ECONOMIC RESEARCH REPORTS

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RR#: 2000-15 **September 2000**

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FAIR DIVISION OF INDIVISIBLE ITEMS[1]

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This paper analyzes criteria of fair division of a set of indivisible items among people whose revealed preferences are limited to rankings of the items and for whom no side payments are allowed. The criteria include refinements of Pareto optimality and envy-freeness as well as dominance-freeness, evenness of shares, and two criteria based on equally-spaced surrogate utilities, referred to as maxsum and equimax. Maxsum maximizes a measure of aggregate utility or welfare, whereas equimax lexicographically maximizes persons' utilities from smallest to largest. The paper analyzes conflicts among the criteria along with possibilities and pitfalls of achieving fair division in a variety of circumstances.

Keywords: Fair division, allocation of indivisible items, Pareto optimality, envy-freeness, lexicographic maximin.

[1] Steven J. Brams acknowledges the support of the C. V. Starr Center for Applied Economics at New York University. Research by Paul H. Edelman was done while he was in the School of Mathematics, University of Minnesota.

1. INTRODUCTION

This paper examines issues of fair division of n indivisible items among m people based solely on their preference orders over the items. It extends the analyses of fair division among people with similar preferences in Brams and Fishburn (2000) and Edelman and Fishburn (2000) to situations with dissimilar as well as similar preferences. We assume, as before, that each person has a most-preferred to least-preferred strict preference order on the n items, that all items have positive value to every person, and no side payments or other transfers are possible.

The set of items is denoted throughout by $S = \{1, 2, \ldots, n\}$, and 2^S is the set of all subsets of S. The m people are indexed by j from 1 to m. We refer to an allocation or distribution of items to people as a *division*, which is an m-tuple $= (A_1, A_2, \ldots, A_m)$ of mutually disjoint subsets of S whose union equals S, where A_j is the subset received by person j. The set of all divisions is denoted by $\mathcal{A}$.

We assume that the preference information available for making a choice from $\mathcal{A}$ consist of an m-tuple $p = (>_1, >_2, \ldots, >_m)$, where $>_j$ denotes person j's linear preference order, or strict ranking, of the items in S from most preferred to least preferred. We refer to p as a *data profile* and will often display it in list format as

$$\begin{array}{lcccc}
1. & a_{11} & a_{12} & \cdots & a_{1n} \\
2. & a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & & & & \\
m. & a_{m1} & a_{m2} & \cdots & a_{mn}
\end{array} \quad ,$$

where each $a_{j1}a_{j2}\cdots a_{jn}$ is a permutation of S and $a_{j1} >_j a_{j2} >_j \cdots >_j a_{jn}$. The set of all data profiles is denoted by P.

The basic task of fair division is to determine one or more divisions in $\mathcal{A}$ for each $p \in P$ that are the 'fairest' or 'most satisfactory' to the persons represented in the data profile. The criteria we use to judge fairness and satisfactoriness include the traditional ones of Pareto-optimality (or Pareto-efficiency) and envy-freeness (Tinbergen (1946), Foley (1967), Varian (1974), Sugden (1984)). We also use a notion of dominance-freeness and notions of equity that reflect how egalitarian a division is.

Most criteria we consider are based on individuals' preference relations on 2^S. However, because the available data about preferences are restricted to preference orders on S, it is necessary to make assumptions about how preferences over subsets relate to preferences over items. We will assume, as in Brams and Fishburn (2000), that each person has a preference-or-indifference weak order ($\succsim_j$) on 2^S that satisfies the qualitative probability axioms of de Finetti

(1931) and Savage (1954) and whose induced order on singletons is the same as $>_j$ on S. We expand on this point in the next section, which also operationalizes the criteria. Section 3 then gives an overview of the rest of the paper.

There have been many other contributions to allocation problems with indivisibilities. They include Crawford and Heller (1979), Demko and Hill (1988), Alkan, Demange and Gale (1991), Tadenuma and Thomson (1991), portions of Brams and Taylor (1996), Bossert (1998), Bogomolnaia and Moulin (1999), Brams and Kilgour (1999), and Herreiner and Puppe (2000). Most of the allocation mechanisms and procedures described therein involve one or more continuous variables, such as infinitely divisible goods, money, or probabilities for randomized allocations. Two that do not are Demko and Hill (1988) and Herreiner and Puppe (2000). Both consider fair division of a finite set of indivisible items among a set of people without assuming the use of side payments or the help of mediators or other exogenous parties.

Demko and Hill propose both deterministic and randomized solution procedures. Their deterministic solution chooses a division that maximizes the minimum utility obtained by the m people, under the assumption that each person's preferences over subsets are represented by a unique probability measure on 2^S, which specifies his or her utilities for various subsets of items. A similar assumption about probability measures, which represent utilities additively over subsets, is made by many others, including Steinhaus (1948), Crawford and Heller (1979) and Alkan, Demange and Gale (1991).

In our finite-S setting, the assumption of a probability measure presumes either a continuous scaling mechanism (DeGroot (1970)) or a set of indifference comparisons between subsets (Fishburn and Roberts (1989)) that are unlikely to hold in practice. Although we relate the terms in a data profile to preference over subsets by means of the qualitative probability axioms, these axioms do not generally imply the existence of a probability measure that represents preferences over subsets.

Another deterministic procedure for allocation that is also designed to make the worst-off person as well off as possible — while ensuring a Pareto-optimal division — is described in Herreiner and Puppe (2000). A variant of their basic procedure that is more likely to yield an envy-free division but might not satisfy Pareto-optimality is noted. Unlike Demko and Hill, Herreiner and Puppe do not assume that individuals' preferences over subsets are represented by probability measures or are related to their rankings over S in any simple way. Instead, they assume that each person has a linear preference order on 2^S. This allows for complementarity and substitutability effects among items and also accommodates the possibility that some people may prefer not to receive certain items. In view of problems of interdependencies that may beset subset evaluations (Farquhar and Rao (1976), Kannai and Peleg (1984), Fishburn (1992), Fishburn and LaValle (1996)), the procedures of Herreiner and Puppe offer a creative way of dealing with subset preferences. On the other hand, the sheer number of subsets (more than a million when $n = 20$), and their presumption of clear preferences between subsets, could detract from the practicability of their procedures.

The use of data profiles as the basis for fair division is thoroughly pragmatic. Even this might tax the judgmental capacities of some people, but it is generally far less demanding than preference assessment under the foregoing procedures. Despite the fact that we relate preference rankings of S to preferences over subsets for the purposes of applying certain criteria and deriving interesting normative conclusions, our approach never elicits preferences over subsets beyond the information supplied by p.

We conclude this introduction by illustrating aspects of our approach that are expanded on later.

EXAMPLE 1.1: Three items are to be distributed to two people on the basis of the data profile

$$\begin{array}{ll} 1. & 1\ 2\ 3 \\ 2. & 1\ 3\ 2\,. \end{array}$$

There are four divisions in which each person receives his or her first or second choice, namely

$$\begin{aligned} &= (1,23) \\ &= (23,1) \\ &= (12,3) \\ &= (2,13)\ , \end{aligned}$$

where $(1,23)$ abbreviates $(\{1\},\{2,3\})$. Other divisions are either Pareto-dominated by another division or are highly inequitable. For example, $(13,2)$ is Pareto-dominated by $(12,3)$ because person 1 prefers 12 to 13 and person 2 prefers 3 to 2. And $(123,\phi)$ is Pareto-optimal but gives all items to person 1.

Divisions and are unconditionally Pareto-optimal, or *Pareto-ensuring*, because neither person can be made better off without hurting the other. We cannot conclude the same thing for and . For example, if both people prefer 1 to 23, or both prefer 23 to 1, then and are

Pareto-optimal. But if person 1 prefers 23 to 1 and person 2 prefers 1 to 23, then Pareto-dominates . When the Pareto-optimality of a division depends on preferences over subsets that are not determined by the data profile in conjunction with the qualitative probability axioms for preferences, we say that the division is *Pareto-possible*. This is the case here for and .

Divisions and , while Pareto-ensuring, are not envy-free. In $= (12, 3)$, person 2 envies person 1 because person 2 prefers 12 to 3. In , person 1 envies person 2. and may or may not be envy-free. If both persons are indifferent between 1 and 23, then and are envy-free. If both prefer 1 to 23, then neither nor is envy-free. And if person 1 prefers 1 to 23 and person 2 prefers 23 to 1, then is envy-free but is not. When the envy-freeness of a division depends on subset preferences, we say that the division is *envy-possible*.

We conclude that and are envy-possible and Pareto-possible, whereas and are *envy-ensuring* and Pareto-ensuring. Moreover, all four divisions are minimally equitable in the sense that each person gets a first or second choice, but and might appear to be more equitable than and because and give the other two items to the person who does not get his or her first choice.

The latter point is supported by assigning values of 3, 2 and 1 to a person's first, second, and third choices, respectively. These values can be viewed as surrogate utilities, based on applications of Laplace's principle of insufficient reason to successive increments of utility and to utility totals. Both persons have value 3 for and , whereas one person has value 5 and the other value 2 for and . Although the sum of the person's values is greater for or than for or , the latter divisions maximize the minimum value a person receives.

It is easily seen that, for all individual additive utilities that are consistent with p, the deterministic solution of Demko and Hill (1988) prescribes or . □

2. PREFERENCES AND CRITERIA

2.1. *Preference Assumptions*

Let $\succsim$ denote a preference-or-indifference relation on 2^S with induced strict preference ($\succ$) and indifference ($\sim$) relations defined by

$$\begin{aligned} A \succ B \quad &\text{if} \quad A \succsim B \quad \text{and} \quad \text{not}\,(B \succsim A)\,, \\ A \sim B \quad &\text{if} \quad A \succsim B \quad \text{and} \quad B \succsim A\,. \end{aligned}$$

Subscript j, as in $\succsim_j$, identifies such a relation for person j. We assume that $\succsim$ (hence every $\succsim_j$) satisfies the following axioms for all $A, B, C \in 2^S$ and all i, $i' \in S$:

Axiom 1: $\succsim$ *is a weak order* (transitive, complete).

Axiom 2: *If* $A \neq \phi$ *then* $A \succ \phi$.

Axiom 3: *If* $(A \cup B) \cap C = \phi$ *then* $A \succ B \Leftrightarrow A \cup C \succ B \cup C$.

Axiom 4: *If* $i \neq i'$ *then* $\{i\} \succ \{i'\}$ *or* $\{i'\} \succ \{i\}$.

Axiom 1 is a traditional order axiom, Axiom 2 is a positive-value assumption, Axiom 3 is a first-order independence or cancellation condition for strict preference, and Axiom 4 says that a person is never indifferent between distinct items. When $n \geq 5$, considerably more demanding axioms than these (Kraft, Pratt and Seidenberg (1957), Fishburn (1996)) are needed (with obvious relaxations in Axioms 2 and 4) for $\succsim$ to be representable by a probability measure μ on 2^S in the sense that $A \succsim B \Leftrightarrow \mu(A) \geq \mu(B)$. We invoke this stronger form later when we note a result which characterizes envy-possible divisions in terms of dominance relations.

Given $\succsim$, define $>_0$ on S by

$$i >_0 i' \quad \text{if} \quad \{i\} \succ \{i'\} \ .$$

Axioms 1, 2 and 4 imply that $>_0$ is a linear order or strict ranking of S with positive value for each item because $\{i\} \succ \phi$. A data profile $p = (>_1, >_2, \ldots, >_m)$ consists of one such ranking for each person, with $i >_j i'$ if $\{i\} \succ_j \{i'\}$.

It follows easily from Axioms 1–3 that $A \succsim B \Leftrightarrow S \setminus B \succsim S \setminus A$ and $A \supset B \Rightarrow A \succ B$. The latter implication is generalized by a dominance relation $>>$ on 2^S defined by

$$A >> B \text{ if } A \neq B \text{ and } |\{i \in A : i \geqq_0 i'\}| \geq |\{i \in B : i \geqq_0 i'\}| \text{ for all } i' \in S \, .$$

It follows from the definition that $A >> B \Leftrightarrow S \setminus B >> S \setminus A$. Note also that $A >> B$ if $A \neq B$ and every $i' \in B \setminus A$ has a different $i \in A \setminus B$ for which $i >_0 i'$. Because $>>$ depends only on $>_0$ (each $>>_j$ depends only on its $>_j$), the dominance relations for the m people are computable from the data profile. Our use of the dominance relations is based on the following lemma (Fishburn (1996)).

Lemma 2.1: *For all distinct* $A, B \in 2^S$:

(i) if $A >> B$ *then* $A \succ B$ *for every* $\succsim$ *that satisfies Axioms 1–4;*

(ii) if not($B >> A$) then $A \succ B$ for at least one $\succsim$ that satisfies Axioms 1–4.

When $n \geq 3$, a linear order $>_j$ on S is consistent with more than one $\succsim_j$ on 2^S. We denote by $W(>_j)$ the set of all weak orders on 2^S that satisfy Axioms 1–4 and have $>_j$ as their induced linear order on S. For example, when $n = 3$ and $1 >_j 2 >_j 3$, $W(>_j)$ has three members according to whether $1 \succ_j 23$, $1 \sim_j 23$, or $23 \succ_j 1$.

We refer to an m-tuple $(\succsim_1, \succsim_2, \ldots, \succsim_m)$ of weak orders on 2^S that satisfy Axioms 1–4 as a *preference profile*. Let $W(p)$ denote the set of all preference profiles that are consistent with data profile $p = (>_1, >_2, \ldots, >_m)$. Then

$$W(p) = W(>_1) \times W(>_2) \times \cdots \times W(>_m) \ .$$

Our analysis takes account of the possibility that any member of $W(p)$ might be the preference profile that induces p.

2.2. *Envy-freeness*

Division $= (A_1, \ldots, A_m)$ is *envy-free for preference profile* $(\succsim_1, \ldots, \succsim_m)$ if

$$A_j \succsim_j A_k \quad \text{for all} \quad j, k \in \{1, \ldots, m\} \ .$$

With respect to data profile p, we say that is

(i) *envy-free* if is envy-free for every preference profile in $W(p)$;

(ii) *envy-possible* if is envy-free for at least one but not all preference profiles in $W(p)$;

(iii) *envy-ensuring* if is envy-free for no preference profile in $W(p)$.

2.3. *Dominance-freeness*

Division $= (A_1, \ldots, A_m)$ is *dominance-free* for $p = (>_1, \ldots, >_m)$ if there are no $j, k \in \{1, \ldots, m\}$ for which $A_k >>_j A_j$. We are not aware that others have used this criterion, but it is clearly relevant to our formulation and is more easily satisfied than envy-freeness or envy-possibleness. Note that, by Lemma 2.1, if is not dominance-free then it must be envy-ensuring. In other words, dominance-freeness is necessary (but not generally sufficient) for envy-freeness or envy-possibleness.

2.4. *Pareto Properties*

Division $= (A_1, \ldots, A_m)$ *Pareto-dominates* division $= (B_1, \ldots, B_m)$ for preference profile $(\succsim_1, \ldots, \succsim_m)$ if

$$A_j \succsim_j B_j \quad \text{for all} \quad j, \quad \text{and} \quad A_j \succ_j B_j \quad \text{for some} \quad j \ .$$

The Pareto dominance relation on $\mathcal{A}$ for any fixed preference profile is asymmetric and transitive and is therefore a strict partial order.

With respect to data profile p, we say that *always Pareto-dominates* if Pareto-dominates for all preference profiles in $W(p)$. In addition, *sometimes Pareto-dominates* [*never Pareto-dominates*] if Pareto-dominates for some but not all [for no] preference profiles in $W(p)$.

We shall find it convenient to say that *strongly dominates* with respect to p if $\neq$ and

$$A_j >>_j B_j \quad \text{or} \quad A_j = B_j \quad \text{for all} \quad j \in \{1, \ldots, m\} \ .$$

In the presence of Lemma 2.1, strongly dominates if and only if always Pareto-dominates (see Lemma 4.1(i)). Strong dominance also entails equal cardinalities, as follows.

LEMMA 2.2: *If strongly dominates with respect to p, then $|A_j| = |B_j|$ for all j.*

PROOF: The strong dominance conditions imply $|A_j| \geq |B_j|$ for all j. Because $\Sigma|A_j| = \Sigma|B_j|$, we have $|A_j| = |B_j|$ for all j. $\square$

We say that, with respect to p, a division is *strongly dominated* if some other division strongly dominates it. We prove later that a division is strongly dominated if and only if it is Pareto-dominated (Theorem 4.11) as defined below.

Let $\mathcal{B}$ and $\mathcal{C}$ be nonempty subsets of $\mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{A}$. Given $\in \mathcal{B}$, is *Pareto-ensuring* in $\mathcal{C}$ for a *preference profile* if it is not Pareto-dominated by another division in $\mathcal{C}$ for that preference profile. Finiteness and Pareto-dominance ensure that every such $\mathcal{B}$ has a Pareto-ensuring division in $\mathcal{B}$ for any given preference profile.

With respect to data profile p, we say that $\in \mathcal{B}$ is *Pareto-ensuring in* $\mathcal{C}$ if it is Pareto-optimal in $\mathcal{C}$ for all preference profiles in $W(p)$. In addition, $\in \mathcal{B}$ is *Pareto-possible* [*Pareto-dominated*] in $\mathcal{C}$ if it is Pareto-optimal in $\mathcal{C}$ for some but not all [for no] preference profiles in $W(p)$. Whenever $\mathcal{C}$ is omitted here, it is understood to be the entire set $\mathcal{A}$ of divisions.

2.5. *Equity Conditions*

In some allocation situations, it is natural to constrain the number of items received by each person. An example for $n \geq m$ is the feasible division set

$$\mathcal{A}_1 = \{ = (A_1, \ldots, A_m) : |A_j| \geq 1 \quad \text{for all} \quad j\} \ .$$

A choice from $\mathcal{A}_1$ ensures that everyone gets something. Another example is the *even-shares* division set

$$\mathcal{A}_e = \{ = (A_1, \ldots, A_m) : |A_j| \leq |A_k| + 1 \quad \text{for all} \quad j, k \in \{1, \ldots, m\}\} \ ,$$

which implies that every $|A_j|$ equals either $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$ and that no person receives more than one item more than any other person. We refer to divisions in $\mathcal{A}_e$ as *even-shares divisions.* When $n = \lambda m$ for a positive integer λ, every even-shares division has $|A_j| = \lambda$ and is an *equal-shares division* (Brams and Straffin (1979)).

Because constraints based only on numbers of items received do not consider preferences, we use surrogate utilities as in Example 1.1 to assess equity, or the egalitarianness of a division. Given $>_j$, let

$$u_j(i) = n - |\{i' \in S : i' >_j i\}| \quad \text{for each} \quad i \in S \ .$$

When $>_j = a_{j1}a_{j2} \cdots a_{jn}$, we have $u_j(a_{j1}) = n$, $u_j(a_{j2}) = n - 1, \ldots, u_j(a_{jn}) = 1$. We extend u_j additively to 2^S by

$$u_j(A) = \sum_{i \in A} u_j(i) \quad \text{for each} \quad A \in 2^S \ ,$$

with $u_j(\phi) = 0$ and $u_j(S) = 1 + 2 + \cdots + n = n(n+1)/2$. We refer to $u_j(A)$ as person j's *point total* for A. The *point-totals vector* for division $= (A_1, \ldots, A_m)$ and data profile $p = (>_1, \ldots, >_m)$ is

$$u(, p) = (u_1(A_1), u_2(A_2), \ldots, u_m(A_m)) \ .$$

Special types of divisions can be defined on the basis of these vectors. We say that $\in \mathcal{A}$ is a *maxsum division* if it maximizes $\Sigma_j u_j(A_j)$, and a *maxmin division* if it maximizes $\min\{u_1(A_1), \ldots, u_m(A_m)\}$. Maxsum divisions focus on aggregate point totals without regard to equity across persons, whereas maxmin divisions ensure as large a point total as possible for the person having the smallest point total.

We extend the equity feature of maxmin lexicographically in a way that resembles lexicographic rules discussed in d'Aspremont and Gevers (1977), Deschamps and Gevers (1978), Blackorby, Bossert and Donaldson (1996), and Luss (1999), among others. Let $U(, p)$ denote the m-tuple whose terms are the $u_j(A_j)$ arranged in nondecreasing order. For example, if $u(,p) = (10, 11, 9)$ for $m = 3$, then $U(,p) = (9, 10, 11)$. Next, for $p \in P$, define $\geq_{lp}$ on $\mathcal{A}$ lexicographically in the natural way on the basis of the $U(,p)$ vectors. That is, when $U(,p) = \; = (r_1, r_2, \ldots, r_m)$ and $U(,p) = \; = (s_1, s_2, \ldots, s_m)$, we have

$$\geq_{lp} \text{ if } =, \text{ or if } \neq \text{ and } r_k > s_k \text{ for the}$$
$$\text{smallest } k \text{ for which } r_k \neq s_k \;.$$

It follows that $\geq_{lp}$ on $\mathcal{A}$ is a weak order, and there is a unique U vector, say $= (t_1, t_2, \ldots, t_m)$, such that $\geq_{lp}$ for all $\in \mathcal{A}$ whenever $U(,p) = .$ We refer to as p's *equimax vector* and to each $\in \mathcal{A}$ for which $U(,p) =$ as an *equimax division.*

According to these definitions, an equimax division first maximizes the point total of a person with the smallest point total. It then maximizes the point total of a person with the next smallest (possibly the same) point total, and so forth.

EXAMPLE 2.3: Suppose three people have the following strict rankings on $S = \{1, 2, \ldots, 6\}$ for p:

1. 1 2 3 4 5 6
2. 2 1 4 3 6 5
3. 6 1 4 2 3 5.

The maximum points for each item, and the people who can attain these maxima, are

6	for	item 1: person 1
6	for	item 2: person 2
4	for	item 3: person 1
4	for	item 4: persons 2 and 3
2	for	item 5: person 1
6	for	item 6: person 3.

The sum of the maximum points is 28. The sum is achieved by any division that gives each item to a person who attains the maximum points for that item. There are two maxsum divisions, $(135, 24, 6)$ and $(135, 2, 46)$, with point-totals vectors $(12, 10, 6)$ and $(12, 6, 10)$, respectively. Both maxsum divisions are dominance-free because every person receives his or her first choice, but we shall see that neither is a maxmin division because their minimum point total of 6 is not as large as possible.

It is not difficult to show that division $= (15, 23, 46)$, with $u(,p) = U(,p) = (8, 9, 10)$, is the unique equimax division. Its point-totals sum, 27, is as large as possible when the minimum point total exceeds 6, and there is no division whose minimum point total exceeds 8. A close second to is $= (13, 24, 56)$ with $u(, p) = (10, 10, 7)$ and $U(, p) = (7, 10, 10)$. Although neither nor is a maxsum division, both are maxsum divisions *within* the set $\mathcal{A}_e$ of equal-shares divisions. □

3. OVERVIEW OF RESULTS

We now preview our subsequent results, based on the preceding criteria. Throughout, the dominance relations $>>_1$ through $>>_m$ correspond to the strict rankings $>_1$ through $>_m$ of a data profile. $= (A_1, \ldots, A_m)$ and $= (B_1, \ldots, B_m)$ denote divisions in $\mathcal{A}$, and parenthetical references cite theorems and examples in later sections.

We have organized our results into three sections. Section 4 focuses on Pareto relations, section 5 emphasizes envy-freeness, and section 6 discusses additional aspects of maxsum and equimax divisions. Our aim is to elucidate possibilities and impossibilities of the criteria separately and in combination.

Section 4 considers characterizations of Pareto-ensuring, Pareto-possible, and Pareto-dominated divisions. The section begins with aspects of Pareto dominance based on the dominance relations $>>_1$ through $>>_m$ (Lemma 4.1). For example, always Pareto-dominates if and only if strongly dominates , and never Pareto-dominates if and only if $B_j >>_j A_j$ for some j. We then note a few types of Pareto-ensuring divisions that assign items to persons sequentially in such a way that all items assigned to j are the highest ranking items in $>_j$ that have not been previously assigned (Lemma 4.2, Theorem 4.3). All divisions formed in this way are either Pareto-ensuring or Pareto-possible (Lemma 4.5). When m divides n, there may be no equal-shares division in $\mathcal{A}_e$ that is Pareto-ensuring in $\mathcal{A}$ (Example 4.4), but some division in $\mathcal{A}_e$ is Pareto-ensuring in $\mathcal{A}_e$ and either Pareto-ensuring or Pareto-possible in $\mathcal{A}$ (Theorem 4.6). Moreover, if if Pareto-ensuring and gives every person at least two items, then must be dominance-free (Theorem 4.7).

There are data profiles for which no maxsum or equimax division is unconditionally Pareto-ensuring (Example 4.9), but every maxsum and equimax division is Pareto-ensuring or Pareto-possible (Theorem 4.10). The final result in section 4 says that a division is Pareto-dominated if and only if it is strongly dominated (Theorem 4.11).

Section 5 observes that is envy-free if and only if $A_j >>_j A_k$ for all $j \neq k$ (Lemma 5.1), which can happen only if is an equal-shares division (Corollary 5.2). When is envy-free with $\lambda = n/m$ and is also Pareto-ensuring, then every person gets his or her first choice, and if $\lambda \geq 2$ then every person gets his or her first $\lambda - 1$ choices (Theorem 5.3). When $\mathcal{A}$ contains at least one envy-free division, at least one such division is not strongly dominated if $m = 2$ (Theorem 5.4), but all can be strongly dominated if $m \geq 3$ (Theorem 5.5). In a similar vein, every dominance-free division is envy-free or envy-possible when $m = 2$ but not when $m \geq 3$ (Theorem 5.6). If all weak orders in a preference profile are additively representable with $m \geq 3$, then a division is envy-free or envy-possible if and only if no A_j is dominated by a convex combination of the other A_k. The absence of convex dominance is sufficient for envy-freeness or envy-possibleness in the context of $W(p)$ (Corollary 5.8).

We conclude section 5 by illustrating possibilities for equal-shares divisions, with $m = 3$ and $n = 6$, for a data profile in which (1) no equal-shares division is envy-free or Pareto-ensuring, (2) some maxsum divisions are equal-shares divisions, all of which are envy-ensuring, (3) some equimax divisions are equal-shares divisions, all of which are envy-possible and Pareto-possible, and (4) one envy-possible equal-shares division is strongly dominated by an equal-shares equimax division (Example 5.9).

Section 6 notes that all maxsum and equimax divisions can be envy-ensuring, and that every maxsum division might give one person all but one item (Lemma 6.1). We then describe data profiles for $m = 2$ in which the maxsum point-totals sum minus the equimax point-totals sum grows quadratically in n (Theorem 6.3). The section concludes with a data profile for which all maxsum and equimax divisions are equal-shares divisions and the maxsum sum exceeds the equimax sum (Theorem 6.4).

4. PARETIAN ANALYSIS

In this section we characterize Pareto-dominance relations with respect to a data profile $p = (>_1, \ldots, >_m)$ in terms of the corresponding dominance relations $>>_1, \ldots, >>_m$. We then describe Pareto-ensuring possibilities and impossibilities for even-shares, maxsum and equimax divisions, among others, and conclude with facts about strong dominance. Throughout, $= (A_1, \ldots, A_m)$ and $= (B_1, \ldots, B_m)$ denote divisions in $\mathcal{A}$.

Lemma 4.1: *Suppose $\neq$. Then, with respect to $p \in P$:*

(i) always Pareto-dominates if and only if strongly dominates ;

(ii) sometimes Pareto-dominates if and only if does not strongly dominate , and not $(B_j >>_j A_j)$ *for all* j*;*

(iii) never Pareto-dominates if and only if $B_j >>_j A_j$ *for some* j*.*

PROOF: (i) Suppose $\neq$ and $A_j >>_j B_j$ or $A_j = B_j$ for every j. Then, for every preference profile $(\succsim_1, \ldots, \succsim_m)$ in $W(p)$, Lemma 2.1 implies $A_j \succsim_j B_j$ for all j with $A_j \succ_j B_j$ whenever $A_j \neq B_j$. It follows that always Pareto-dominates . If not$(A_j >>_j B_j$ or $A_j = B_j)$ for some j, then Lemma 2.1 implies that $B_j \succ_j A_j$ for some $\succsim_j$ in $W(>_j)$, so does not always Pareto-dominate .

(ii) It follows from Lemma 2.1 that the conditions of (ii) imply that Pareto-dominates for some but not all preference profiles in $W(p)$.

(iii) If $B_j >>_j A_j$ then $B_j \succ_j A_j$ by Lemma 2.1 for every $\succsim_j$ in $W(>_j)$, so there is no preference profile in $W(p)$ at which Pareto-dominates . If not$(B_j >>_j A_j)$ for every j, then (ii) or (i) obtains. $\square$

Our first result for Pareto-ensuringness notes that the most inequitable divisions are always Pareto-ensuring.

LEMMA 4.2: *Every division that gives all items to one person is Pareto-ensuring in* $\mathcal{A}$ *with respect to every* $p \in P$*.*

PROOF: If has $A_j = S$, and $\neq$, then $A_j >>_j B_j$, so never Pareto-dominates by Lemma 4.1(iii). $\square$

Several ensuing results use divisions constructed sequentially for a given ordering of the m people, say $1, 2, \ldots, m$ for definiteness. Let $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an m-tuple of nonnegative integers that sum to n. We then define $T(\lambda_1, \lambda_2, \ldots, \lambda_m)$ for a given $p = (>_1, >_2, \ldots, >_m)$ as the division in which A_1 is the set of the top (most-preferred) λ_1 items in $>_1$ and, for $j = 2, \ldots, m$, A_j is the set of the top items in $>_j$ that are not already in $A_1 \cup \cdots \cup A_{j-1}$. For example, if $p = (123456, 154362, 326145)$, then $T(2, 2, 2) = (12, 45, 36)$ and $T(3, 1, 2) = (123, 5, 46)$.

The next result counterbalances Lemma 4.2 with the fact that some minimally equitable divisions are also Pareto-ensuring. In particular, $\mathcal{A}_1$ for $n \geq m$ always has such divisions.

THEOREM 4.3: *If* $n < m$*, then* $T(1, \ldots, 1, 0, \ldots 0)$ *is Pareto-ensuring in* $\mathcal{A}$ *with respect to every* $p \in P$*. If* $n \geq m$*, then* $T(n - m + 1, 1, 1, \ldots, 1)$ *is Pareto-ensuring in* $\mathcal{A}$ *with respect to* *every* $p \in P$.

PROOF: Let p be given. Suppose $n < m$. Let $= T(1, \ldots, 1, 0, \ldots, 0)$, so $|A_j| = 1$ for $j \leq n$ and $|A_j| = 0$ for $j > n$. If has $|B_j| = 0$ for some $j \leq n$, then $A_j >>_j B_j$ and we conclude by Lemma 4.1(iii) that never Pareto-dominates .

Consequently, $\in \mathcal{A}$ can Pareto-dominate for some preference profile in $W(p)$ only if $|B_j| = |A_j|$ for every j. But Pareto-dominance would require $B_1 = A_1$, else $A_1 >>_1 B_1$, and $B_2 = A_2$, else $A_2 >>_2 B_2$, and so forth, so that $=$, a contradiction. We conclude that never Pareto-dominates for every $\in \mathcal{A}$, so is Pareto-ensuring in $\mathcal{A}$.

Now suppose $n \geq m$. Let $= T(n - m + 1, 1, 1, \ldots, 1)$. If either $|B_j| = 0$ for some $j > 1$ or $|B_1| < n - m + 1$, then never Pareto-dominates . And if $|B_1| = n - m + 1$ and $|B_j| = 1$ for all $j > 1$, the supposition that Pareto-dominates for some preference profile in $W(p)$ leads, as above, to the contradiction that $=$. It follows that is Pareto-ensuring in $\mathcal{A}$. $\square$

When $n \leq m + 1$, Theorem 4.3 shows that the set $\mathcal{A}_e$ of even-shares divisions always has divisions that are Pareto-ensuring in $\mathcal{A}$. This need not be true when $n \geq m + 2$.

EXAMPLE 4.4: Let $p = (1234, 1234)$, so $n = m + 2 = 4$. $\mathcal{A}_e$ has six divisions, and for each there is at least one preference profile in $W(p)$ for which it is Pareto-dominated by another division in $\mathcal{A}$:

$(12, 34)$ and $(13, 24)$ can be Pareto-dominated by $(234, 1)$;
$(14, 23)$ can be Pareto-dominated by $(23, 14)$;
$(23, 14)$ can be Pareto-dominated by $(14, 23)$;
$(24, 13)$ and $(34, 12)$ can be Pareto-dominated by $(1, 234)$. $\square$

Our next main result, Theorem 4.6, shows that $\mathcal{A}_e$ always contains a division that is Pareto-ensuring in $\mathcal{A}_e$ and either Pareto-ensuring or Pareto-possible in $\mathcal{A}$. We precede it by a construction which, given a division , produces a preference profile for which is Pareto-ensuring or Pareto-possible.

Given $p = (>_1, \ldots, >_m)$ and $= (A_1, \ldots, A_m)$, let $f(j) = 0$ if $A_j = \phi$, and otherwise let $f(j)$ be the least-preferred item in A_j according to $>_j$. For each j, let v_j be a positive and strictly decreasing (in $>_j$) real-valued function on S such that, when $f(j) > 0$,

$$v_j(f(j)) = 1$$
$$v_j(i) < 1 + 1/n \quad \text{for} \quad i >_j f(j)$$
$$\sum_{\{i : f(j) >_j i\}} v_j(i) < 1/2\,.$$

Define $v_j(A)$ as $\sum_{i \in A} v_j(i)$ for all j and all $A \in 2^S$, and let $\succsim_j$ be the weak order on 2^S that satisfies

$$A \succsim_j B \Leftrightarrow v_j(A) \geq v_j(B) \ .$$

We denote by $W(p,)$ the set of preference profiles $(\succsim_1, \ldots, \succsim_m)$ constructible in this manner.

LEMMA 4.5: *For all $p \in P$ and all m-tuples $(\lambda_1, \ldots, \lambda_m)$ of nonnegative integers that sum to n, $T(\lambda_1, \ldots, \lambda_m)$ is either Pareto-ensuring or Pareto-possible in $\mathcal{A}$.*

PROOF: Given p, let $\ = T(\lambda_1, \ldots, \lambda_m)$ and let X be a preference profile in $W(p,)$. If some division Pareto-dominates for X, then some with $|B_j| = 0$ whenever $|A_j| = 0$ Pareto-dominates for X. Suppose is such a division. Let $j_1 < j_2 < \cdots < j_q$ be the j's with $f(j) > 0$. Then $B_{j_1} \supseteq A_{j_1}$, for otherwise $v_{j_1}(A_{j_1}) > v_{j_1}(B_{j_1})$ and $A_{j_1} \succ_{j_1} B_{j_1}$. Proceeding sequentially, Pareto-dominance also requires $B_{j_2} \supseteq A_{j_2}$, $B_{j_3} \supseteq A_{j_3}, \ldots, B_{j_q} \geq A_{j_q}$. But then $B_{j_k} = A_{j_k}$ for $k = 1, \ldots, q$, so we get the contradiction that $\ = \ $. Hence no division in $\mathcal{A}$ Pareto-dominates for X, so is Pareto-ensuring or Pareto-possible in $\mathcal{A}$ with respect to p. □

THEOREM 4.6: *Suppose $n \geq m + 2$ and $p \in P$. Let $n = \lambda m + \beta$ with $\lambda \in \{1, 2, \ldots\}$ and $\beta \in \{0, 1, \ldots, m-1\}$. If $\beta = 0$ let $\ = T(\lambda, \ldots, \lambda)$, and if $\beta > 0$ let $\ = T(\lambda+1, \ldots, \lambda+1, \lambda, \ldots, \lambda)$. Then is Pareto-ensuring in $\mathcal{A}_e$ and is Pareto-ensuring or Pareto-possible in $\mathcal{A}$ with respect to p.*

PROOF: The final assertion for $\mathcal{A}$ follows immediately from Lemma 4.5. To prove that is Pareto-ensuring in $\mathcal{A}_e$, suppose first that $\beta = 0$. If $\ \in \mathcal{A}_e$ Pareto-dominates for some preference profile in $W(p)$, we require $B_1 = A_1$ (else, with $|B_1| = |A_1|$, $A_1 >>_1 B_1$), $B_2 = A_2$ (else, given $B_1 = A_1$, $A_2 >>_2 B_2$),..., so $\ = \ $, a contradiction. Hence is Pareto-ensuring in $\mathcal{A}_e$.

Suppose $\beta > 0$, so $\ = T(\lambda + 1, \ldots, \lambda + 1, \lambda, \ldots, \lambda)$ with $|A_j| = \lambda + 1$ for persons $1, \ldots, \beta$. Suppose $\ \in \mathcal{A}_e$ Pareto-dominates for some preference profile in $W(p)$. Then $B_1 = A_1$, for if either $|B_1| = \lambda$ or $|B_1| = \lambda + 1$ and $B_1 \neq A_1$ then $A_1 >>_1 B_1$. Similarly, $B_2 = A_2, \ldots, B\beta = A\beta$. Then, for every $j > \beta$, membership in $\mathcal{A}_e$ implies that $|B_j| = \lambda$ and we get $B_{\beta+1} = A_{\beta+1}$ (else $A_{\beta+1} >>_{\beta+1} B_{\beta+1}$)$, \ldots, B_m = A_m$. Therefore $\ = \ $, a contradiction. It follows that is Pareto-ensuring in $\mathcal{A}_e$. □

Our next result shows that a Pareto-ensuring division that gives every person at least two items is dominance-free. Although dominance-freeness does not guarantee envy-freeness or envy-possibleness when $m \geq 3$ (see Theorem 5.6), it seems likely that a dominance-free division will not be envy-ensuring.

THEOREM 4.7: *Suppose $m \geq 2$, $n \geq 2m$, and $p \in P$. If $\in \mathcal{A}$ is Pareto-ensuring with respect to p and has $|A_j| \geq 2$ for all j, then A is dominance-free.*

PROOF: Suppose $\in \mathcal{A}$ has $|A_j| \geq 2$ for all j and is *not* dominance-free. Assume without loss of generality that $A_1 >>_2 A_2$. Let i be the first item in $>_2$ that is in A_1. Then $i >_2 k$ for all $k \in A_2$ because $A_1 >>_2 A_2$ and A_1 and A_2 are disjoint. By Lemma 2.1, $\{i\} \succ_2 A_2$ for some weak order in $W(>_2)$. Also, because $|A_2| \geq 2$, $(A_1 \cup A_2) \setminus \{i\} \succ_1 A_1$ for some weak order in $W(>_1)$. Let $B_1 = (A_1 \cup A_2) \setminus \{i\}$, $B_2 = \{i\}$, and $B_j = A_j$ for $j \geq 3$. It follows that sometimes Pareto-dominates , so is not Pareto-ensuring. In other words, if is Pareto-ensuring then it must be dominance-free. $\square$

We continue with results for maxsum and equimax divisions. The following lemma notes the simple structure of the set of maxsum divisions. Recall that $u_j(i) = n - |\{i' \in S : i' >_j i\}|$.

LEMMA 4.8: *Given $p \in P$, let*

$$\begin{aligned} x_i &= \max_j u_j(i) \quad \textit{for all} \quad i \in S\,, \\ M_j &= \{i : u_j(i) = x_i\} \quad \textit{for all} \quad j \in \{1, \ldots, m\}\,. \end{aligned}$$

Then is a maxsum division if and only if $A_j \subseteq M_j$ for all j.

PROOF: Clearly, $\max \Sigma_j u_j(A_j) = \Sigma_i x_i$. If $A_j \subseteq M_j$ for all j then $\Sigma_j u_j(A_j) = \Sigma_i x_i$, so is a maxsum division. If $A_j \not\subseteq M_j$ for some j, then $i \in A_j \setminus M_j$ has less than x_i points in $\Sigma_j u_j(A_j)$, so $\Sigma_j u_j(A_j) < \Sigma_i x_i$. $\square$

We precede a proof that all equimax and maxsum divisions are Pareto-ensuring or Pareto-possible by showing that there might be no Pareto-ensuring equimax or maxsum division.

EXAMPLE 4.9: Let $p = (12345, 31452)$. Then $M_1 = \{1, 2\}$ and $M_2 = \{3, 4, 5\}$, so, by Lemma 4.8, the unique maxsum division is $(12, 345)$ with point-totals vector $(9, 10)$ and $\Sigma_i x_i = 19$. Clearly, $(12, 345)$ is also the unique equimax division. But it is Pareto-dominated by $(245, 13)$ if $45 \succ_1 1$ and $1 \succ_2 45$, so $(12, 345)$ is not Pareto-ensuring. $\square$

THEOREM 4.10: *For every $p \in P$, every equimax division and every maxsum division is Pareto-ensuring or Pareto-possible.*

PROOF: Given p, let Y be the preference profile in $W(p)$ that has $A \succsim_p B \Leftrightarrow u_j(A) \geq u_j(B)$

for every j. Suppose is an equimax division. If Pareto-dominates for Y, then $u_j(B_j) \geq u_j(A_j)$ for all j, and $u_j(B_j) > u_j(A_j)$ for some j, contrary to as an equimax division. Therefore, no other division Pareto-dominates for Y, so is Pareto-possible or Pareto-ensuring. A similar result clearly holds for Y if is a maxsum division. □

We conclude our Paretian analysis with a proof that is strongly dominated with respect to p if and only if it is Pareto-dominated with respect to p. In the proof, we say that $\succsim_j$ in $W(>_j)$ is a *superdecreasing weak order for* $>_j$ if, for all distinct A and B in 2^S,

$$A \succ_j B \text{ if the most-preferred item by } >_j \text{ in } (A \setminus B) \cup (B \setminus A) \text{ is in } A \setminus B\,.$$

When $>_j = a_{j1}a_{j2}\cdots a_{jn}$, this can be represented additively as $A \succsim_j B \Leftrightarrow \sum_{i\in A} w_j(i) \geq \sum_{i\in B} w_j(i)$, with $w_j(a_{jn}) > 0$ and $w_j(a_{jq}) > nw_j(a_{j,q+1})$ for $q = 1, \ldots, n-1$.

Theorem 4.11: *For all $p \in P$ and $\in \mathcal{A}$, the following are mutually equivalent:*

(i) is strongly dominated;

(ii) there is a list $A\alpha$, $A\beta$, $A\gamma, \ldots, A\mu$ of two or more different components of , and $a_j \in A_j$ for each $j \in \{\alpha, \beta, \gamma, \ldots, \mu\}$ such that

$$a\alpha > \beta a\beta > \gamma a\gamma \cdots > \mu a_\mu > \alpha a\alpha\ ;$$

(iii) is Pareto-dominated.

Proof: We show that (ii)⇒(i), (i)⇒(iii) and (iii)⇒(ii).

(ii)⇒(i). Assume (ii). Take $B\alpha = (A\alpha \cup \{a\mu\}) \setminus \{a\alpha\}, \ldots, B\gamma = (A\gamma \cup \{a\beta\}) \setminus \{a\gamma\}$, $B\beta = (A\beta \cup \{a\alpha\}) \setminus \{a\beta\}$, and $B_j = A_j$ for $j \not\in \{\alpha, \beta, \gamma, \ldots, \mu\}$. Then strongly dominates .

(i)⇒(iii). If strongly dominates then, by Lemma 2.1, Pareto-dominates for every preference profile in $W(p)$.

(iii)⇒(ii). Assume (iii). Let $Z = (\succsim_1, \ldots, \succsim_m)$ with $\succsim_j$ a superdecreasing weak order for $>_j$ for every j. Let $\neq$ Pareto-dominate for Z. With $K = \{k : B_k \neq A_k\}$, we have $\cup_K B_k = \cup_K A_k$, $|K| \geq 2$, and $B_k \succ_k A_k$ for all $k \in K$. By the choice of Z, the most-preferred item in $(B_k \setminus A_k) \cup (A_k \setminus B_k)$ for $>_k$ is in $B_k \setminus A_k$. Denote this item by b_k. Assume without loss of generality that $K = \{1, 2, \ldots, r\}$. Because each b_k is in an $A_j \setminus B_j$ for some $j \neq k$ in $\{1, 2, \ldots, r\}$, there is a partition $(A_1^*, A_2^*, \ldots, A_r^*)$ of $\{b_1, b_2, \ldots, b_r\}$ with $A_k^* \subseteq A_k \setminus B_k$ for each

k such that

$$\begin{array}{c} b_1 >_1 A_1^* \\ b_2 >_2 A_2^* \\ \vdots \\ b_r >_r A_r^* \quad , \end{array}$$

where $b_k >_k A_k^*$ means that $b_k >_k x$ for every x, if any, in A_k^*. With $x <_k y \Leftrightarrow y >_k x$ and $b_k \in A_{g(k)}^*$, we have

$$b_1 <_{g(k)} b_{g(k)} <_{g(g(k))} b_{g(g(k))} <_{g(g(g(k)))} \cdots$$

and must obtain a cyclic arrangement for (ii) with $\{\alpha, \beta, \gamma, \ldots, \mu\} \subseteq \{1, 2, \ldots, r\}$. $\square$

5. ENVY-FREE ANALYSIS

We now bring aspects of dominance-freeness and envy-freeness into the picture, beginning with a characterization of envy-freeness.

LEMMA 5.1: *Division is envy-free with respect to $p \in P$ if and only if*

$$A_j >>_j A_k \textit{ for all distinct } j \textit{ and } k \textit{ in } \{1, \ldots, m\}\,.$$

PROOF: We use Lemma 2.1. If $A_j >>_j A_k$ for all $j \neq k$ then $A_j \succ_j A_k$ for all $j \neq k$, so there is no envy. If not$(A_j >>_j A_k)$ for some $j \neq k$ then $A_k \succ_j A_j$ for some $\succsim_j$ in $W(>_j)$. $\square$

The condition of Lemma 5.1 is very demanding. It requires $n = \lambda m$ for some integer λ and can hold only for equal-shares divisions.

COROLLARY 5.2: *If is envy-free for any $p \in P$, then $|A_j| = |A_k|$ for all $j, k \in \{1, \ldots, m\}$.*

PROOF: $A_j >>_j A_k$ implies $|A_j| \geq |A_k|$, and $A_k >>_k A_j$ implies $|A_k| \geq |A_j|$. The corollary's conclusion then follows from Lemma 5.1. $\square$

An argument similar to that used in the proof of Theorem 4.7 shows that if an envy-free division is also Pareto-ensuring, then every person gets his or her first choice. Moreover, if $\lambda = n/m \geq 2$, then every person gets his or her first $\lambda - 1$ choices.

THEOREM 5.3: *Suppose $m \geq 2$, $p \in P$, and division $\in \mathcal{A}$ is envy-free and Pareto-ensuring with respect to p, where $|A_j| = \lambda = n/m$ for all j. Then every person gets his or her first* *choice, and if $\lambda \geq 2$ then every person gets his or her first $\lambda - 1$ choices.*

PROOF: Assume that is envy-free. By Corollary 5.2, $|A_j| = |A_k|$ for all j and k. Let $\lambda = n/m$, so $|A_j| = \lambda$ for all j. If $\lambda = 1$, then envy-freeness implies that every person must get his or her first choice. Suppose $\lambda \geq 2$. Assume without loss of generality that person 1's strict ranking is $12\cdots n$, and suppose that person 1 does not get item i, where $i \leq \lambda - 1$, whereas person 2 (for definiteness) gets item i. Let X be the items in A_1 that person 1 prefers less than i. Then $|X| \geq 2$. Let have $B_1 = (A_1 \cup \{i\}) \setminus X$, $B_2 = (A_2 \cup X) \setminus \{i\}$, and $B_j = A_j$ for all $j \geq 3$. Then, as in the proof of Theorem 4.7, person 1 might prefer B_1 to A_1, and person 2 might prefer B_2 to A_2. Consequently, sometimes Pareto-dominates , so is not Pareto-ensuring. It follows that if is Pareto-ensuring then every person must get his or her first $\lambda - 1$ items. □

That an envy-free division can be strongly dominated and therefore Pareto-dominated (Theorem 4.11), is true of $= (1346, 2578)$ when p is

1. 1 2 3 4 5 6 7 8
2. 5 6 7 8 1 2 3 4 .

Here is strongly dominated by $(1234, 5678)$. However, when $m = 2$ and there are envy-free divisions, at least one such division must be Pareto-possible or Pareto-ensuring. This is an immediate corollary of Theorem 4.11 and the following result. It is true also, as shown by an example in Brams, Edelman and Fishburn (2000), that an envy-free and Pareto-ensuring division need not be an equimax division when $m \geq 3$.

THEOREM 5.4: *If $m = 2$, $p \in P$, and $\mathcal{A}$ contains at least one envy-free division with respect to p, then at least one of those divisions is not strongly dominated.*

PROOF: Let $A^c = S \setminus A$. Suppose $m = 2$ and $= (A, A^c)$ is envy-free with respect to p, so $A >>_1 A^c$ and $A^c >>_2 A$. If $= (B, B^c)$ strongly dominates , then, with the use of $C >> D \Leftrightarrow D^c >> C^c$, we have

$$B >>_1 A >>_1 A^c >>_1 B^c, \quad \text{so} \quad B >>_1 B^c \quad \text{by transitivty}\,,$$

and

$$B^c >>_2 A^c >>_2 A >>_2 B, \quad \text{so} \quad B^c >>_2 B \quad \text{by transitivity}\,.$$

Therefore is also envy-free. It follows from finiteness and transitivity of the $>>_j$ that some envy-free division is not strongly dominated. □

The conclusion of Theorem 5.4 can fail when $m \geq 3$.

THEOREM 5.5: *If $m \geq 3$, there are data profiles with unconditionally envy-free divisions all of which are strongly dominated.*

PROOF: The smallest example that verifies Theorem 5.5 for $m = 3$ has $n = 6$ with data profile

$$\begin{array}{ll} 1. & \mathbf{1}\ 2\ \mathbf{3}\ 4\ 5\ 6 \\ 2. & \mathbf{4}\ 3\ \mathbf{2}\ 1\ 5\ 6 \\ 3. & \mathbf{5}\ 1\ 2\ \mathbf{6}\ 3\ 4\,. \end{array}$$

With the aid of Lemma 5.1 and Corollary 5.2, it is easily seen that $= (13, 24, 56)$, shown above in boldface, is the only envy-free division. However, it is strongly dominated by $= (12, 34, 56)$, which is envy-possible but not envy-free because person 3 might prefer 12 to 56.

The reasoning which shows that is the unique envy-free division extends easily to larger m. We illustrate it for $m = 4$ with data profile

$$\begin{array}{ll} 1. & \mathbf{1}\ 4\ \mathbf{6}\ 2\ 3\ 5\ 7\ 8 \\ 2. & \mathbf{2}\ 5\ \mathbf{4}\ 1\ 3\ 6\ 7\ 8 \\ 3. & \mathbf{3}\ 6\ \mathbf{5}\ 1\ 2\ 4\ 7\ 8 \\ 4. & \mathbf{7}\ 1\ 4\ 5\ \mathbf{8}\ 2\ 3\ 6\,. \end{array}$$

Suppose $= (C_1, C_2, C_3, C_4)$ is envy-free. By Corollary 5.2, $|C_j| = 2$ for all j. By Lemma 5.1, C_j must contain person j's first choice and exclude j's last choice, so $j \in C_j$ for $j = 1, 2, 3$, and $C_4 = 78$. Then $C_4 >>_4 C_1$ requires $C_1 = 16$, and $C_3 >>_3 C_1$ requires $C_3 = 35$. Therefore $(16, 24, 35, 78)$ is the only envy-free division. But it is strongly dominated by $=$ $(14, 25, 36, 78)$. □

Clearly, unquestioned adherence to envy-freeness is untenable, because it can force the choice of a strongly dominated division. In the preceding proof, we note that the dominating divisions, and , are the unique maxsum and equimax divisions for the two examples. Because a strongly dominated division can be neither a maxsum nor an equimax division, the envy-free divisions, and , do not satisfy these properties.

We now consider other aspects of envy-freeness. Recall that is dominance-free if no j

and k have $A_k >>_j A_j$. Dominance-freeness is clearly necessary for envy-freeness or envy-possibleness, but it need not be sufficient.

THEOREM 5.6: *If $m = 2$, every dominance-free division is envy-free or envy-possible with respect to p. If $m \geq 3$, there are data profiles with dominance-free divisions that are envy-ensuring.*

PROOF: If $m = 2$ and $= (A_1, A_2)$ is dominance-free, then some preference profile in $W(p)$ has $A_1 \succ_1 A_2$ and $A_2 \succ_2 A_1$.

For $m = 3$, let p be given by

1. **1 2** 7 9 3 4 8 5 6
2. 2 3 **4 5 6 7** 1 9 8
3. 1 2 **3** 7 5 6 4 **8 9**.

Division $= (12, 4567, 389)$ is dominance-free, but it is envy-ensuring because person 3 always envies either person 1 or 2. Suppose to the contrary that $\succsim_3 \in W(>_3)$ satisfies

$$\begin{aligned} 389 &\succsim_3 12 \\ 389 &\succsim_3 4567\,. \end{aligned}$$

We also have $145 >>_3 389$ and $267 >>_3 389$, so $145 \succ_3 389$ and $267 \succ_3 389$. Transitivity gives

$$\begin{aligned} 145 &\succ_3 12 \\ 267 &\succ_3 4567\,. \end{aligned}$$

However, cancellation (Axiom 3) implies both $45 \succ_3 2$ and $2 \succ_3 45$, a contradiction to weak order.

Similar examples for $m \geq 4$ are easily constructed by enhancements of the preceding one. $\square$

Because dominance-freeness is necessary but not sufficient for envy-possibleness or envy-freeness when $m \geq 3$, we are interested in a condition stronger than dominance-freeness that is sufficient as well as necessary to preclude envy-ensuringness. We give such a condition only for the set of preference profiles whose $\succsim_j$'s are additively representable, but note also that this condition is sufficient with respect to $W(p)$.

Let $W(p)_{add}$ denote the set of all preference profiles $(\succsim_1, \ldots, \succsim_m)$ in $W(p)$ in which each

$\succsim_j$ is additively representable as

$$A \succsim_j B \Leftrightarrow \sum_{i \in A} w_j(i) \geq \sum_{i \in B} w_j(i) \ ,$$

where $w_j(a_{j1}) > w_j(a_{j2}) > \cdots > w_j(a_{jn}) > 0$ when $>_j = a_{j1}a_{j2}\cdots a_{jn}$. For $A \in 2^S$ and $i \in S$, let

$$A(i) = \begin{cases} 1 & \text{if } \ i \in A \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, for division $\ = (A_1, \ldots, A_m)$ and $j \in \{1, \ldots, m\}$, we say that A_j *is dominated by a convex combination of the other* A_k if there exist numbers

$$\mu_k \geq 0 \quad \text{for all} \quad k \in \{1, \ldots, m\} \setminus \{j\}$$

with

$$\sum_{k \neq j} \mu_k = 1\,,$$

such that, when $>_j = a_{j1}a_{j2}\cdots a_{jn}$,

$$\sum_{i=1}^{q} \sum_{k \neq j} \mu_k A_k(a_{ji}) \geq \sum_{i=1}^{q} A_j(a_{ji}) \quad \text{for} \quad q = 1, \ldots, n\,,$$

with strict inequality for at least one q.

LEMMA 5.7: *If $m \geq 3$, $p \in P$, and $\ \in \mathcal{A}$, then there is an additively representable weak order $\succsim_1$ in $W(>_1)$ for which*

$$A_1 \succsim_1 A_j \quad \textit{for} \quad j = 2, \ldots, m$$

if and only if A_1 is not dominated by a convex combination of A_2 through A_m.

PROOF: The proof is essentially the same as the proof of Theorem 4 in Edelman and Fishburn (2000). □

The following corollary is an easy consequence of Lemma 5.7 and $W(p)_{add} \subseteq W(p)$.

COROLLARY 5.8: *Suppose $m \geq 3$, $p \in P$, and $\ \in \mathcal{A}$. Then is envy-possible or envy-free in the context of $W(p)_{add}$ if and only if no A_j is dominated by a convex combination of the other A_k. In the context of $W(p)$, is envy-free or envy-possible if no A_j is dominated by a convex combination of the other A_k.* □

Example 2 in Edelman and Fishburn (2000) notes that division $\ = (\{1,2,3\}, \{6,7,8,9,10,11\}, \{4,5,12,13\})$ is dominance-free for the same preferences data profile

1. **1 2 3** 4 5 6 7 8 9 10 11 12 13
2. 1 2 3 4 5 **6 7 8 9 10 11** 12 13
3. 1 2 3 **4 5** 6 7 8 9 10 11 **12 13**

but that A_3 is dominated uniquely by the convex combination of A_1 with $\mu_1 = \frac{2}{3}$ and A_2 with $\mu_2 = \frac{1}{3}$. It follows that every additively representable $\succsim_3$ in $W(>_3)$ has a w_3 that satisfies

$$\frac{2}{3}w_3(A_1) + \frac{1}{3}w_3(A_2) > w_3(A_3) \ ;$$

hence, $A_1 \succ_3 A_3$ or $A_2 \succ_3 A_3$ for all preference profiles in $W(p)_{add}$. It seems plausible, however, that some $\succsim_3$ in $W(>_3)$ that is not additively representable can have $A_3 \succsim_3 A_1$ and $A_3 \succsim_3 A_2$, in which case we obtain an example with convex dominance and envy-possibleness. The example was designed to have $\mu_1 \neq \mu_2$, so the approach in the proof of Theorem 5.6 could not be used to derive a contradiction to $A_3 \succsim_3 A_1$ and $A_3 \succsim_3 A_2$ on the basis of Axiom 3 and Lemma 2.1.

We conclude this section by illustrating aspects of equal-shares divisions for a situation in which such a division is likely to be chosen.

Example 5.9: Let p be given by

1. 1 2 3 4 5 6
2. 1 4 2 3 5 6
3. 2 4 1 6 5 3 .

Interesting features of p include:

(a) no equal-shares division is envy-free or Pareto-ensuring;

(b) two equal-shares divisions are maxsum divisions, and both are envy-ensuring and Pareto-possible. Question marks are used below to indicate divisions that might Pareto-dominate others;

(c) there are two equimax divisions, and both are equal-shares divisions that are envy-possible and Pareto-possible;

(d) there are two other envy-possible equal-shares divisions besides the equimax divisions, and one of these is strongly dominated by an equimax division.

We list the six equal-shares divisions of (b), (c) and (d) along with their point totals and other comments. Verification of the properties claimed in (a)–(d) can be aided by prior results, including Lemma 4.5, Theorems 4.10 and 4.11, and Lemma 5.1. The two maxsum divisions of (b) are Pareto-ensuring in $\mathcal{A}_e$. This example well illustrates, we believe, the difficulty of declaring a division "most fair," which is a matter we return to at the end of the paper.

	division	point totals	sum	comments
(b)	(13, 45, 26)	(10, 7, 9)	26	$13 >>_2 45$. Pareto-dominated by $(345, 1, 26)$?
(b)	(35, 14, 26)	(6, 11, 9)	26	$14 >>_1 35$. Pareto-dominated by $(2, 14, 356)$?
(c)	(15, 34, 26)	(8, 8, 9)	25	Pareto-dominated by $(34, 15, 26)$?
(c)	(23, 15, 46)	(9, 8, 8)	25	Pareto-dominated by $(15, 23, 46)$?
(d)	(34, 15, 26)	(7, 8, 9)	24	Pareto-dominated by $(15, 34, 26)$?
(d)	(15, 23, 46)	(8, 7, 8)	23	strongly dominated by $(15, 34, 26)$.

6. MAXSUM AND EQUIMAX DIVISIONS

Among other findings, we have proved one positive and one negative result for maxsum and equimax divisions, namely that (1) every maxsum and equimax division is Pareto-ensuring or Pareto-possible (Theorem 4.10), and (2) there are data profiles with envy-free divisions, none of which is a maxsum or equimax division (Theorem 5.5). We note also that there are data profiles, for example $p = (123, 123, 123)$, for which all divisions are envy-ensuring, so we know that maxsum and equimax divisions cannot guarantee envy-freeness or envy-possibleness. Because all data profiles have maxsum and equimax divisions, and such divisions are often very attractive, we conclude our technical analysis with further remarks about them.

We begin with the observation that maxsum divisions can be very inequitable.

LEMMA 6.1: *For every $m \geq 2$ and all $n \geq 3$, there are data profiles all of whose maxsum divisions give one person all but one item.*

PROOF: Let $p = (12\cdots n, n12\cdots n-1, n12\cdots n-1, \ldots, n12\cdots n-1)$. Then every maxsum division has $A_1 = \{1, 2, \ldots, n-1\}$. $\square$

Given p, let $M(p)$ denote the maximum point-totals sum over all divisions in $\mathcal{A}$, and let $E(p)$ be the point-totals sum of the equimax vector for p. Clearly, $M(p) \geq E(p)$, and Example 1.1 is the smallest example for which $M(p) > E(p)$. Examples 2.3 and 5.9 also have $M(p) - E(p) = 1$. A small example with $M(p) - E(p) = 2$ follows.

EXAMPLE 6.2: Let p be given by

$$\begin{array}{ll} 1. & 1\,2\,3\,4 \\ 2. & 3\,4\,1\,2 \\ 3. & 1\,3\,2\,4. \end{array}$$

Then $M(p) = 4+3+4+3 = 14$. The unique equimax division is $(24, 3, 1)$ with equimax vector $(4, 4, 4)$, so $E(p) = 12$. $\square$

This raises the question of how large $M(p) - E(p)$ can be. We have verified by computer enumeration that, for $m = 2$, $\max[M(p) - E(p)] = 1$ for $n \in \{3, 4\}$, $\max = 2$ for $n \in \{5, 6\}$, $\max = 3$ for $n = 7$, and $\max = 4$ for $n = 8$. At $n = 8$, two different types of data profiles give $\max = 4$. We denote these as types I and II, and note a representative data profile for each:

$$\begin{array}{lllllllllll} \text{type I:} & 1. & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \qquad M(p) = 46 \\ & 2. & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 & \qquad E(p) = 42\,, \end{array}$$

$$\begin{array}{lllllllllll} \text{type II:} & 1. & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \qquad M(p) = 48 \\ & 2. & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & \qquad E(p) = 44\,. \end{array}$$

Our next theorem shows that, for $m = 2$, $\max[M(p) - E(p)]$ grows quadratically in n. It can also be shown for $m = 2$ that $\max[M(p) - E(p)]$ can be no greater than about $0.065n^2$ for $n \geq 6$, but we will not prove this here. In any event, because $\max M(p)$ itself is always less than $(3/4)n^2$, the lower bound on $\max[M(p) - E(p)]$ in the following theorem is as large as possible in order of magnitude.

THEOREM 6.3: *When $m = 2$, there is an integer n_0 such that* $\max[M(p) - E(p)] > 0.039n^2$ *for all $n \geq n_0$.*

PROOF: We use data profiles like the type II representative for $n = 8$. Given $m = 2$ and $n \geq 8$, let t be an integer near $n/4$ and let $p(t)$ be the following data profile:

$$\begin{array}{cccccccccc} 1. & 1 & 2 & \cdots & t & t+1 & \cdots & n-t & n-t+1 & \cdots & n \\ 2. & t+1 & t+2 & \cdots & 2t & 2t+1 & \cdots & n & 1 & \cdots & t & . \end{array}$$

By Lemma 4.8, the maxsum division $= (A_1, A_2)$ has

$$A_1 = \{1, \ldots, t\} \quad \text{and} \quad A_2 = \{t+1, \ldots, n\}\,.$$

Let $(t) = (E_1, E_2)$ be an equimax division for $p(t)$. If $i \leq t$, $j > t$, $i \in E_2$, and $j \in E_1$, then $(E_1 \cup \{i\}) \setminus \{j\} \succ_1 E_1$ and $(E_2 \cup \{j\}) \setminus \{i\} \succ_2 E_2$, which contradicts (t) as equimax. Because

E_1 must contain items from A_2, we conclude that $A_1 \subset E_1$. Therefore

$$\begin{aligned} E_1 &= A_1 \cup B \\ E_2 &= A_2 \setminus B \,, \end{aligned}$$

where B is a nonempty subset of A_2.

Let k be the largest integer for which $u_1(A_1 \cup C) < u_2(A_2 \setminus C)$ when $C = \{t+1, \ldots, t+k\}$. Then it must be true that $|B| \geq k$. Otherwise, we would have

$$u_1(E_1) < u_1(A_1 \cup C) \quad \text{because} \quad C >>_1 B$$

and

$$u_2(E_2) > u_2(A_2 \setminus C) > u_1(A_1 \cup C) \ ,$$

in which case $(A_1 \cup C, A_2 \setminus C)$ would be equimax-superior to (E_1, E_2), a contradiction. It follows that with k as defined here, we have $M(p) - E(p) = |B|t \geq kt$.

When the first k items in A_2 are transferred to person 1, so that $C = \{t+1, t+2, \ldots, t+k\}$, person 1's total for $A_1 \cup C$ is $(t+k)[n + (n-t-k+1)]/2 = (t+k)(2n-t-k+1)/2$, and person 2's total for $A_2 \setminus C$ is $(n-t-k)[(n-k-1+1)+(t+1)]/2 = (n-t-k)(n-k+t+1)/2$. The k value that equalizes the totals therefore satisfies

$$(t+k)(2n-t-k+1) = (n-t-k)(n-k+t+1) \,.$$

We solve this quadratic equation in k to obtain the equalization value k^* given by

$$k^* = n - \frac{t-1}{2} - \left[\frac{n(n+1)}{2} + \left(\frac{t+1}{2}\right)^2\right]^{1/2} \quad .$$

When k^* is not an integer, its use below actually understates the decrease from the maxsum total. The total decrease in the maxsum total with the use of k^* is

$$tk^* = t\left\{n - \frac{t-1}{2} - \left[\frac{n(n+1)}{2} + \left(\frac{t+1}{2}\right)^2\right]^{1/2}\right\} \leq M(p(t)) - E(p(t)) \,.$$

We simplify this by omitting its -1 and $+1$ terms:

$$tk^* \doteq t\left\{n - \frac{t}{2} - \left[\frac{n^2}{2} + \left(\frac{t}{2}\right)^2\right]^{1/2}\right\} \,.$$

As explained at the end of the proof, our final result is not affected by the simplification.

We now choose t, which will be near $n/4$, to maximize tk^*. For convenience, let $t = 2\lambda n$. When the right side of the preceding equation is differentiated with respect to t (or λ), we find that it is maximized when its derivative vanishes, i.e., when

$$n - \lambda n - \left(\frac{n^2}{2} + \lambda^2 n^2\right)^{1/2} = \lambda n \left[1 + \frac{\lambda n}{\left(\frac{n^2}{2} + \lambda^2 n^2\right)^{1/2}}\right].$$

This reduces to the cubic equation $16\lambda^3 - 4\lambda^2 + 8\lambda - 1 = 0$, whose relevant root is $\lambda_0 = 0.1290$. The corresponding t value is

$$t_0 = 2\lambda_0 n = 0.258n\,.$$

The value of $t_0 k^*$ by the simplified form is

$$\begin{aligned} t_0 k^* &= 0.258n \left\{ n - 0.129n - \left[\frac{n^2}{2} + (0.129n)^2\right]^{1/2} \right\} \\ &= 0.0393n^2\,. \end{aligned}$$

By decreasing this final figure, say to $0.039n^2$, and by taking n suitably large, simplification discrepancies become irrelevant, and we conclude for all sufficiently large n that

$$M(p) - E(p) > 0.039n^2$$

for some two-person data profile p on n items. □

Our final observation is that $M(p) > E(p)$ is possible even when all maxsum and equimax divisions are equal-shares divisions. We leave open the question of whether $M(p) - E(p)$ can become arbitrarily large for fixed m in such cases.

THEOREM 6.4: *There are data profiles with $M(p) > E(p)$ whose maxsum and equimax divisions are all equal-shares divisions.*

PROOF: Consider p given by

1. 1 2 3 4 5 6 7 8 9
2. 4 1 2 3 6 8 5 7 9
3. 5 1 7 2 3 9 4 6 8 .

The unique maxsum division (Lemma 4.8) is clearly $(123, 468, 579)$, with point-totals vector $(24, 18, 20)$ and $M(p) = 62$. There is also a unique equimax division, namely $(136, 248, 579)$, with equimax vector $(20, 20, 20)$ and $E(p) = 60$. Equimax verification follows from the fact that $60 \leq E(p) \leq 61$ according to $(20, 20, 20)$, and $M(p) = 62$ for a non-equimax division. With an equimax division, $E(p) \in \{60, 61\}$ requires $4 \in A_2$, $\{5, 7, 9\} \subseteq A_3$ and $\{6, 8\} \cap A_3 = \phi$. To get $u_1(A_1) \geq 20$ and $u_2(A_2) \geq 20$, we require $\{1, 2, 3\} \subseteq A_1 \cup A_2$, so $A_3 = \{5, 7, 9\}$. We then need $1 \in A_1$, $2 \in A_2$, $6 \in A_1$ and $8 \in A_2$. $\square$

7. DISCUSSION

Our purpose has been to analyze criteria for fair division of indivisible items among people when their revealed preferences consist of rankings of items and no side payments are allowed. The criteria we considered include refinements of Pareto optimality and envy-freeness as well as dominance-freeness, evenness of shares, and two criteria based on point totals of equally spaced surrogate utilities. The first of those two, maxsum, maximizes the sum of the persons' point totals. The second, equimax, maximizes the smallest point total, then maximizes the next-smallest point total subject to first maximization, and so forth.

Although maxsum divisions tend to have high aggregate utility and good Pareto-optimal properties, they can be very inequitable and induce unnecessary envy. While equimax divisions may have slightly smaller point-totals sums, they also do well with respect to Pareto optimality and, in addition, tend to be more equitable, engender less envy, and give people approximately even shares.

When a situation requires an even-shares division, equimax and perhaps maxsum within the even-shares set could identify good possibilities. Other constraints, such as giving each person at least one item near the top of his or her ranking, can be combined with other criteria to select a reasonably fair division that honors the constraints.

Situations may differ significantly in the criteria judged to be important as well as in the types of data profiles that may arise. As a consequence, different allocation schemes will be more suitable for some situations than for others. Because of this, our objective has been to elucidate the promises and pitfalls of different criteria, both singly and in combination, as a guide to the evaluation and selection of fair divisions under a variety of circumstances.

We conclude that important criteria can conflict, as Example 5.9 readily demonstrates. In the absence of an equal-shares envy-free or Pareto-ensuring division in this example, there is a choice between two different maxsum and two different equimax divisions. It is evident not only from this example but from our general analysis that fair division is an intellectually demanding problem. Because it has, in addition, far-reaching real-world consequences (Brams and Taylor (1999)), its theoretical foundations are surely worthy of a sustained effort of building and reconstruction.

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