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SCALE ECONOMIES, THE REGIONS OF MULTIPLE TRADE EQUILIBRIA, AND GAINS FROM ACQUISITION OF INDUSTRIES

BY

Ralph E. Gomory
and
William J. Baumol

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C. V. STARR CENTER FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

SCALE ECONOMIES, THE REGIONS OF MULTIPLE TRADE EQUILIBRIA, AND GAINS FROM ACQUISITION OF INDUSTRIES[1]

Ralph E. Gomory[2] and William J. Baumol[3]

"It is probably correct to say that economies of scale tend to be ignored in theoretical [trade] models...for the simple reason that the theoretical difficulties are considerable....That this is a poor reason for excluding them is evident, especially if it is true that they constitute one of the principal sources of international trade." (Chipman [1965 p. 737] as quoted in Ethier [1979 p. 1]).

I. Introduction

In contrast with the unique equilibria that characterize a classical international trade model, scale economies models of any substantial size characteristically yield an enormous number of equilibria. These equilibria form a systematic and significant pattern that seems never to have been recognized before. In this paper it will be shown these equilibrium points occur not just anywhere but lie in a well defined region of a graph that is described presently. We will discuss the shape of that region and its economic meaning. Our emphasis throughout will be on the entire array of equilibria rather than individual equilibrium points. The regions of equilibrium points reveal (1) that there are substantial ranges of outcomes in which there is direct conflict between the interests

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[1]This paper is intimately related to Gomory [1991b] which laid out the basic features of the model we use here. This paper provides new results fundamental for the use of the analysis for policy purposes.

[2]The Alfred P. Sloan Foundation.

[3]Professor of economics, Princeton and New York Universities and Director, C. V. Starr Center for Applied Economics, N.Y.U. We are extremely grateful for helpful comments to Jagdish Bhagwati, Alan Blinder, Avinash Dixit, David Dollar, Gene Grossman, Paul Krugman, Robert Willig and Edward Wolff. For a brief history of the origins of this paper and the significant contributions of Herbert Scarf see Appendix A.

of countries that trade with one another, (2) that there exists a region of outcomes where one country's dominance is so substantial that it damages not only the other country but also itself, by depriving its customer country of purchasing power, (3) that outcomes worse than autarky are not exceptional, but are a part, and sometimes a substantial feature, of any model with a larger number of traded goods.

Despite the importance in international trade of industries which appear to be characterized by scale economies, and the significant contributions to the subject provided by a number of prominent economists, the Ricardian model, characterized by diminishing or constant returns to scale still seems to dominate thinking on trade issues among nonspecialists.

Even in 1965 when Chipman wrote, a good deal was understood about the theoretical consequences of scale economies for trade theory, and the significant contributions that have appeared since the mid 1980's have added greatly to this knowledge. From this work (references are provided presently) we know that scale economies tend to preclude uniqueness in trade equilibria, and that the multiple equilibria are apt to include a number that are locally stable. These stable equilibria may well contain some that are mutually disadvantageous, sometimes substantially so, relative to some other equilibria, some even worse than autarky. However, the literature has generally worked with small models, typically composed of two countries and two goods, which do not have many of the properties of the fully array of equilibria.

The approach we take here is quite different. Because of the technical progress made in Gomory [1991b] we can now deal directly with *large* Ricardian models having economies of scale. This analysis, while consistent with the earlier results, will also reveal significant new aspects of international trade. These new results, together with the characteristics of scale economies equilibria recognized by earlier writers -- the tendency toward specialization and multiple equilibria, many Pareto dominated by others -- depict a world very different from the unique and mutually beneficial equilibrium that can plausibly be expected in a regime of scale diseconomies. It describes a world in which conflict of national interests plays a much larger part. It also suggests that a policy of *laissez-faire*, which recommends itself in a Ricardian world, at least requires reexamination in one where scale economies and the attendant divergence of national interests are important.

The analysis offered here leads to several primary conclusions:

1. In the presence of economies of scale of the sort that will be specified presently, when accompanied by high sunk entry costs of a nation into a particular industry, the set of equilibria for a model of international trade is best described not by means of a few typical equilibrium points but in terms of a *region* in which these points lie and which they tend to *fill up*.

2. The number of equilibrium points increases indefinitely as the number of commodities in which they trade increase. These equilibrium points tend to fill up the solution region in the sense that, given any arbitrarily selected point, P, in the region, with a sufficient number of commodities traded an equilibrium point will appear within any preselected distance (however small) from P.

3. In this model the benefits of trade are real and often very large[4], but market forces do not achieve them automatically. In fact, unlike the classical case of constant or diminishing returns, the gains to a country can be negative. But they can be increased, and generally to a very high level, when a country adds to the number of industries it captures. However, a country *always* loses out by capturing too large a share of the world's industries.

4. Under many circumstances there are *externalities* to success or failure in a given industry in the sense that the country as a whole benefits from that success, or loses from that failure.

5. Some stable outcomes are good for both countries, some are good for one and poor for the other, and some can be *poor for both*. In particular, the world can easily end up in a *stable* equilibrium point that is highly disadvantageous to a less affluent country. It may be extremely difficult for that country to move out of such an equilibrium, and a direction of movement toward a better equilibrium point is likely not to be obvious either to a central planner or to the private capital market.

6. Achievement of maximal utility by one country can easily entail an equilibrium that is

[4] In this model benefits from trade have two sources connected with economies of scale. The first is the saving when a single country supplies the world market for some good. This usually helps both countries. The second, which is often very large, flows from use by one country of an entrenched economy of scale to hold onto an industry despite high wages, and despite the fact that the other country would be the cheaper producer if it were able to overcome the difficulties of entry. This type of benefit usually helps only one country, at the expense of the other. An example of this type is provided in section XI.

very poor for the other.

7. There is a substantial region of outcomes in which the interests of trading countries conflict, particularly because, unlike the diminishing-returns case, one nation can gain at the expense of another by taking one or more industries away from the latter.

II. On Earlier Discussions of Scale Economies and Trade Equilibria

The previous literature has, in effect, employed three different models of the nature of the scale economies. As has repeatedly been recognized, while each of these corresponds to some real and probably significant phenomena of reality, they require markedly different analytic methods and vary considerably in the conclusions they generate.

The first of these scenarios, one of several to which the analysis in this paper applies (see Section III, below), assumes that firms are perfectly competitive, that they operate under conditions of constant or diminishing returns to scale, and that the scale economies are produced by *externalities* that benefit the firms within a single industry in a given country. Under such circumstances prices will, of course, be set at levels that yield zero profits (sometimes referred to as "average cost pricing" though that is undefined for the multi-product firm). This model implies that there will be strong forces making for specialization and non uniqueness of equilibrium. Examples of writings using this approach include Kemp [1969] and Ethier [1982]. The argument goes back to Marshall's *Principles* and was a source of controversy, particularly soon after its formulation. For a good review of the history see Chipman [1965 p.740ff]).

The second widely-used scale economies model assumes them to be internal to the firm. As is well known, this leads us to expect that the market will be monopolistic or subject to monopolistic competition, and unless the markets are perfectly contestable, it is likely to entail non-zero profits. Helpman and Krugman [1985] have been the leading users of this approach, and have produced extremely valuable and illuminating results with its aid (see also, e.g., Krugman [1979] and Helpman [1984]).

The third scale economies model, rather rarely used, also assumes perfect competition and externalities. However, in contrast to the first of the models, it is assumed here that the externalities are not a function of the size of the output of the industry within a single country. Rather, those externalities are generated by the industry's output *world wide*. For obvious reasons, in such a case the tendency for extreme specialization within individual countries disappears and some of the policy problems that characterize the case of externalities within particular countries also vanish. This approach was, apparently, first suggested by Viner [1937], and it has more recently been investigated systematically and effectively by Ethier [1979].

The importance of the scale economies case has long been recognized, as is indicated by the Chipman quotation at the beginning of this paper. Still, until about 1980, with the exception of a few discussions that have become classics (e.g., Matthews [1949/50], Meade [1952] and Kemp [1969]) the subject was often presented as an afterthought--almost as a minor complicating amendment to the standard analysis.

"The conclusions of this literature are characterized by multiple equilibria with extensive specialization." (Ethier [1982 p. 1243-1268]). It was also recognized that the specialized equilibria tend to be stable (in the Marshallian sense; see, e.g., Kemp's discussion). In addition, it was observed that some of the equilibria are apt to be relatively detrimental to both countries (e.g., Meade p. 38) and are likely to leave at least some of the trading countries worse off than they would be under autarky. In these circumstances, it was argued by Graham, the economies that were harmed by trade would benefit from protection that withdrew them from trade in the pertinent commodity--a contention that itself stimulated a substantial literature.

The 1980's brought with them a flowering of the literature. Notably, the invaluable work of Helpman and Krugman and that of Grossman and Helpman expanded the frontiers of the field and added considerable illumination. Much of this work proceeded in directions very different from those of the current article. The role of imperfect competition in international trade, including the implications of contestability of those markets were examined with care and sophistication. The pertinent dynamics were explored extensively.

The hallmark of our departure from previous work, as already noted, is our ability to deal directly with large commodity set problems and with our resulting shift in focus from a small set of equilibrium points to the region that is needed to encompass them as the number of traded commodities grows. In so doing we build on the work of Gomory [1991b], which we interpret and extend materially.

III. General Structure of the Model

Our model rests primarily on the first of four basic assumptions:

(1) in the presence of economies of scale, entry into an industry by

the country in which that industry did not previously exist, and in the face of an opposing entrenched industry, is difficult.

In addition, we assume that

(2) the assumption that economic profits are zero, so that national income can be equated to the earnings of the one scarce input, labor;

(3) in autarky each country will produce for itself a market clearing positive quantity, however small, of each of the commodities encompassed in the model, including those it purchases from abroad when trade occurs;

(4) each firm and industry produces only a single product, so average cost can be defined and calculated, in the absence of fixed and common outlays on any multiplicity of products.

The only assumption likely to seem troubling are is (2), entailing zero profits despite the presence of scale economies. However, much of previous trade theory has, following Marshall, worked with models incorporating both these features. As has already been noted, one of the scenarios to which our analysis applies is Marshall's world of perfect competition with scale economies absent from the firm, but present for the industry within any one country. These industry scale economies are generated by externalities that do not cross the nation's borders. Though a review of the trade literature indicates that this is the most widely used of the scale economies constructs it has always aroused controversy. It is sometimes suggested that this case rarely arises except where specialized labor is most effectively trained by experience on the job and the labor force is immobile internationally. We are convinced that this view of the matter greatly underestimates the externalities of proximity of sets of industries whose activities lend support to one another. The situation is most dramatically illustrated by "...the picture of James Watt stuffing soaked rags in the gaps between his pistons and cylinders in an effort to prevent loss of steam until Wilkinson's boring mill provided him with reasonably accurate cylinders" (Rosenberg [1976 p.199]). Here it must be remembered that canon-maker John Wilkinson, located less than 100 miles from Watt and Wilkinson, profited substantially from the success of one another's work, the demand for the latter's boring machine stimulated by the market for Boulton and Watt engines. The presence of mutually supportive activities in close proximity seems frequently to be a requirement for success. The modern semiconductor industry, or indeed any complex manufacturing industry, is completely dependant on a host of specialized and experienced suppliers and services, whose absence greatly complicates the start-up of an industry and whose presence contributes greatly to efficiency. In this case, then, we can indeed have a range of scale economies for the industry, constant returns for the firm, perfect competition and, hence, zero profits.

However, that is not the only case in which our analysis holds.

Another scenario that yields sufficient conditions for the validity of our analysis is one with the following three characteristics:

1. The production function of the firms in the industry entail scale economies up to some intermediate scale of operation. Beyond this minimum efficient scale, returns either remain constant for some interval (the average cost curves are "flat bottomed") or quickly begin to diminish.

2. As a result, the industry contains a multiplicity of firms. These firms do not collude. Their vigorous competition drives economic profits close to zero.

3. Entry into the industry by a country which initially is not producing any of the industry's outputs is difficult. This is presumably so because the entry entails heavy sunk investments, with all their associated costs.

The empirical evidence indicates that the case of flat bottomed average cost curves is common. Here, the *firm's* average cost curve declines sharply and substantially up to some level of output, and then has a flat bottom for a considerable range of output thereafter, beyond which the average cost curve turns upward (curve $AC_1AC'_1$ in Figure 1a). With the cost curve of every firm having the same shape, in the absence of externalities the *industry's* average cost curve, AC_1RS in Figure 1b, will replicate the single firm's declining average costs at its leftward end. It will then become horizontal, because as demand shifts outward new firms will be able to enter only when volume is sufficient to enable them to operate at or beyond their own minimum efficient scale.[5] If the downward sloping portion of the cost curve corresponds to significant outlays that

[5] This is strictly true if the range of outputs with diminishing average costs is narrower than the flat bottomed portion of the firm's average cost curve. Otherwise, the flat portion of the industry's average cost curve will be interrupted by relatively narrow segments in which the curve rises and then falls back to the horizontal. These interrupting intervals grow increasingly

are fixed and sunk, then there will clearly be a considerable barrier that must be overcome when such an industry is first launched in some economy.[6] Nevertheless, despite this entry barrier, the multiplicity of firms actually present in the industry can generate sufficient competitive pressure to keep profits close to their competitive level.

If, in addition, there are proximity externalities, these can add substantially to the size of the sunk investment necessary for a country to launch itself into the industry because, to succeed, firms in the industry require the presence of ancillary enterprises that supply specialized inputs, research, training and other support activities to the industry. We can even assume that the ancillary firms cannot survive without the presence of the industry, while the industry cannot succeed without the availability of the services of these ancillary firms. In this scenario, then, there are mutual external benefits from the entry of firms into the new industry and entry of firms into the ancillary activities. These external benefits can grow with the size of the industry and thereby extend the range of output over which the industry's average cost curve slopes downward, despite the zero slope of the average cost curve of each firm near its equilibrium point, though that is not necessary for the workings of our model.

IV. The Basic Diagram

It is useful to begin our analysis with a brief glance at the graph that is our main analytical instrument, though many of its properties will not be clear until later in the discussion. Figure 2a is such a diagram for the case of two countries (1 and 2) and 13 traded goods.[7] The horizontal axis measures the share of the total income of the two countries that accrues to Country 1, that is, it represents $Z_1 = Y_1/(Y_1+Y_2)$, where Y_i is any measure of national income in Country i. For each trade equilibrium point one can calculate the value of Z_1, and given (ordinal) utility functions for each of the two countries, one can then determine U_1 and U_2, the utility level each country

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narrow as industry output expands, and they ultimately disappear. On all this see Baumol, Panzar and Willig [1988 pp. 32-40].

[6] The requirement that entry be difficult, and the possible role of sunk costs in making this so, indicates why we cannot use the assumption that the markets are contestable to ensure that the zero-profit condition is satisfied.

[7] This figure is based on the data from Table 1. The method of calculation of both the boundary curves and the equilibrium points is explained later in this paper.

derives in that equilibrium.[8] U_1 is then measured on the right-hand vertical axis, and U_2 on the left-hand vertical axis.

The graph contains four curves, which are labelled U_{1max}, U_{1min}, etc. The U_1 and U_2 curves obviously pertain to Country 1 and Country 2 respectively. U_{1max} is to be interpreted as the upper boundary of the equilibrium points, E_1, for Country 1, that is, as the *upper utility frontier* and U_{1min} as the *lower utility frontier*. The point labelled Aut_1 represents the utility level country I achieves under autarky.

The graph also shows a considerable number of dots between U_{2min} and U_{2max}. These are actual equilibrium points generated by computer for the two country-13 good case. These dots illustrate how the equilibrium points tend to fill the region of equilibria -- the region between the upper and lower utility frontiers.

The shape of the region of specialized equilibria and its boundaries that is seen in Figure 2a recurs in every model with reasonable parameter values in the dozens of cases we have analyzed. Later, we will provide rigorous theorems and intuitive explanations indicating the generality of these shape. As mentioned in the introduction it has a number of significant implications. Figure 2a displays a range of values of Z (that between a and c in Figure 2a) in which the upper boundary of Country 1 is ascending toward the right while that of Country 2 is descending. This range, which has no counterpart in a world of scale diseconomies, is one in which the interests of the two trading countries are apt to be diametrically opposed -- that is, a move which increases the utility of one of the countries will generally reduce that of the other. We note also that the point A which gives maximal utility to Country 2 confines the utility of Country 1 to a rather low lying range; this is again a reflection of the underlying conflict.

Second, as the graph shows, beyond some turning point (Z_1 = c for Country 1 in Figure 2a) a country suffers a steady *loss* in utility as the share of the world's industries that it dominates

[8] The use of community indifference curves has a long history. It is described in Chipman [1965], pp. 690-698 who also examines the analytical issues to which these curves give rise. There are clear logical problems entailed in such an aggregation of preferences, but on operational grounds the device continues in widespread use. Obviously, the existence of a community indifference map for all practical purposes entitles one to proceed with the aid of an ordinal utility function of that same aggregated community on the premise that the requirements of integrability are satisfied.

over the other country increases. The reason for this rather surprising phenomenon, as we will see, is that as Country 1's share of industries rises, the other country becomes an ever poorer customer for its exports (increasingly impoverished Country 2 can provide ever less of its own commodities to Country 1 in exchange for 1's exports). Finally, we see that as a country approaches complete dominance of world trade its utility approaches the utility it obtains under autarky (point AUT 1 for Country 1 in the figure).

Figure 2a suggests the very strong gains from trade that can occur in a scale-economies setting. The high utility outcomes for Country 2 are *very* much higher than the autarky level.[9] In addition most (though not all) of the utility outcomes for Country 2 lie above its autarky level. However, the same can not be said for Country 1. The boundaries of the equilibrium region for Country 1 are seen in the background of Figure 2a, and it is clear that roughly half the region lies below its autarky level. We see that neither mutual gain nor loss from trade is automatic in this model.

We will also find later, (Figures 8a to 8d), that the presence of some industries with scale *diseconomies* serves to truncate the region of equilibria, and that as the share of industries with diseconomies increases, the range in which the upper frontiers are opposite in slope contracts . As the diseconomies industries approach 100 percent of total demand the region of solutions contracts into the single point of the classical diseconomies theory. There remains no interval in which the interests of the two countries may be in direct opposition in the manner that was just described. In other words, the analysis appears to show that scale economies systematically exacerbate the opportunities for rancorous economic rivalry among nations.

It is important to recognize that little of this can be deduced from a traditional two-commodity model that yields only two specialized equilibrium points E_a and E_b, in Figure 2b. For those two points cannot possibly indicate the shape of the region that emerges quite clearly with as few as, say, six traded commodities, with its 256 specialized equilibrium points.

The logic and derivation of the shapes of the frontiers will be explored in the discussion

[9] Strictly speaking, since our utility functions are interpreted as ordinal, the height of a point in the figure is arbitrary. Still, it is clear intuitively that scale economies enhance the maximal gains from trade. For trade permits increased specialization by countries, and where there are scale economies specialization reduces real costs.

that follows. The objective in introducing the graph at this point has only been to make more concrete the notion of the equilibrium *region* that is the focus of our analysis, and to offer a preview of the direction in which we will be heading. However before describing this new approach fully, we will first review the more traditional two-good models of economies and diseconomies of scale.

V. The Cases of Scale Economies and Diseconomies: Some Recapitulation

We begin with a very brief review of the scale diseconomies model, ignoring the case of constant returns because of its familiarity. In our model, with its simplifying assumption that there is only one input, labor, it is easy to show that if two commodities are both produced under conditions of scale diseconomies (increasing marginal input cost) the production frontier must be concave (downwards) -- that is, there must be diminishing returns to specialization in either of those two goods. Similarly, if both goods are produced under conditions of scale economies (decreasing marginal input use), the production frontier will generally be convex[10].

In the classical case of diminishing returns, following Ricardo's parable, EE' and PP', the production frontiers of England and Portugal, are both concave, because there are diminishing marginal returns to increased specialization in either wine or cloth production in either country. If there is trade and it is optimal for England to produce the wine-cloth combination given by some point S, then it will be efficient for Portugal to be at a point S' on its frontier (if one exists), where the slope of PP' is the same as the slope of EE' at S. This must be so because if the two slopes were unequal one country could increase its wine output more than the other could by shifting enough labor out of cloth production to forego one yard of cloth, and so it would be more efficient for that country to produce a larger share of the world's wine and for the other to produce a larger share of cloth.

Thus, the scale diseconomies case is characterized by interior equilibrium points and

[10]In an unpublished note Professor Avinash Dixit has shown that in the multi-input case, while a convex frontier can arise only where there are economies of scale, one can have scale economies even if the production frontier is concave. Thus, the scale economies phenomena with which this article is concerned can arise even where portions of the frontier are concave or the frontier is concave throughout.

production of the same commodity by different countries. It is also well known that in this case if markets are perfectly competitive and externalities are absent the market mechanism automatically drives the world economy toward an equilibrium that is efficient, Pareto optimal, and in which every nation gains from its participation in trade. The world never finds itself stuck in an equilibrium that is inefficient and damaging to the social welfare.

The writings mentioned earlier showed that matters become quite different when we turn to our focal case, that of scale economies. Here we first deal, for simplicity, with that in which England has constant returns but Portugal has scale economies[11] (Figure 3). Portugal's production frontier, PP', is now concave as a result of the premise that the marginal opportunity cost of wine, $\partial c_p / \partial w_p$, decreases as it shifts more of its labor into wine production, while EE' is a line segment. The line segment PRP' connecting the endpoints of PP' has an absolute slope smaller than that of EE', suggesting that if there were complete specialization by both countries, Portugal should produce only wine; that is, comparing only specialized points, Portugal has a comparative advantage in wine production (OP'/OP > OE'/OE, meaning that Portugal gives up less cloth output per unit of added wine production than England does). Yet, if one happens to start off at a point such as A, at which Portugal is producing a substantial amount of cloth and, as a result, as shown by the large absolute slope of PP' at A, Portugal is enjoying relatively large marginal scale economies in cloth production at that point, the market will push Portugal toward local maximum point P at which Portugal produces no wine at all. This shows how scale economies can deprive a country entirely of competitiveness in the commodity in which it enjoys an overall comparative advantage. England is driven toward low or zero cloth production and Portugal toward low or zero wine production despite the fact that global comparative advantage goes precisely the other way.

Another way to understand the logic of the issue is to recognize that in the presence of scale diseconomies it is the ratio of *marginal* products that determines the pertinent comparative

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[11]Clearly this means that technology, in the broadest sense, must differ between the two countries. In the literature it is more common to begin with the simplifying premise that there are scale economies in the production of one of the two goods included in the model, with constant returns to scale in the production of the other good.

advantage. But this can differ markedly from the ratio of *average* products over the entire relevant range that underlies the average comparative advantage pertinent to efficiency under a specialized equilibrium in the scale economies case, so that the marginal and average productivity ratios can lead to very different equilibria.

Once it is locked into an equilibrium such as just described, there is very little a country can do to get out of it with the aid of moderate steps. Figure 4, again, shows the case where Portugal has been trapped into serving as specialist in cloth production, despite its global comparative advantage as a wine supplier. Here, the budget line PB shows Portugal's purchase options when it devotes all its resources to cloth production, turning out OP of cloth. The absolute value of the slope of PB is, of course, the ratio, p_w/p_c, of the international prices of cloth and wine which, for the moment, are assumed to be fixed. Thus, Portugal can sell some of its cloth and get a mix of the two goods such as that at point J, or it can sell all of its cloth, getting OB of wine in exchange. This is far less wine than Portugal could produce for itself by moving all of its labor into wine and so moving to point P'. But any modest reallocation of its resources, say, to point K on PP', will only make things worse, because at K it gets both less wine and less cloth than it does by remaining specialized in cloth and then trading at current world prices.

In a world model with significant and ubiquitous economies of scale, specialized production is likely to play the dominant role. For, if the two countries were both to produce some commodity G, their outputs must be such that the marginal cost of G in Country 1 must be the same as that in Country 2. But with scale economies, if any disturbance were to lead to an expansion of the output of G in Country 1, with no matching rise in its output in Country 2, the marginal cost in 1 would normally fall below that of 2, and the world market would be launched on the path toward a local maximum in which 1 produced all of G. While this argument is only approximate it is enough to suggest why specialized production plays so central a role in the analysis that follows. It will consequently be convenient to have the

Definition: A <u>perfectly specialized equilibrium</u> is one in which no commodity is produced at the same time by more than one country.

It seems intuitively clear that every perfectly specialized equilibrium will be a local maximum, for the reasons illustrated in Figure 4. Moreover, it should be clear that when n commodities are produced in a two-country world, the number of perfectly specialized equilibrium local maxima will be 2^n-2, since (excluding the two extreme cases where either country produces nothing for export) there will be exactly two options in the choice of producer country for each commodity. It is, of course, possible that some of these equilibria will not be feasible because one of the countries may have too small a labor force to produce and satisfy total world market demand for each and every one of the commodities that a particular equilibrium point assigns to it. However, this infeasibility case is in fact ruled out by our (rather mild) assumption that each country's labor force is sufficiently large to produce, in a state of autarky, *some* positive quantity of each of the commodities in the model and to satisfy domestic demand for each such item at a market clearing price. This point is dealt with explicitly in section VII (Theorem 7.1) where the formal equilibrium model is discussed.

Thus, even when the number of commodities, n, is quite moderate, the number of local optima can be *very* large. Moreover, many if not most of them will entail competitive lock-in and sacrifices of utility for at least one country. It would appear that, once caught at such a stable point, no country could be sure of the direction in which to move in order to improve its circumstances significantly. The geometric analogue of the calculation problem would appear to be that of an exploration party located in a foggy crater whose rim has 2^n-2 hillocks of varying height, the explorers being determined to get to the highest point in the area. The question, then, is whether in the circumstances of our problem anything can be done to provide an orderly array of the available local maxima, one that offers some guidance to someone attempting to move the economy toward a markedly better state of affairs. We will see, next, that this is, indeed, possible.

VI. The Basic Construct: The National Utility Frontiers

For this purpose we require a diagram that plots the utilities of the two countries in the different equilibria, as a function of Z_1, Country 1's share of the total national income of the two countries. This graph has already been introduced (Figure 2a). We assume that for each country it is possible to combine the preferences of its inhabitants into an (ordinal) social utility function that does not, however, permit comparison with the magnitude of total utility in the other country. This premise, of course, takes us only a small step beyond the common use of community indifference curves in international trade models. We will normalize these utilities so that at their maxima their value is unity. Besides serving as indicators of the state of welfare, these utility functions also determine the demand functions for each of the n commodities in each country.

Each perfectly specialized equilibrium is characterized by a vector of integer variables, x_i, where $0 \leq x_i \leq 1$, and x_i is to be interpreted as the share of world output of commodity i produced by Country 2. Such an equilibrium will yield a pair of social utility values, U_1, U_2 for the two countries, and these can be plotted as a pair of points lying directly above the corresponding value of Country 1 relative income, Z_1 (e.g., points R and S at $Z_1 = a$ in Figure 2a).[12]

A utility-maximizing calculation (explained below) for each country for each value of Z_1 then yields two upper utility frontiers, U_{1max} and U_{2max}, indicating a utility level which the country in question cannot exceed at a given value of Z_1. We will also find that each country has its lower utility frontier, which is somewhat similar in shape to the upper frontier. Together, the two frontiers for a given country bound the region of perfectly specialized equilibrium points for that country.

Because of the integer character of the x variables (the finite number of perfectly specialized equilibria) not every point on an upper utility frontier or in a neighborhood immediately below it will be an equilibrium point. However, as n, the number of commodities, increases toward infinity, it can be shown in Gomory [1991b] that corresponding to any point r on such a frontier, there will be an equilibrium point that lies within any preassigned distance δ from r, however small a value of δ that is selected.

Before introducing the formal model we will say a few words about non-specialized equilibria. Non-specialized equilibria do exist in this model. Indeed, they are very numerous, far outnumbering the specialized equilibria, and **while most are unstable,** some of them can even, in a very plausible sense, be stable. They are not confined by the lower boundary curve, but they are always below the upper one. A typical graph containing the non-specialized equilibria as

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[12]Note that in our model, in full employment equilibrium, the level of Country 1's relative wage w_1/w_2, must increase monotonically with Z_1, that country's relative income. For w_1L_1 and w_2L_2 are the respective total incomes of the two countries, and so $1/Z_1 = (w_1L_1 + w_2L_2)/w_1L_1 = 1 + w_2L_2/w_1L_1$. With L_1, L_2 given the result follows at once.

smaller dots and the specialized ones as larger dots appears in Figure 5. An analysis of non-specialized equilibria appears in Gomory [1991b]. Despite the interesting properties of these intermediate equilibria it is the specialized equilibria that in fact determine the shape of the boundary curves and we will focus upon them in what follows.

VII. Equilibria in the Formal Model

We now provide a more detailed description of the determination of the model's equilibria. We will assume that each country has a well defined utility which is Cobb-Douglas form. Thus, as is well known, the demand-determined expenditure for the ith good is $d_{i,j}Y_j$ with Y_j the national income of Country j, and when $d_{i,j}$ is the exponent in Country j's utility function of $y_{i,j}$ the quantity of good i consumed in Country j.

The production functions $f_{i,j}$ are assumed to have economies of scale in the sense of declining average costs, i.e., $l_{i,j} > l'_{i,j}$ implies $f_{ij}(l_{ij})/l_{ij} > f_{ij}(l'_{ij})/l'_{ij} > 0$. We also assume a zero derivative at the origin, meaning roughly that some minimum level of activity is required before any output can be produced.

Since we will emphasize the array of equilibria, rather than any one equilibrium point, we will use the *normalized* national income variables $Z_j = Y_j/(Y_1+Y_2)$ as they enable us to plot equilibria corresponding to a wide range of national incomes in a finite part of the Z-U plane. We also introduce the key will use the key variables x that will enable us to describe the various equilibria in a rather uniform way. $x_{i,j}$ is defined to be the fraction of the world outlay on the ith good that is spent for the portion of the ith good made in Country j, so that in a specialized equilibrium either $x_{i,j}=0$ or $x_{ij}=1$.

By a zero-profit equilibrium point we will mean an assignment of industries among the two countries (that is, a set of non-negative $x_{ij,}$ $x_{i,1}$, $x_{6,1} + x_{i,2}=1$), a price vector p_i, a set of wage rates w_j, and an allocation $l_{i,j}$ of each country's labor supply L_j among the industries in which that country is a producer, in which (1) the supply of each good equals the demand for it, (2) any industry with non zero output earns zero profit, (3) the demand for labor in each country equals the total quantity of labor supplied and (4) for either country the values of the exports and imports are equal. These conditions will now be discussed in turn.[13]

(1) The supply of the each good equals its demand. We therefore have, with demand determined by the Cobb-Douglas utility,

$$(7.1)\quad p_i \sum_j f_{i,j}(\, l_{i,j}\,) = \sum_j d_{i,j} Y_j.$$

(2) Each producing industry earns zero profit.

$$(7.2)\quad p_i\, f_{i,j}(l_{i,j}) = w_j l_{i,j} \quad for\ f_{i,j}(l_{i,j})>0,$$

where since labor is the only input, $w_j l_{ij}$ is the total cost of industry i in Country j.

Before completing the set of equilibrium requirements it is convenient to rearrange the preceding equilibrium equations and rewrite them in terms of the variables Z_j and $x_{i,j}$. Equations (7.1) and (7.2) together link demand through supply to the wage bill. In terms of the $x_{i,j}$ this relation of demand governed expenditure on good i to the wage bill is

$$(7.3)\quad x_{i,1}(d_{i,1}Y_1+d_{i,2}Y_2)=w_1 l_{i,1}$$
$$(7.4)\quad x_{i,2}(d_{i,2}Y_1+d_{i,2}Y_2)=w_2 l_{i,2}$$

with, $0 \leq x_{i,j} \leq 1$ and $x_{i,1}+x_{i,2}=1$. Or if we introduce the normalized national incomes $Z_j=Y_j/(Y_1+Y_2)$, these equations become $x_{i,}(d_{i,1}Y_1+d_{i,2}Y_2) = w_{jj}L_j l_{i,j}/L_j = Y_j l^*_{i,j}$, or dividing through by Y_1+Y_2,

$$(7.3a)\quad x_{i,1}(d_{i,1}Z_1+d_{i,2}Z_2)=l^*_{i,1}Z_1$$
$$(7.4a)\quad x_{i,2}(d_{i,2}Z_1+d_{i,2}Z_2)=l^*_{i,2}Z_2.$$

Here the $l^*_{i,j}$ are normalized labor quantity variables, $l^*_{i,j}=l_{i,j}/Lj$ representing the fraction of the labor force in Country j employed in making product i.

Similarly, (7.1) together with the definition of the $x_{i,j}$ imply

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[13] We will not actually discuss the fourth condition because it is easily shown that this condition is automatically satisfied when the other three conditions hold.

$$(7.5) \quad p_i\, f_{i,1}(l_{i,1}) = x_{i,1}(d_{i,1}Y_1 + d_{i,2}Y_2)$$
$$(7.6) \quad p_i\, f_{i,2}(l_{i,2}) = x_{i,2}(d_{i,1}Y_1 + d_{i,2}Y_2).$$

Equations (7.3a-4a) and (7.5-6) are equivalent to (7.1) and (7.2).

(7.5-6) have a rather special character. If both countries are producers of i equations (7.5-6) yield a condition on the $x_{i,j}$

$$(7.7) \quad \frac{f_{i,1}(l_{i,1})}{f_{i,2}(l_{i,2})} = \frac{x_{i,1}}{x_{i,2}}$$

i.e., the ratio of the outputs of good i by the two countries is equal to the ratio of their shares of world expenditure on i. This implies that the unit cost (and hence the price) must be the same in both countries. If, however, there is only one producer, *i.e. one of* $x_{i,1}$, x_{i2} *equals zero and the other equals unity,* then there is really only one equation. That equation then gives p_i without any further restriction on the $x_{i,j}$. In words, if there is only one producer, that producer's output, whatever it is[14], determines the price.

We come, finally, to the third equilibrium condition, which is the requirement

(3) the assignments of labor, the $l_{i,j}$, must be a partition of the labor forces of the two countries. That is, $\Sigma_i l^*_{i,j} = \Sigma_i l_{ij}/Lj = 1$.

If we sum (7.3a) and (7.4a) and use this last condition we get

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[14]This argument assumes that the output is non-zero. To show that it is, we use our assumption that both countries are producers of all goods at a positive level when in a state of autarky. To see the connection assume that Country 1 is the sole producer of good i. Then the amount of labor it uses to meet the world demand is obtained from (7.3). Using $Y_j = w_j L_j$, $l_{i,1} = d_{i,1}L_1 + d_{i,2}(w_2/w_1)L_2$, which for any exchange rate is always $\geq d_{i,1}L_1$ the amount of labor used by Country 1 in autarky. But if the autarky amount of labor produces a positive output so will the larger amount $l_{i,1}$. Country 1 obtains the increase in $l_{i,1}$ over its autarky quantity from the labor released from the industries j as they are taken over by Country 2 when one departs from the state of autarky.

(7.8) $(\sum_i d_{i,1}x_{i,1})\, Z_1+(\sum_i d_{i,2}x_{i,1})\, Z_2= Z_1$

(7.9) $(\sum_i d_{i,1}x_{i,2})\, Z_1+(\sum_i d_{i,2}x_{i,2})\, Z_2= Z_2.$

These equations tell us simply that each country's (relative) income is given by the sum of domestic and foreign expenditures for its own products. It is easily shown that (7.8), which involves only $x_{i,1}$ and (7.9), which involves only $x_{i,2}$, are linearly dependent, and if the $x_{i,1}$ and Z satisfy (7.8) then the $x_{i,2}$ and Z satisfy (7.9).

We will occasionally refer to either (7.8) or the equivalent equation (7.9), as the zero excess labor equation, because if x and $Z=(Z_1,Z_2)$ satisfy (7.8) and (7.9) then the $l^*_{i,j}$ they generate through (7.3a) and (7.4a) will satisfy $\Sigma_i l^*_{i,j} = 1$, and, conversely, if an (x,Z), and $l^*_{i,j}$ satisfy (7.3a) and (7.4a) and $\Sigma_i l^*_{i,j} = 1$, they satisfy (7.8) and (7.9).

To summarize, what we have shown is that the necessary and sufficient condition for (x,Z) to determine an equilibrium point is that (x,Z), and the $l^*_{i,j}$ derived from them, should satisfy (7.8-9), (7.5-6) and (7.3a-4a). In other words, one can determine an equilibrium by selecting <u>any</u> set of values for (x,Z) and testing them to see if they satisfy the relationships listed in the preceding sentence. If these (x,Z) values do meet the requirements they are the basis for an equilibrium and one can proceed from those values and the production function, the supply demand equations, etc. to find the corresponding values of the remaining variables, p_i, l_{ij}, w_i, $y_{i,j}$.

Let us consider this process in more detail. For an arbitrary production pattern x (with the x not necessarily integer) we can find a Z(x) that satisfies (7.8) because (7.8) is a linear equation. This x and Z(x) are then put into (7.3a-4a). We can then solve for the $l^*_{i,j}$. This leaves (7.5-6) or (7.7) which must be checked to determine whether they are satisfied by the proposed solution.

Some x will satisfy these equations, most won't. Those x that do are the equilibrium x.[15] On the other hand, as we remarked above, if only one country produces a given good, i.e., whenever one of $x_{i,1}$ or $x_{i,2}$ equals 0 and the other equals 1, (7.5-6) is satisfied automatically. For example if $x_{i,2}$ is zero, so Country 2 produces none of good i, then the quantity of labor, $l_{i,2}$ used on good i in Country 2 is also zero, and (7.6) becomes 0=0, whatever the value of p_i is. (7.5) then determines a p_i that automatically satisfies both equations. This gives us a result which serves as the foundation for much of our theory.

Theorem 7.1: Any set of *integer, (i.e. 0,1) x is always an equilibrium.* That is, *any* perfectly specialized solution must be an equilibrium.

We will use this result[16] extensively together with the important theorem that follows.

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[15]In economic terms we have shown that for any x there is a relative national income Z, or, equivalently, an exchange rate w_1/w_2, at which the quantity of labor required by the production pattern x is exactly the amount of labor available in each country. What distinguishes equilibria from the other x is that equilibria satisfy the additional requirement that when there are two producers they produce at equal cost, or, if there is only one producer, that the one producer has a strictly positive output.

[16]The reader's intuition may well be troubled at this point because the proof that each and every one of the 2^n - 2 specialized solutions is an equilibrium seems a bit too easy. Suppose a specialized solution, call it E, assigns n-1 of the world's n goods to Country 1 as the sole producer of these items. How can we be sure that Country 1's labor force will be sufficiently large to satisfy world demand for these items? Cannot the solution then be infeasible?

The answer is that every such solution *must* be feasible under our assumption that Country 1 produces *some* (possibly very small) quantity of each of the n goods when the world is in a state of autarky (Norway would even produce a few mangoes in hothouses), and that these output quantities are sufficient to satisfy domestic demands for those goods at equilibrium prices.

This has already been shown formally in the text. To capture the spirit of the reason for the feasibility of equilibrium E consider a heuristic account of the transition from the autarky equilibrium, A, to equilibrium E. In this transition process, Country 2 takes over from Country 1 all production of one of the goods, call it good n. This releases some of Country 1's labor force for use elsewhere. Meanwhile, Country 2 as exporter of only the single good n will obtain very little of Country 1 currency, and so can afford to import very little of goods 1,...,n-1 from Country 1. At a suitable exchange rate and suitable relative prices for good n and the other products, the Country 1 labor force released from production of good n when the world moves from solution A to E will then suffice to meet Country 2's very modest import demands for

Together these two theorems tell us that in a world with scale economies a country that acquires an industry tends to hold on it.

Furthermore,

Theorem 7.2: The perfectly specialized equilibria of Theorem 7.1 will have the property of local (Marshallian) stability.

This is readily demonstrated and has been discussed in several earlier writings (see particularly Ethier [1979 p. 14]). Intuitively it follows from the extreme convexity near the origin of our production function which is ensured by the assumption that the function has a zero derivative at the origin, so that a non producer of a commodity, i, (quantity=zero) who attempts its production will always have, for low levels of output, a higher average cost than the producer with a positive output of that good. Therefore, the former non-producer's average cost will exceed the current price (which is equal to the competitor's average cost). Consequently, the former non-producer must earn a negative profit at current prices if it undertakes sufficiently low quantities of production. On the Marshallian dynamics premise, this will lead to a fall in output of i in that country.

We will work primarily with perfectly specialized equilibria, and this emphasis will be justified by the results. We start by writing an expression for the utility of a perfectly specialized equilibrium. Any x and Z determine the quantity of labor in each industry through (7.3a-4a). Once the amount of labor is known, the output quantities produced in each country can be determined using the production functions $f_{i,j}(l_{i,j})$. We use $q_{i,j}((x,Z))$ to represent these output quantities, and these output quantities, as we will see next, enable us to determine utility.

The total amount of the ith good supplied by both producers together is $Q_i(x,Z)=q_{i,1}(x,Z)+q_{i,2}(x,Z)$. Whatever the price p_i may be, this total amount of output is split between the consumers in the two countries in proportion to their monetary outlay, as given by their demand curves. Thus, with Cobb-Douglas utility the fraction of the total that country 1 gets is $F_{i,1}=d_{i,1}Z_1 \ / \ (d_{i,1}Z_1 + d_{i,2}Z_2)$, the fraction Country 2 gets is $F_{i,2}=d_{i,2}Z_2 \ / \ (d_{i,1}Z_1 + d_{i,2}Z_2)$, the quantity Country 1 gets is $y_{i1} = F_{i,1}Q_i$, and the quantity Country 2 gets is $y_{i2} = F_{i,2}Q_i$. With the

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goods 1,...,n-1 in equilibrium.

quantities known, we can obtain the resulting total utility.

As we are using Cobb-Douglas utility, the utility U_1 for Country 1 is given by

$$(7.10) \quad \ln U_1(x,Z) = u_1(x,Z) = \sum_i d_{i,1} \ln F_{i,1}(Z) Q_i(x,Z).$$

This expression can be quite complicated in its dependence on x. Fortunately, for perfectly specialized equilibria (only) this is equivalent to an expression that is linear in the x

$$(7.11) \quad Lu_1(x,Z) = \sum_i x_{i,1} d_{i,1} \ln F_{i,1}(Z) q_{i,1}(1,Z_1) + x_{i,2} d_{i,1} \ln F_{i,1}(Z) q_{i,2}(1,Z_2)$$

as can be seen by trying out any individual term for $x_{i,1}=1$ and $x_{i,1}=0$.

VIII. The Formal Analysis of Utility Frontiers

We can now easily describe how one calculates the upper and lower utility frontiers that bound the region of perfectly specialized equilibria.

First, to obtain all of the perfectly specialized equilibria it is only necessary to carry out the following steps: (a) insert integer x's in (7.8), (b) from (7.8), find the Z(x), (c) use x and Z(x) in (7.3a-4a) to obtain the labor quantities, $l_{i,j}$, (d) use these in the production functions to obtain the quantities produced and (e) from these, obtain the utilities. The plots of the equilibrium points we have seen in Figures 2a and 2b were produced in this way.

However, since the number of these equilibria grows as 2^n, it rapidly becomes difficult to compute them all and to deal with whatever general characteristics they exhibit. Instead, we will concentrate on the boundaries of this large array of solutions (upper and lower utility frontiers).

We define the curve $B_1(Z)$[17] in the Z-U plane, the upper utility frontier for Country 1, by

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[17] Here we only describe one of the two boundary approaches introduced in Gomory [1991b]. The advantages of the *linear* programming approach described here is the ease with which it produces a relatively smooth boundary, and one that seems to be theoretically tractable. The *integer* programming approach, not described here, produces a boundary that is both tighter and more jagged and is harder to analyze. However the integer approach is better at finding the integer (i.e., equilibrium) points near that boundary. Both approaches provide distinct and useful ways of thinking about the problem. Generally speaking integer programming of some sort (even if it is only rounding of linear programming) is needed to deal with actual equilibrium points, while both methods will produce a boundary.

$$\text{(8.1)}\quad B_1(Z) = Max_x\ Lu_1(x,Z)$$
$$\textit{subject to}\quad \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}x_{i,2} = Z_2,$$

That is, we maximize the linear expression for utility, (7.11) subject to the constraint, (7.9), that Country 2's income is equal to the demands for its goods by the two countries together. We could equally well have used as our constraint the equivalent equation for the income of Country 1.

Now if (x,Z(x)) is a perfectly specialized equilibrium it must satisfy (7.9) and therefore be among the x's considered in the maximization of $U_1(Z(x))$. This means that the utility of x is $\leq$ $B_1(Z(x))$ and therefore $B_1(Z)$ is an upper boundary of the array of perfectly specialized solutions.

$B_1(Z)$ is, fortunately, very easy to compute. While this is certainly suggested by (8.1) it becomes more apparent if we write the objective function in (8.1) in terms of the $x_{i,2}$ only. Using $x_{i,1}+x_{i,2}=1$ to eliminate the $x_{i,1}$ in (7.11) gives

$$\text{(8.2)}\quad Lu_1(x,Z) =$$
$$\sum_i d_{i,1}\ln F_{i,1}(Z)q_{i,1}(1,Z) + \sum_i x_{i,2}d_{i,1}\ln\frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)},$$

so we can put the maximization problem (8.1) in a form involving the $x_{i,2}$ only

$$\text{(8.3)}\quad B_1(Z) = Max_x\ Lu(x,Z) = P_1(Z) + \sum_i x_{i,2}d_{i,1}\ln\frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)}$$
$$\textit{subject to}\quad \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}\ x_{i,2} = Z_2,$$

with $P_1(Z)$ representing the first sum in (8.2). There is of course an equivalent form of the problem involving the $x_{i,1}$ only, (we will refer to this as (8.3a)) and our choice of the $x_{i,2}$ as the variables to appear explicitly is an arbitrary one. What we have now, for any Z, is a linear programming problem in the $x_{i,2}$ with only one constraint equation and upper bounds, $x_{i,2} \leq 1$, on the $x_{i,2}$. Such a problem is trivially solvable by ordinary linear programming, or by even simpler techniques.[18] The boundary is the set of points, Z, $B_1(Z)$, obtained for all values of Z. This boundary calculation is both simple and rapid. A 27-industry model requires about three minutes of calculation on an IBM PS2 Model 80, and the required computations grow very slowly with problem size. These are the calculations that provide us with the boundary curves in the various figures.

A lower boundary BL_1 (the lower utility frontier) for the array of perfectly specialized equilibria can be determined in exactly the same way by *minimizing* the objective function in (8.1) as (8.3). This is how the lower boundary curves in the various figures were obtained.

IX. Further Properties of the Upper Boundary

So far we have only shown that $B_1(Z)$ is somewhere above the array of specialized equilibria, although our figures suggest a much closer connection. We will now report two theorems that make the nature of that closer connection explicit for non-specialized equilibria. We refer to Gomory [1991b] for proofs.

Theorem 9.1: Let x be any equilibrium solution, *whether specialized or not*. Let Z(x) be the corresponding Z and $u_1(x,Z)$ the utility of x to Country 1, then

$$u_1(x,Z) \leq B_1(Z).$$

So all the equilibrium points, not just the specialized ones, lie under the upper boundary curve. Non-specialized equilibria can, however, lie under the lower boundary curve.

Next we turn to the issue of the stability of the specialized equilibria. It is the shape of the production function of industry i in Country j for low quantities of labor that determines the *stability* of a perfectly specialized solution. If the production function gives zero output for labor quantities from zero almost to the autarky level of output and then shoots up steeply, we will have a very stable equilibrium point since a large quantity of input is required for the industry in question to become even remotely competitive. For the same reason, if the output rises steeply near the origin the point will be less stable. However, if the remaining portions of the pertinent

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[18]There is a definite order in which the variables can be introduced in the problem which is borrowed from the extensive literature on the knapsack problem. Some economic meaning of this as well as computational detail is covered in Gomory [1991b].

functions are identical they will yield the same boundary curve, since, as we will explain next, the shape and position of the boundary curve is unaffected by the portion of the production function corresponding to small quantities of labor. It follows that a given boundary is compatible with either relative stability or instability of the equilibria.

To show that the position of the boundary is unaffected by portions of the production functions corresponding to small quantities of labor we first note that in the maximization problem (8.3) yielding the boundary curve, (a) the production functions $f_{i,j}(l_{i,j})$ only appear in the objective function, not in the constraint equation, and (b) they only appear there in the terms $q_{i,j}(1,Z)$. In $q_{i,j}(1,Z)=f_{i,j}(l_{i,j})$ the labor quantities are the quantities required to satisfy total world demand, and, as was shown in Footnote 14, these are always greater than the labor quantities required by the producing country in autarky. *Thus the form of the production function for quantities of labor lower than those under autarky does not in any way affect the shape of the boundary curve.*

Next we consider whether or not $B_1(Z)$ lies near the actual equilibrium points. The answer is that they do. Gomory [1991b] provides an answer expressed in terms of the parameters of the maximization problem, estimating the distance from any point of $B_1(Z)$ to an equilibrium point. We will state here, and then explain, one consequence of those estimates.

Theorem 9.2: Let n be the number of commodities available for trade and let P_n be a sequence (with x increasing) of n-commodity problems with bounded parameters. Then, for any Z' and any ϵ there is an n sufficiently large that any point on the line segment between $B_{n,1}(Z)$ and $BL_{1,n}(Z)$, the nth upper and lower boundaries, will have an integer equilibrium point within ϵ in both coordinates.

This theorem requires a certain amount of explanation. The main assertion is that under the conditions given the entire space between the upper and lower boundary curves becomes filled in with perfectly specialized equilibria as the sequence of problems encompasses more and more industries. This means, among other things, that the boundaries get to be increasingly good representations of the set of equilibrium points as there will be points pressing right up against them. This process is already quite advanced in Figure 2a.

Theorem 9.2 refers to a sequence of n-industry problems with an increasing number of industries. The bounded parameters of the theorem entail the requirement that as more and more industries are included each one accounts for a smaller and smaller percentage of total expenditure.

We have now shown how the boundaries can be calculated and have indicated that they are a rather good descriptions of the equilibrium set. In fact, we have asserted that if a problem becomes sufficiently large these are the only possible boundaries. To take a further step we must introduce a new concept: the Classical Linear Output Level.

X. The Classical Linear Output Level

The classical linear output level is a reference point $Z_1 = Z_c$, which, roughly speaking, is the level of relative income at which full employment will be achieved in both countries if the vector x is selected so as to assign each country the production of those and only those goods that it can supply more cheaply than the other country at the relative wage rates determined from Z_C.

More precisely let $S_1(Z_1)$ be the set of goods made more cheaply[19] by Country 1 *as sole producer* ($w_1 l_{i1}/q_{i1} < w_2 l_{i2}/q_{i2}$ for any i in S_1) at relative national income level Z_1 and the relative wage rates fixed by that value of Z_1. Let $S_2(Z)$ be the corresponding set for Country 2. For Z_1 sufficiently small (very low wage in Country 1), S_1 will not be empty. Let $L^*_1(Z)$ represent the total demand for Country 1's labor, if products are always assigned entirely to Country 1 when it is the less costly producer[20]

The expression in the footnote confirms that $L^*_1(Z)$ is obviously greater than unity (excess demand for labor in country 1) for Z_1 near zero so that there is a very low relative wage in Country 1 (see footnote 12 above) and then $L^*_1(Z)$ decreases monotonically with increasing Z_1 (increasing wage) and eventually becomes less than unity. The steady decrease in demand for Country 1's labor has one basic cause: i.e., the rise of real wages in Country 1, and the consequent rise in the price of its products as its share of the world's export industries grows. This, in turn, has two effects: First, for any fixed set, S_1, the demand for County 1's labor decreases monotonically with increasing Z_1, as we can see from the equation. Second, as Z_1

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[19] That is, at lower <u>average</u> cost. In our model the average cost of any product at any given output level is well defined, given the absence of fixed and common costs for any two or more products.

[20] This is given by

$$L^*_1(Z) - \sum_{i\in S_1(Z_1)} l^*_{i,1} - \sum_{i\in S_1(Z_1)} (d_{i,1}+d_{i,2}\frac{Z_2}{Z_1}).$$

increases the set $S_1(Z)$ only loses members. Such a loss causes a discontinuous downward jump in L^*_1. The values of Z_1 for which $L^*_1(Z_1) > 1$ form an open interval starting at $Z_1=0$ with endpoint[21] $Z_c<1$. At $Z_1=Z_c$ we can have, depending on circumstances, $L^*_1(Z_c)=1$ or $L^*(Z_c)<1$.

We will use the phrase "below the classical level" for $Z_1<Z_c$. Above the classical Level the demand for Country 1's labor is less than the supply, but the demand for Country 2's labor exceeds the supply. The behavior at Z_c itself and the related notion of Classical Point, are both explained in Gomory [1991b].

There are several useful equivalent ways of looking at the set S_1 and hence at the Classical Level. The first we have already given, $i\epsilon S_1(Z) <=> (w_1 l_{i1}/q_{i1} < w_2 l_{i2}/q_{i2})$. However since $w_1 l_1$ is the wage bill when Country 1 is the sole producer and $w_2 l_2$ is the wage bill when Country 2 is the sole producer are both are equal to the world expenditure on good i they and so are equal to each other. So $i\epsilon S_1(Z) <=> q_{i,1} > q_{i,2}$. But $q_{i,1}>q_{i,2} <=> d_{i,1} \ln q_{i,2}/q_{i,1} < 0$ and this expression is the ith term in the boundary maximization problem (8.3). This linkage to the terms in the boundary maximization problem will be used below.

XI. The Shape of the Upper Utility Frontier and its Economic Implications

Using the concept of the Classical Level, we will now discuss several critical characteristics of the upper utility frontier $B_1(Z)$ that have appeared in all the figures: (1) It rises monotonically from $Z_1=0$ to Z_C, passing through the autarky level on the way; (2) it remains above the autarky level to the right of Z_C; (3) the curve has a maximum that always lies to the right of Z_C, except where (in what may be characterized as cases of extreme discontinuity) the maximum lies at Z_c itself; (4) it returns the utility of Country 1 to its autarky level, U_1^A, as Z_1 approaches 1. The fact that the maximum of the utility frontier generally lies to the right of Z_c will prove to be of critical significance for application of our results.

For this analysis we will need two lemmas that relate the terms in the linearized utility

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[21] The numerical value of Z_c is obtained as follows: (1) Choose $Z'=.5$, (2) with Z' chosen all the $q_{i,j}(1,Z')$ are known so (3) compute $L^*_1(Z.)$. (4a) If $L^*_1(Z') > L_1$ the demand for labor in Country 1 is too large so $Z. < Z_c < 1$. (4b) If $L^*_1(Z') < L_1$ the demand for labor in Country 1 is too small so $Z' > Z_c > 1$. (5) Choose the midpoint of the interval containing Z_c to be the new Z' and repeat. At each iteration Z_c is confined in an interval of half the previous size.

function (7.11) to $y^1_{i,a}$, the amount of the ith good Country 1 receives in autarky.

Lemma 11.1: $F_{i,1}q_{i,1}(1,Z) > y^1_{i,a}$

This says that when Country 1 is sole producer of good i it always gets more of that good than it would in autarky.

Lemma 11.2: If $i \in S_2(Z)$ (or, equivalently, if $q_{i,1}(1,Z) < q_{i,2}(1,Z)$), then $F_{i,1}q_{i,2}(1,Z) > y^1_{i,a}$. Here Country 2 is the producer but it is also the lower cost producer, and the assertion is that once again the quantity obtained by Country 1 is greater than its autarky amount.

The proof of both lemmas is given in Appendix B. An immediate consequence of these lemmas is

Theorem 11.1: Let x be a perfectly specialized equilibrium point with associated relative national income Z(x). Then for each good i, if Country 1 is either (a) the actual producer, or (b) the higher cost producer, then at equilibrium point x, Country 1 gets more of that good than it would in autarky.

An equivalent statement is that at x Country 1 gets more of the ith good than in autarky unless Country 2 is both the producer and the higher cost producer.

If the conditions of the theorem are not met Country 1 can get more or less of the ith good than in autarky. However an examination of the argument in Appendix B shows that for very small economies of scale outcomes worse than autarky will be common.

Behavior of the Frontier Above the Classical Level $Z_1 > Z_c$.

Now we will connect these concepts with properties of the boundary curve for $Z_1 > Z_C$. Our first result is that the boundary $B_1(Z)$ above the Classical Level is also about autarky in terms of utility.

Theorem 11.2: If U_1^A is the utility to Country 1 at autarky, then $B_1(Z) \geq U_1^A$ for all Z_1 above the Classical Level. A similar result holds for Country 2 *below* the Classical Level.

For proof see Appendix B.

Next we will discuss equilibria near the boundary curve. To each point Z', $B_1(Z')$ of the boundary curve we will associate the equilibrium point which is obtained by rounding[22] the solution x of (8.3) for that Z'. We will call the new all integer x, $x^1(Z')$ and refer to it as the associated point.

Our second result is that the associated point is an equilibrium point that lies to the right of Z', and that it gives Country 1 more of every single good than it obtains in autarky. More formally

Theorem 11.3: $Z_1(x^1(Z')) > Z'_1 > Z_C$ and $x^1_{i,j}=1 \Rightarrow F_{i,1}q_{i,j}(1,Z(x^1)) > y^1_{i,a}$.

Proof: Since an $x_{i,2}$ was rounded *down* x^1 will underutilize the labor of Country 2 at Z'. To expand the demand for labor in Country 2 to match the labor supply, as is required in equilibrium, we need a lower wage rate in Country 2. This is equivalent to an increase in Z_1 to a larger value, $Z_1(x^1)$. (This point is covered more carefully in Gomory [1991b]). Since $Z_1(x^1)$ is to the right of Z' it follows that all goods in $S_2(Z')$ are in $S_2(Z(x^1))$. Consequently all positive $x_{i,2}$ are in $S_2(Z(x^1))$ and the result follows from Theorem 11.1.[23]

Gomory [1991b] also shows that the associated point for each Z' approaches its boundary point as the problem becomes large, so that, in a large problem, *all* the boundary points to the right of Z_C will have nearby points that yield high utility relative to autarky.

This benign result is obtained despite the following observation: for all equilibria to the right of the Classical Level and, therefore, for the associated points in particular, Country 1 will always be the producer of at least one good that could have been made (at that Z) at a lower cost by Country 2. More precisely:

Theorem 11.4:[24] If x is a perfectly specialized equilibrium point with $Z_1(x) > Z_C$, then for some

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[22] Since we now need equilibria, not boundary points, we move from linear programming to integer programming by the time honored device of rounding. The non-integer variable $x_{i,2}$ must be rounded *down* to 0, and the corresponding $x_{i,1}$ rounded *up* to 1. Since there is only one equation in (8.3) there is at most one non-integer variable to be rounded.

[23] Theorem 11.1 can also be used in conjunction with the integer programming/inequality approach of Gomory [1991b] to obtain even stronger gains from trade results.

[24] **Proof:** This is almost a restatement of the definition of the Classical Level. Let C_1 be the set of goods produced by Country 1. If there is no i in C_1 that is also in S_2 we would have C_1 included in $S_1(Z(X))$, the set of goods made more cheaply by Country 1. Since $Z_1(x) > Z_C$, the demand for Country 1's labor would then be less than $L^*_1(Z)$ which is $< L_1$. But since x is an

i, $x_{i,1} = 1$ and $i \epsilon S_2(Z(x))$.

Another important attribute of $B_1(Z)$ is that it always descends toward the autarky level of utility for Country 1 after Z_1 passes the Classical Level and increases toward unity. The intuitive reasoning behind this has been described in an earlier section. A careful proof can be found in Gomory [1991b].

The Portion of the Frontier Below the Classical Level, i.e., $Z_1 \leq Z_c$.

We now prove another of the characteristic properties of the boundaries: that for a broad class of production functions the boundaries rise monotonically from $Z_1 = O$ to the Classical Level.

Theorem 11.5: For problems with production functions having marginal to average productivity ratios $l_{i,j}f'_{i,j}(l_{i,j})/f_{i,j}(l_{i,j}) \leq 1 + 1/Z_C$, and with identical demand parameter values, $d_{i,1} = d_{i,2}$, $B_1(Z)$ and $BL_1(Z)$ are both monotone increasing for $0 \leq Z_1 \leq Z_C$.

The restriction on the functions is very mild and includes almost any function that one would desire in this setting. The evaluation of the functions takes place at the labor quantity $l_{i,j}$ used by Country j as the sole producer of the ith good, and so is unaffected by the shape of the curve at low labor quantities. The proof is given in Appendix B.[25] The monotone increase in the boundary curve is a feature of all our examples to date, most of which do not satisfy the identical demand hypotheses of the theorem. This suggests that the monotonically property is likely to hold for a wider class of problems. This view is supported by a careful reading of the relevant part of Appendix B. We now have the obvious

Theorem 11.6: Whenever $B_1(Z)$ has the monotonicity property just described its maximum point will necessarily lie at or above the Classical Level.

Corresponding to the monotone increase of $B_1(Z)$ below the Classical Level is the monotone

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equilibrium point, the quantity of Country 1's labor needed to make the products in C_1 is exactly L_1. This contradiction ends the proof.

[25] Here we may note that, since Z_C is < 1, $1 + 1/Z_C$ is never less than 2 and often close to 3. Thus, the restriction on the production functions in Theorem 11.5 can be interpreted to mean that the marginal product of labor divided by the average product of labor $\leq 1 + 1/Z_c \geq 2$. And, as usual, the production functions are required to be of the form assumed here *only above their autarky labor quantities*; below that their form is arbitrary.

decrease of $B_2(Z)$ *above* Z_C. This monotone decrease means that any sequence of equilibria above Z_C with increasing Z_1 will produce a sequence of utility values for Country 2 that are confined in a range (between the upper and lower boundaries for Country 2) whose upper end is constantly decreasing. One example of such a sequence is a sequence of points near $B_1(Z)$ that approaches the maximum point from the left.

This completes our discussion of the main general characteristics of the boundary curves. However there remains another important point: that utility maximization by one country may well be detrimental to the other.

In our experimental observations the maximum point for Country 1 has two properties.

1) While Country 1 clearly benefits by being near a U^1 maximizing equilibrium point, at such a value of Z_1 the utility to Country 2 is usually low, typically near and often below the autarky level, U^2_A, of Country 2.[26]

2) All or almost all of the goods made by Country 1 at an equilibrium point x, Z(x) near the maximum are in $S_2(Z)$. That is, at the corresponding exchange rate most of the goods made by Country 1 would be made more cheaply if they were produced by Country 2.

Because of Theorem 11.1 (applied to Country 2 rather than Country 1) circumstances very close to those in Property 2) have to hold to make possible the low utility for Country 2 described in 1). While we cannot prove that these properties hold generally, their plausibility and intelligibility is increased by examination of an easily understood special case.

Competition Between Identical Countries.

In the special case where country size, productive capabilities, and tastes are identical in the two countries we can obtain unambiguous conclusions because this case permits a solution in explicit algebraic form.

We therefore consider two countries with the same size of labor force, $L_1 = L_2$, with identical demand parameters, $d_{i,1} = d_{i,2}$ and with identical production functions which we will assume are of the form $e_i l^\alpha$ for the ith good. Note that α, which we will assume is ≥ 1, is taken

[26] These effects are exacerbated when Country 2 is the larger country. This is because in this model, as in other models of international trade, the larger country benefits less from trade than the smaller one. For an analysis of this effect see Gomory [1991b].

to be fixed and identical for all goods. We then have the formula for the utility frontier[27] which provides a derivation of formula (11.3) more direct than that in Gomory [1991b]

$$(11.3)\ B_1(Z)-U^A{}_1Z_1\ \{(\frac{1}{Z_1})^{Z_1}(\frac{1}{(1-Z_1)})^{(1-Z_1)}\}^{\alpha}.$$

A similar expression holds for $U_2^A(Z)$ with $U_2^AZ_2$ replacing $U_1^AZ_1$. Note that many different models having different $d_{i,1}$ and different e_i will have different equilibrium points but will produce the same boundary curve. There are only two pertinent parameters, U_1^A and α, and the demands and the production function coefficients affect the boundary curve only through the value of U_1^A. The Classical Level for all models is located at $Z_1=.5$ (equal wage rates) as one would expect from the symmetry of the case. An illustrative example with 7 goods and $\alpha=1.5$ is shown in Figure 6a. In all these models the upper and lower boundaries coincide (as we can see from the derivation) and all perfectly specialized equilibria lie on the boundary curve.

We now turn to the analysis of this interesting special case. Clearly in Fig. 6a we can see the region of conflict that extends from the maximum of Country 1 to the maximum of Country

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[27] For in this case

$$q_{i,j}(1,Z)-f_{i,j}(l_{i,j})-f_{i,j}(l^a{}_{i,j})(\frac{l_{i,j}}{l^a{}_{i,j}})^{\alpha}-f_{i,j}(l^a{}_{i,j})(\frac{1}{Z_1})^{\alpha}.$$

To derive a formula for the utility frontier we turn to the maximization problem (8.3) and see that $q_{i,2}\ /\ q_{i,1} = (Z_1/Z_2)^{\alpha}$ which is the same for all i. Using this and the fact that any feasible x satisfies the equation in (8.3), we quickly see that for fixed Z the value of the objective function is $P_1 + Z_2\ln(Z_1/Z_2)^{\alpha}$ for *all* feasible x. So all such x maximize the linearized utility. In particular, all integer x must lie on the boundary curve.

Now substituting for $q_{i,1}$ in P_1 gives

$$P_1-\sum_i d_{i,1}\ln f_{i,1}(l^a{}_{i,j})(\frac{1}{Z_1})^{\alpha}$$

so the expression for the linearized utility is

$$u_1(x,Z)-\ln U^A{}_1+\alpha\ln\frac{1}{Z_1}+Z_2\alpha\ln\frac{1}{Z_2}+Z_2\alpha\ln Z_1.$$

2. Both curves also exhibit the return to autarky phenomenon. Also, in any of these models at any equilibrium point above the Classical Level, *all goods made by Country 1 can be made more cheaply by Country 2.* This follows immediately from the identical production functions and the higher wage rate of Country 1. It follows that all points to the right of the Classical Level automatically satisfy the conditions of Theorem 11.1 and Country 1 obtains more of every good at these outcomes than it would in autarky.

Certainly Property 2 holds in an extreme form because at the maximum Country 1 is not the cheaper producer of anything.

Turning next to Property 1 we can also show, using the formula (11.3), that in *all* these models the utility to Country 2 of the point corresponding to the maximum of $B_1(Z)$ is always lower than the autarky point for $\alpha < 1.8$. An equilibrium with utility lower than autarky for Country 2 can be seen clearly in Figure 6a, where $\alpha=1.5$. This suggests that the poor outcome for Country 2 corresponding to utility maximization by Country 1, which we referred to above as *Property 1*, is not an oddity attributable to special choice of data, but *appears extensively even in the simplest cases.*

The behavior of the model for different exponents α also illustrates the two aspects of economies of scale mentioned in the introduction. The first or benign aspect can be seen by noting that the expression

$$\left(\frac{1}{Z_1}\right)^{Z_1}\left(\frac{1}{(1-Z_1)}\right)^{(1-Z_1)}$$

is always greater or equal to one. Therefore increasing α in (11.3) increases the utility of *both* Country 1 and County 2 for all Z. The curve in Fig. 6a simply shifts upward everywhere.

On the other hand we can illustrate the less benign power of economies of scale by setting $\alpha=1$. To stabilize the resulting equilibria, we will assume that $f(l)=0$ for some initial interval. The resulting curve, which still has its characteristic form, is shown in Figure 6b. What is striking about Figure 6b is that the autarky levels appear to be obtained by both countries exactly at the Classical Level. This is confirmed by noting that if $Z_1=Z_c=.5$ is substituted in (11.3) the resulting utility for Country 1 is $2^{(\alpha-1)}U_1^A$, with a similar result for U^2. Consequently for $\alpha=1$ both Countries are at the autarky level for $Z_1 = Z_c = .5$. Here we have competition in its most extreme form. At all points to the right of the Classical Level Country 1 gets more of every single good than it does in autarky, while Country 2 gets less utility than it does in autarky. In fact, equation (11.2) of Appendix B shows that Country 2 will get *less of every single good of which it is not the producer* than it would in autarky, while for those goods of which it is the producer it gets *exactly the autarky amount.*

Clearly the situation is reversed to the left of the autarky level.

XII. Pareto Optimality and the Relation Between Near-Boundary Equilibria for the Two Countries

As we have just noted, there is a region (call it R*) in our graph in which B_1, the boundary for Country 1, slopes upward as Z increases, while B_2 slopes downward for the same range of values of Z (the region between Z=a and Z=c in Figure 2a). As was already observed, it is tempting to conclude that the segments of B_1 and B_2 in R* constitute the set of Pareto optimal solutions, because it would appear that in this region any change in Z that increases $U_1(Z)$, the maximum utility of Country 1 at Z, must entail a decrease in $U_2(Z)$, the corresponding magnitude for Country 2. However, that inference is incorrect. It does not follow because, as we will now show, an equilibrium point that yields a utility for Country 1 equal to or nearly equal to the Country 1 upper boundary value, B_1 at the corresponding value of Z will not, in general, entail a utility anywhere near B_2 for Country 2. Indeed, there may be few if any equilibria which simultaneously place both countries near their respective utility boundaries.

To demonstrate this we focus upon equilibria that in some sense are represented by the points on B_1. These boundary points are produced by x that maximize (8.3) for each Z. These x are not themselves equilibrium points. However the convergence theorems of Gomory [1991b] show that for large problems there are equilibria arbitrarily near these x with the same x_i in all components but one.[28]

[28] But we must remember that there can be other equilibria that can also be nearby and have a different structure and therefore a different, and possibly better, value to Country 2. This indicates that determination of the point of highest utility to Country 2 among the points that are on or near the boundary of Country 1 for a given Z is itself a maximization problem. Clearly, we touch here on the notion of efficiency which we plan to examine further in another paper.

We proceed by plotting a curve, $B_{S2}(Z)$ that plots $L_2(x,Z)$, the linearized utility of Country 2 using for every Z those x that solve (8.3) and provide the upper boundary of Country 1. We will call the resulting curve the shadow boundary, $B_{S2}(Z)$, of Country 2. Points on $B_{S2}(Z)$ represent a Country 2 utility level that can be always be approached by perfectly specialized equilibria that simultaneously almost attain the upper boundary to Country 1's utility. Loosely speaking, it is the best Country 2 can hope for when Country 1 is near its upper utility frontier.[29]

We conclude that an equilibrium that places Country 1 near its utility frontier may entail a solution for Country 2 well below the latter's upper boundary. In our model, with its special assumptions, it is easy to prove that there is a major exception: that where the two countries have identical demand functions.

Theorem 12.1: With identical demand functions $B_2(Z) = B_{S2}(Z)$.

Proof: With identical demands the objective functions of the two countries in either (8.2) or (8.3) are identical. It follows that the x that maximizes (8.2) also maximizes (8.3) and produces $B_2(Z)$ in the utility function $u_2(x,Z)$. However this is also the definition of the shadow boundary.

But when the demand functions are not identical, as shown in Figure 7, the shadow boundary may well lie considerably below $B_2(Z)$. This and other examples illustrate that, often, $B_{S2} < B_2$. Very roughly speaking, it appears that there is more conflict between the utility maximization of the two countries (for fixed Z) as their demand curves becomes less similar.

The lone general exception to this (i.e., the only exception not dependent on special assumptions about the utility and production functions) occurs at the Classical Level, where $B_2(Z)$ and $B_{S2}(Z)$ always coincide.

Theorem 12.2: $B_2(Z_C)=B_{S2}(Z_C)$.

As in the symmetric demand theorem we need only show that the same x maximizes both (8.2) and (8.3) for $Z=Z_C$. This result is demonstrated in Gomory [1991b].

XIII. Mixed Scale Economies and Diseconomies

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[29] However, any equilibrium point near $B_1(Z)$ we cannot exclude the possibility that there are other nearby equilibria *near* $B_1(Z)$ that do much better for Country 2 (but worse for Country 1). In Figure 7 we show such equilibria. These were obtained by solving the related integer maximization problem which gives specialized points very near B_1. Note that some of them do much better for Country 2 than does the shadow boundary.

The two ends of Figure 2a depict extreme situations. Near either vertical axis one of the countries has been shut out of most trade activities, with the other nation having coopted almost every product for itself. A more detailed analysis shows that in more extreme cases all the inhabitants of one country are apt to be engaged in the production of one or two types of goods which they trade, at disadvantageous exchange rates, for the many goods made by their trading partner.

The two features of our model that have led us to so extreme a picture are the assumptions that the production of each and every commodity is characterized by scale economies (that are internal to the national industry but external to the firm), and the assumption that all products are actually traded internationally and are therefore subject to international competition. Then there are likely to be among the perfectly specialized solutions some in which most of the commodities are produced exclusively by one of the two countries, and the rest follows.

In reality, at least two features lead to a result that is more moderate. First, each country produces a number (usually a substantial number) of commodities such as personal services and housing which, for all practical purposes, are not exported and to which our analysis does not pertain. When a country has lost all of its traded products it can and will continue to produce nontraded goods. This results in a diagram in which the zero point of utility in the figures must be reinterpreted to represent the utility contributed by the non-traded sector of the economy. For a country 30 percent of whose output is traded and 70 percent is not, this zero level may constitute a large proportion of the country's total utility, with the graphs representing the possible additions to utility above this quantity that can derive from of international trade.

Second, and more important for our discussion, there are a number of industries that reach the stage of diminishing returns at a relatively low level of output. Agriculture is the most obvious example, but non uniform clothing manufacture, and a wide variety of other products probably also share this attribute. Because, as we have seen, such products invite the simultaneous activity of a multiplicity of small suppliers, the country that is driven out of all activities with scale economies will continue to find a market for its export of goods produced under conditions of diseconomies of scale. Those countries will be driven to specialize in the export of agricultural products, primary materials, textile manufactures and the like--all the products that the LDCs do actually offer in reality. Because of the nature of diseconomies of scale, this will not drive the industrialized economies out of such fields altogether. Instead, countries of both types may continue indefinitely to be sources of both sorts of products.

Next, let us describe briefly a more formal analysis of the mixed economies-diseconomies model. The formal requirements for an equilibrium point in the mixed economies-diseconomies case are almost the same as in the pure economies case.

We continue to use variables $x_{i,1}$ and $x_{i,2}$ to split the total world production of good i between the two countries. If we divide the index i into those industries for which there are diseconomies (the set D) and the i for which there are economies (the set E) we derive for the goods in E the same relations as before between, (7.3) and (7.4), expenditure and wage payments, and for these goods we also obtain once more the pricing relationships (7.5) and (7.6).

For i in D, however, we obtain the different relationships

$$(13.1)\quad x_{i,1}(d_{i,1}Y_1+d_{i,2}Y_2)=w_1 l_{i,11}+\pi_{i,1}$$
$$(13.2)\quad x_{i.2}(d_{i,2}Y_1+d_{i,2}Y_2)=w_2 l_{i,2}+\pi_{i,2}$$

which permit rents, $\pi_{i,1}$ and $\pi_{i,2}$. These equations are accompanied by the price to marginal cost relations

$$(13.3)\quad p_i\, f_{i,1}(l_{i,1})=w_1$$
$$(13.4)\quad p_i\, f_{i,2}(l_{i,2})=w_2.$$

As in the pure economies case, we have the zero excess labor relations (7.8) and (7.9), which play the same role here in ensuring that the labor allocation that emerges from the choice of the x and Z is a partition of the actual labor force. Of course in (7.8-9) the Z representing the normalized national income includes both the wage receipts and the rents earned in each country. The main new role of these equations (7.8) and (7.9) relates to the equations (13.1) and (13.2) involving the variables $x_{i,1}$, $i \epsilon D$, to which we will refer as the vector x_D. Once Y is specified in (13.1), or (13.2) the marginal pricing conditions uniquely determine the x_D. This is the behavior we expect from the classical model.

Next, consider how the presence of some diminishing returns products affects our two basic calculations, determination of perfectly specialized equilibria, and determination of the boundaries.

First, consider the perfectly specialized equilibria. If we use x_E to denote the $x_{i,1} \in E$, it is still true that choice of *any integer x_E yields a determinate equilibrium point.* But now it is necessary, in addition, to compute the corresponding x_D, which was not present in the pure integer case. Once an x_E is selected, it is possible to choose a Z_l arbitrarily and its value will then determine the x_D. But, the result will, in general, not be an equilibrium. For within arbitrary Z_l the vector $x=(x_E,x_D)$ will generally under or overutilize the labor force of one country or the other, so Z_l needs to be chosen, as in the pure integer case, to make the quantity of labor demanded a partition of the labor force. As before this means choosing Z_l to satisfy (7.8-9).

Selection of the Z_l is now more complicated than in the pure integer case. In the latter, once the x_E was chosen we simply solved a linear equation for the w_l . Here we must choose Z_l and the resulting x_D and the resulting Z_l solve (7.8-9). There is no doubt that there *is* an appropriate Z_l since there are low wage rates for Country 1 that can cause the positive $x_{i,1}$ from x_E alone to overutilize the labor of Country 1 and also high wage rates that can always cause its under- utilization. This means there is a Z_l that will cause the quantity of labor demanded to equal L_1. This Z_l is found by trying out wage rates w, which determine the labor quantities $l_{i,j}$, $i \epsilon D$, and the rents $\pi_{i,j}$. Z_l is then known, and we can put Z_l values in (7.8-9) and correcting iteratively, the correction depending on whether the current Z_l has resulted in over or underutilization.

For the boundary calculation we proceed as before with a series of optimization problems for each Z. Once Z is chosen, however, the x_D is determined[30]. This can then be entered into the zero excess labor equation, leaving only the x_E to be determined by maximization, as in (8.3). In economic terms, once the wage levels are set, the amounts produced in each country in the diseconomies industries are determined by the marginal cost conditions, and the amounts of labor used in each country in those industries is known. What is left of the labor force in each country is available for the economy's scale economy industries.

Also, when the x_D are known, the quantity of each good produced in those industries is known, and therefore the contribution of these goods to the utilities of both countries. The

[30] Although the vector x_D is determined uniquely once the Z is specified, calculating it in the presence of profits does require an iterative procedure, although it is one that seems to go very rapidly.

maximization problem (8.3) then becomes

$$(13.5)\quad Max_x\ Lu(x,Z)\ -P_1(Z)+\sum_{i\epsilon E} x_{i,2}d_{i,1}\ln\frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)}$$

$$\sum_{i\epsilon E}(d_{i,1}Z_1+d_{i,2}Z_2)\ x_{i,2}-\ Z_2-\sum_{i\epsilon D}(d_{i,1}Z_1+d_{i,2}Z_2)x_{i,2}.$$

Once Z_1 is chosen, the right hand side of the equation in (13.5) is a known constant. The term $P_1(Z)$ in the objective function now includes, in addition to its previous content, the contribution to utility of the goods from the decreasing returns industries.

The results of our analysis are illustrated by Figures 8a to 8d. These are the utility-frontier diagrams corresponding to economies in which diminishing returns products constitute successively larger shares of the total. In Figure 2a, which we have already discussed, all products are characterized by scale economies. In Figure 8a the share of diminishing returns products is 25 percent. In succeeding graphs this proportion rises successively to 50, 75 and, finally, 95 percent of the total. In every figure but the last the utility frontiers retain their characteristic hill shape. They continue to have a negatively sloping segment, and there continues to be a range of values of Z_1 toward the middle of the graph throughout which the utility of Country 1 rises while that of Country 2 declines. But as scale diseconomies dominate the world economy increasingly, the range over which the frontiers extend grows increasingly narrow, and the range over which utility values extend also diminishes.[31] Finally, when the world is exclusively devoted to the production of items with scale diseconomies the frontiers of the two economies degenerate into an single <u>and common</u> point. The reason, of course, is that all countries then share in the production of every good so that there is no scope for an array of equilibria in which one country captures a disproportionate share of commodities entirely for itself at the expense of the other country. There is a unique equilibrium point toward which market forces drive all producers. That equilibrium point has all the beneficent welfare properties documented by the Arrow-Debreu theorem. This

[31] This is because at some Z the diseconomies industries alone consume the entire labor force of one country or the other, and for Z beyond that no solution is possible.

point, in sum, depicts the triumph of the invisible hand.

But in a world where scale economies are present to any substantial degree such a happy ending is by no means assured.

XIV. Toward Policy

The preceding analysis provides the foundation for the seven conclusions listed in our introduction. It also suggests the desirability of a number of policy approaches very different from those implied by the standard models that rest on the assumption of universal diseconomies or constant returns to scale. In addition to our earlier list, the following conclusions emerge:

1. In the presence of scale economies deliberate measures by a country that enable it to capture an industry from another country can have a considerable payoff. (This is in contrast to the scale diseconomies world in which there is a presumption against interference with the market).

2. The private market may fail to achieve these benefits because the required investments can be very large and entail enormous jumps. Private investors may fail to take advantage of the opportunity for the reasons usually cited in the economic literature:

- because the risks and uncertainties are enormous
- because the private risks may be greater than the social risks and
- because a considerable portion of the payoff may take the form of external benefits that accrue to the nation as a whole rather than to the industry itself, so that firms lack the incentive to invest as much as the public interest requires.

3. The practical difficulties of encouraging the requisite large moves can hardly be overestimated. They include:

- Political pressures that all too often bias governments toward futile attempts to salvage dying industries rather than seeking out emerging industries with genuine promise;
- Political pressures that push toward protectionism even when successful participation in the markets of the world can clearly be superior to the isolation of a home market from which foreigners have been excluded.
- The degree of real coordination between governments and industry required for the design of rational programs to which, at least in some countries,

neither party is accustomed;

- The difficulty of choosing beneficial directions of movement with the help of unaided judgment. That is why a model such as the one provided in this article, which permits rough exploration of the range of possible equilibria, and offers some guiding qualitative principles, promises to be helpful.

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Appendix A

The origins of this paper may be of some interest, since they are somewhat unusual. My co-author, who for many years was a senior corporate executive and Director of Research of IBM, had in the course of his work been exposed to many of the competitive problems of the computer industry. He made many trips to Japan and saw at first hand the remarkable development of Japanese industry. He became convinced that there were elements in that development that were not well captured by the more widely known economic arguments about international trade, and that these elements were linked to the advantages of incumbency and to economies of scale. He believed that the mathematics most directly relevant to these topics was integer programming, the subject which he had pioneered 30 years ago (he is generally regarded as the father of integer programming), and whose connections with economics he and I had worked on together at that time (1960). He planned, upon his retirement from IBM in 1989 to pursue these directions and although he had considerable education in economics, he realized that he needed an economist as a partner.

Gomory was fortunate in having as an old friend Herbert Scarf, Sterling Professor of Economics at Yale. The two had known each other since graduate school days. Scarf had long been convinced of the importance of economies of scale for the understanding of economic reality and of the resulting invalidity of many standard results of welfare economics and general equilibrium theory. The likelihood that under scale economies competitive equilibria do not exist, that profitability of a new activity no longer implies that it must increase welfare, and other such disturbing observations, had led him to explore whether cooperative game theoretic analysis might serve as a replacement for the competitive paradigm. He was able to prove that this was unfortunately not correct and that increasing returns could all too easily result in an economy with an empty core.

Scarf came to visit Gomory when the latter was recovering from an illness, and their conversation turned to economies of scale. The two decided to write something expository together on what economic theory had to say about trade in the presence of economies of scale. This attempt was an education for Gomory about economic equilibria from one of the world's great masters of that subject, and eventually turned from an expository paper into the beginnings of the research paper, (Gomory [1991b]). Although Scarf's and Gomory's interests gradually diverged, Scarf remained an enormously helpful and interested influence on the development of the work.

When a version of the mathematical framework was complete and a summary of it had been submitted for publication, (Gomory [1991a]), Gomory once more felt the need for an economist partner, this time someone oriented toward the policy implications of the emerging work. Recalling our pleasant previous association Gomory decided to try renewing our partnership of some 30 years ago. Given the importance of the topic, the innovative approach, and my own happy memories of our previous work together I accepted with delight. The present paper is the result. W.J.B.

Appendix B: Proofs of Some Theorems in The Text

Proof of Lemma 11.1: To establish lemmas 11.1 and 11.2 we will write put the expression $F_{i,1}q_{i,j}$, which represents the amount of the ith good Country 1 receives if Country j is the producer, into a form which shows the relation to autarky. First we rewrite $F_{i,1}$

$$F_{i,1}=\frac{d_{i,1}Z_1}{d_{i,1}Z_1+d_{i,2}Z_2}.$$

$$\textit{But from (7.3a), } L_1\frac{(d_{i,1}Z_1+d_{i,2}Z_2)}{Z_1}=l_{i,1}(1,Z).$$

$$\textit{Since } d_{i,1}L_1=l^a_{i,1} \textit{ we have}$$

$$(11.1)\ F_{i,1}q_{i,j}=\left(\frac{l^a_{i,1}}{l_{i,1}}\right)f_{i,j}(l_{i,j}).$$

In (11.1) $l^a_{i,1}$ is the labor required by Country 1 in autarky to produce the ith good.

Next we add and then subtract terms $f_{i,1}(l_{i,1})$ and $f_{i,1}(l^a_{i,1})/\ l^a_{i,1}$ to obtain a form that highlights the differences from autarky.

$$(11.2)\ F_{i,1}q_{i,j}=f_{i,1}(l^a_{i,1})+\{l^a_{i,a}(\frac{f_{i,1}(l_{i,1})}{l_{i,1}}-\frac{f_{i,1}(l^a_{i,1})}{l^a_{i,1}})\}+\{\frac{l^a_{i,1}}{l_{i,1}}(f_{i,j}(l_{i,j})-f_{i,1}(l_{i,1}))\}.$$

In (11.2) the first term is the amount obtained in autarky, the second adds the effect of economies of scale if Country 1 produces the world supply, and the third corrects for the fact that Country j is the producer.

Proof of Lemma 11.1: Since $j=1$ the third term in (11.2) is zero. Because of economies of scale the second term will be positive if $l_{i,1}$ is $> l^a_{i,1}$. However the producing country does use more labor when it makes the world supply than it does in autarky, so the inequality holds. (This also follows by writing (7.3a) in the form $l_{i,1}=L_1(d_{i,1}+d_{i,2}Z_1/Z_2)$ which is always greater than $d_{i,1}L_1$).

The proof of Lemma 11.2 is immediate, since the second term in (11.2) is unchanged and the third term is now positive.

Proof of Theorem 11.2: For $Z_1 > Z_c$ the production of the goods in $S_2(Z)$, that is to say the goods for which Country 2 is the lower cost producer, can employ more than the total labor of Country 2. Consequently the excess labor equation in (8.3) can be satisfied by using at a positive level only $x_{i,2}$ with $i \in S_2(Z)$. From the remark at the end of the section on the Classical Level, this means that the equation in (8.3) can be satisfied using only those $x_{i,2}$ that have positive coefficients in the objective function. Since the boundary calculation (8.3) is a maximization problem, *only these $x_{i,2}$ will be used at a positive level in the maximizing x.*

The ith term of the linearized utility in (7.11) is

$$d_{i,1}\{x_{i,1}lnF_{i,1}q_{i,1}(1,Z)+x_{i,2}lnF_{i,1}q_{i,2}(1,Z)\}.$$

Now, we have just stated that every term in the linearized utility with *positive* $x_{i,2}$ is one for which Country 2 is the cheaper producer. Using this and applying our two lemmas to the two parts of this term shows that each part is $>$ ln $y^1_{i,1}$. Using $x_{i,1}+x_{i,2}=1$ we see that the ith term is $>$ $d_{i,1}$ln $y^1_{i,a,}$. But this last is the utility provided by the ith good in autarky. It follows that the sum over all i is also $> U_1^A$. This ends the proof.

Proof of Theorem 11.5: We will show the monotone character of $B_1(Z)$ below the classical level by looking at its derivative.

$B_1(Z)$ is the solution of a linear programming problem, the maximization problem (8.3). We will have different bases (in the linear programming sense) for different Z. The basis is quite simple, there is only one equation, except for the non-negativity conditions and the upper bounds, so there is only one basic variable, this is the variable that is (or can be) non-integer. It is reasonably intuitive, and shown in Gomory [1991b] that there are only a finite number of basis changes in the interval $0 \leq Z_1 \leq 1$. As Z_1 increases these occur when one of the $x_{i,2}$ reaches its upper bound or when two terms in the objective function are about to cross in value and have just reached equality. At these points the derivative of $B_1(Z)$ can have a discontinuity.

For our immediate purpose we need only deal with Z' that are not points where the basis changes. We can therefore compute the derivative at Z' under the assumption that the non-basic variable is $x_{k,2}$ for some interval around Z', and that all the other $x_{i,2}$ are fixed at 0 or 1 throughout the interval.

Using our assumption of symmetric demands, $F_1(Z)=Z_1$ so (7.11) is a sum of terms that are individually

$$d_{i,1}lnZ_1\{x_{i,1}\ln\ q_{i,1}(1,Z)+x_{i,2}\ln\ q_{i,2}(1,Z)\}.$$

We can differentiate this with respect to Z_1 if we bear in mind that the derivative of $x_{i,j}$ with respect to Z_1 is zero except for $i=k$, where (7.8) and (7.9) give us

$$\frac{dx_{k,1}}{dZ_1}-\frac{1}{d_{k,1}} \qquad \frac{dx_{k,2}}{dZ_1}--\frac{1}{d_{k,1}}$$

and that the derivative of $q_{i,j}(1,Z)=f_{i,j}(l_{i,j})$ is obtained using (7.3a-b) to link $l_{i,j}$ to Z_1.
The result is

$$\frac{d}{dz_1}\ln f_{i,j}(l_{i,j}) - \frac{f'_{i,j}(l_{i,j})}{f_{i,j}}\{-\frac{l_{i,j}}{Z_j}\frac{dZ_j}{dZ_1}\}$$

Adding up the terms and writing $\alpha_{i,j}$ for the "generalized exponent" $f'_{i,j}\, l_{i,j} / f_{i,j}$ gives

$$\frac{1}{Z_1}+\sum_i (-\alpha_{i,1}\frac{x_{i,1}d_{i,1}}{Z_1})+\sum_i (\alpha_{i,2}\frac{x_{i,2}d_{i,2}}{Z_2})+\ln\ \frac{q_{i,1}}{q_{i,2}}$$

This expression contains three terms, each of which has an economic interpretation. The first term derives from the increase in $F_{i,1}$ with Z_1, and reflects the increasing share of every good obtained by Country 1 with increasing Z_1. The two sums are more complex, the first reflects the decrease in utility due to the fact that increasing Z_1 means higher wages and less labor in each of the industries in which Country 1 is the producer. The second is the same term for Country 2 reflecting the fact that Country 2's goods have become cheaper. the term f'l/f enters because it measures the sensitivity of each industry's output to changes in labor quantity. We refer to it as the generalized exponent because for a production function of the form l^{α} it is in fact α. The third term reflects the change in utility directly due to the fact that Country 1 is producing the ith good rather than Country 2. This term will be positive whenever Country 1 is acquiring an industry where it is the cheaper producer and negative otherwise.

Using once more $d_{i,1}=d_{i,2}$ (7.8) and (7.9) give us

$$\sum_i \frac{x_{i,1}d_{i,1}}{Z_1}-1 \quad \sum_1 \frac{x_{i,2}d_{i,2}}{Z_2}-1$$

So that each of the two sums can be regarded as weights totalling 1 attached to the $\alpha_{i,j}$.

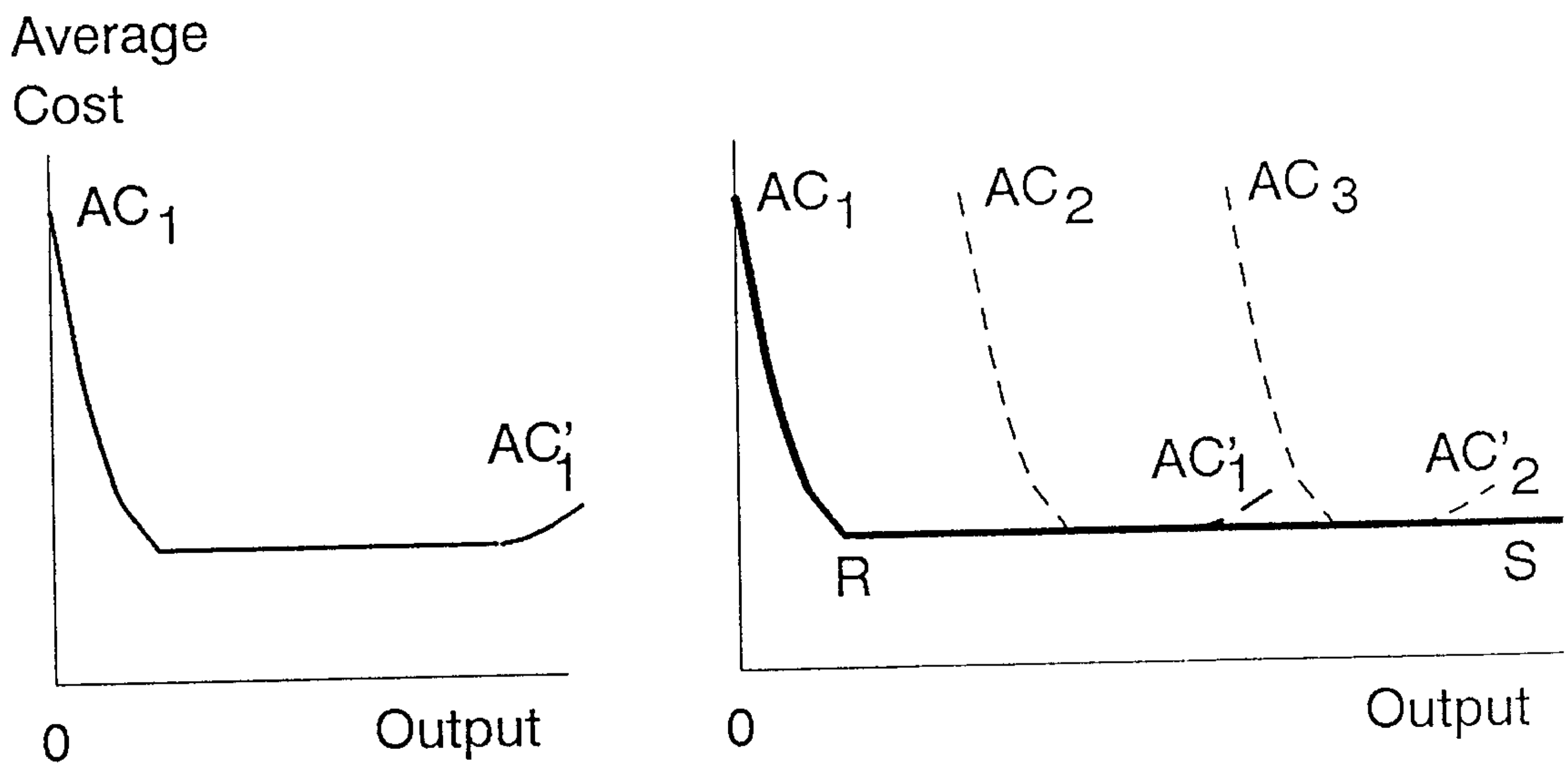
The first sum gives us an average α_1 and the second an average α_2. The assumption of the theorem is that $\alpha_{i,j} < 1 + 1/Z_C$, so this must hold for the averages as well. Also, for any economies of scale production function $\alpha_{i,j} \geq 1$. So we will have, *whenever* $q_{i,1} > q_{i,2}$

$$\frac{1}{Z_1}+(\alpha_2-\alpha_1)+\ln\frac{q_{i,1}}{q_{I,2}}\geq\frac{1}{Z_C}+(\alpha_2-\alpha_1)\geq\frac{1}{Z_C}+1-(1+\frac{1}{Z_C})\geq 0$$

i.e. a nonnegative derivative, which is the result required for the theorem. We next discuss the ratio $q_{i,1}/q_{i,2}$. Returning to the various equivalent definitions of the classical level, we know that for the upper boundary calculation the chosen i will have a positive objective function term to the left of the Classical Level, since there are enough of these terms to use up the labor supply, only these will be chosen by the maximization algorithm.(i.e. Country 1 will chose to add only industries in which is the cheaper producer).

For different reasons the same result holds for the minimization calculation that gives BL_1. In the minimization calculation the variables are chosen in reverse order of density, the most negative ones first. Since, to the left of the Classical Level, there are not enough of these to use up the labor supply, in solving the knapsack problem the last variable chosen, which is the non-integer one, will have to have positive density (i.e. to minimize utility Country 1 will attempt to choose industries where it is the poorer producer, but there are not enough of these to go around). This establishes the theorem.

We note finally that this result applies to points where there is no change of basis only. This is enough for the proof of the theorem as it excludes only a finite number of points. At these other points we may well have right and left derivatives that are different and therefore a discontinuity in the derivative.



1a. the Firm

1b. the Industry

Figure 1.

File costdf2a.drw 4 May '92

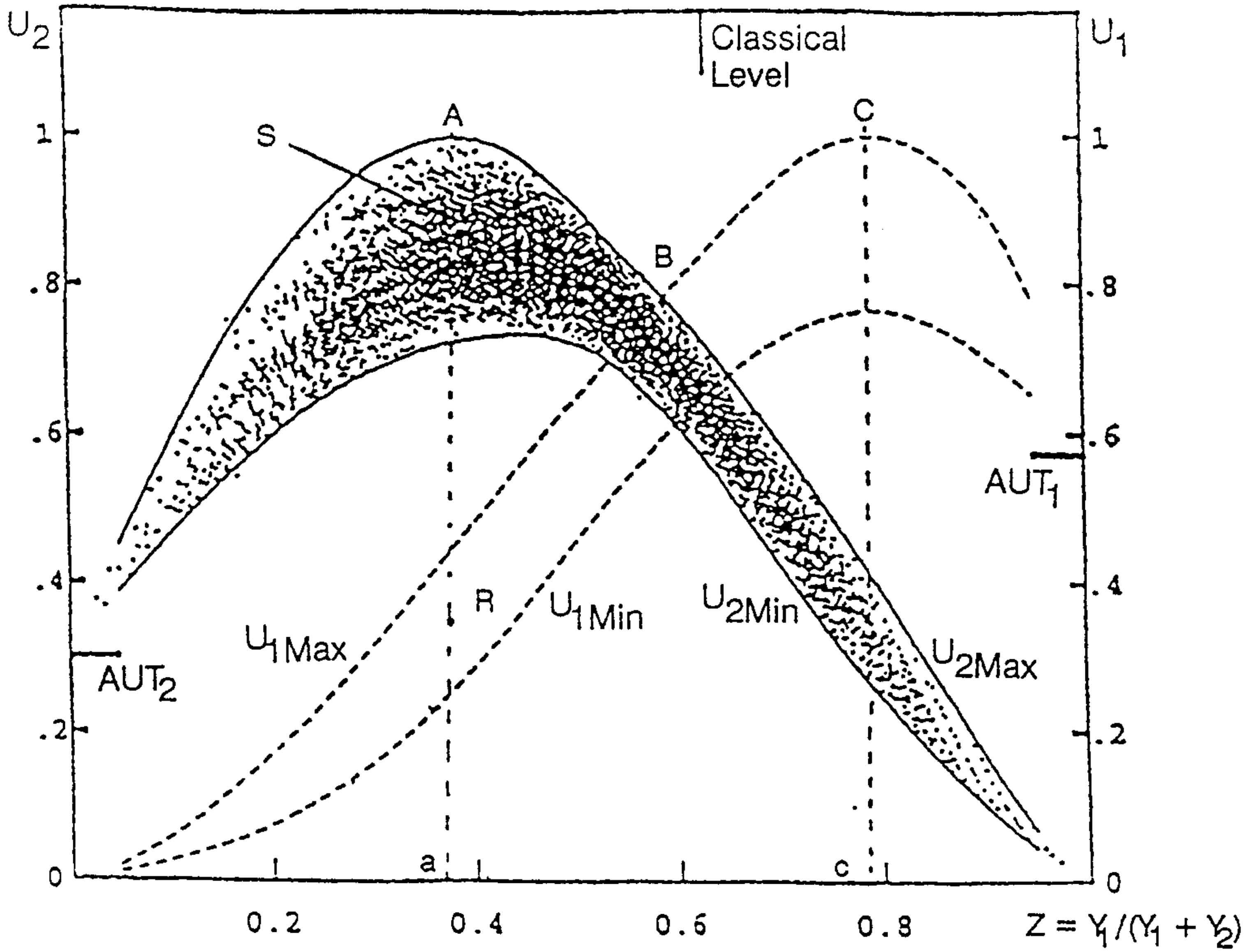
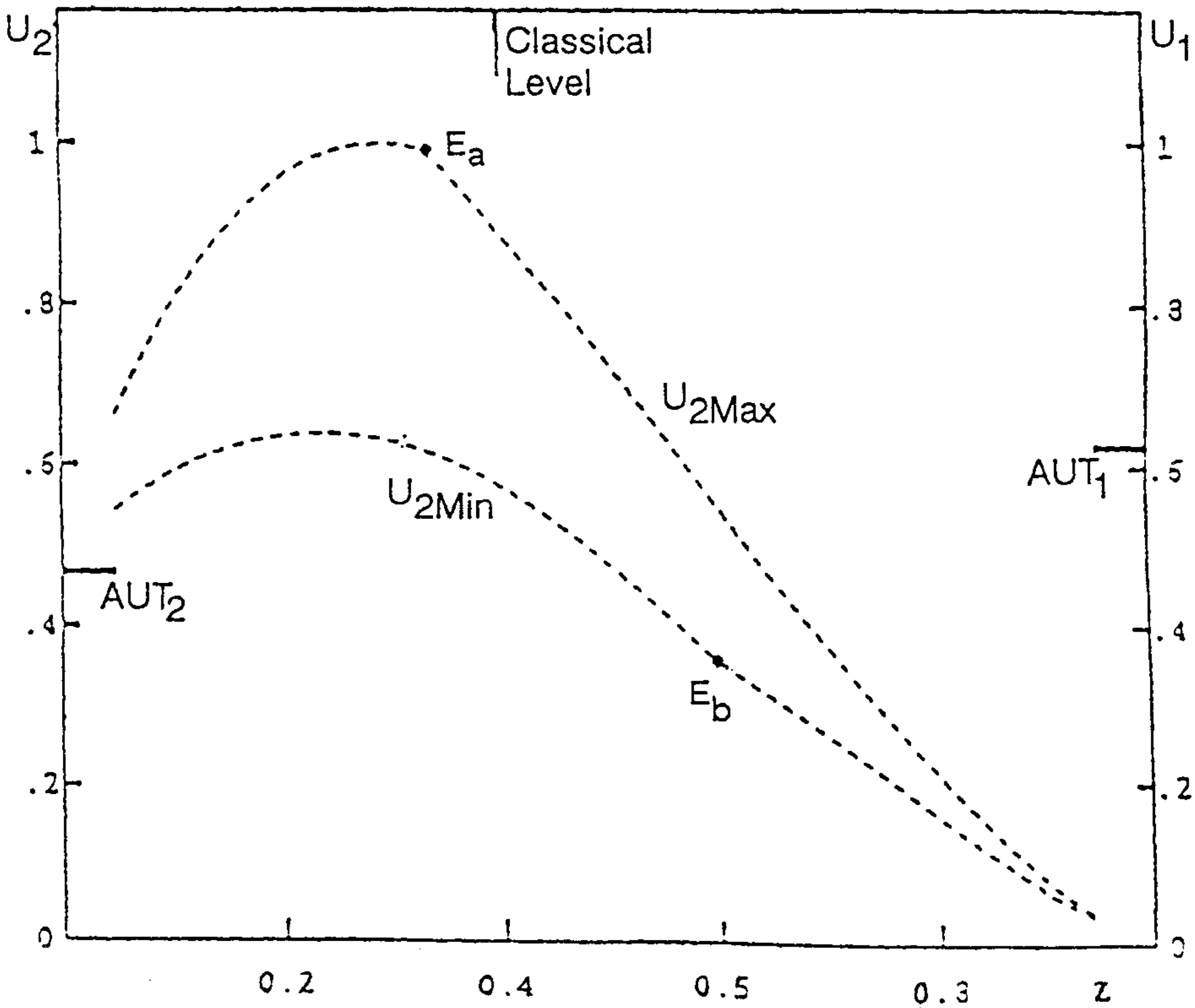

U2
U1
Classical
Level
A
C
S
B
1
.8
.6
.4
.2
0
AUT1
AUT2
R
U1Min
U2Min
U1Max
U2Max
a
c
0.2
0.4
0.6
0.8
Z = Y1/(Y1 + Y2)

Figure 2A.



U2
U1
Classical
Level
Ea
U2Max
U2Min
AUT1
AUT2
Eb
1
.8
.6
.4
.2
0
1
.8
.5
.4
.2
0
0.2
0.4
0.5
0.3
z

Figure 2B.

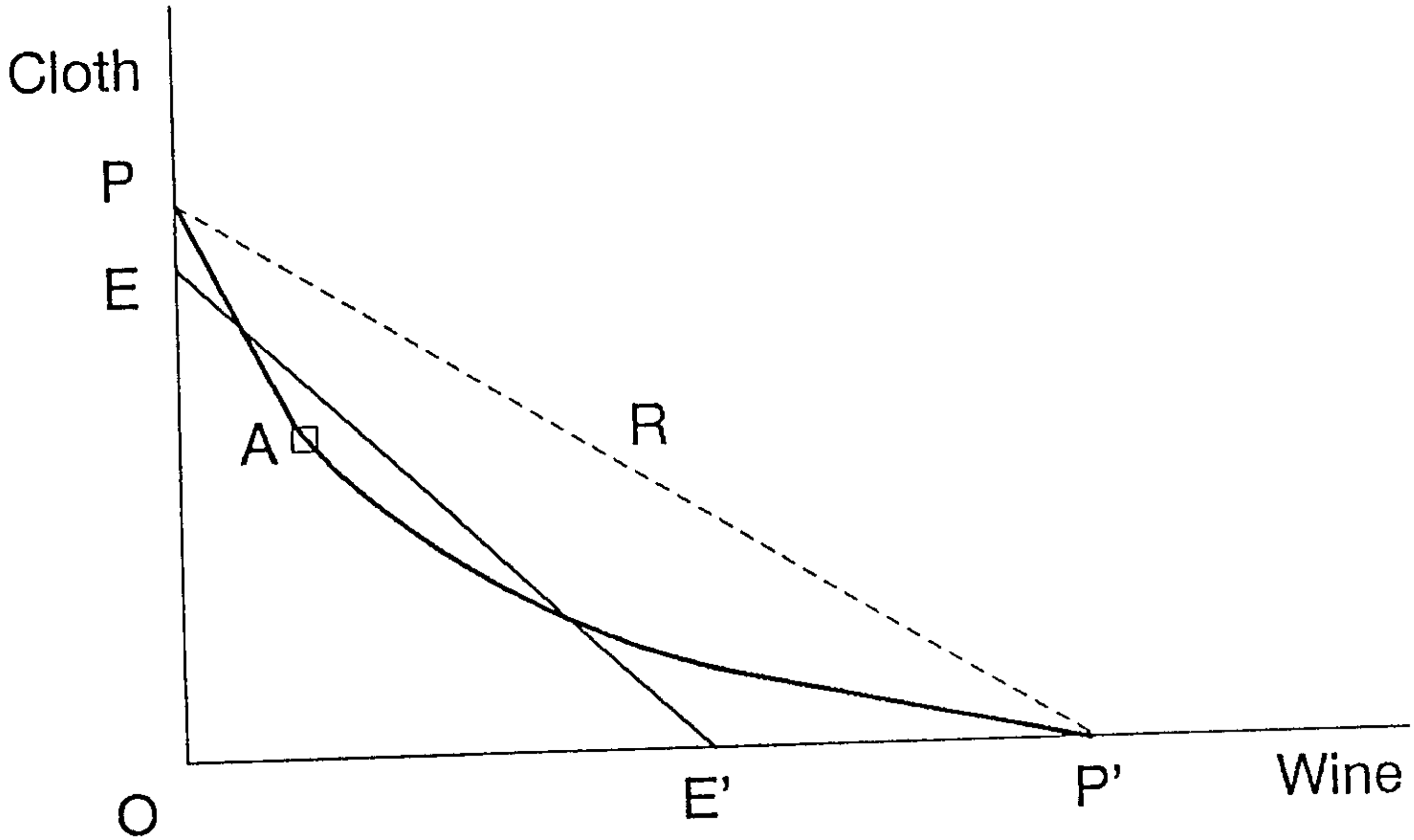


Figure 3.

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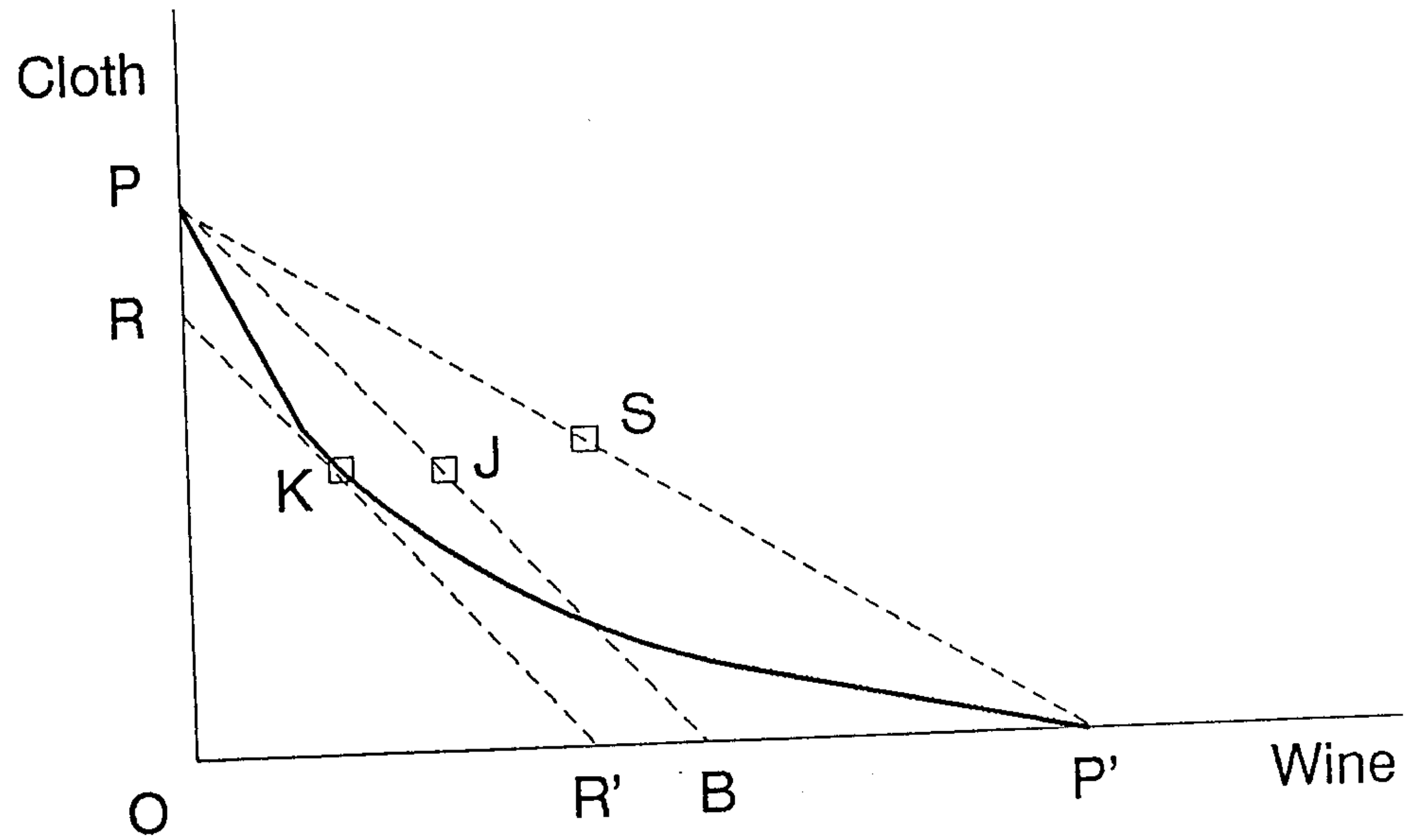
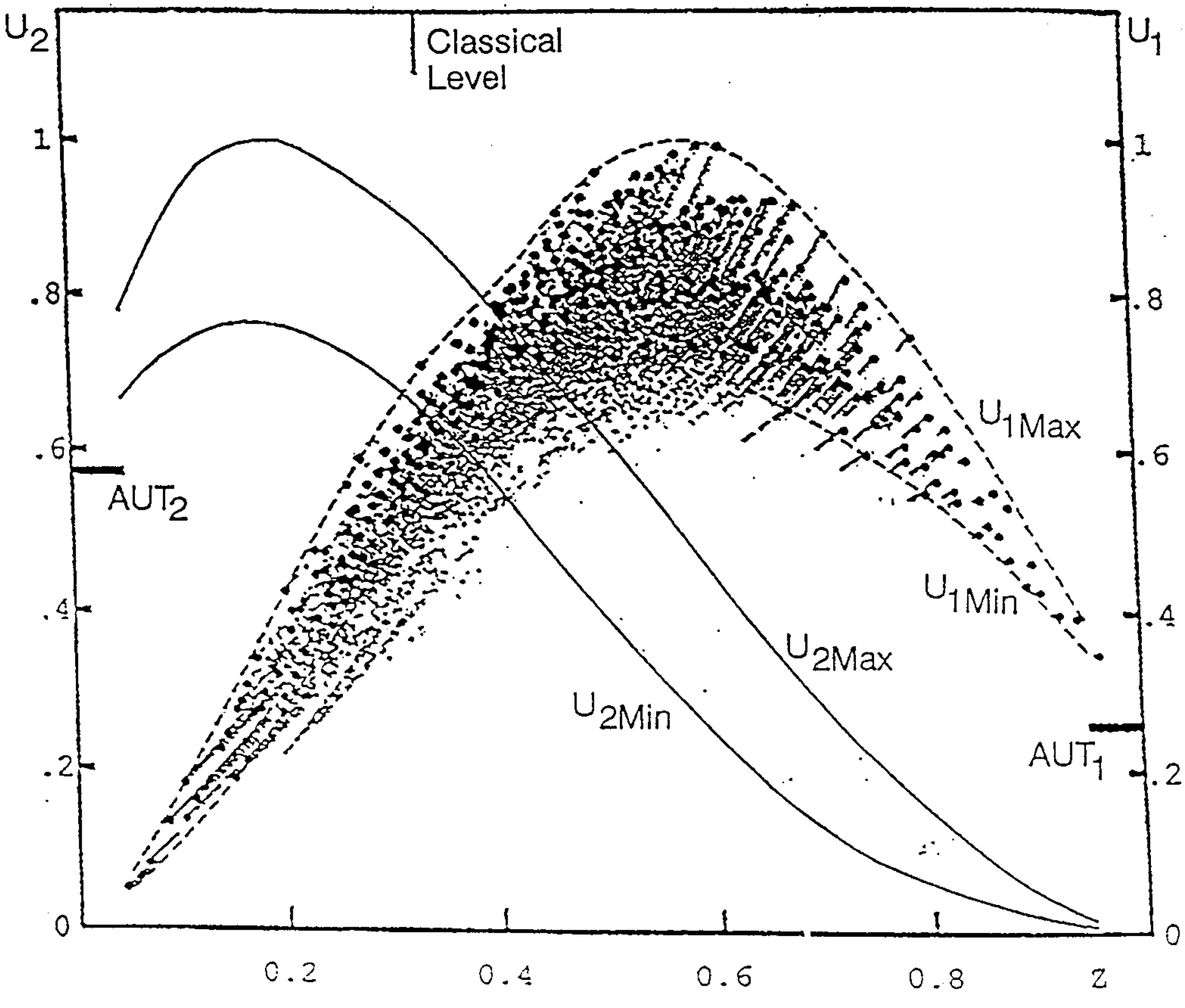


Figure 4.

File: comptrf3.drw 10 March '92



U2
Classical
Level
U1
1
.8
.6
.4
.2
0
AUT2
AUT1
U1Max
U1Min
U2Max
U2Min
0.2
0.4
0.6
0.8
Z

Figure 5

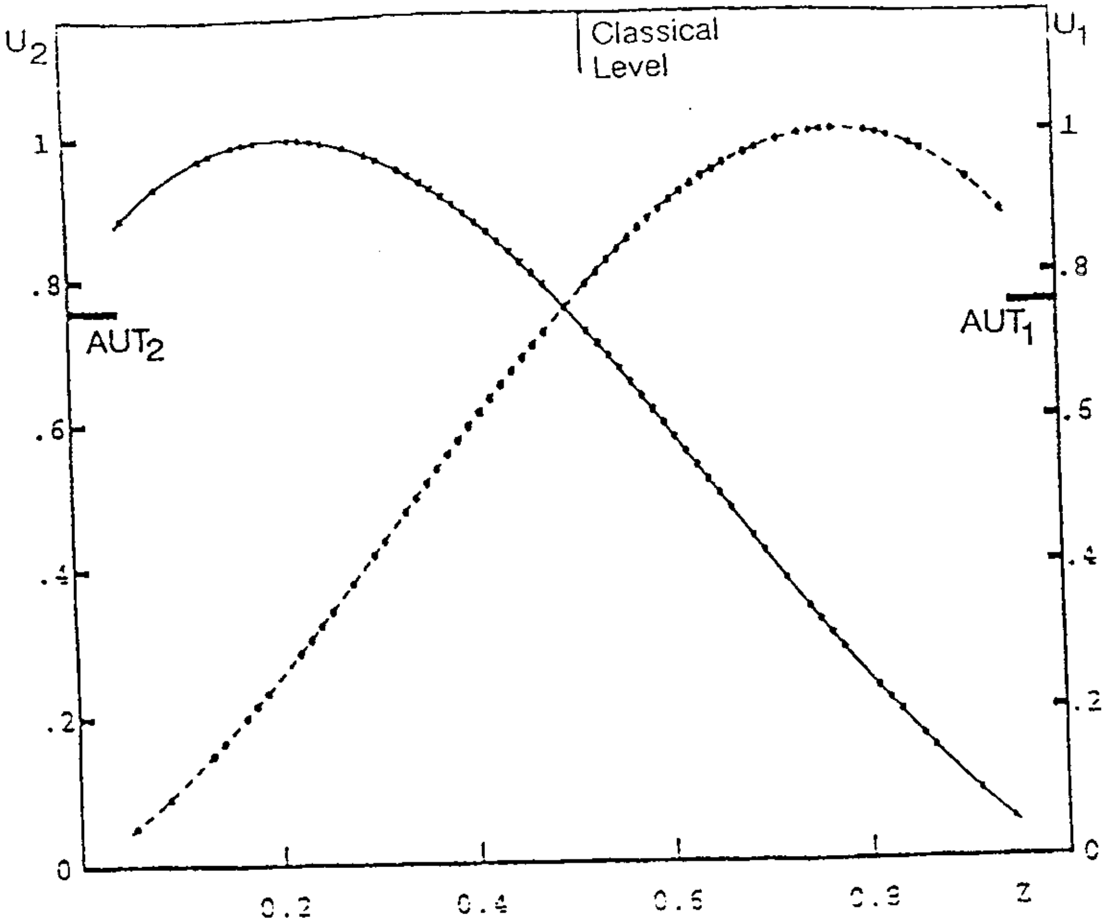


Figure 6A

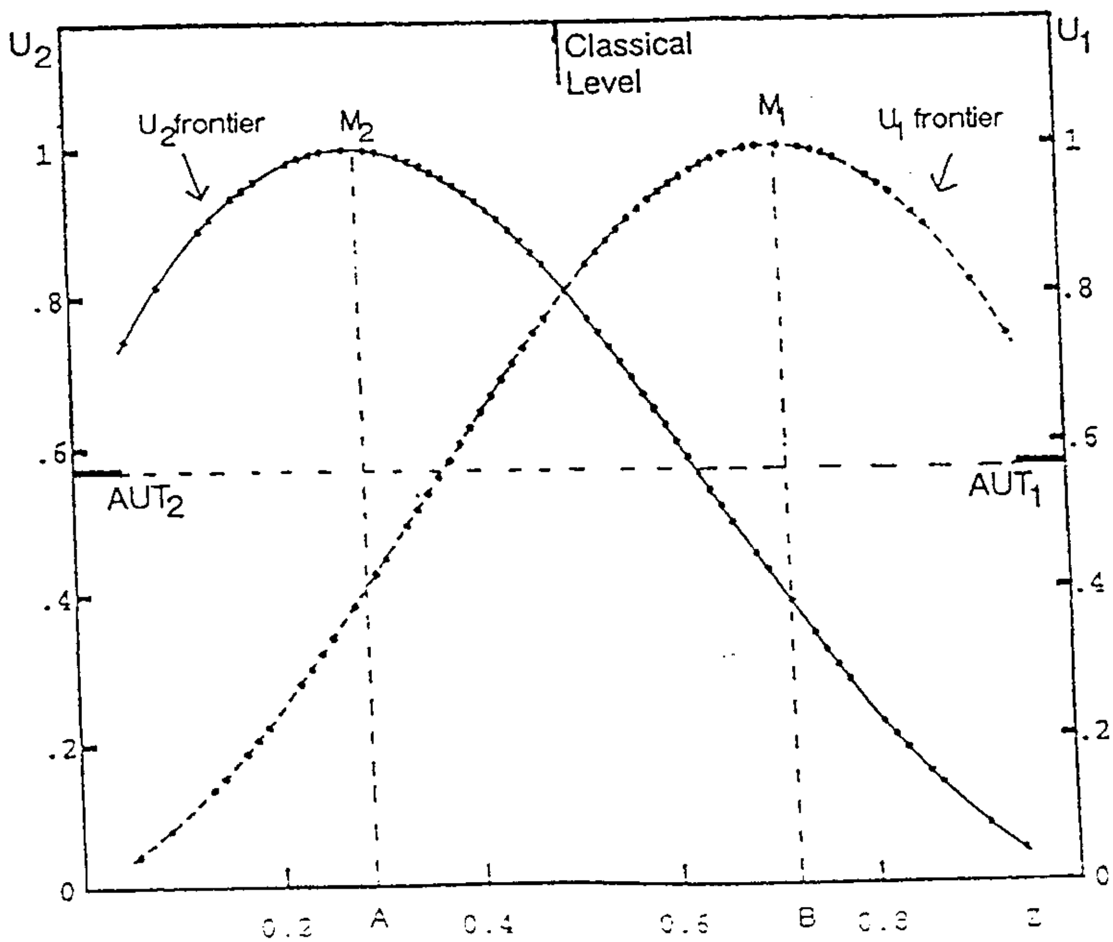
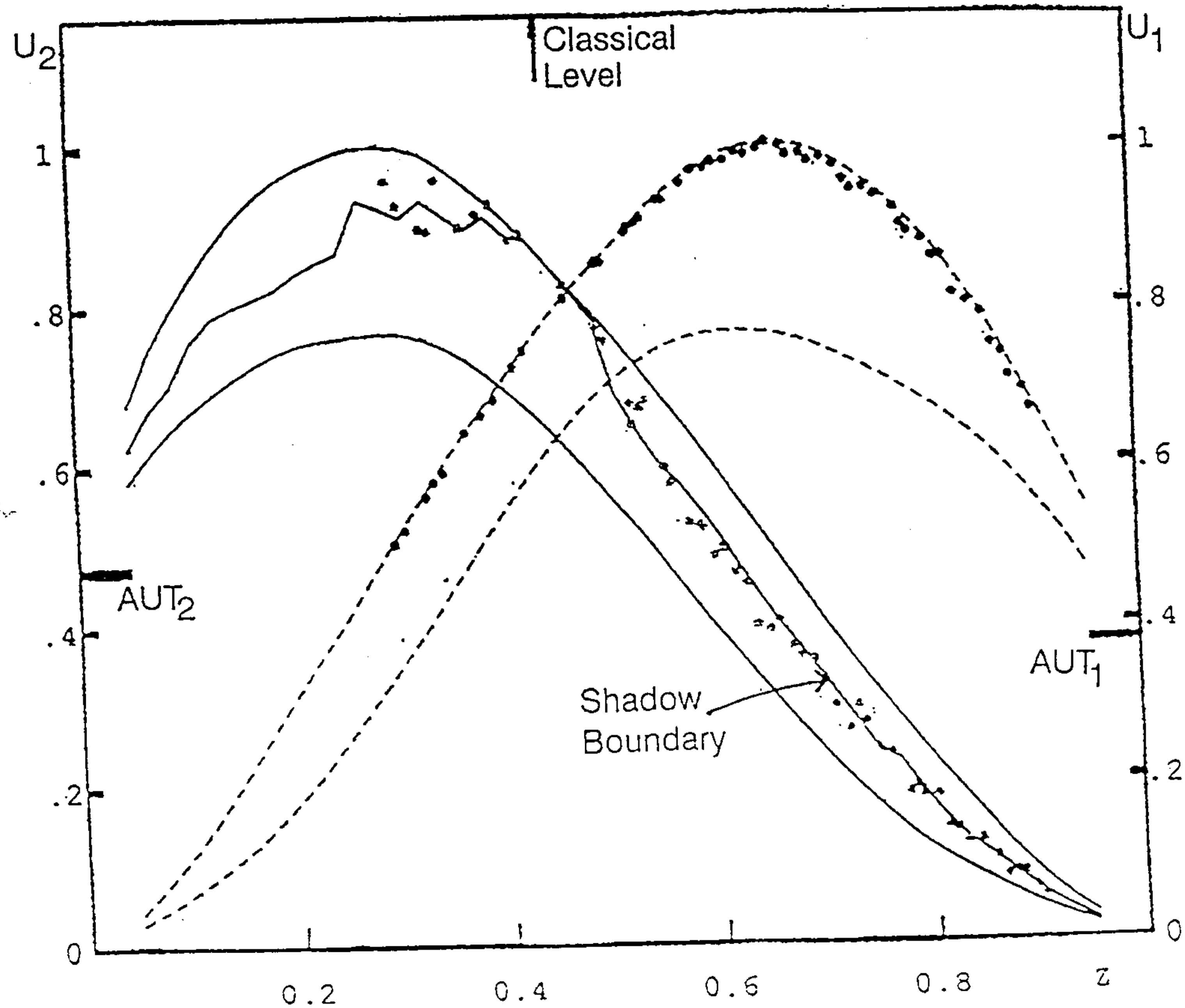


Figure 6B



U2
U1
Classical Level
1
.8
.6
.4
.2
0
AUT2
AUT1
Shadow Boundary
0.2
0.4
0.6
0.8
z

Figure 7

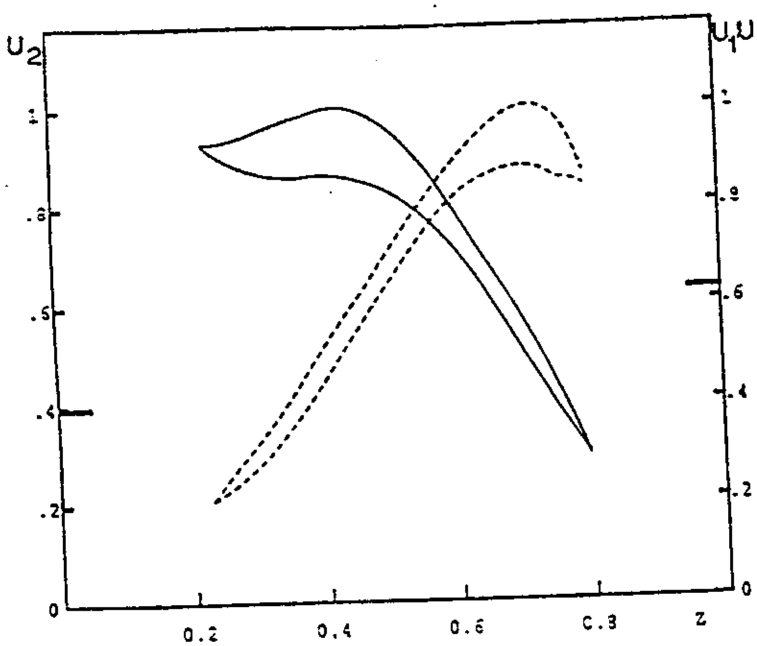


Figure 8A

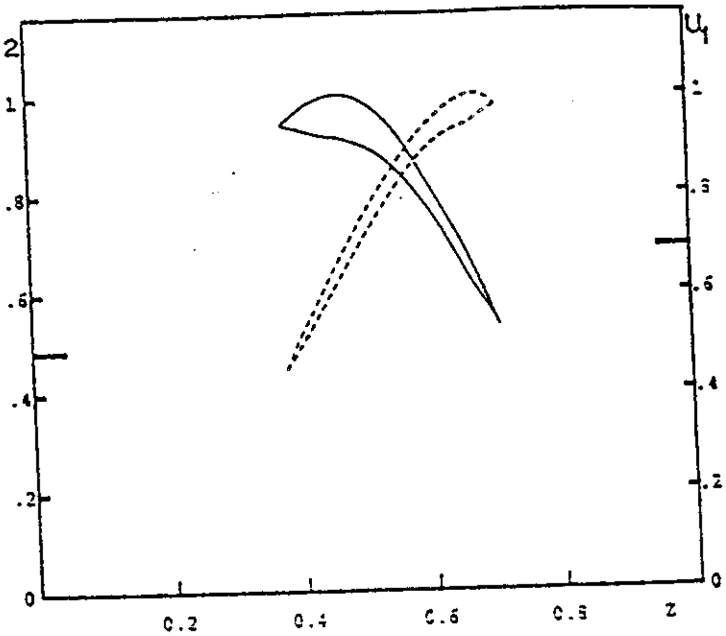


Figure 8B

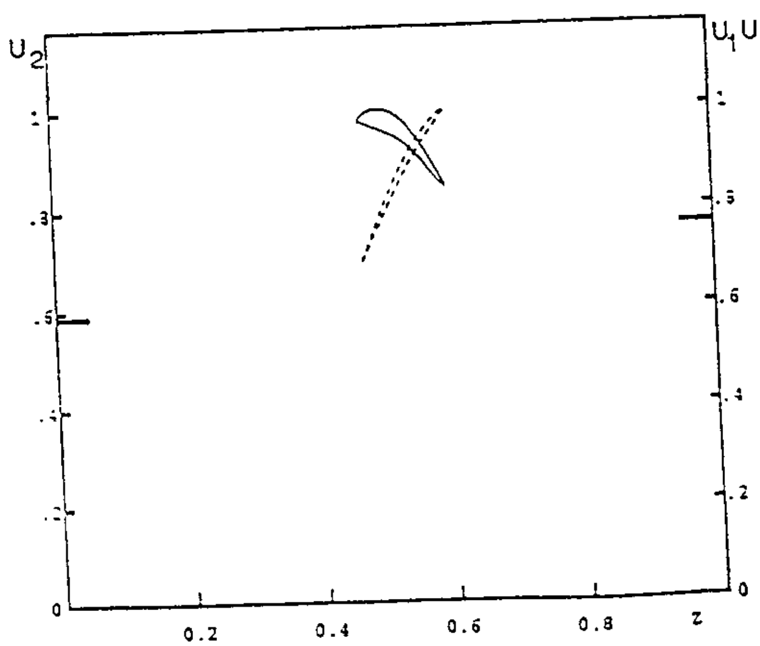


Figure 8C

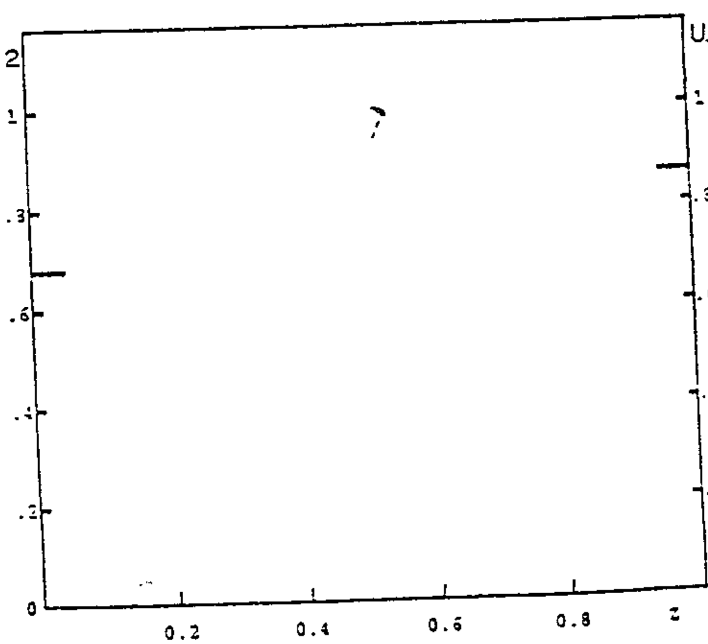


Figure 8D

TABLE 6.1

PRODUCTS	1	2	3	4	5	6	7	8	9	10	11	12	13
C1 Dem. Coef.	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20	0.13	0.17	0.11	0.25
C2 Dem. Coef.	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07	0.12	0.05	0.09	0.17
Production Exponents	1.00	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61	1.40	1.30	1.20	2.01
C1 Prod. Coef.	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50	1.10	0.90	1.20	1.01
C2 Prod. Coef.	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30	1.20	0.70	0.82	1.11

C1 = Country 1 C2 = Country 2
Dem. Coef. (demand coefficient) = d_{ij} in demand function $d_{ij} Y_j$ for commodity i, country j
Prod. Coef. (production coefficient) = e_{ij} in output = $e_{ij}\, l_{ij}$
C1 - Labor Supply 9 C2 - Labor Supply 5 Production Function $e_{ij}\, l^{p_i}$