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ON EFFICIENCY AND COMPARATIVE ADVANTAGE IN TRADE EQUILIBRIA UNDER SCALE ECONOMIES

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Abstract: On Efficiency and Comparative Advantage in Trade Equilibria Under Scale Economies

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This paper studies the efficiency properties of the many international trade equilibria that exist in a world of scale economies. We show that such equilibria can be efficient even if they violate comparative advantage. The paper derives sufficient conditions for efficiency and provides two new efficiency concepts that may be more illuminating that the classical measure in the scale economies case. One is a measure called local efficiency that applies to an equilibrium that is more efficient than any solution attainable by small changes in any of the variables. The other measure is a metric that quantifies the degree to which an equilibrium falls short of classic efficiency. The paper also shows how the efficient frontier can be constructed in the scale economies case and describes some of its properties.

On Efficiency and Comparative Advantage in Trade Equilibria Under Scale Economies

William J. Baumol* and Ralph E. Gomory**

1. Introduction

This paper studies the efficiency properties of the many international trade equilibria that exist in a world of scale economies. We will find anomalies such as equilibria that are efficient even though they violate the rules of comparative advantage. We also will introduce new concepts such as a measure of degree of efficiency (or inefficiency) of equilibria that are not efficient in the usual sense.

The recent burst of writings on the role of scale economies in trade theory, entailing work by Ethier [1979, 1982], Helpman and Krugman [1985], Grossman and Helpman [1991] and others, has added considerably to the insights provided by earlier writers such as Matthews [1949-50], Meade [1952], Chipman [1965] and Kemp [1969]. They have led us to see that the nature of the equilibria predicted by a model incorporating constant or decreasing returns is very different from that in a world of scale economies. The recent work of Gomory [1991, 1992] and Gomory and Baumol [1992] serves to differentiate those cases even further. It is now well established in the literature that:

1. While in the scale diseconomies case[1] there is generally a unique equilibrium that calls for a categorical assignment of industrial activities among producing countries, the scale economies paradigm invariably offers a multiplicity of equilibria. Indeed, even in a world of only two countries and n commodities, the number of the most pertinent class of equilibria will grow, roughly, as 2^n, i.e., as the number of country-commodity combinations that assign each product to a single country.

2. In the presence of pervasive scale economies there is reason to expect the equilibrium that emerges from market forces to have a tendency toward specialization, with no commodity produced simultaneously in two countries. Any such specialized assignment of products to countries will, in general, constitute an equilibrium configuration, and any such equilibrium will generally be stable locally, meaning that it is difficult to move from one equilibrium to another either through deliberate policy or automatic action of market forces. Starting from these propositions, it will be shown here that

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[1] For brevity we will henceforth take this term to include the case of constant returns unless otherwise noted.

1. Many of the locally-stable equilibria that arise under scale economies will *not* satisfy the requirements of economic efficiency, let alone those of Pareto optimality. However, there will be cases in which *all* of the many possible equilibria are efficient, and others in which, over substantial regions, no equilibrium may meet the requirements of economic efficiency.

2. The comparative advantage criterion does *not* always have to be satisfied for an assignment of products among countries to be efficient, and where it does hold it is the ratio of *average* rather than *marginal* productivities that governs.

3. As these results illustrate, in the scale economies case, economic efficiency, a global concept, may not be as illuminating or as well behaved as it is in other circumstances. Accordingly, two related concepts, *λ-efficiency*, a measure of nearness to efficiency, and *local efficiency*, will be formulated, and their implications explored.

4. All specialized equilibria in a world of scale economies will be shown to be locally efficient, that is, no *small* changes in the allocation of inputs can yield an increase in some output quantities without a reduction in the amount of some other output.

5. We will provide calculations and arguments indicating that, especially in models with a large number of goods, many of the great multiplicity of equilibria present will be nearly efficient, although they will not be efficient.

2. The Case of Scale Economies: Preliminary.

Though the correspondence is imperfect, it will be convenient to begin here by expressing the type of returns to scale in terms of the concavity-convexity of the producing country's production frontier -- its locus of efficient points for domestic production -- showing the maximum output of any one of its products that it can turn out, given the outputs of each of its other products and the quantity of each input that it has available. If the production frontier is concave (downward) we will say that it is subject to scale diseconomies, that is, increasing specialization entails decreasing output returns to further assignment of inputs to the product in which specialization occurs. Similarly, convexity of the frontier will be identified with scale economies.[2]

The most striking contrast of the scale economies case with the more commonly studied cases of constant returns or scale diseconomies is the predominating role of specialized equilibria (rather than multicountry production of every traded good, as under diminishing returns), the large number of these specialized equilibria, and their inherent stability. Though no formal

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[2] The connection is easy to see in the two-product case where, as is well-known, the absolute value of the slope of the frontier is the ratio of the marginal products of the bundle of inputs subject to transfer from one good to the other. If the inputs are homogeneous and the production processes of the two goods are independent of one another, then as we move from left to right, the output of good Z on the vertical axis will fall and the output of the other good, X, will rise. If this leads to a fall in the marginal product in Z and a rise in the marginal product in X, as we may interpret scale economies to imply, then the absolute slope of the production frontier must fall, implying that it will be convex. However, in an unpublished note Avinash Dixit has shown that concavity can arise even under economies of scale, and that scale economies are necessary but not sufficient for convexity.

proofs will be offered here, brief intuitive discussion of these results is in this case a reliable indicator of their logic.

For more than three decades it has been known that trade in goods produced under conditions of intra-national scale economies (that is, scale economies in the industry operating within the borders of each single country that produces the traded commodities) tends to result in specialized equilibria in which no good is produced in more than a single country. The reason is obvious: *ceteris paribus*, the country that happens to capture a large share of the market for the good will attain a scale economies cost advantage over its rivals in production of the items, and will be able to drive its competitors from the market. The more of the market it captures, the greater its cost advantage.

There are, of course, many such specialized assignments of goods among countries. For two countries and n commodities (excluding the two cases where one of the countries exports nothing) there will clearly be 2^n -2 such assignments. How many of them will be equilibria? The answer is that *all* of them will if a few straightforward assumptions are satisfied. Any that do not impose infeasible production requirements will be equilibria as the markets adjusts prices, exchange rates and relative outputs to clear the market, under the usual premises for the workings of the market mechanism.

Moreover, in a plausible dynamic model such as that used to investigate Marshallian or Walrasian stability it is easy to show that each such equilibrium is likely to be stable **locally**. This is so because any country that seeks to leave such an equilibrium must begin to supply some good it was not producing before. But if it enters the market on a small scale, it will find itself at a hopeless disadvantage relative to the incumbent in the field. Thus, incremental departure from such an equilibrium is very difficult, and success is apt to require the assemblage and risk of large amounts of capital.

Let us turn, now, to the subject of the current article, the economic efficiency of the many equilibria that scale economies generate.

3. Toward the Study of Efficiency in Trade Under Scale Economies

Despite the information that only an n-good model can provide, there is much to be learned from the two-good case, for some of its lessons do carry over to the n-commodity case. For brevity, the efficiency conditions for a pair of commodities in an n-commodity world will be referred to as the two-good efficiency conditions. By this we mean that there is no pair of goods i (produced by country A) and k (produced by country B) whose production, (using the same labor forces as before in each country), can be switched between the two countries, so that after the switch more of one good is being produced and no less of the other.

We define (global) *economic efficiency* in resource allocation in the usual way, as an allocation such that it is impossible to increase further the output of any one commodity without a simultaneous reduction in the outputs of some other commodities. Scale economies, however, introduce several complications into the analysis of efficiency. First, there is the role of specialized equilibria, which means that the shift from one such equilibrium to another can involve a discontinuous jump in the allocation of resources rather than a gradual movement in that allocation. Second, because the quantities of inputs available to smaller and larger economies may differ considerably, a shift in assignment of commodity i from a large country to a small may unavoidably entail a reduction in the quantity of i that it is feasible to produce, so that one cannot compare the efficiency of the two equilibria by measuring the change in output of the other good, k, holding the quantity of i constant. That is, the comparison cannot be carried out because the required constancy of i production is just not feasible.

We begin with a result that demonstrates the pertinence of the two-good efficiency conditions to an n-commodity world:

Theorem 3.1. In an n-good world, in the absence of externalities and intergood production complementarities (positive or negative), the two-good efficiency conditions for each and every pair of goods constitute *necessary* conditions for efficiency in the allocation of resources among the n goods.

The proof is trivial given the preceding definition. If the two-good efficiency requirements are violated by good i (produced by country A) and good k (produced by country B) then by transferring production of the goods from one country to the other and leaving the remainder of resource allocation totally unchanged one can increase the output of at least one of the goods without reducing the output of any other item.

In what follows we will consider nonspecialized resource allocations as well as those that are specialized, even though the former may not be equilibria, or if they are, they may be unstable. For ease in following, the discussion is framed in terms of Ricardo's two countries (England and Portugal) but substituting for his cloth and wine two more "high-tech" products, computers and Walkman radios, in whose production we may expect scale economies. In such a two-good, two-country model it must be remembered that there are always exactly two perfectly specialized assignments. Portugal can produce all the Walkmans and England all the computers, or the reverse can be true.

Figure 1 provides the pertinent information. PP"P' and EE"E' are the production frontiers for Portugal and England. The two specialized solutions are (E',P) and (P',E). In the first of these England produces all the world's Walkmans and Portugal produces all of the computers, while in the second of these the allocation is reversed. Figure 2 reproduces the two frontiers from the previous graph and shows the specialized solution points as $S = (w_e, c_p)$ and $S' = (w_p, c_e)$, where w_e is the number of walkmans produced in England, etc. In this case, where one of the frontiers, PP', lies entirely above the other, EE', neither equilibrium point dominates the other. However, in Figure 3, where the two frontiers intersect, the one specialized point $S = (w_p, c_e)$ clearly dominates the other, S', so that the latter must be inefficient. We can generalize this result:

Theorem 3.2. If two countries' production frontiers in two commodities are everywhere continuous and intersect one another at an odd number of points then one of the two specialized equilibria will dominate the other. However, if the frontiers do not intersect or have an even number of intersections neither specialized equilibrium will dominate the other.

Proof. If the number of intersections is odd, on the horizontal axis one of the curves will have a larger abscissa than the other and on the vertical axis the other curve will have a larger ordinate than the first. Thus, for example, $w_p > w_e$ but $c_e > c_p$, that is, Portugal can produce more Walkmans than England if one or the other specializes in Walkmans production, but England can produce more computers than Portugal if one or the other specializes in computers. Then, the equilibrium (w_p, c_e) in which each country specializes in its field of superiority will clearly dominate the other specialized equilibrium in which each country produces only the good at which it is the inferior producer.

If, however, the curves have 0,2,4,... or any other even number of intersections, then one of the two curves must have *both* the larger end-point abscissa and the larger end-point ordinate. Thus, for example, we can have $w_p > w_e$ and $c_p > c_e$. Thus, since any specialized solution corresponds to one end point of each of the two countries' frontiers, neither of the specialized solutions (w_p, c_e), (w_e, c_p) will dominate the other, since one element of each must be larger than the corresponding element of the other.

4. Increase in Number of Equilibria that Can be Efficient Because of Scale Economies

We turn to the theorem that describes the efficiency properties of the specialized equilibria, and begin to explore the relationship to the comparative advantage criterion for efficiency in the two-good case. For this purpose it is convenient to return to Figure 1. There, for the two production frontiers we have drawn in the linear chords PLP' and ELE' connecting the end points of the frontiers. As is well-known, the slope of the frontier at any point is the ratio of the marginal products, for the two goods, Walkmans (w) and computers (c), of the inputs that are subject to transfer from one of those goods to the other. That is, using I to represent quantity of input, we have at any point along one of the frontiers, say, PP"P', $- dc/dw = (dw/dI)/(dc/dI)$. Similarly, we can measure the ratio of the average products *over the entire length of such a frontier*, by the slope of the chord, PLP', that is, $- \Delta c/\Delta w = (\Delta w/\Delta I)/(\Delta c/\Delta I) = AP_{wk}/AP_{ck}$, the ratio of the average productivity of country k in Walkmans production relative to its productivity in computers.

Theorem 4.1. Let the two products be c and w. Where there is a difference between the two countries in the ratios AP_{wk}/AP_{ck}, then one of the specialized equilibria will always be efficient. The other will generally not be efficient in the linear case, whether or not the one dominates the other. However, in the scale economies case they can both be efficient.

To prove Theorem 4.1 it is convenient to begin with two obvious lemmas. Let us call the *linearized world production frontier* the set of efficient points for the total outputs of the two countries that would apply if the true country frontiers were the chords PLP' and ELE'. In contrast, let us refer to the corresponding frontier derived from the true (convex) production frontiers, PP"P' and EE"E', for the two countries as the *actual world production frontier.* Then we have,

Lemma 4.1. The intersection of any given vertical line and the actual world production frontier either lies closer to the origin than the intersection of that same line with the linearized production frontier, or the two intersections must coincide.

Proof. By its convexity, the true production frontier for each country lies below the chord connecting the two endpoints of the frontier.

Hence, every output combination that is feasible with the true country frontiers located and shaped as they are would also be feasible if the two (linear) chords PLP' and ELE' were the true country frontiers.

Lemma 4.2. The two actual specialized equilibrium points in the world w,c space are identical with the points that would represent the two specialized equilibria if the two chords PLP' and ELE' were the true country frontiers.

Proof. The positions of the specialized equilibrium points depend exclusively on the positions of the endpoints of the two country frontiers. But the endpoints of PLP' and PP"P' coincide by construction, and the same is true of ELE' and EE"E'.

To prove Theorem 4.1 we observe

Lemma 4.3. Any specialized equilibrium point, S, that is efficient in the linearized case must also be efficient in the scale economies case.

Proof. If S is undominated by any feasible point in the linearized case, by Lemma 4.1 it cannot be dominated by any feasible point in the actual (scale economies) case. That is, if either specialized point lies on the linearized world production frontier it must also lie on the actual world production frontier by our two lemmas, and so that point must be efficient.

The proof that one of the equilibria must be efficient, as Theorem 4.1 asserts, follows for the linear case and, hence, by Lemma 4.3, for the scale economies case, from the observation that point S will be efficient for the linearized model if it is the optimal solution to the single constraint linear program

(4.1) $$\text{Maximize } c_e + c_p$$

subject to

(4.2) $$w_e + w_p \geq k$$

(4.3) $$c_e \geq 0, \; c_p \geq 0, \quad w_e \geq 0, \; w_p \geq 0$$

where $w_e = a_e - b_e c_e$ and $w_p = a_p - b_p c_p$, a_e, b_e , a_p, and b_p, all constant, are the equations of the linearized production frontiers of England and Portugal, respectively. The constraint inequality in (c_e, c_p) space is easily shown to be linear in the two variables, thus:

(4.4) $$c_e + c_p \, (b_p / b_e) \leq (a_e + a_p - k) / b_e.$$

Thus, the feasible region in (c_e, c_p) space is triangular with corners (0,0) $(c_e^*, 0)$ and $(0, c_p^*)$, with the values of the c* determined from (4.4). Clearly, at least one of the latter two basic solutions (both of them, obviously, specialized) must, then, be efficient, as was to be shown.

To complete the proof of Theorem 4.1 we must provide an example in which both specialized solutions are efficient. An extreme case of scale economies offers the simplest example. Suppose there are two products, M and N and that equipment to produce them comes only with large capacity so that it costs no more in total to produce, say, 100 units than to produce a single unit. Production of M interferes with production of N, so that only one of them will be offered by a supplier. Moreover, the two suppliers differ in their production capacities for N and M. An example is easy to imagine. Consider two airlines, one flying a 737 (capacity about 150 passengers) and one a 747 (about 350 capacity) along a given route. By law, a plane can carry only nonsmokers or only smokers, but not both. Moreover, suppose that if one carries smokers one needs aircleaning equipment that reduces capacity by 20 seats. Once a plane flies, the marginal cost of another passenger is zero, until capacity is reached. The possible equilibrium outputs are

Airplane	Nonsmokers	Smokers
737	150	130
747	350	330

Table I. Total passengers carrying capacities.

This yields two possible specialized equilibria in which transportation is offered to both types of passengers: one in which smokers (M) are carried on the 737 and nonsmokers (N) on the 747, and vice-versa. The output vectors for the two equilibria, S' and S are S' (350 N, 130 M) and S= (150 N, 330 M). Both of these specialized equilibria are, clearly, efficient because no increase in number of one of the two types of passengers carried is possible without either a rise in number of flights (input use) or a fall in number of the other type of passenger.

Figure 4 shows the two equilibria (S and S'), the 747 production frontier (FOF'), the 737 Frontier (TOT') and the feasible region for the combined outputs (assuming both types of passenger were offered service), region OF'S'VSF'. Points S and S' are patently both efficient, thus completing the proof of Theorem 4.1.

We will also see presently that one of the equilibria, S', violates comparative advantage.

5. Marginal vs Average Standards for Comparative Advantage

For the diminishing returns case standard analysis tells us that given a solution in which $(dc_e/dI)/(dw_e/dI) > (dc_p/dI)/(dw_p/dI)$, efficiency can be improved by increasing England's output of computers and Portugal's output of walkmans. Here, the criterion of movement toward efficiency is what we can describe as "*marginal* comparative advantage," i.e., comparative advantage expressed in terms of first derivatives of the production function. But in the linear case for any pair of goods the ratios of average and marginal product of any input are clearly equal to one another, and are equal to the slope of the production frontier. Thus, in the linear case comparative advantage can be expressed in terms of either the ratio of average products or that of marginal products. Moreover, this observation, together with the linearization of the scale economies graphs that has just been discussed indicates that the comparative advantage criterion pertinent for scale economies must be expressed in terms of *average* products such as $\Delta w_p/\Delta I$, where the increments are averaged over the entire length of the frontier. For the perfectly specialized points are the end points of the frontier, and the pertinent comparative advantage ratio is clearly given by the slope of the line segment connecting them. Indeed, Theorem 7.1 below will confirm that it is satisfaction of the average comparative advantage criterion that in the case of scale economies constitutes a sufficient condition for efficiency.

We now have immediately from Figure 4

Theorem 5.1. Under scale economies an efficient specialized equilibrium need not satisfy the comparative advantage rule. In other words, the comparative advantage criterion is no longer a necessary condition for efficiency.

The proof by counterexample is trivial. The chords, FF' and TT', respectively joining the endpoints of the 747 and 737 production frontiers differ from one another in their slopes, since 330/350 > 130/150. Then, clearly, only one of the two specialized equilibria, e.g., that in which the 747 carries only smokers and the 737 carries only nonsmokers, will satisfy the comparative advantage conditions. But both specialized equilibria are efficient in the case shown in Figure 4. (End of proof). We will presently provide a slightly broader type of counterexample that can also serve as proof of Theorem 5.1.

Substitution of average for marginal concepts as standards of optimality occurs elsewhere in the presence of scale economies. This is not the place to attempt to generalize the observation. However, we may note the reason why scale economies enhance the position of average criteria. The point is that increasing returns make for corner solutions-- for values of the variables that are either zero or as large as the constraints of the problem permit, with intermediate values tending to be ruled out. The search for solutions then becomes a matter of choice among the alternative extreme values of the variables, with discontinuous jumps from the one polar value to the other characterizing the analytic process. Clearly, in comparing such widely separated alternatives, the marginal yield of a small move away from one and toward another does not constitute the pertinent information. Rather, one needs a datum that characterizes the yield promised by the entire discontinuous jump from one candidate solution to another, a datum that can perhaps be compared with other available moves to determine which of them promises the larger payoff. Figures such as average product, that is, the product of an input bundle averaged over the entire pertinent range of the variable at issue, turn out, apparently often, to serve that role where economies of scale are prevalent.

6. The Remainder of the Efficient Frontier for the Two-Good Case

The proof of Theorem 4.1 leads us directly to the construction of the entire linearized world production possibility frontier. We first take up the case of the single dominant specialized equilibrium (Figure 3). The analysis can easily be extend to show for the linearized model that if Portugal has the comparative advantage in walkman production then in the constraint $w_p - w_e \geq k$ if $k < w_p^*$, where w^*_p is Portugal's largest feasible w output, efficiency requires $w_e = 0$. Hence, the segment of the linearized world frontier in the region $0 \leq w_p \leq w_p^*$ will simply be C*LS, which is Portugal's linear frontier, PLP', shifted directly upward, parallel to its original position, until its lowest point (its right-hand end) coincides with the efficient specialized equilibrium point, S. A completely symmetrical argument shows that for $w > w_p^*$ the frontier will be ELE', England's frontier, shifted rightward to SLW*, just far enough that its highest (and leftmost) point coincides with S. In the figure the *actual* world production frontier will be the scalloped curve C*CSDW* which is obtained by moving the two *convex* production frontiers for the individual countries parallel to their original positions until they reach point S.

Matters are a bit more complicated where, as in Figure 2, neither of the specialized equilibria dominates the other. From either specialized point one can move rightward or leftward (upward) by increasing the output of the good whose production is zero at that specialized point, and reducing the other output correspondingly. For example, at point S in Figure 5 w_p, Portugal's w output, is zero. Hence, one can move to the right from S along curve SW*,which is obtained by a rightward shift of Portugal's production frontier PP'. Similarly, from point S' where $w_e = 0$ one can move rightward along S'W*, which is obtained by moving England's frontier, EE', rightward until it meets S'. In this way we obtain two possible world production frontiers, C*SW* and C*S'W*. In the linear case it is not difficult to show that the pseudo world frontier through the inefficient specialized point must always lie inside the true frontier through the efficient specialized equilibrium point. However, where the production frontiers of the two countries are convex the two candidate frontiers can conceivably intersect (e.g., at points T and T' in Figure 5) and then there will be intervals in which one of the curves provides the corresponding portion of the efficiency frontier and other intervals in which the other curve does so.

We also see, as noted earlier, that in the case shown both specialized equilibria, S and S', are efficient, even though one of them does not satisfy the comparative advantage inequality. A bit of experimentation confirms that this case will arise when and only when two requirements are satisfied: (i) the production frontiers of the two countries must not intersect, or must have an even number of intersections, so that neither specialized point dominates the other (Theorem 3.2); and (ii) the production frontiers of the two countries must be sufficiently convex. The latter requirement is, of course, the geometric equivalent of strong economies of scale. This second requirement gives us an intuitive explanation of the possibility that a location equilibrium can be efficient even when it violates comparative advantage. The reason, simply, is that where scale economies are sufficiently powerful they can enable a specialized solution, S, that violates comparative advantage, nevertheless, to yield outputs larger than any other arrangement represented by a point in the neighborhood of that specialized solution point. If the second specialized solution point, S' -- the one that does satisfy the comparative advantage rule -- is sufficiently different from S to prevent S' from dominating S, then it should be clear why they can both be efficient. In short, the point is that scale economies can bring cost advantages to a specializing country even sufficient to overcome a considerable source of inefficiency, such as a location assignment that allocates industries to the countries that have a comparative *dis*advantage in their operation.

7. n-Commodity Efficiency: Sufficiency of Comparative Advantage

With this background on the two good case we are now prepared for an analysis of the general n-good model. In this section it will be shown that if an equilibrium satisfies the (average product) comparative advantage rule this is sufficient to guarantee the efficiency of that equilibrium. We will start by discussing discuss efficient ways to produce a given quantity Q_i of each of n goods, and prove that inefficiency of an output vector means that it can be produced using less that the available quantity of labor.

If we use an amount of labor $l_{i,j}$ to produce the ith good in Country j the resulting production pattern must satisfy the labor quantity inequalities[3]

(7.1) $$\sum_i l_{i,1} \leq L_1 \quad \text{and} \quad \sum_i l_{i,2} \leq L_2.$$

Here L_1 and L_2 are the labor supplies in the two countries.

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[3]It is these labor quantity restrictions that lay behind the equations such as $w_e = a_e - b_e c_e$ and $w_p = a_p - b_p c_p$, that appeared in our analysis of the two good model. In dealing with the n-good model we need to bring the labor restriction directly into the discussion.

If the Q_i are an *efficient* set of goods, then any production pattern that produces the Q_i must have equality in both the inequalities of (7.1). Otherwise the unused labor the inequalities represent could make a set of goods that would strictly dominate the Q_i.

For a given i, one way to produce Q_i is to have it made solely in Country j. Then $Q_i=f_{i,j}(l'_{i,j})$ where $f_{i,j}$ is the production function for good i in Country j, and $l'_{i,j}$ and is the amount of labor required in Country j to produce Q_i when Country j is the sole producer. Once the Q_i and the $l'_{i,j}$ have been chosen, we can introduce variables $y_{i,j}$ that generate a related family of production patterns. We will use an amount of labor $l_{i,j}= y_{i,j}l'_{i,j}$ in Country j to make good i. We restrict the $y_{i,j}$ by $0\leq y_{i,j}\leq 1$ and $y_{i,1}+y_{i,2}=1$. If all the $y_{i,j}$ are integer this production pattern will produce exactly the quantities Q_i. If some $y_{i,j}$ are not integer, i.e., not 0 or 1, the amounts of labor $y_{i,1}l'_{i,1}$ and $y_{i,2}l'_{i,2}$ at work in the two countries in those industries will generally *not* be sufficient to produce a total of Q_i because of economies of scale.

If we choose the $y_{i,j}$ arbitrarily, the resulting amounts of labor will not necessarily satisfy the labor constraints (7.1). So for feasibility of the production patterns we need the conditions:

(7.2) $$\sum_i y_{i,1}l'_{i,1}\leq L_1 \quad \text{and} \quad \sum_i y_{i,2}l'_{i,2}\leq L_2.$$

(7.1) and (7.2) are closely connected. From any production pattern $l_{i,j}$ we can produce an associated $y_{i,j}$ by defining $y_{i,j} = f_{i,j}(l_{i,j})/Q_i$. We will use this $y_{i,j}$ to produce the "associated" production pattern $y_{i,j}\, l'_{i,j}$. Clearly, since $Q_i \geq f_{i,j}\,(l_{i,j})$, $0 \leq y_{i,j} \leq 1$ and $y_{i,1}+y_{i,2}=1$. Also, since $Q_i=f_{ij}(l'_{ij})$, by definition of l'_{ij} $y_{i,j}l'_{i,j}=f_{i,j}(l_{i,j})l'_{i,j}/f_{i,j}(l'_{i,j}) \leq f_{i,j}(l_{i,j})l_{i,j}/f_{i,j}(l_{i,j})=l_{i,j}$, where the inequality is due to economies of scale. Since we have $y_{i,j}l'_{i,j}\leq l_{i,j}$ we see that: (1) if the production pattern $l_{i,j}$ satisfies (7.1) then $y_{i,j}l'_{i,j}$ satisfies (7.2); (2) if $l_{i,j}$ underuses the labor of one of the countries, so does the associated production pattern $y_{i,j}l'_{i,j}$; (3) if $l_{i,j}$ is a perfectly specialized production pattern, then the original production pattern and the associated production pattern are the same, i.e., $l_{i,j}=y_{i,j}l'_{i,j}$, where $y_{i,j}=1$ for the producing country and $y_{i,j}=0$ for the nonproducing country.

If Q is *not* efficient, then there is some set of $l_{i,j}$ that make the quantities Q_i exactly while underusing the labor of at least one of the countries. As we mentioned under (2) above this is true of the corresponding y as well. If y underuses the labor of Country 2 we can always change y to produce underutilization in Country 1. A positive $y_{i,1}$ can be slightly decreased and the corresponding $y_{i,2}$ slightly increased to maintain $y_{i,1}+y_{i,2}=1$ without violating the labor constraint on Country 2. The decrease in $y_{i,1}$ results in underutilizarion of labor in Country 1. Consequently we have:

Lemma 7.1

If Q is *not* efficient there is a y and a production pattern $y_{i,j}l'_{i,j}$ that underutilize the labor of Country 1.

One way to look for such a y is to consider the minimization problem:

(7.3) $$L* = \text{Min}_y \sum_i y_{i,1}l'_{i,1} \quad \text{subject to}$$
$$\sum_i y_{i,2}l'_{i,2}\leq L_2.$$

If L^* turns out to be less than L_1 we will have found such a y.

We can rewrite (7.3) in terms of only the $y_{i,2}$ by using $y_{i,1}=1-y_{i,2}$. This gives us $L^*=\Sigma_i l'_{i,1} - Max_y \Sigma_i y_{i,2} l'_{i,1}$. Using L^{**} for $Max_y \Sigma_i y_{i,2} l'_{i,1}$ gives:

(7.4)
$$L^{**} = Max_y \sum_i y_{i,2} l'_{i,1}$$
$$\sum_i y_{i,2} l'_{i,2} \leq L_2$$

The condition $L^{**} = {}_i l'_{i,1} - L_1$ is equivalent to $L^*=L_1$. This maximization problem is equivalent to the minimization problem (7.3). Its intuitive meaning is that we start the calculation with everything being made in Country 1 (all $y_{i,2}=0$) and then shift the maximum possible amount into Country 2.

The maximization problem that appears in (7.4) is the familiar knapsack problem. Its solution is obtained by making the variables $y_{i,2}$ positive in the order of the corresponding ratios $l'_{i,1}/l'_{i,2}$, the largest ratio first, until the inequality in (7.4) is satisfied as an equality. Each variable is taken in turn and increased from 0 until it is 1, or until the inequality in (7.4) is satisfied as an equality. When the latter occurs the remaining $y_{i,2}$ are set to 0 and the algorithm ends. This will result in at most one non-integer $y_{i,2}$. If there are no non-integer $y_{i,2}$, then the $y_{i,j}$ provide a specialized production plan. This background leads us to the following Theorem:

Theorem 7.1. (Average Product) Comparative Advantage Sufficient for Efficiency.
If a specialized production plan $l_{i,j}$ satisfies the labor inequalities as equalities, and has ratios $l'_{i,1}/l'_{i,2}$ that satisfy $l'_{i,1}/l'_{i,2} \geq l'_{k,1}/l'_{k,2}$ whenever i is produced in Country 2 and k is produced in County 1, then that production plan is efficient.

Proof: The corresponding production plan consists of setting $y_{i,2}=1$ (and $y_{i,1}=0$) for the goods produced in Country 2, and $y_{i,2}=0$ (and $y_{i,1}=1$) for the goods produced in Country 1. Because of the condition on the ratios of the $l_{i,j}$ in Theorem 7.1 , this y is exactly the one that would be produced by solving the maximization problem in (7.4). This maximization problem is equivalent to the minimization problem in (7.3) so this $y_{i,j} l'_{i,j}$, which is in fact the same as the original production plan, must minimize the labor used in Country 1. One of the assumptions of the theorem is that this specialized production plan satisfies the labor inequality of Country 1 as an equality. Since it also minimizes the use of Country 1's labor there can be no $y_{i,j} l'_{i,j}$ that underutilizes the labor of Country 1, so Lemma 7.1 applies. This proves the theorem.

Since by definition $Q_i = f(l'_{i1}) = f(l'_{i2})$ then, clearly, $f(l'_{ij})/l'_{ij}$ is the average product of labor in production of good i in country j, AP_{ij}. So the inequality in Theorem 7.1 is just the standard comparative advantage criterion, $AP_{i1}/AP_{i2} \geq AP_{k1}/AP_{k2}$, and the theorem tells us simply that any equilibrium that satisfies these relations for every pair of outputs must be efficient.

Clearly there is a symmetric statement of Theorem 7.1 in which the roles of Country 1 and Country 2 are interchanged.

We can now use Theorem 7.1 to discuss some special cases to indicate how wide a range of possibilities arises under scale economies. These are the cases mentioned in the introduction that illustrate the variety of outcomes that can occur when the classical efficiency concept is used in the scale economies model.

Identical Production Functions: We assume $f_{i,1}(l)=f_{i,2}(l)$ for all i. Any specialized equilibrium point provides a set of Q_i, the quantities actually produced at that equilibrium. It also provides $l_{i,j}$ that satisfy the equations requiring at equilibrium that the total labor supply be used exactly. Finally the identical $f_{i,j}$ give us $l_{i,1}/l_{i,2}=1$ for all i, so that the conditions of Theorem 7.1 are satisfied. This shows that for countries with identical production functions, *all* specialized *equilibria are efficient.*

To explain the next two examples we must allude briefly to some of the results of Gomory [1991,1992] and Gomory and Baumol [1992]. In these papers we showed not only that the n-good model has 2^n-2 specialized equilibria, but also that if we calculate for each equilibrium its relative national income $Z_1=Y_1/(Y_1+Y_2)$ and its (Cobb-Douglas) utility U_1 for Country 1, and then plot the points (Z_1,U_1) in the Z_1-U_1 plane, the resulting 2^n-2 points lie in a well defined region of the plane. This region has a characteristic shape and well defined upper and lower boundaries that can be computed. We also showed that the region between the upper and lower boundaries tends to fill up solidly with these equilibria as n increases.

A similar statement can be made about a Z_2-U_2 plane, if we choose to plot the equilibria with $Z_2 = Y_2/(Y_1+Y_2)$ as the horizontal axis and U_2, the Cobb-Douglas utility for Country 2 as the vertical axis. In Figure 6, (and later in Figures 7,8, and 9), we combine the (Z_1,U_1) and (Z_2,U_2) plots. In Figure 6 the dark dots are the points (Z_1,U_1) and the lighter dots the points (Z_2,U_2). The horizontal axis is Z_1 if read from left to right, and $Z_2=1-Z_1$ if read from right to left. Figure 6 represents the equilibria for an 11 industry model.

With this background we can now discuss the next examples.

Production functions $e_{i,j}l_{i,j}^{\alpha}$: Note that in this case we are assuming that the exponent α is the same in all the production functions. At any equilibrium with its Q_i and $l_{i,j}$ the resulting $l'_{i,1}/l'_{i,2}$ ratios are $l'_{i,1}/l'_{i,2}=(e_{i,2}/e_{i,1})^{(1/\alpha)}$. If we arrange the ratios $l'_{i,1}/l'_{i,2}$ in descending order, that order is the same as the order of the ratios $e_{i,2}/e_{i,1}$. If in addition we now assume that the model has identical demand parameter values, $d_{i,1}=d_{i,2}$, this is the same order that the variables $y_{i,2}$ were chosen in solving the knapsack problem to get the upper boundary for these equilibria in [Gomory 1992] and Gomory and Baumol [1992] There the ordering was determined by the ratios $q_{i,2}/q_{i,1}= (e_{i,2}/(w_2)^{(1/\alpha)}/(e_{i,1}/w_1^{(1/\alpha)})=(e_{i,2}/e_{i,1})^{(1/\alpha)}(w_1/w_2)^{(1/\alpha)}$ with w_1 and w_2 fixed. This yields the identical ordering. It follows that for equilibria lying on that boundary the ratio condition of Theorem 7.1 holds, and they are efficient. By reversing the roles of Country 1 and Country 2 in the argument above we get a similar conclusion for Country 2's upper boundary. So we can state that under the assumptions *of the case under discussion all equilibria lying on the upper boundaries are efficient.*

Production Functions $e_{i,j}l_{i,j}^{\alpha i}$: Given any equilibrium point and the resulting Q_i, we can find, as above, that the $l_{i,1}/l_{i,2}$ ratios are given by the expression $(e_{i,2}/e_{i,1})^{(1/\alpha i)}$. However, these ratios are no longer in the order of the $e_{i,2}/e_{i,1}$ because the exponent α_i depends on i. However the ordering of the ratios, whatever that turns out to be, is independent of the actual Q_i values and therefore is the same for all the different equilibrium points. If we select from among the 2^n-2 equilibria those that have that order, i.e. choose $x_{i,2}=1$ for the first m in the ordering and $x_{i,2}=0$ for the rest, then the resulting equilibrium will satisfy the conditions of and be efficient. This gives

us n-1 efficient equilibria. These equilibria are shown in Figure 7 for a 27 good model.
Ratios that depend on the Q_i. One might think at this point that there is always some string of efficient equilibria to be obtained by finding the equilibria whose ordering corresponds to the ordering of the $l_{i,1}/l_{i,2}$. However, what has simplified our work to this point is that the $l_{i,1}/l_{i,2}$ ratios have been independent of Q_i, while in general the $l_{i,1}/l_{i,2}$ depend also on the size of the Q_i and therefore on which country is the producer at the equilibrium point. This dependence can in fact exclude the possibility of efficient equilibrium points over wide ranges of relative national incomes. In Appendix A we give an example in which there are no efficient specialized equilibria in the equilibrium region of the Z_1-U_1 plane for $Z_1 > .5$. While this example is somewhat artificial it does show the detail sensitive nature of the classical efficiency concept in this setting. It therefore seems worthwhile to consider other concepts that may generalize the classical concept to this new setting better than does direct adoption of the classical definition.

8. Local Efficiency: the Efficiency Costs of Nonspecialization

We will call the $l_{i,j}$ *locally efficient* if, roughly speaking, there is no nearby $l_{i,j}$ that provides more than the quantities Q_i. More precisely $l_{i,j}$, is locally efficient if there is some ε such that $|l_{i,j}-l_{i,j}| < \varepsilon$ implies that $f_{i,1}(l_{i,1})+f_{i,2}(l_{i,2}) \leq Q_i$ for all i.

Local efficiency still contains the idea that generates interest in the concept of efficiency, It still asks whether or not a better arrangement, one that generates larger quantities of goods than the Q_i, is possible. However, in a *local* efficiency test only nearby arrangements are considered, and this restriction may have a certain degree of common sense about it. Using local efficiency the behavior of our large numbers of equilibria becomes much more coherent.

To get that more coherent picture we need the concept of an *almost specialized production pattern* (aspp). This simply means that with the exception of at most one i, $l_{i,1} > 0$ implies $l_{i,2} = 0$ and $l_{i,2} > 0$ implies $l_{i,1} = 0$. We will also assume as part of the definition that for positive $l_{i,j}$, $f_{i,j}(l_{i,j}) > 0$, i.e., that we do not have positive labor input and zero output at that point, as this leads to automatic inefficiency. Note that a perfectly specialized production pattern is always an almost specialized production pattern. With this definition we can state the main theorem.

Theorem 8.1

A sufficient condition for $l_{i,j}$ to be locally efficient is that the $l_{i,j}$ be an almost perfectly specialized production pattern.

This has the important corollary:

Corollary: The production pattern of any perfectly specialized equilibrium point is locally efficient.

Proof of the theorem: The theorem is almost proved if we demonstrate the following lemma:

Lemma 8.1

If $l_{i,j}$ is an aspp, then for ε sufficiently small, any changes $\delta_{i,1}$ that strictly decrease the total labor quantity in Country 1, while maintaining the total output of both countries at Q_i, will result in a strict increase in the labor used in Country 2.

That the lemma is plausible can be seen in the following way: if we attempt to decrease the labor $l_{i,1}$ on the one non-specialized good, we must increase the amount of labor $l_{i,2}$ in Country 2 to maintain the output of the ith good. If we then try to avoid an increase in total labor use in Country 2 we must decrease the $l_{j,2}$ in some other industry. However in this industry Country 2 is the sole producer, so any decrease in $l_{j,2}$ will result in a *very large* increase in $l_{j,1}$ because this industry is starting from scratch in Country 1. Now we no longer have a decrease in Country 1's total labor use. This type of reasoning can be extended to every possible situation and that is done in Appendix B.

If we accept Lemma 8.1, then to prove the theorem we need only observe that if there is a $l'_{i,j}$ that makes at least Q_i, in every industry using the same labor supply, and makes strictly more of Q_j, then by contracting the production of the jth good until it is exactly Q_j we will underuse the labor of one of the two countries. If we suppose that it is Country 1, then $l_{i,j}-l'_{i,j}$ would give us a set of $\delta_{i,j}$ that underuse the labor of Country 1, and which make the Q_i while not increasing the use of labor in Country 2. This contradicts the lemma and proves the theorem.

That the aspp condition of Theorem 8.1 is not too arbitrary can be seen from the following theorem:

Theorem 8.2

If the production functions $f_{i,j}$ have increasing derivatives (rising marginal products of labor), then *aspp is a necessary and sufficient condition* for local efficiency.

Examples of production functions with increasing derivatives are all production functions of the form $e_{i,j}l^{\alpha_{i,j}}$, with $\alpha_{i,j} \geq 1$, or any production function of that form preceded by an interval of zero output. An example of a reasonable production function *not* meeting the criterion is a production function that starts out 0, then increases sharply, and then becomes linear with a positive slope that is less than the slope in the preceding steep portion. The proof of Theorem 8.2 is given in Appendix C.

Since efficiency implies local efficiency, Theorem 8.2 has as an immediate corollary a result about classical efficiency:

Corollary: If the production functions $f_{i,j}$ have increasing derivatives there are *no efficient equilibria with more than one shared industry*.

This result helps to explain our emphasis on specialized and near specialized (aspp) equilibria.[4] In the case of increasing derivatives they are the only equilibria with even a chance

[4]It is easy to show that in the simple linear Ricardian model with two goods and two countries almost all efficient equilibria will be aspp, but *not* perfectly specialized, as the standard textbook examples suggest. This is readily seen by inspection of the world linearized frontier in Figure

of being efficient. For under scale economies, failure to specialize when demand conditions permit entails a considerable sacrifice of efficiency.

Section 9: λ-efficiency.

Although all our specialized equilibria are locally efficient, we are still interested in their efficiency in a more global sense. Now, however, we will not ask whether the point is efficient or not efficient but rather ask how bad or how good it is from an efficiency standpoint. We would like a quantitative rather than a binary answer.

For this purpose we introduce λ-efficiency, a measure closely related to what has gone before. Consider any set of Q_i and its production pattern $l_{i,j}$. We will look for other production patterns $l^*_{i,j}$ that make the same set of Q_i, possibly using less of the total labor of the two countries, i.e. $\Sigma_{i,j} l^*_{i,j} < \Sigma_{i,j} l_{i,j}$. We define $\lambda = (\min \Sigma_{i,j} l^*_{i,j})/(\Sigma_{i,j} l_{i,j})$ where the minimization extends over all production patterns $l^*_{i,j}$ that make the Q_i and satisfy (7.1). We define λ to be the λ-efficiency of Q. Clearly $\lambda = 1$ coincides with the classical notion of efficiency, but now we can describe a point as having a λ-efficiency of .9 or .77 or whatever, thus measuring the percent of the total workforce required to make the original set of goods when they are made in the most efficient possible way. In other words, 1-λ is the proportion of the available labor wasted by the inefficiency of the equilibrium under study. We will next discuss ways of finding the λ associated with a production pattern $l_{i,j.}$ We will not be able to determine the magnitude of lambda precisely, but will be able to obtain workable upper and lower bounds.

We will exploit the calculation in (7.3) or the its equivalent form (7.4) keeping in mind that these calculations do not minimize over all production patterns but over a related problem (see the discussion about the relation of these problems on page 10). In (7.3) we minimized the total quantity of labor used by Country 1 while not exceeding the labor supply in Country 2. We will also need the completely symmetrical calculation in which we minimize the quantity of labor used in Country 2 while not exceeding the labor supply of Country 1.

We will change the objective function, no longer minimizing the quantity of labor used in Country 1, as in (7.3), but rather, we will choose the $y_{i,j}$ to minimize the total labor used in both countries to make the Q_i. We then obtain the following result:

Theorem 9.1

If we assign the production of the ith good to the country that produces it with the least labor, then: (1) if the labor supply in neither country is exceeded this assignment is the total labor-minimizing outcome, (2) if the labor supply of Country 2 is exceeded then the calculation in (7.3) gives the total labor-minimizing outcome, (3) if the labor supply of Country 1 is exceeded then the calculation that is symmetric to (7.3) gives the total labor-minimizing outcome.

The proof is given in Appendix D.

As we mentioned earlier, for a non integer $y_{i,j}$, the labor amounts $y_{i,j}\, l'_{i,1}$ and $y_{i,2}\, l'_{i,2}$ may together not produce the total quantity Q_i of the ith good. Hence the total labor-minimizing

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3.

outcome obtained from Theorem 9.1 uses no more labor than would an actual labor-minimizing pattern and it may use less. Consequently the outcome of this calculation gives us an *underestimate* λ_u of the λ-efficiency of the $l_{i,j}$ that produce the Q_i.

We would also like to have an overestimate. We will in fact have two. To explain the first let us assume that we are in case (2) of Theorem 9.1 . In (7.3) let us solve the minimization problem, restricting ourselves to *integer* $y_{i,j}$. If the resulting production pattern does not require more than the labor L_1 available in Country 1, then the resulting $l_{i,j}=y_{i,j}l'_{i,j}$ will actually produce the Q_i while satisfying the constraints (7.1). In this case the total labor used is an *overestimate* of the total labor required, since it is always possible that there is a better production pattern involving one non-integer $y_{i,j}$. This gives us an overestimate of λ. If the production pattern obtained requires more of Country 1's labor than L_1 we must take the overestimate of λ to be 1. There is a completely symmetric treatment if we are in case (3) of Theorem 9.1 . This first overestimate seems generally to work better on small problems, while the second (next paragraph) is simpler and better suited to large problems.

The second overestimate is based on the underestimate of Theorem 9.1 Let imagine that in calculating the underestimate in case (2) of Theorem 9.1 the ith industry has the non-integer variable $y_{i,j}$. Then simply increase the amount of labor Country 1 has in the ith industry until (1) the amount Q_i is produced between the two countries or (2) the total labor supply of Country 1 is exhausted. In case (1) we have a way of producing the Q_i and hence an overestimate of the total labor requirement and of λ, in the second case we take the overestimate to be 1. The over estimate produced in this fashion we will name λ_o. Clearly $\lambda_u \leq \lambda \leq \lambda_o$.

Evidently the extra labor added to Country 1 must be $\leq l'_{i,1}$ which is the amount that would enable Country 1 to make Q_i all by itself. This means that for an equilibrium point, where the labor of both countries is fully utilized, the gap between λ_u and λ_o is less than $l'_{i,1}/(L_1+L_2)$. It is intuitively plausible that for large problems with many industries adding up to large total labor forces any $l'_{i,j}$ will be small compared to L_1+L_2, and hence the gap between λ_u and λ_o will be very small. This suggests that for large problems both λ_u and λ_o will converge to λ. The only difficulty in proving this result is to find a proper definition of "large", and to take into account the fact that even in large problems there are equilibria in which one country, say Country 1 produces almost all goods while Country 2 produces very few. Such equilibria may still have in Country 2 ratios $l_{i,2}/(L_1+L_2)$ that are large even in a very large problem. A proper definition of large is given in the beginning of the proof in Appendix E, and these special equilibria are taken care of by a restriction on $Z_1=Y_1/(Y_1+Y_2)$. Assuming the restrictions of Appendix E, for any model with n industries we have:

Theorem 9.2

For any ε there is an n large enough that for all equilibria with relative national incomes $\varepsilon < Z_1 < 1-\varepsilon$, $\lambda_o-\lambda_u<\varepsilon$.

We can now use these estimates to look at the efficiencies of the various equilibria populating the regions of equilibrium . In Figure 8 we show the .98 equilibria in an 11 good model whose λ_u is $>.98$. In Figure 9 we show the 449 equilibria whose $\lambda_o>.98$. In these figures, and in other similar figures that we have examined, there is no particular tendency for the equilibria near the middle (or more precisely near the Classical Level) to be more efficient than the rest. There is, however, some tendency for the more efficient points to be near the upper boundary. This tendency is much more pronounced in the middle region than at either the right or left ends.[5]

However taking the region as a whole we may ask how efficient *on the average* are these equilibria. In Figure 10 (a cumulative distribution) we have considered all the equilibria of the 11 good model. The height y of the upper curve gives the percent of the 2048 equilibria whose λ_u is $\leq$ x. The lower curve is a similar plot for λ_o. A plot of actual efficiency, if we could obtain it, would lie in between. The two vertical bars are the average λ_u on the left and the average λ_o on the right. The average efficiency lies in between, i.e., between .91 and .93. Considering the range of production efficiencies in the model parameters[6] the high average efficiency is a little surprising.

In Figure 11 we have the same plot as that in Figure 10 but this time we have taken a random sample of 2000 points from a larger (27 good) model. The convergence of the two curves and the improved estimates of the average efficiency are exactly what we would expect from Theorem 9.2 .

Section 10. Conclusion

As promised in the introduction we have shown the wide range of states of efficiency, in the classical sense, that are possible for the many equilibria that exist under economies of scale. We have discussed examples in which all 2^n-2 specialized equilibria are efficient, and examples in which there are very few efficient equilibria. We have introduced two new efficiency measures, local efficiency and λ-efficiency that seem to function mo

re uniformly in the economies of scale setting while retaining many of the properties that give interest to the notion of efficiency. More generally, we have seen how much the roles of efficiency and comparative advantage differ between the worlds of scale economies and diseconomies.

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[5]At least a partial explanation of this tendency is the following: It is possible to show that for an equilibrium that is not "strong" in the sense of Gomory and Baumol [1992], the underestimate λ_u can never be 1, and can be very much less. Strong equilibria are therefore much more plausible candidates for high λ values, especially in large problems. It can also be shown, as it appears in some of the figures in Gomory and Baumol [1992], that the strong equilibria are near the upper boundary near the center but can be anywhere at the far right or left ends of the figures.

[6]In the 11 country model the production function for the ith industry in the jth country was $e_{i,j}l^{\alpha i}$. The α_i were between 1 and 2. The 11 values for the $e_{i,1}$ were (1.00, 1.02, 0.70, 0.94, 1.24, 0.60, 0.70, 0.77, 0.50, 1.10, 0.90) and the 11 values for the $e_{i,2}$ were (0.52, 0.71, 0.91, 0.92, 1.01, 1.23, 1.30, 1.02, 0.30, 1.20, 0.70). The 27 country model was similar.

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Appendix A: A Case With No Efficient Specialized Equilibria in a Substantial Region

Let us choose as production functions for the first good $f_{1,1}=l^{\alpha 1}$ and $f_{1,2}=l^{\alpha 2}$ with $\alpha_1 > \alpha_2$. For the remaining goods we can choose any production functions that are identical in the two countries.. We assume that $L_1=L_2$, and that we have symmetrical demand parameter values $d_{i,1}=d_{i,2}$ for all goods. We also specify that the demand parameter for the first good, $d_{1,1}$ is $.5/L_1$.

We will now consider the efficiency of specialized equilibrium points x with $Z_1>.5$. Case(1) *All equilibrium points with* $x_{1,1}=1$. The amount of labor used in Country 1 as sole producer of good 1 is, from Gomory [1992], or Gomory and Baumol [1992] ,$l'_{1,1}=d_{1,1}L_1/Z_1 =.5/Z_1<1$. This amount of labor produces an amount Q_1 of the first good and to produce that quantity in Country 2 would require an amount of labor $l'_{1,2}=(l'_{1,1})^{\alpha 1/\alpha 2}$. Since $\alpha_1 > \alpha_2$ and $l'_{1,1}<1$ we have $l'_{1,2} < l'_{1,1}$, so Country 2 could make that quantity with less labor. Case(2) *All equilibrium points with* $x_{1,2}=1$. The amount of labor used in Country 2 as sole producer is $l'_{1,2}=d_{1,2}L_2/Z_2 =.5/Z_2=.5/(1-Z_1)>1$. This amount of labor produces an amount Q_1 of the first good which would require an amount of labor $l'_{1,1}=(l'_{1,2})^{\alpha 2/\alpha 1}$ in Country 1. Since $\alpha_2<\alpha_1$ and $l'_{1,2}>1$ this shows that $l'_{1,2}> l'_{1,1}$, so Country 1 could make that quantity with less labor. In both cases, and therefore at all equilibria with $Z_1>.5$, the producing country is the less efficient producer of good 1.

Now at any of these equilibria let us take as the $y_{i,j}$ in (7.3) $y_{i,j}=1-x_{i,j}$. If Country 1 was the producer of good 1 at the equilibrium point, as well as some other set of goods, all these goods will be now be assigned to Country 2 in the efficiency calculation. There they will require less labor than they did in Country 1 because all goods but the first require the same labor and the amount required by the first has decreased. Since the two countries are the same size the labor of Country 2 is *underutilized*. Similarly, using the $y_{i,j}$ will exactly utilize the full Country 1 labor supply since *all* the goods require the same amount of labor in the efficiency calculation as in equilibrium,. It follows that the equilibrium point x was not efficient.

Clearly this reasoning can be repeated for the equilibria where Country 2 is the producer of good 1. It follows that in this model there are no specialized equilibria with $Z_1>.5$ that are efficient.

Appendix B. Proof of Lemma 8.1

Let us imagine that the non-specialized good is good 1. Then the hypothesis is

B(1.1)

$$\delta_{1,1}+\sum_{\delta_{i,1}>0}\delta_{i,1}+\sum_{\delta_{i,1}<0}\delta_{i,1}=\delta_1<0.$$

These changes induce changes in the $l_{i,2}$ that keep the total output of each good at Q_i. The change in labor quantity used in Country 2 is

B(1.2)

$$-\mu\delta-\sum_{\delta_{i,1}>0}\tau_i\delta_{i,1}-\sum_{\delta_{i,1}<0}\sigma_i\delta_{i,1}=\delta_2$$

where the μ,τ,σ are positive coefficients measuring the magnitude of the changes in Country 2 required exactly to maintain production of the Q_i. For ε sufficiently small μ will approach the ratio of the of the derivatives $f'_{1,1}(l_{1,1})$ and $f'_{1,2}(l_{i,2})$ which is some fixed non zero number. The τ_i will however all approach 0, since the only way to add labor to a specialized $l_{i,1}$ in Country

1 while maintaining the total Q_i unchanged is to add it to an $l_{i,1}$ that is 0. If it were added to a non-zero $l_{i,1}$ there would be no way to reduce the $l_{i,2}$ since it is already 0. However since the productivity of $f_{i,1}(\delta_{i,1})$ approaches 0, $\tau_i\delta_{i,1}$, the corresponding change in County 2 approaches 0, so all $\tau_i\to 0$. By similar reasoning we determine that all $\sigma_i\to\infty$.

If we define $\tau = \max_i \tau_i$ and $\sigma = \min_i \sigma_i$ we have

B(1.3)
$$\delta_2 \geq -\mu\delta_{1,1} - \sum_{\delta_{i,1}>0} \tau\delta_{i,1} - \sum_{\delta_{i,1}<0} \sigma\delta_{i,1}$$

then using B(1.1) to substitute for the first sum in B(1.3) gives

B(1.4)
$$\delta_2 \geq (-\mu+\tau)\delta_{1,1} + (\tau-\sigma)\sum_{\delta_{i,1}<0} \delta_{i,1} + \tau\delta_1$$

while using B(1.1) to substitute for the second sum in B(1.3) gives

B(1.5)
$$\delta_2 \geq (-\mu+\sigma)\delta_{1,1} + (\sigma-\tau)\sum_{\delta_{i,1}>0} \delta_{i,1} + \sigma\delta_1$$

If $\delta_{1,1} < 0$ then every term in B(1.4) is nonnegative for sufficiently small ε and the term involving δ_1 is strictly positive. If $\delta_{1,1} > 0$ then every term in B(1.5) is nonnegative and the term involving δ_1 is strictly positive. This establishes that δ_2 is positive and thus proves the lemma.

Appendix C. Proof of Theorem 8.2

Let us assume that there are two (or more) industries with production in both countries. Let us designate them as industries j and k. The labor employed in the two countries in the jth industry is $l_{j,1}$ and $l_{j,2}$, and in the kth $l_{k,1}$ and $l_{k,2}$. If in Country 1 we shift a very small amount of labor δ from industry j to industry k we must make a corresponding *increase* $(f'_{j,1}(l_{j,1})/\ f'_{j,2}(l_{j,2}))\delta$ in $l_{j,2}$ to maintain the production of the amount Q_j. Similarly, the increase in the labor in industry k in Country 1 results in a *decrease* $\delta(f'_{k,1}/\ f'_{k,2})$ in the labor required in Country 2 to make the amount Q_k. If $f'_{k,1}/\ f'_{k,2} \geq f'_{j,1}/f'_{j,2}$ we can continue shifting labor while decreasing (or at worst maintaining) the total labor used in Country 2. Under the assumptions of Theorem 8.2 , $f'_{k,1}$ / $f'_{k,2}$ will increase and $f'_{j,1}/f'_{j,2}$ will decrease so the shifting can continue while maintaining the production of both goods and strictly *decreasing* the amount of labor used in Country 2, until production levels reach 0 in one of the industries in one of the countries. This shows that the original production pattern was not efficient. Clearly, if the original labor shift resulted in an increase in the labor required in Country 2 the reverse shift should be employed and the argument will go forward as before. This proves that, with the assumptions on the derivatives of the production functions given in Theorem 8.2, more than one split product is inconsistent with efficiency. This establishes the theorem.

Appendix D. Proof of Theorem 9.1

Let us assume that the country whose labor is overutilized im the assignment is Country 2. Let us change the objective function in (7.3) to obtain a new problem in which we minimize $\Sigma_{i,j} y_{i,j} l'_{i,j}$ subject to the labor constraints in both countries. If we use the relation $y_{i,1}+y_{i,2}= 1$ to eliminate the $y_{i,1}$ in the objective function we obtain as the maximization problem equivalent to (7.4) :

$$\text{Max}\sum_i (l_{i,1}-l_{i,2})y_{i,2} \quad \text{subject to}$$

$$\sum_i l_{i,2}y_{i,2}\le L_2$$

$$\sum_i l_{i,1}y_{i,1}\le L_1.$$

Let us disregard the second inequality for the moment. Then this maximization problem is the same as (7.4) except for the objective function. The solution method is the same, fill up the inequality by increasing some of the $y_{i,2}$ from 0 to 1, choosing those with the best density. The density in (7.4) was $l'_{i,1}/l'_{i,2}$; here it is $1-(l'_{i,2}/l'_{i,1})$. Also here there will be negative terms in the objective function whenever $l'_{i,2} > l'_{i,1}$. These must not be used in the maximization as they will decrease the objective function. However, the fact that Country 2 is overutilized in the assignment of Theorem 9.1 means that there are enough $l'_{i,2}\le l'_{i,1}$ to fill up the inequality, so that we can go on increasing successive $y_{i,2}$ with positive coefficients in the objective function until the first inequality is satisfied as an equality. Although the ratios in (7.4) and in this problem are different they lead to exactly the same choice of successive $y_{i,2}$ because the largest $l'_{i,1}/l'_{i,2}$ also gives the largest $1-(l'_{i,2}/l'_{i,1})$. Consequently the $y_{i,2}$ values in the maximizing solutions are the same.

There remain one point to cover. We must be sure that the second inequality in B(1.6), which we have disregarded until now, is satisfied. Now the minimizing $y_{i,2}$ are known to minimize the labor in Country 1 subject to the first constraint in B(1.6), and that minimum must be $\le L_1$ since the original equilibrium point provides a solution to (7.4) with value L_1. It follows that the second inequality is satisfied, and this proves Theorem 9.1 .

The total labor minimizing value that is obtained is clearly L_2 plus the minimized labor of Country 1.

Appendix E. Proof of Theorem 9.2

The assumptions we will make are:

1) **The model and demands:** The n-industry model in autarky consists of n industries each having a labor supply $l^a_{i,j}$. The size of the labor force in the jth country is obtained by adding up the labor used in each industry so $L_j^n=\sum_{i=1,n}l^a_{i,j}$. Determining the industry size in autarky in fact determines the demand parameters $d_{i,j}^n$ since with Cobb-Douglas demand we have $d_{i,j}^nL_j^n=l^a_{i,j}$.
2) **Industry Size:** There is a limited size range for each industry in autarky, i.e., $l_{min}\le l^a_{i,j}\le l_{max}$. This implies that $nl_{min}\le L_j^n\le nl_{max}$.
3) **Relative Productivity**: If Country 1 uses an amount of labor $l_{1,k}$ to produce some quantity of the kth good then Country 2 will require some amount $l^*_{k,2}$ to produce the same quantity. We assume that if, for all i, all the $l_{i,1}$ are confined by an inequality $l_{i,1} \le L^1$ there is some quantity L^{*1} such that the corresponding $l^*_{i,2}$ satisfy $l^*_{i,2} \le L^{*1}$. The purpose of this restriction is to rule out unbounded productivity ratios between the two countries in industries of bounded size. We make the same assumption with the roles of the two countries interchanged, designating the two constants that appear by L^2 and L^{*2} .

From Theorem 9.1 we know that the minimization of total labor calculation is equivalent to minimizing the labor used in either Country 1 or Country 2. Let us assume that we are minimizing the labor used in Country 1, subject to the labor constraint of Country 2 . The underestimate of Country 1's labor obtained from the knapsack calculation can be written as

$$l_u = l_1 + y_{k,1} l'_{k,1}$$

where l_1 is the all the labor required in Country 1 except for that required for industry k, the industry divided between the two countries. Similarly, using the second method of calculation we get an overestimate of the labor required in Country 1 which is:

$$l_o = l_1 + l*_{k,1} \quad \text{where} \quad l*_{k,1} \quad \text{is determined by} \quad f_{k,1}(l*_{k,1}) + f_{k,2}(y_{k,2} l'_{k,2})) = Q_i.$$

Clearly $l*_{k,1} \leq l'_{k,1,}$ the labor required for Country 1 to make the quantity Q_k of the kth good all by itself, so $0 \leq l_o - l_u \leq l'_{k,1}$. If we can bound $l'_{k.1}$ with a fixed bound for all n we will have the convergence of λ_o and λ_u since their difference is $\leq l'_{k,1} / (L_1 + L_2)$ and $(L_1 + L_2)$ grows indefinitely with n.

Now in our equilibrium model, described in Gomory [1992] or Gomory and Baumol [1992] the amount of labor used by the sole *producing* country j in equilibrium is $l'_{k,j} = L_j(d_{k,1}Z_1 + d_{k,2}Z_2)/Z_1$. If we apply this to the n industry model and, as in Theorem 9.2 , confine ourselves to a range of Z_1, $\varepsilon \leq Z_1 \leq 1$, and use the assumption 1) above to bound the $d_{k,j}$ and assumption 2) above to bound L_1 and L_2, we find that in the nth model, the producing country, country j, uses an amount of labor $l'_{k,j} \leq nl_{max}(l_{max}/nl_{min} + l_{max}/nl_{min})(1/(1-\varepsilon)) = (l^2_{max}/l_{min})(1/(1-\varepsilon) = L(\varepsilon)$. This bound is independent of n, so it holds for any industry in any model in our sequence. Also, the bound does not refer to Country 1 or Country 2 so we would have obtained the same bound had we been minimizing the labor of Country 2. What we have shown is that if the range of Z_1 is bounded away from the endpoints, the amount of labor used by the producing country in any one industry in equilibrium is also bounded, while the labor forces in the two countries grow indefinitely with n. If we knew that Country 1 was the producing country for industry k in the equilibrium we are analyzing we would have the necessary bound on $l'_{k,1}$, but we must also take into account the possibility that Country 1 is the non-producing country. It is at this point that we use assumption 3).

If Country 2 is the producing country, let us take L^2 to be $L(\varepsilon)$. Then assumption 3) gives us a corresponding $L*^2$ which bounds $l'_{k,1}$. So in either case we have bounded $l'_{k,1}$, and since $L_1 + L_2$ grows indefinitely, we have $\lambda_o - \lambda_u -> 0$ as n increases. This proves Theorem 9.2 .

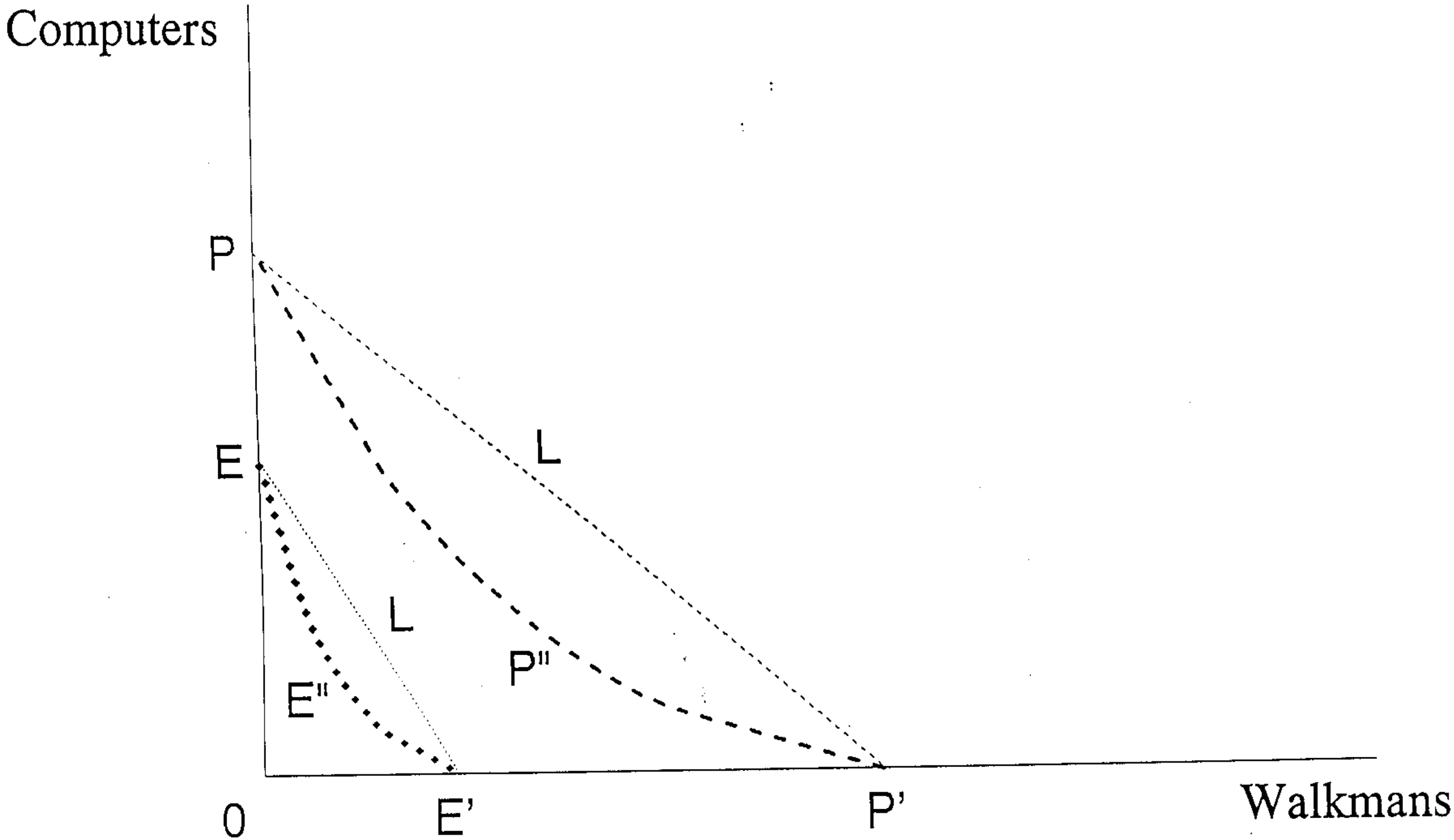

Figure 1. Convex Production Frontiers and Their Chords

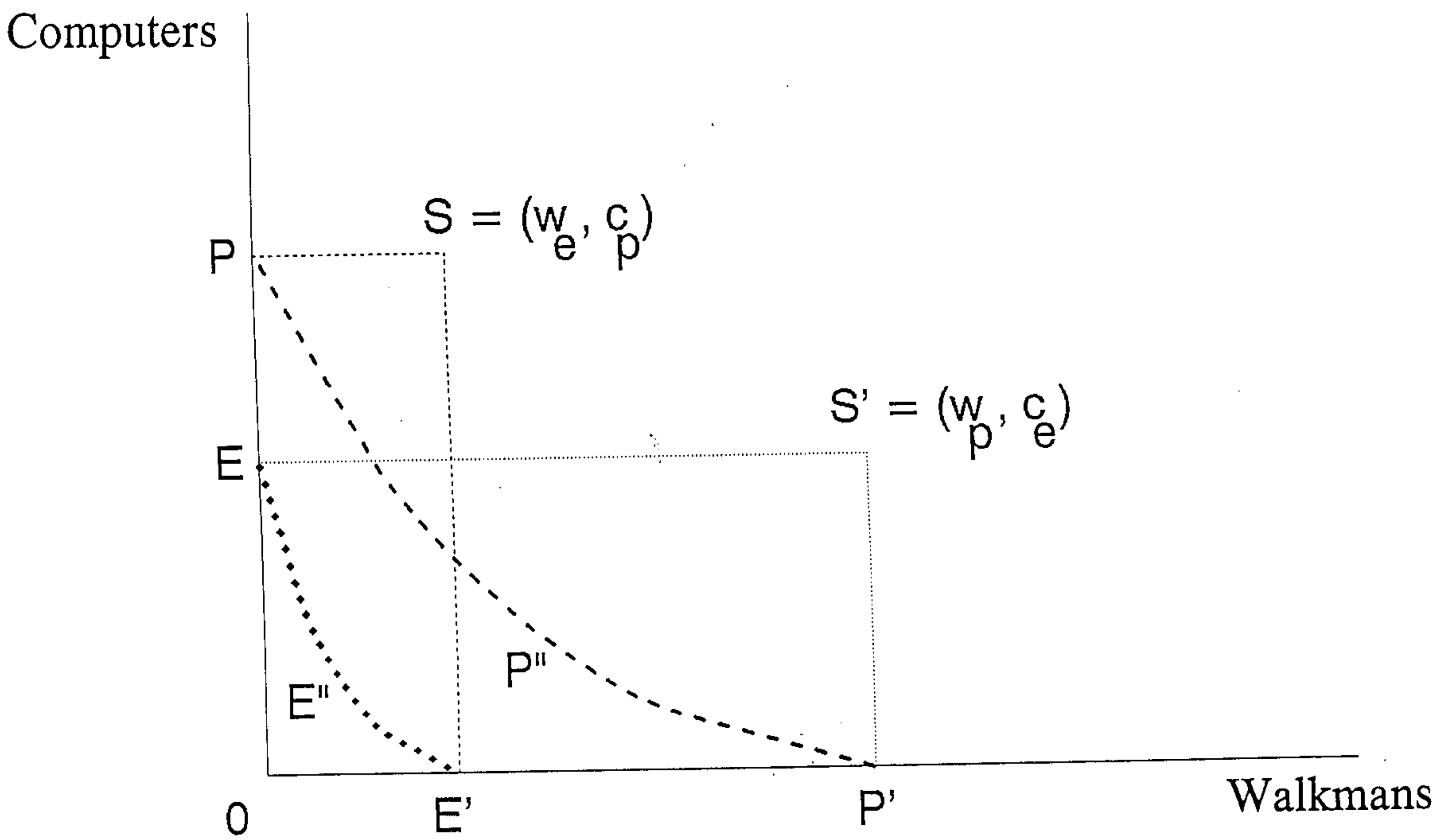

Figure 2. Neither Specialized Equilibrium Dominates the Other

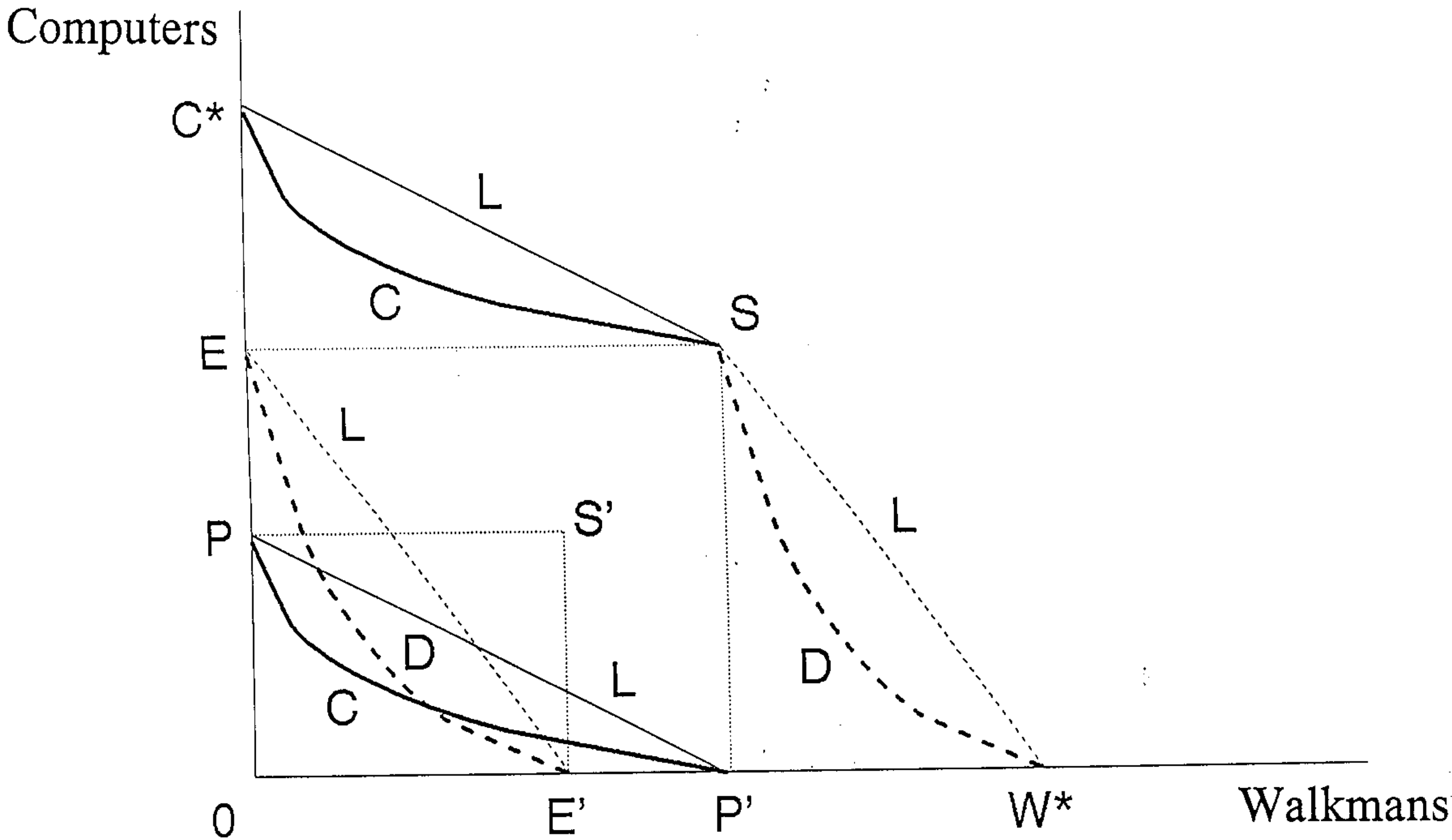

Figure 3. One Specialized Equilibrium Dominates the Other

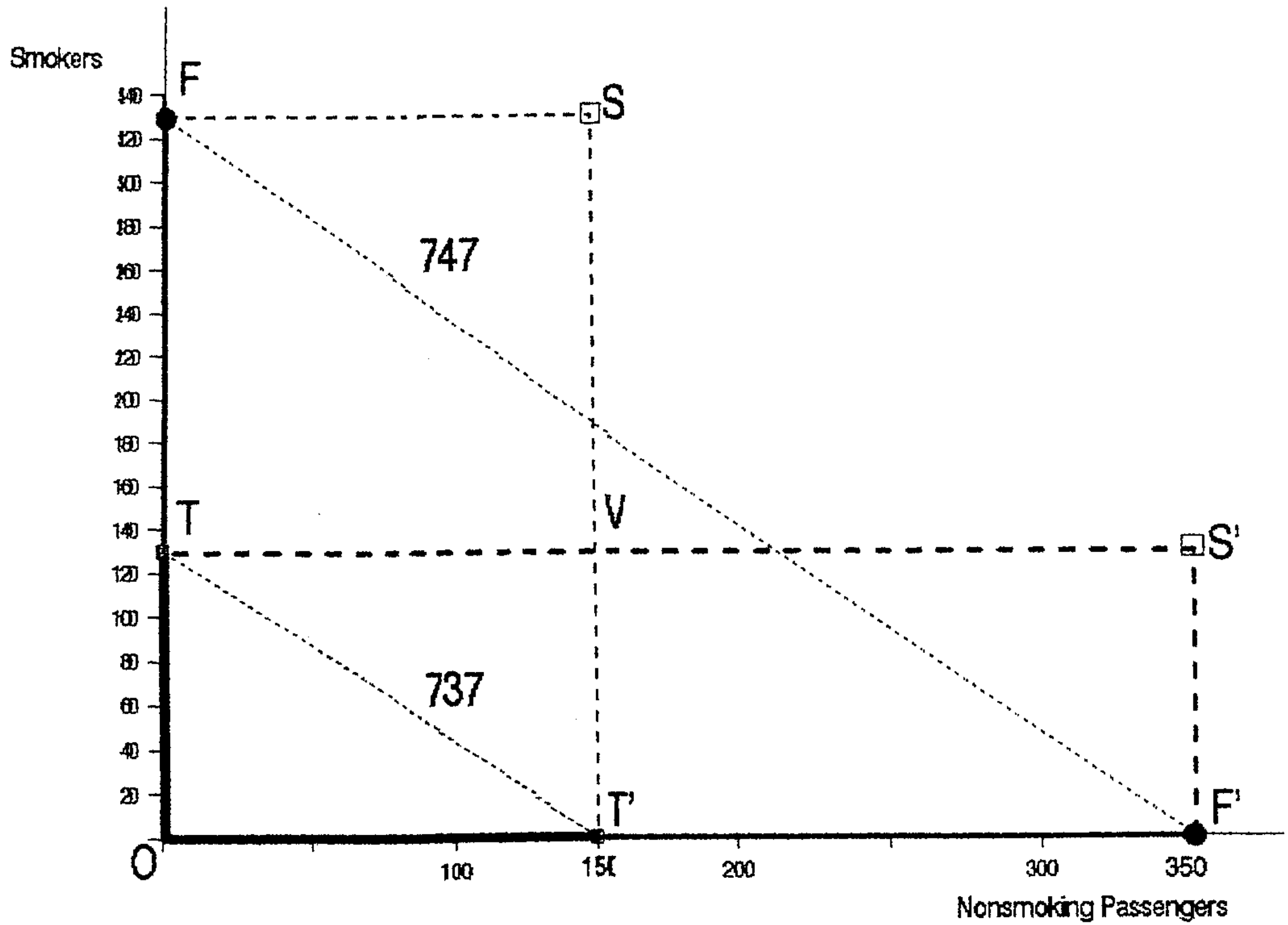

Figure 4. Two Efficient Equilibria With Extreme Scale Economies

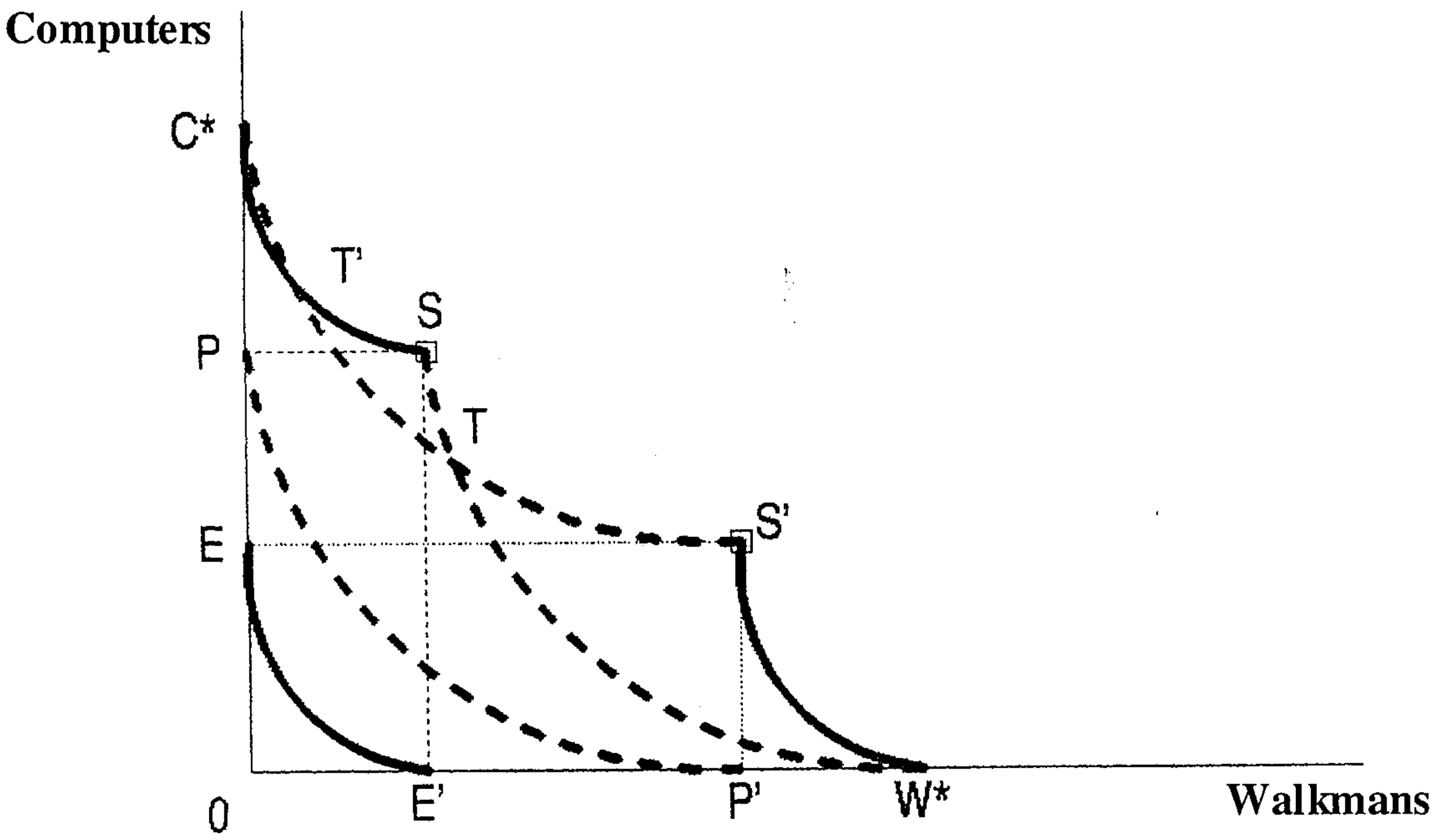

Figure 5. Two Efficient Specialized Equilibria

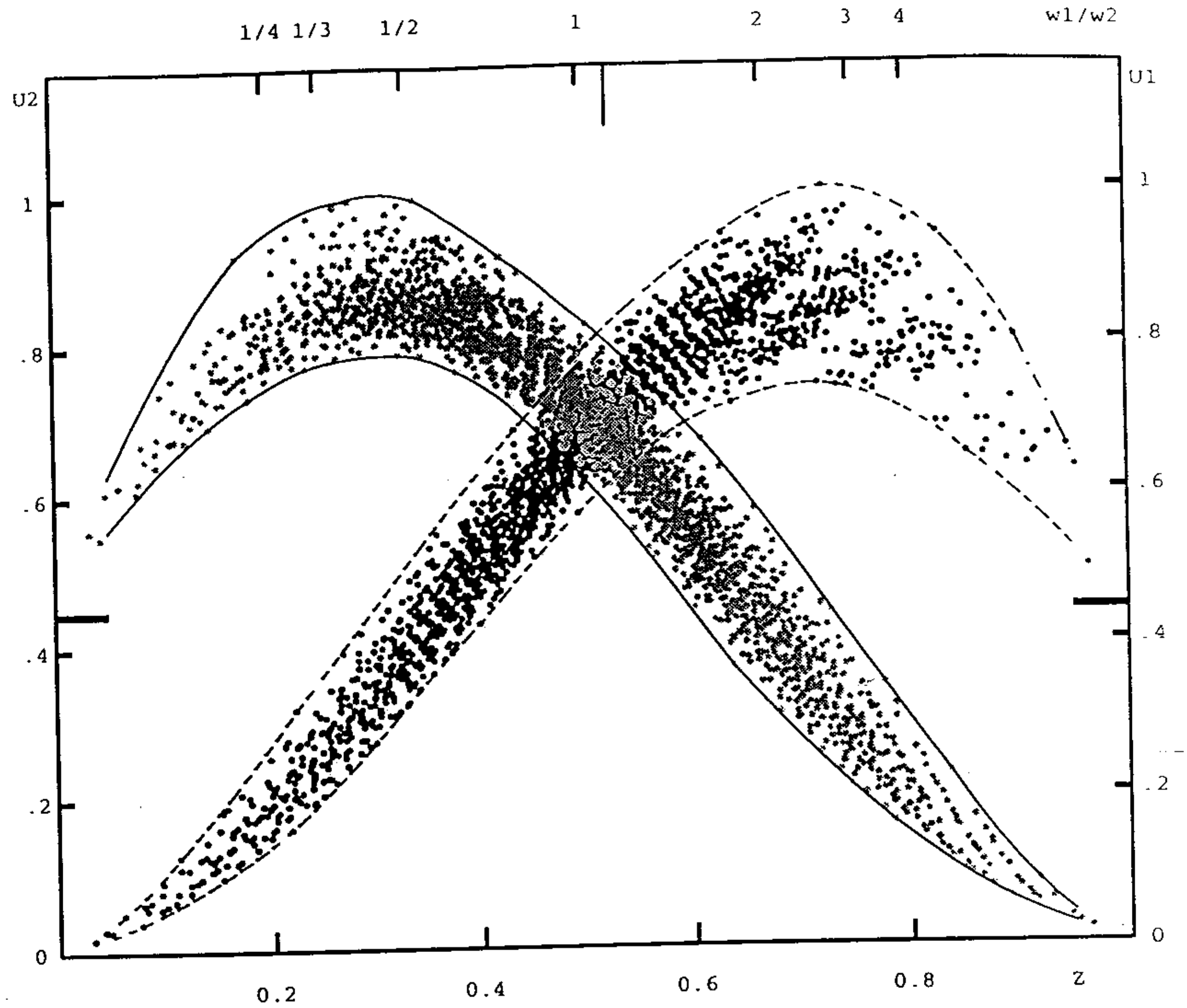

Figure 6. Regions of Equilibria from an 11 industry model

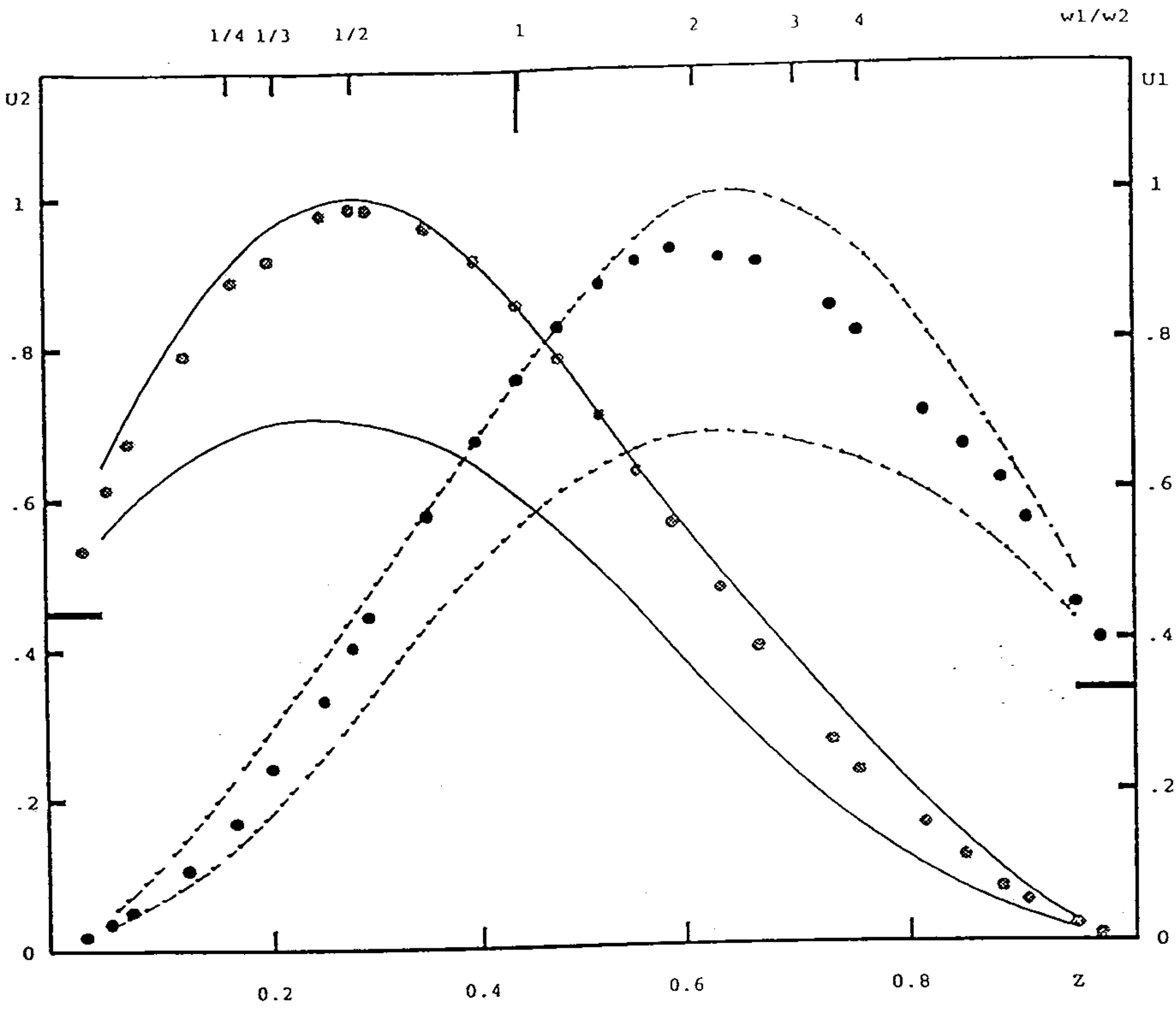

Figure 7. The Equilibria of Theorem 7.1

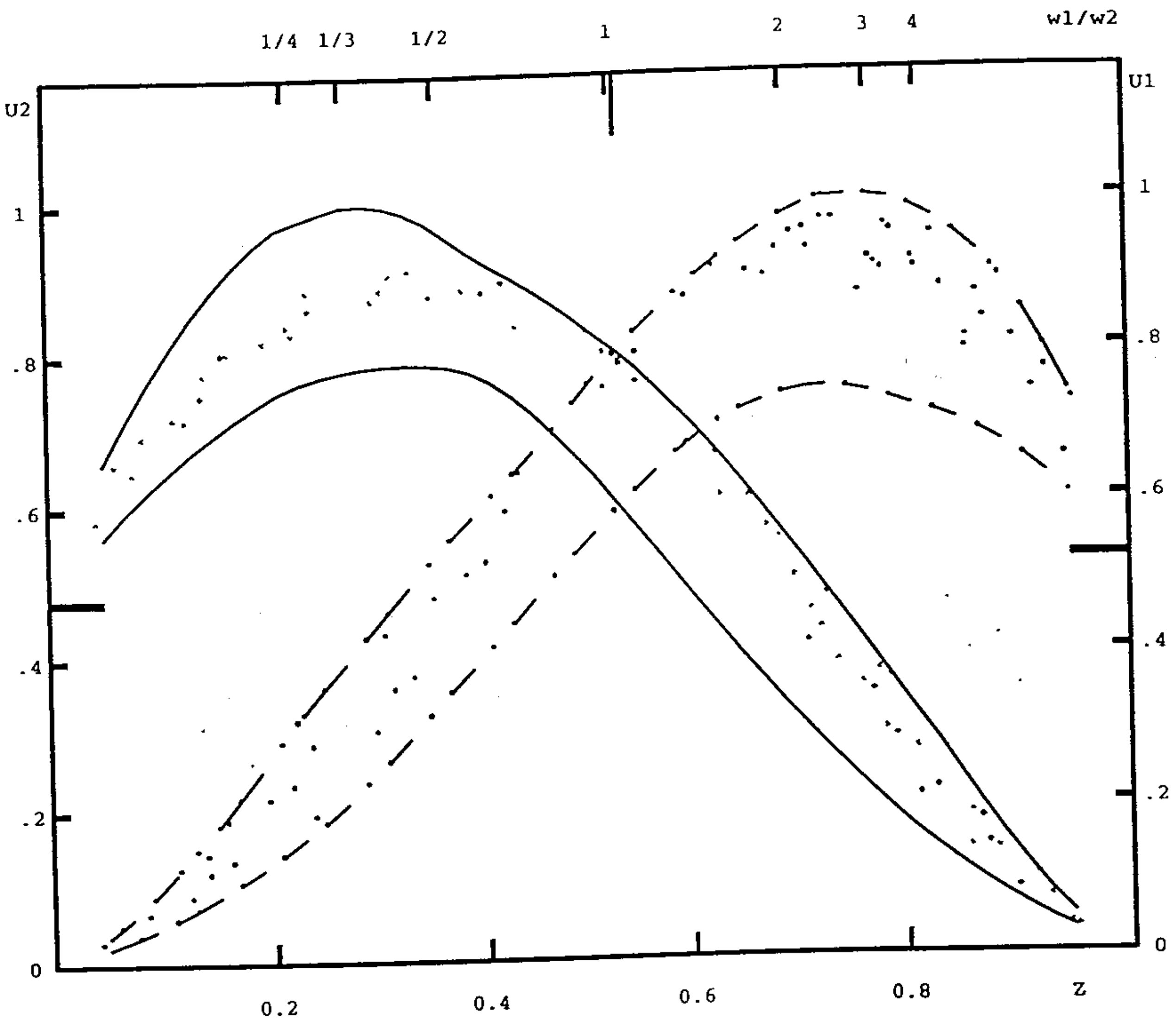

Figure 8. Equilibria with $\lambda_u > .98$

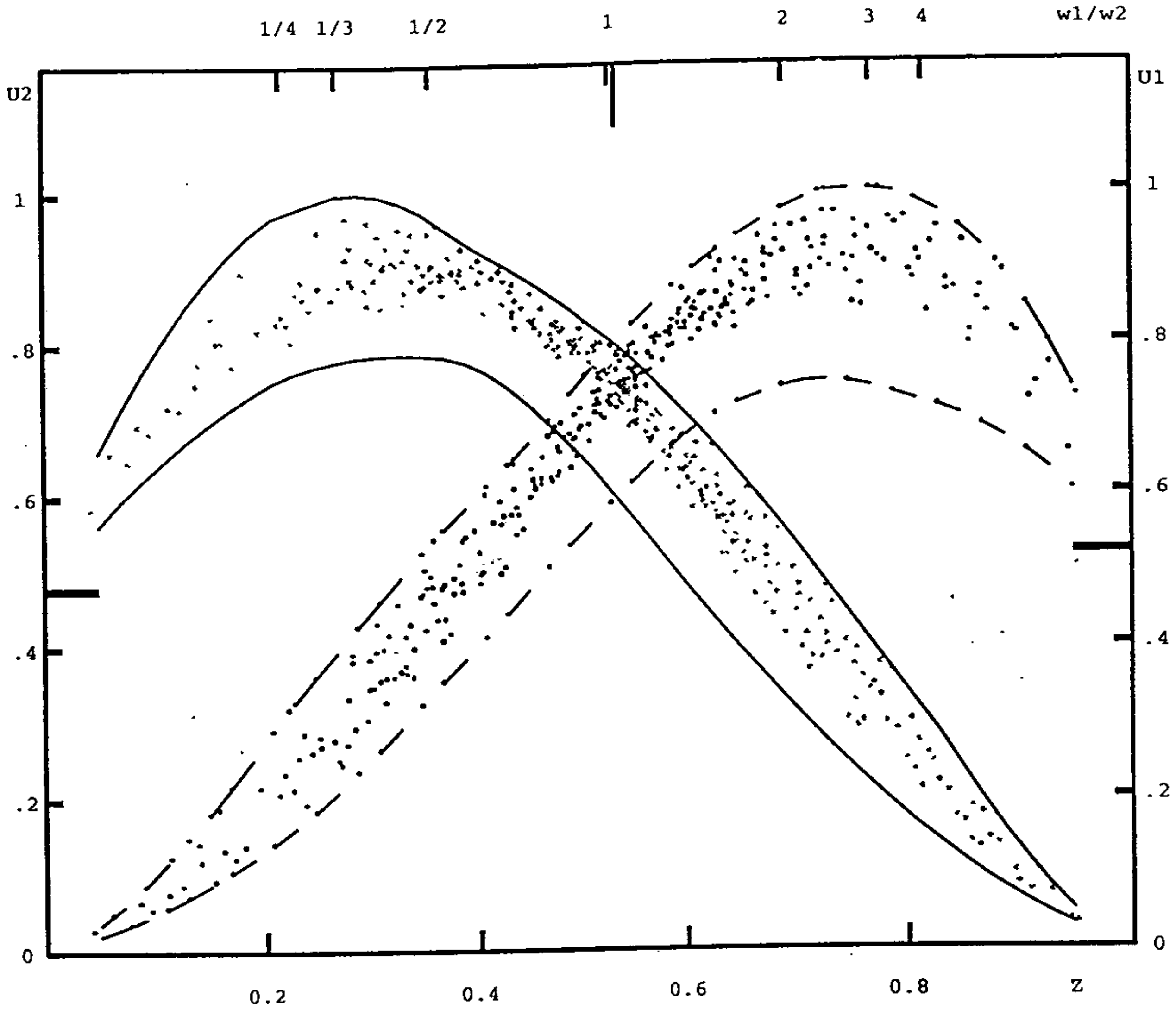

Figure 9. Equilibria with $\lambda_o > .98$

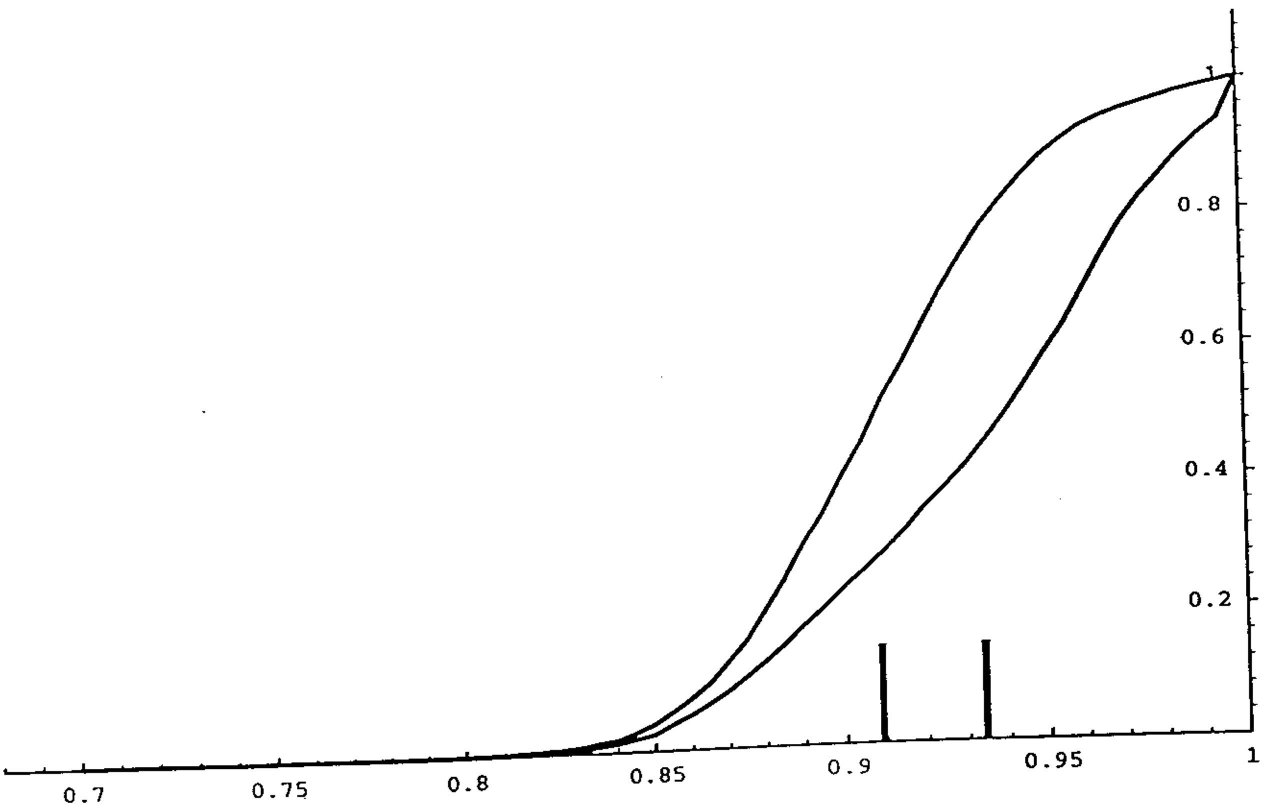

Figure 10. λ_u and λ_o for an 11 good model

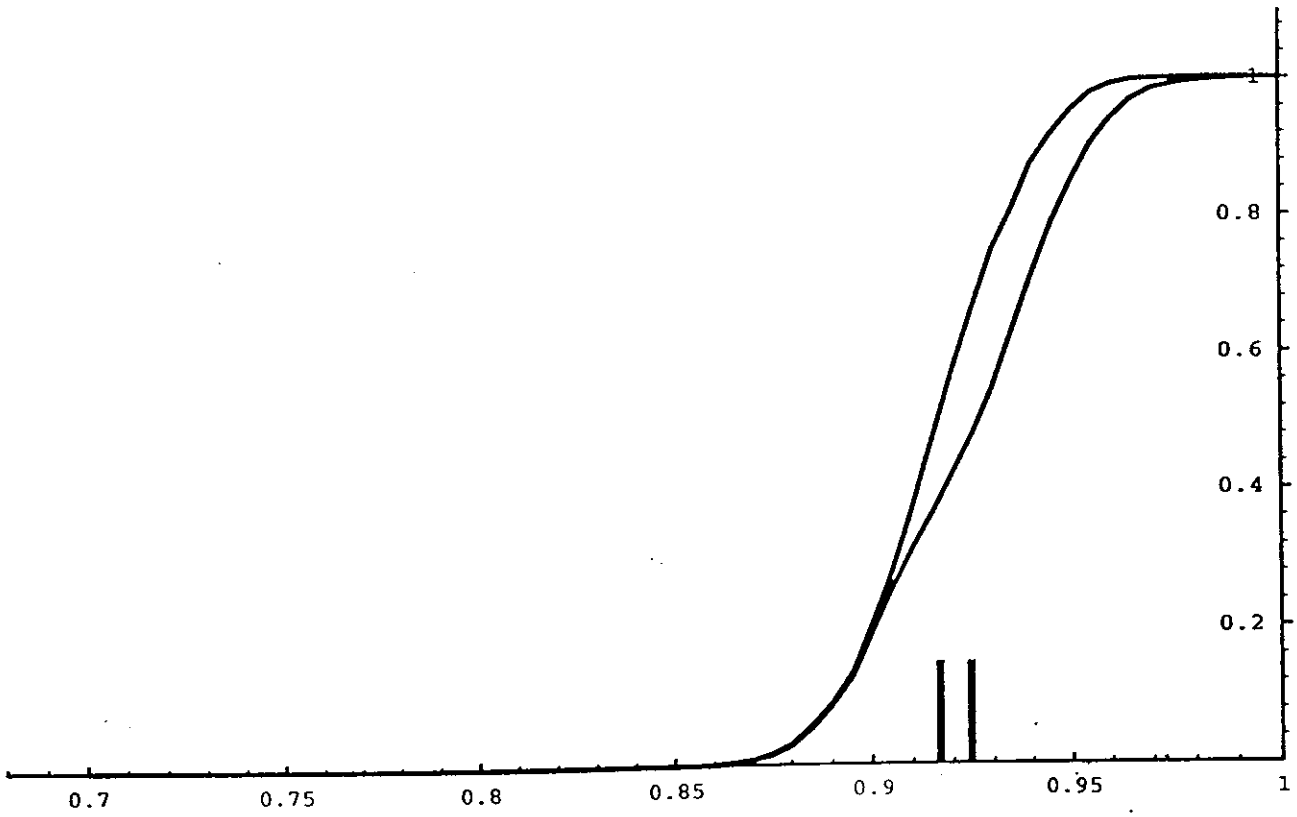

Figure 11. λ_u and λ_o for a sample from a 27 good model