i \rightarrow Q_i$ such that $q_i(\omega_0) = L_i(\tau_i(\omega_0))$ at each $\omega_0 \epsilon \Omega_0$ (η_i-a.e). (One should consult Nyarko (1993b) for the details, and in particular see Lemma 9.6 of Nyarko (1993b) for a proof that $q_i^\infty(\omega_0)$ obeys the "probabilistic coherence condition" mentioned in (3.2) of this paper so that $q_i(\omega_0)$ does indeed belong to Q_i (for η_i-a.e. ω_0); and see Lemma 9.5 of Nyarko (1993b) for the verification of the fact that under the assumption that $\{\pi_i\}_{i\epsilon I}$ are mutually absolutely continuous, the above construction is independent of the versions of the conditional probabilities $\eta_j(. \mid \tau_j)$ used.) We define, using the notation of (3.11), the measure

$$\bar{\mu}_i \equiv [[\pi_i \otimes L(\tau)] \otimes D(\tau)] \otimes P_\gamma \tag{11.8}$$

to be the measure over $T x \Omega$ with the following properties: (a) the marginal over T is equal to π_i; (b) the conditional over Q given τ assigns probability one to the vector $L(\tau) \equiv \{L_i(\tau_i)\}_{i\epsilon I}$; (c) the distribution over F given any $(\tau,q)\epsilon TxQ$ is equal to $D(\tau)\equiv\Pi_{j\epsilon I}D_j(\tau_j)$; and (d) the distribution over Z^∞ given any (τ,q,f) is equal to P_γ of section (3.6) where $\tau=\{\{(\theta_i,\tau_i^\#)\}_{i\epsilon I},\theta_0\}$ and $\gamma=(\theta_0,f)$. Then it should be clear that $\bar{\mu}_i(.\mid\tau_i)$ represents the belief over Ω of agent i of type τ_i in the pseudo BNE under the assumption that the ex ante beliefs $\{\pi_j\}_{j\epsilon I}$ and the decision functions $\{D_j\}_{j\epsilon I}$ are "common knowledge."

__(ii) μ_i is a subjective ex ante belief:__ Let μ_i be the marginal of $\bar{\mu}_i$ on Ω. Recall again our terminology from (6.1): given any measure η over a cartesian product XxY, $\eta(dx)$ denotes the marginal of η on X. Also recall the definition of the product operation $\otimes$ of (3.11). By integration (actually "disintegration") or equivalently by iterated conditioning, it should be clear that we may write μ_i as follows:

$$\mu_i = [[\mu_i(dq_i)\otimes\mu_i(df_i\mid q_i)]\otimes\mu_i(d(q_{-i},\theta_0,f_{-i})\mid q_i)]\otimes P_\gamma. \qquad (11.9)$$

From Nyarko (1993b, (9.7)) it may be shown that at η_i-a.e. ω_0, our earlier construction of $q_i(\omega_0)=(\theta_i(\omega_0),q_i^\infty(\omega_0))$, is "coherent" in the following sense: the probability distribution of the random variable (q_{-i},θ_0,f_{-i}) on Ω_0 induced by $\eta_i(.\mid q_i(\omega_0))$ (the conditional of η_i given $q_i(\omega_0)$) is equal to the measure $P_i(q_i^\infty(\omega_0))$ of section (3.2). This in turn implies that for μ_i-a.e. $q_i=(\theta_i,q_i^\infty)$, $\mu_i(d(q_{-i},\theta_0,f_{-i})\mid q_i)= P_i(q_i^\infty)$. By comparing (11.9) and (3.12), we see that μ_i is indeed an ex ante subjective belief for agent i over the state space Ω.

__(iii) μ_i obeys (6.2):__ Now take as given the "intrinsic meaning" of the characteristic; i.e., fix the mappings $m_i:\Omega\rightarrow T_i$ and define $m(\omega)\equiv\{\theta_0,\{m_i(\omega)\}_{i\epsilon I}\}$. Assume that the pseudo BNE respects the definition m: i.e., assume that for all $i\epsilon I$, $\bar{\mu}_i(\{(\tau,\omega)\epsilon Tx\Omega\text{: } m(\omega)=\tau\})=1$. Recall that μ_i is the marginal of $\bar{\mu}_i$ on Ω. We proceed to show that $\{\mu_i\}_{i\epsilon I}$ is a T-based BNE.

Fix any agent i and a characteristic τ_i. Let $q_i(\tau_i)\equiv L_i(\tau_i)$ where $L_i:T_i\rightarrow Q_i$ is the function defined in part (i) and write $q_i(\tau_i)=(\theta_i(\tau_i),q_i^{\infty}(\tau_i))$; $q_i(\tau_i)$ is the Savage-Bayesian type of that agent of characteristic τ_i in the pseudo BNE. From the definition of a pseudo BNE agent i's choice of a behavior strategy is optimal given that agent's belief about the behavior strategy choices of other agents. From the coherence property mentioned in part (ii) above, this belief is the same as that generated by the measure $P_i(q_i^{\infty}(\tau_i))$. In particular, for η_i-a.e. $(\tau_i,f_i)\epsilon T_i x F_i$, f_i maximizes agent i's utility $U_i(.,q_i(\tau_i))$ (see (3.9)) when agent i's Savage-Bayesian type is $q_i(\tau_i)$. From the definition of μ_i this in turn implies that for μ_i-a.e. $(q_i,f_i)\epsilon Q_i x F_i$, f_i maximizes $U_i(.,q_i)$. Hence (6.2i) holds.

To prove the rest of (6.2) we make the following observations: First, m_i may be considered a random variable on $T x\Omega$ by defining $m_i(\tau,\omega)$ to be equal to $m_i(\omega)$. Since the pseudo BNE is assumed to respect the intrinsic definition of a characteristic, $m_i=\tau_i$ $\bar{\mu}_i$-a.e. So conditioning on m_i is equivalent to conditioning on τ_i ($\bar{\mu}_i$-a.e.). Second, $q_i=L_i(\tau_i)$ by construction so conditioning with respect to the pair (q_i,τ_i) is equivalent to conditioning with respect to only τ_i (again $\bar{\mu}_i$-a.e.). Third, since μ_i is the marginal of $\bar{\mu}_i$ over Ω, both measures agree over random variables defined over Ω. Finally, if π_i and π_j are mutually absolutely continuous, then so are $\bar{\mu}_i$ and $\bar{\mu}_j$. Hence, all of the above observations which are true $\bar{\mu}_i$ a.e. are also true $\bar{\mu}_j$-a.e.

To prove (6.2ii) we may therefore argue as follows: $\mu_i(df_i \mid q_i,m_i)=\bar{\mu}_i(df_i \mid q_i,m_i) = \bar{\mu}_i(df_i \mid q_i,\tau_i) = \bar{\mu}_i(df_i \mid \tau_i) = \bar{\mu}_i(df_i \mid m_i) = \mu_i(df_i \mid m_i)$. To prove (6.2iii) we have the following related arguments: $\mu_i(df_j \mid q_i,m_j)=\bar{\mu}_i(df_j \mid q_i,m_j)=\bar{\mu}_i(df_j \mid q_i,\tau_j)=D_j(\tau_j)$ (by definition of $\bar{\mu}_i$) $= \bar{\mu}_j(df_j \mid \tau_j)=\bar{\mu}_j(df_j \mid m_j)=\mu_j(df_j \mid m_j)$. Condition (6.2iv) follows similarly. Hence $\{\mu_i\}_{i\epsilon I}$ is a T-based BNE.

11.10. Appendix C. Details of the Re-Parametrization in (8.2).

Suppose we are given the ***original*** model with primitives specified by $<\Theta,F,Z^{\infty},\{\mu_i\}_{i\epsilon I}>$. We shall now construct a ***new*** model (specified by a hat, $\hat{}$), where we change the attribute vector to specify in addition the behavior strategy chosen by the agent; i.e., where $\hat{\Theta}_i \equiv \Theta_i x F_i$ for all $i \epsilon I$. We will show that with this re-parametrization there is a homomorphism between the space of Harsanyi Types and the space of Savage-Bayesian types.

For each $i \epsilon I$ we may construct the ***space*** of hierarchies of beliefs over $\hat{\Theta} x F$, $\{\hat{Q}_i^k\}^{\infty}_{k=1}$, in the manner described in (3.2) (by setting Y_i of that section equal to $\hat{\Theta}_i x F_i$). Define $\hat{Q}_i$ to be the resulting set of Savage-Bayesian types of agent i and set $\hat{Q} = \Pi_{i \epsilon I} \hat{Q}_i$. Define $\hat{\Omega} \equiv \hat{Q} x \Theta_0 x F x Z^{\infty}$; this is the state space for the new model. Define for each $i \epsilon I$, $\psi_i^1 : Q_i^1 \rightarrow \hat{Q}_i^1$ by setting for each $q_i^1 \epsilon Q_i^1$ and $\hat{M} \subseteq \hat{\Theta}_{-i} x F_{-i}$,

$$\psi_i^1(q_i^1)(\hat{M}) \equiv q_i^1(M) \quad \text{where } M \equiv \{(\theta_{-i}, f_{-i}) \epsilon \Theta_{-i} x F_{-i} : (\theta_{-i}, f_{-i}, f_{-i}) \epsilon \hat{M}\}. \qquad (11.11)$$

The range of ψ_i^1 in $\hat{Q}_i^1$ is the set of first order beliefs in the new model such that agent i "reads off" agent j's attribute vector to infer her behavior strategy. Let us proceed inductively. Given $\psi_j^r : Q_j^r \rightarrow \hat{Q}_j^r$ for each $j \epsilon I$ and for some integer r, define $\psi_i^{r+1} : Q_i^{r+1} \rightarrow \hat{Q}_i^{r+1}$ by setting for each $q_i^{r+1} \epsilon Q_i^{r+1}$ and $\hat{M} \subseteq \hat{Q}^r_{-i} x \hat{\Theta}_{-i} x F_{-i}$

$$\psi_i^{r+1}(q_i^{r+1})(\hat{M}) \equiv q_i^{r+1}(M) \quad \text{where } M \equiv \{(q_{-i}^r, \theta_{-i}, f_{-i}) \epsilon Q_{-i}^r x \Theta_{-i} x F_{-i} : (\psi_{-i}^r(q_{-i}^r), (\theta_{-i}, f_{-i}), f_{-i}) \epsilon \hat{M}\}. \qquad (11.12)$$

Next define $\psi_i : Q_i x F_i \rightarrow \hat{\Theta}_i x \Pi^{\infty}_{r=1} \hat{Q}_i^r$ by setting $\psi_i((\theta_i, q_i^1, q_i^2, \ldots), f_i) = (\hat{\theta}_i, \hat{q}_i^1, \hat{q}_i^2, \ldots)$ where $\hat{\theta}_i \equiv (\theta_i, f_i)$ and $\hat{q}_i^r \equiv \psi_i^r(q_i^r)$ for each r. One may check that the sequence $\psi_i(q_i, f_i)$ obeys the probabilistic coherence condition mentioned in (3.2) so belongs to $\hat{Q}_i$. Define the measure $\hat{\mu}_i$ over $\hat{\Omega}$ by setting for each $\hat{M} \subseteq \hat{\Omega}$, $\hat{\mu}_i(\hat{M}) \equiv \mu_i(M)$ where $M \equiv \{\omega = (q, \theta_0, f, z^{\infty}) \epsilon \Omega : (\psi(q,f), \theta_0, f, z^{\infty}) \epsilon \hat{M}\}$. $\hat{\mu}_i$ is agent i's ex ante belief in the new model.

Define $\hat{Q}_i^*$ to be the range of ψ_i in $\hat{Q}_i$. Then $\hat{Q}_i^*$ is the subset of $\hat{Q}_i$ where the hierarchies of beliefs of agent i in the new model obey the "common knowledge of literacy condition" mentioned in (8.2). The mapping $\psi_i: Q_i x F_i \rightarrow \hat{Q}_i^*$ can easily be seen to be a homomorphism between $Q_i x F_i$ and Q_i^*. A Harsanyi type for agent i in the ***new model*** specifies agent i's attribute vector $\hat{\theta}_i \equiv (\theta_i, f_i)$ as well as a hierarchy of beliefs over $\hat{\Theta} \equiv \Theta x F$. A Savage-Bayesian type $q_i = (\theta_i, q_i^\infty)$ in the ***original model*** specifies agent i's attribute vector, θ_i, as well as a hierarchy of beliefs over the set $\Theta x F \equiv \hat{\Theta}$ (but does not specify agent i's behavior strategy). Hence, the ***space*** of Harsanyi types in the ***new*** model, $\hat{H}_i$, is equal to the cartesian product $Q_i x F_i$ of agent i's space of Savage-Bayesian types and her space of behavior strategies. The mapping ψ_i therefore defines a homomorphism between the space of Harsanyi types of the ***new model,*** $\hat{H}_i \equiv Q_i x F_i$, and the set of Savage-Bayesian types in the ***new model*** which obey the "common knowledge of literacy condition," $\hat{Q}_i^*$.

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