ECONOMIC RESEARCH REPORTS

A NEW CLASS OF SOLUTIONS TO DYNAMIC PROGRAMMING PROBLEMS ARISING IN GROWTH THEORY AND APPLICATIONS TO DYNAMIC GAMES

by

Jess Benhabib
and
Aldo Rustichini

R.R. # 90-12 March, 1990

C. V. STARR CENTER
FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

A New Class of Solutions to Dynamic Programming Problems Arising in Growth Theory and Applications to Dynamic Games

by

Jess Benhabib
Department of Economics
New York University
New York, NY 10003

AND

Aldo Rustichini
Department of Economics
Northwestern University
Evanston, IL 60208

* We would like to thank the C.V. Starr Center at New York University for research facilities and technical assistance.

ABSTRACT

We provide exact solutions for a class of stochastic dynamic programming problems in growth theory involving pairs of constant relative risk aversion utility functions and multi-sector CES technologies. This generalizes the solutions for the well-known case of logarithmic utility coupled with Cobb-Douglas production functions. We are also able to incorparate depreciation schemes through a vintage capital approach We then study applications of our results to dynamic games JEL Classfication Numbers: 020, 026, 111. Key words: Dynamic programming, dynamic games, growth theory

1. Introduction

Many problems in economic theory can be formulated in terms of dynamic programming. Since general analytic solutions are not available, one often uses particular functional forms amenable to analytical solution. In finance, intertemporal problems of portfolio choice involve linear budget constraints and some reasonable functional forms for preferences can be chosen to obtain explicit solutions. In growth theory however, the production functions forming the constraint set are typically nonlinear About the only type of problem that can be explicitly solved in the growth context is the one with logarithmic preferences and Cobb-Douglas technology, and such a specification of technology is not compatible with less than a hundred percent depreciation rate (This is so because less than full depreciation adds a linear component to the production function.) A successful application of this framework to a stochastic multisector growth model is given by Long and Plosser 1983]

In this paper we propose to extend the class of solvable specifications in growth problems to pairs of constant relative risk aversion utility functions coupled with CES production functions. The results generalize the log utility and Cobb-Douglas production specification in several ways. First, in the single capital good case where the production function is subject to a stochastic shock following a Markov process, the propensities to save are not constants but random variables that depend on realizations of the shock Nevertheless the propensities are still independent of the stock of capital Second, in the multisector case where elasticities of substitution differ across goods, the propensities to save, the equilibrium quantity of labor and the proportions of factors allocated across goods also depend on the capital stock vector Finally, we can allow for a variety of vintages of capital that differentiates equipment by age, and in this manner we can introduce depreciation schemes along the lines suggested by Radner [1966] Explicit solutions can be obtained without recourse to quadratic approximations and the model can then be simulated on the computer

In section three we apply the solution techniques of the earlier sections to dynamic games of the type investigated by Levhari and Mirman [1980]. We are interested in trigger strategy equilibria where punishments for deviating from cooperation consist of reversion to stationary equilibria. A particular type of powerful punishment consists of extreme stationary equilibria where agents consume all that they can, thereby exhausting the stock In the log utility Cobb-Douglas specification such extreme actions lead to equilibrium states of zero consumption and infinitely negative utility In our framework elasticities of substitution smaller than unity generate much less drastic (and maybe more plausible) trigger strategies. The enforcability of cooperative behavior from some initial states but not from others becomes an issue of particular interest, which we can study in some detail We also investigate and discuss the possibility of "switching equilibria" studied in Benhabib and Radner [1988], where cooperative behavior is eventually but not immediately enforced along an equilibrium path. The parameterization of our model by the substitution elasticities turns out to be quite useful for a heuristic understanding of the nature of trigger strategy equilibria, as discussed at the end of section three

The last section offers tentative conjectures about policy functions corresponding to more general specifications than the ones that we have considered

An analytical solution for the simple stochastic case when capital lasts two periods.

For ease of exposition we will start with a simple stochastic growth model The representative agent has a separable utility function given by $U(c,L) = Ac^{1-\epsilon} + W(1-L) = u(c) + W(1-L)$ where $A = A'/(1-\epsilon)$, A', $\epsilon > 0$ and W is a concave function representing the utility of leisure Total time endowment is normalized to one, L is labor and c is consumption

The production function is of CES type We will treat depreciation along the lines suggested by Radner 1966], who adopted a vintage structure. New equipment k turns into depreciated one-period old equipment $\mu_1 k$ after one period, into two-period old equipment $\mu_2\mu_1 k$ after two periods and so on $(1- \mu_i)$ is the depreciation rate for equipment that is i-1 years old Thus any arbitrary depreciation scheme is possible. The treatment of production however differentiates this scheme from the standard aggregative treatments The production function is of the CES type, given by

$$y = z\left[a_1 k_1^{1-\epsilon} + a_2 k_2^{1-\epsilon} + (1-a_1-a_2)L^{1-\epsilon'}\right]^{1/(1-\epsilon')}$$

where ϵ' a_1, $a_2 > 0$, k_1 is new equipment and k_2 is one-year old equipment The multiplicative factor z can be taken as the observed shock which follows some arbitrary stochastic process. For simplicity we will assume that z follows a stationary first order Markov process so that $z_{t+1} = z_t^{\lambda} e_t$, where e_t is lognormally distributed for all t and $0 \leq \lambda < 1$.

For simplicity we also assume that capital lasts two periods only, that it depreciates at rate $(1-\mu_1)$ at the end of the first period, and that new and old machines enter production as separate capital goods, rather than as a simple weighted aggregate. This specification implies that new and old machines are complements rather than substitutes. (For a more traditional specification where capital goods of different vintages are substitutes, see Benhabib and Rustichini [1989].) In this framework, a_1 and a_2 can be chosen to reflect the relative importance of each vintage in the process of production. This formulation follows Radner [1966], except that the production function is Cobb-Douglas and the utility function is logarithmic in the Radner formulation. Since factors enter multiplicatively in the Radner formulation, positive production requires all vintages to be present in positive amounts.

Analytic solutions for dynamic programming growth problems are known for some special cases. The basic cases are the constant relative risk aversion utility functions for consumption (or more generally, those of the HARA type) coupled with linear technologies, generally used for portfolio problems; the simple linear utility functions coupled with arbitrary production functions; and the well-known log utility coupled with Cobb-Douglas production. If we let ϵ_u and ϵ_ρ be the elasticities of substitution for the constant-relative risk aversion utility functions and the CES production functions respectively, the diagram below represents what is already known and what we propose to add The vertical axis (linear technology), the horizontal axis (linear utility), and the point (1,1) represents cases that we already know how to solve. The diagonal represents the new cases for which we propose a solution

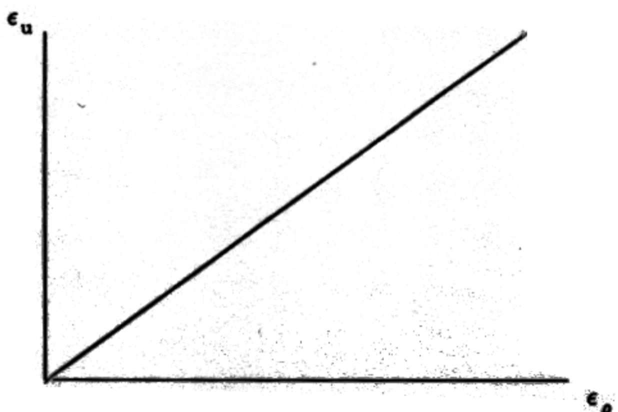

Figure 1

(The multisector case requires a generalization and will be presented later.) Therefore we set the elasticity of substitution for utility given by ϵ, equal to the elasticity of substitution for production, given by ϵ' The dynamic programming problem then becomes, given the current equipment level k_1, the one-year old equipment level k_2, and the realization of the shock z_1,

$$V(k_1,k_2,z_1) = \max_{c,L} Ac^{1-\epsilon} + W(1-L) + \delta EV\Big\{z_2\Big[a_1k_1^{1-\epsilon} + a_2K_2^{1-\epsilon} + (1-a_1-a_2)L^{1-\epsilon}\Big]^{1/(1-\epsilon)} - c, \mu k_1, z_2\Big\}$$

subject to $0 \le c \le z_2\Big(a_1k_1^{1-\epsilon} + a_2k_2^{1-\epsilon} + (1-a_1-a_2)L^{1-\epsilon}\Big)$ and $0 \le L \le 1$. The expectation involving the random variable z_2 is taken with respect to the realization of z_1.[1]

[1] Sufficient conditions for the value function to be well-defined would

We briefly outline the computations used to derive the policy function. Throughout the paper, we will restrict our attention to characterizing interior solutions. Straightforward modifications needed for characterizing solutions on the boundary are left to the reader. The first order conditions with respect to consumption and labor are given by

$$A(1-\epsilon)c^{-\epsilon} = \delta EV_1'$$

$$\frac{dW}{dL} = \delta EV_1 \cdot MPL_1$$

where MPL_j is the marginal product of labor in the jth good: in the one sector model used above of course j=1. Similarly MPK_{ij} will denote the marginal product of the equipment of vintage i in the production of good j. The notation $'$ and $''$ denote the number of periods ahead at which the variable is evaluated. Hence V_1' is evaluated at the values that variables take on next period, V_1'' at values two periods from now and so on. Differentiating V we obtain:

$$V_2 = \delta EV_1' \cdot MPK_{21} = A(1-\epsilon)c^{-\epsilon} \cdot MPK_{21}$$

$$V_1 = \delta EV_1' \cdot MPK_{11} + \mu\delta EV_2' = A(1-\epsilon)c^{-\epsilon} \cdot MPK_{11} + \mu\delta EA(1-\epsilon)(c')^{-\epsilon} \cdot MPK_{21}'.$$

---

require some assumptions to assure boundedness. It can be shown (see Benhabib and Rustichini [1989]) that the total value will be bounded if $(a_1)^{1/\delta}$, $\delta < 1$ and $\theta < 1$ where θ is the autoregressive component of a first-order stochastic process for z: that is $z_{t+1} = z_t^{\theta}$.

Updating and substituting into the first order condition for consumption we get the equation:

$$A(1-\epsilon)c \quad = \delta A(1-\epsilon)E\left[(c')^{-\epsilon}\cdot MPK_{11}'\right] + \mu\delta^2 A(1-\epsilon)E\left[(c'')^{-\epsilon}\cdot MPK_{21}''\right]$$

equation has the standard interpretation that the marginal utility of a consumption unit today equals the discounted sum of utilities that it will produce in the future. We note that for the CES case MPK_{i1} $z^{1-\epsilon}\, a_i(y/k_i)^{\epsilon}$. Let the candidate solution for the policy function be given by $c = \lambda y$. Then $k_1' = (1-\lambda)y$ and $k_2'' = \mu k_1' = \mu(1-\lambda)y$. Substituting these into equation (*) and solving for λ at time zero we obtain:

$$(1-\lambda) = \left[a_1\delta E(z_1{}^{1-\epsilon}|z_0) + a_2\delta^2\mu^{1-\epsilon}E(z_2{}^{1/\epsilon}|z_0)^{1-\epsilon}\right]$$

For the case of log utility and Cobb-Douglas production where $\epsilon=1$ and no vintages ($a_2=0$), this reduces to the standard result that the marginal propensity to consume is equal to $(1-a_1\delta)$, a constant. We also obtain that the labor supply at time zero is the solution to:

$$\frac{dW}{dL} = A(1-\epsilon)(1-a_1-a_2)z_0^{1-\epsilon}\lambda^{-\epsilon}L^{-\epsilon}$$

Choosing a function W(1-L) yields the quantity of labor supply.

The solution to the dynamic programming problem is therefore given by the expressions for the marginal propensity to save and the labor supply each period which are random variables independent of the capital stocks

We will see below that this will no longer be the case when there is more than one produced capital good. (Note of course that if z's are iid, then the marginal propensity to save is constant for the one-sector model presented above.)

The dynamics of growth can also be represented by a simple linear difference equation. We have $k_{t+1} = (1-\lambda)y_t$. Let $x_t = k_t^{1-\epsilon}$ for all t. Then the dynamics are given by

$$x_{t+1} = \left[z_t(1-\lambda)\right]^{1-\epsilon}(a_1x_t + \mu^{1-\epsilon}a_2x_{t+1} + (1-a_1-a_2)L)$$

where L is the constant optimal labor supply.

We should also note that it is possible to slightly extend the above case to a HARA utility function. Consider for simplicity the case without the vintage (a_2=0). Then let $u(c) = A(c+\eta)^{1-\epsilon}$ where η is a constant and let the corresponding CES production function be given by

$$y_t = z_t\left[a_1(k+\gamma)^{1-\epsilon} + (1-a_1)L^{1-\epsilon}\right]^{1/(1-\epsilon)}$$

where $\gamma = -\eta$. Then it is easily shown that the optimal policy is $c = \lambda y + d$ where $\lambda = 1-(\delta a_1)^{1/\epsilon}$ and that $d = -\eta = \gamma$. Thus a solution for this case requires utility and production function pairs for which not only the substitution elasticites are identical but for which the constants η and γ add up to zero. In such a case either the production function or the utility function will not be well defined for c or k values that are sufficiently low.

The value function $V(k_1, k_2, z_1)$ can also be computed for the problem presented above. We postpone the derivation of the value function to the section on applications to game theory below.

3. An analytic solution for a two-sector model when capital lasts three periods.

In this section we derive the optimal policies for a two-sector model where both goods are consumable and where capital goods last for three periods. Formally, this means that there are six capital goods that enter each production function. Furthermore, the allocation of each capital good to the production of the two new goods must be determined. We denote by f_{i1} the proportion of capital good i allocated to the production of good 1: it follows that the proportion $(1-f_{i1})$ is allocated to the second good. f_{0i} denotes the proportion of the allocation of labor to the production of the first good. Since the stochastic case was covered for the simpler model in the previous section and would carry over in precisely the same manner, for simplicity of notation we initially restrict our attention to the deterministic case. The dynamic programming problem can then be formulated as

$$V(k_1,k_2,k_3,k_4,k_5,k_6) = \text{Max } A_1c_1^{1-\epsilon_1} + A_2c_2^{1-\epsilon_2} + W(g_t - L)$$

$$+\delta V\left\{\left\{\left[\alpha_{11}(f_{11}k_1)^{1-\epsilon_1} + \alpha_{21}(f_{21}k_2)^{1-\epsilon_1} + \alpha_{31}(f_{31}k_3)^{1-\epsilon_1} + \alpha_{41}(f_{41}k_4)^{1-\epsilon_1}\right.\right.\right.$$

$$\left.\left.\left. + \alpha_{51}(f_{51}k_5)^{1-\epsilon_1} + \alpha_{61}(f_{61}k_6)^{1-\epsilon_1} + \alpha_{01}(f_{01}L)\right]^{1-\epsilon_1}\right\}^{1/(1-\epsilon_1)} - c_1,\right.$$

$$\Big\{\alpha_{12}\Big[(1-f_{11})k_1\Big]^{1-\epsilon_2} + \alpha_{22}\Big[(1-f_{21})k_2\Big]^{1-\epsilon_2} + \alpha_{32}\Big[(1-f_{31})k_3\Big]^{1-\epsilon_2}$$

$$+ \alpha_{42}\Big[(1-f_{41})k_4\Big]^{1-\epsilon_2} + \alpha_{52}\Big[(1-f_{52})k_5\Big]^{1-\epsilon_2} + \alpha_{62}\Big[(1-f_{62})k_6\Big]^{1-\epsilon_2}$$

$$+ \alpha_{02}\Big[(1-f_{01})L\Big]^{1-\epsilon_2}\Big\}^{1/(1-\epsilon_2)} - c_2,\ \mu_1 k_1,\ \mu_2 k_2,\ \mu_3 k_3,\ \mu_4 k_4\Big\}$$

where $\sum_{i=0}^{6}\alpha_{i1} = \sum_{j=0}^{6}\alpha_{j2} = 1$ and where the maximization is with respect to f_{ij}'s, c_1, c_2 and L. The constraints are $0 \le f_{ij} \le 1$, $0 \le L \le g_t$ and $0 \le c_s \le y^s$, where y^s is the output of good s, s = 1, 2, given by the corresponding CES production function. g_t represents the endowment of time, which may be growing and is not necessarily a constant. $(1-\mu_i)$ represents the depreciation of the ith capital good. Note that k_i this period becomes $\mu_i k_i = k_{i+2}$ in the subsequent period.

As in the one-sector case of the previous section, we require the elasticity of substitution of the utility of consumption to be equal to the elasticity of substitution of the corresponding production function. In this two-sector case or in its multisector generalization however, we do not require the elasticities to be equal to each other across goods: thus we can have $\epsilon_1 \neq \epsilon_2$. This is a major difference from the multisector formulations using log utility and Cobb-Douglas production for which all elasticities must equal unity.

After some manipulation, the first order conditions for the consumption goods can be written as follows:

$$A_1(1-\epsilon_1)c_1^{-\epsilon_1} = \delta A_1(1-\epsilon_1)(c_1')^{-\epsilon_1}\cdot MPK_{11}' + \delta^2\mu_1 A_1(1-\epsilon_1)(c_1'')^{-\epsilon_1}\cdot MPK_{31}'' + \delta^3\mu_3\mu_1 A_1(1-\epsilon_1)(c_1''')^{-\epsilon_1}\cdot MPK_{51}'''$$

$$A_2(1-\epsilon_2)c_2^{-\epsilon_2} = \delta A_2(1-\epsilon_2)(c_2')^{-\epsilon_2}\cdot MPK_{22}' + \delta^2\mu_2 A_2(1-\epsilon_2)(c_2'')^{-\epsilon_2}\cdot MPK_{42}'' + \delta^3\mu_4\mu_2 A_2(1-\epsilon_2)(c_2''')^{-\epsilon_2}\cdot MPK_{62}'''$$

The interpretation of the above conditions is standard, and as before: the marginal utility of consumption in the current period equals the discounted sum of utilities that a unit of a capital good can produce during its life. Note that MPK_{ij} denotes the marginal product of capital good i in the production of good j.

If we set $c_1 = \lambda_1 y^1$ and $c_2 = \lambda_2 y^2$, we can simplify the first order conditions for the whole system to the following equations:

$$(1-\lambda_1)^{\epsilon_1} = \delta\alpha_{11}f_{11}^{-\epsilon_1} + \delta^2\alpha_{31}\mu_1^{1-\epsilon_1}f_{31}^{-\epsilon_1} + \delta^3\alpha_{51}(\mu_1\mu_3)^{1-\epsilon_1}f_{51}^{-\epsilon_1}$$

$$(1-\lambda_2)^{\epsilon_2} = \delta\alpha_{22}f_{21}^{-\epsilon_2} + \delta^2\alpha_{42}\mu_2^{1-\epsilon_2}(1-f_{41})^{-\epsilon_2} + \delta^3\alpha_{61}(\mu_4\mu_2)^{1-\epsilon_2}(1-f_{61})^{-\epsilon_2}$$

$$\frac{A_1(1-\epsilon_1)\lambda_2^{\epsilon_2}}{A_2(1-\epsilon_2)\lambda_1^{\epsilon_1}} = \left\{\frac{\alpha_{i2}f_{i1}^{\epsilon_1}}{\alpha_{i1}(1-f_{i1})^{\epsilon_2}}\right\} k_i^{\epsilon_1-\epsilon_2} \qquad \text{for } i = 0, 1, \ldots, 6$$

$$W'(g_t - L) = \frac{A(1-\epsilon_1)}{\lambda_1^{\epsilon_1}} \cdot \frac{\alpha_{01}}{f_{01}^{\epsilon_1}} \quad L^{-\epsilon_1}$$

where $k_0 \equiv L$. The above 10 equations can then be solved for the 10 variables f_{i1} (i = 0, 1, ..., 6), c_1, c_2 and L.

We first note that unless $\epsilon_1 = \epsilon_2$, the solution will depend on the state vector $(k_1, k_2, .., k_6)$. Thus λ_1, λ_2, f_{i1} and L are in general not independent of the state variable. Second, an explicit solution is not generally possible so that a non-linear solver will be required to solve the above equations at every step along the optimal path. The solution will nevertheless be exact and will not involve approximations. Given the parameters of the system, the optimal paths for the capital goods and the evolution of λ_1, λ_2, f_{i1} and L can be computed.

We can easily modify the above derivations along the lines of the previous section to introduce stochastic shocks and allow the shocks to differ across goods. For example, for the case where the two capital goods last only one period ($\mu_1=\mu_2=\mu_3=\mu_4=0$) and z_j^i is the multiplicative shock to the ith good in period j, the optimal policies are given by the six equations:

$$(1-\lambda_1) = \left\{\left\{\alpha_{11}\delta E\left[(z_1^1)^{1-\epsilon_1} \,|\, z_0^1\right]\right\}^{1/\epsilon_1}\right\}/f_{11}$$

$$(1-\lambda_2) = \left\{\left\{\alpha_{21}\delta E\left[(z_1^2)^{1-\epsilon_2} \,|\, z_0^2\right]\right\}^{1/\epsilon_2}\right\}/(1-f_{21})$$

$$Q \equiv \frac{A_1(1-\epsilon_1)\lambda_1^{-\epsilon_1}}{A_2(1-\epsilon_2)\lambda_2^{-\epsilon_2}} = \left\{\frac{\alpha_{12}\left[(z_0^2)^{1-\epsilon_2}\right]f_{11}^{\epsilon_1}}{\alpha_{11}\left[(z_0^1)^{1-\epsilon_1}\right](1-f_{11})^{\epsilon_2}}\right\} \cdot k_1^{\epsilon_1-\epsilon_2}$$

$$Q \equiv \left\{\frac{\alpha_{22}\left[(z_0^2)^{1-\epsilon_2}\right](1-f_{21})^{\epsilon_2}}{\alpha_{21}\left[(z_0^1)^{1-\epsilon_1}\right](f_{21})^{\epsilon_1}}\right\} \cdot k_2^{\epsilon_2-\epsilon_1}$$

$$Q = \left\{ \frac{\alpha_{02}\left[(z_0^2)^{1-\epsilon_2}\right](1-f_{01})^{\epsilon_2}}{\alpha_{01}\left[(z_0^1)^{1-\epsilon_1}\right](f_{01})^{\epsilon_1}} \right\} \cdot L^{\epsilon_2-\epsilon_1}$$

$$W'(g_t - L) = \left\{ \frac{A_1(1-\epsilon_1)\alpha_{01}(z_0^1)^{1-\epsilon_1}}{\lambda_1^{\epsilon_1}(f_{01}L)^{\epsilon_1}} \right\}$$

Note again that policies will be independent of stock levels if $\epsilon_1=\epsilon_2$. The above framework also allows a portfolio analysis of the effects of differential changes in the riskiness and in the mean returns of the two assets, which we do not pursue further in this paper.

4. Applications to Dynamic Games.

In this section we apply the results of the previous section to theory of dynamic games. The basic model is that of Levhari and Mirman [1980], extended to allow for history-dependent trigger strategies. The model has two players, whose total utilities are given by

$$\sum_0^\infty U(c^i)\beta^t \quad , \qquad i = 1,\ 2.$$

The accumulation equation for capital is

$$k_{t+1} = f(k_t) - c_t^1 - c_t^2 \qquad , \ k_0 \text{ given.}$$

Of course, the constraints c^1, c^2, $k \geq 0$ must hold at all times. We define $y_t = f(k_t)$.

The strategy of player i will be represented by a function which maps the state (y_t, y_{t+1}) into consumption c_t^i : $c_t^i = h^i(y_t, y_{t+1})$. This strategy represents a limited amount of history dependence, which for our purposes will be sufficient. We also define the special Markov or stationary strategies as $c_t^i = h_s^i(y_t)$. A Nash equilibrium will be represented by a pair of strategies, one for each player, such that neither player can improve his total payoff by a unilateral change in his strategy at any point in the game.

We set the utility function of player i to be $(1/(1-\epsilon)(c^i)^{1-\epsilon}$ and the production function to be $y = f(k) = (\alpha_1 k^{1-\epsilon} + \alpha_0 \ell^{1-\epsilon})^{1/(1-\epsilon)}$, where $\alpha_0 = 1-\alpha_1$. Here ℓ is the total labor supply so we can assume that each agent is endowed with $^1/_2\ \ell$. We note that since there is no disutility of labor, it cannot be an equilibrium strategy for either player to withold any labor, even if the opposing player were to withold his labor, as long as the player can capture some part of the total output. Since the total supply of labor is fixed, we can treat ℓ as a constant.

We now compute the cooperative solution, assuming each agent is given $(^1/_2)$ weight. The value function is given by:

$$V(k) = \max_{c^1, c^2} \left\{ {}^1/_2\left[1/(1-\epsilon)(c^1)^{1-\epsilon}\right] + {}^1/_2\left[1/(1-\epsilon)(c^2)^{1-\epsilon}\right] + \beta V\left[(\alpha_1 k^{1-\epsilon} + \alpha_0 \ell^{1-\epsilon})^{1/(1-\epsilon)} - c^1 - c^2\right]\right\}$$

The solution for c^i is given by

$$(c^i)^{-\epsilon} = \beta\alpha_1 \left[(c^i)'\right]^{-\epsilon} (y'/k')^{\epsilon}$$

where primes denote variables evaluated one period ahead. If we let c^i λy and then note that $k_1' = (1-2\lambda)y$, we obtain as a solution:

$$\lambda = 1/2\left[1-(\beta\alpha_1)^{1/\epsilon}\right]$$

Since in equilibrium, as can be ascertained from first order conditions, $c^1 = c^2$ at each moment in time, the value V(k) will also correspond to the total utility that a single player obtains under cooperation If we set $V(k) = sk^{1-\epsilon} + I$, we obtain (since $c^1 = c^2$)

$$V(k) = sk^{1-\epsilon} + I = (1/(1-\epsilon))\lambda^{1-\epsilon}(\alpha_1 k^{1-\epsilon} + \alpha_0 \ell^{1-\epsilon})$$

$$+ \beta s\left[(1-2\lambda)^{1-\epsilon}(\alpha_1 k_1^{1-\epsilon} + \alpha_0 \ell^{1-\epsilon})\right] + \beta I.$$

Equating coefficients and solving, we obtain

$$s = \frac{(1/(1-\epsilon))\lambda^{1-\epsilon}\alpha_1}{1 - \beta(1-2\lambda)^{1-\epsilon}\alpha_1} = \left[1-(\beta\alpha_1)^{1/\epsilon}\right] \quad \alpha_1 (1/2)^{1-\epsilon}(1/(1-\epsilon)$$

$$I = s(\alpha_0/\alpha_1)\left[1/(1-\beta)\right]\ell^{1-\epsilon}$$

Of course the consumption strategies required by cooperation are not equilibrium strategies unless they can be sustained by some threats

We now compute an equilibrium in stationary strategies. Each players' value function is defined as

$$V_s(k) = \max_c \left\{(1/(1-\epsilon))c^{1-\epsilon} + \beta V\left[(1-\lambda_s')(\alpha_1 k_0^{1-\epsilon}+\alpha_0 \ell^{1-\epsilon})^{1/(1-\epsilon)} - c\right]\right\}$$

where λ_s' is the proportion of output appropriated by the opposing player who is following a stationary strategy. If we set the stationary strategy of the player as $c = \lambda_s y$, we can compute the stationary equilibrium solution with the same technique used for the cooperative case We obtain the two equations

$$(1-\lambda_s-\lambda_s')^{\epsilon} = \beta\alpha_1(1-\lambda_s')$$

$$(1-\lambda_s'-\lambda_s)^{\epsilon} = \beta\alpha_1(1-\lambda_s)$$

which imply that $\lambda_s' = \lambda_s$ where λ_s is the solution to

$$(1-2\lambda_s)^{\epsilon} = \beta\alpha_1(1-\lambda_s).$$

The value function for a player in a stationary equilibrium will be given by

$$V_s(k) = s_s k^{1-\epsilon} + I_s$$

where

$$s_s = \frac{\left[1/(1-\epsilon)\right]\lambda_s^{1-\epsilon}\alpha_1}{1 - \beta\alpha_1(1-2\lambda_s)^{1-}}$$

$$I_s = \left[1/(1-\beta)\right](\alpha_0/\alpha_1)\ell^{1-\epsilon}s_s .$$

Since cooperation dominates the stationary equilibrium for all k, it follows that $s \geq s_s$, $I \geq I_s$.

The above analysis for stationary equilibria can also be generalized to the case of multiple capital goods, using the same techniques as in the previous section. The consumption propensity for good i would be given by λ_s^i which solves $(1-2\lambda_s^i) = \beta\alpha_{ii}(1-\lambda_s^i)$, where α_{ii} is the coefficient of the capital good i in the CES production function of the i'th good. This still leaves the question of the allocation of each capital stock to the production of the various goods. If one player was assigned the task, his dominant strategy would be to choose the allocations efficiently, as in the previous section. The derivation of the allocation proportions f_{ij} would also essentially follow the rules of the previous section.

We now turn to trigger strategies. Let the strategy of each player be defined by

$$c_t^i = \begin{cases} \lambda y_t & \text{if} \quad k_t = (1-2\lambda)y_{t-1} \\ \lambda_s y_t & \text{otherwise} \end{cases}$$

Thus player i follows the cooperative strategy if both himself or the opposing player has followed the cooperative strategy in the previous period. Otherwise he reverts to the stationary strategy. The value of optimally defecting from cooperation is therefore given by

$$V_{SD} = \underset{c}{\text{Max}} \left\{ (1/(1-\epsilon))c^{1-\epsilon} + \beta s_s \left[(\alpha_1 k_1^{1-\epsilon} + \alpha_0 \ell^{1-\epsilon})^{1/(1-\epsilon)}(1-\lambda) - c \right]^{1-\epsilon} + \beta I_s \right\}$$

of course that the trigger strategies defined above imply that if a defection takes place once, the players will revert to stationary strategies forever The optimal defection value for consumption is given by

$$c = (1-\lambda)\lambda_{SD} y$$

$$\text{where } \lambda_{SD} = \frac{\left[\beta s_s (1-\epsilon)\right]^{-1/\epsilon}}{1 + \left[\beta s_s (1-\epsilon)\right]^{-1/\epsilon}}$$

The value of defection, $V_{SD}(k)$ is given by

$$= s_{SD} k^{1-\epsilon} + I_{SD}$$

where

$$s_{SD} \quad \left[(1/(1-\epsilon)\right]\lambda_{SD}{}^{1-\epsilon}(1-\lambda)^{1-\epsilon}\alpha_1 + \beta s_s (1-\lambda)^{1-\epsilon}(1-\lambda_{SD})^{1-\epsilon}\alpha_1$$

$$I_{SD} = \left[1/(1-\beta)\right]\left[\alpha_0 \ell^{1-\epsilon}/\alpha_1\right] s_{SD}$$

The issue that arises in this case is whether V and V_{SD} intersect: that is, can cooperation be enforced with trigger strategies from some states but not for others? In particular we can inquire if cooperation is feasible for $k \geq \bar{k}$ ($k \leq \bar{k}$) but not for $k < \bar{k}$ ($k > \bar{k}$) where $\bar{k} > 0$ is some critical value. These possibilities can be ruled out by simple inspection of s, s_{SD}, I and I_{SD}. If $s > s_{SD}$, then it follows that $I > I_{SD}$, and conversely, if $s < s_{SD}$, it follows that $I < I_{SD}$, so that V and V_{SD} do not intersect. We can express this with the following Lemma:

Lemma 1: If cooperation is enforcable (unenforcable) under the threat of reverting to the stationary equilibrium from some k, it is enforcable (unenforcable) from all k.

Another type of trigger strategy involves much stronger punishments, where a player tries to consume all the output that he can. The best response for the opposing player is to also consume as much as he can. We will assume that if players try to consume more than the available output they will share what is available equally, unless one player tries to consume less than half of what is available In such a case that player gets the full amount he is aiming for, and the opposing player gets the remainder. Of course in equilibrium neither player will be satisified to see the other player get more than half of the output and thereby exhaust what is available. Strategies for which agents try to consume all the available stock will be termed extreme More formally, a trigger strategy involving extreme punishments can be expressed as

$$c_t = \begin{cases} \lambda y_t & \text{if} \quad k_t = (1-2\lambda)y_{t-1} \\ (1-\lambda')y_t & \text{otherwise} \end{cases}$$

where λ' is the proportion of output consumed by the other player. Of course the sharing rule states that if $\lambda' \geq 1/2$, then the realized consumption $c_t = (^1/_2)y_t$

The value of defecting from cooperation under the threat of an extreme punishment is given by

$$V_D(k) = \underset{c}{\mathrm{Max}}\Big\{\Big[1/(1-\epsilon)\Big]\cdot c^{1-\epsilon} + (\beta/(1-\epsilon))\Big\{1/2\Big[\alpha_1\Big[(1-\lambda)(\alpha_1 k^{1-\epsilon} + \alpha_0 \ell^{1-\epsilon})^{1/(1-\epsilon)} - c\Big]^{1-\epsilon} + \alpha_0 \ell^{1-\epsilon}\Big]^{1/(1-\epsilon)}\Big\}^{1-\epsilon} + \Big[\beta^2/(1-\beta)\Big]\Big[1/(1-\epsilon)\Big](1/2)^{1-\epsilon}\alpha_0 \ell^{1-\epsilon}\Big\}$$

The last term on the right-hand side represents the total payoff to the players after the stock of capital has been exhausted at the end of the second period Note that this is only feasible if $\epsilon < 1$ since otherwise no output can be produced without capital, as in the Cobb Douglas case. Furthermore, if $\epsilon \geq 1$ the utility of zero consumption will be minus infinity so that defection will never become attractive

The optimal consumption to initiate defection in this case is given by

$$c = \lambda_D(1-\lambda)y$$

where $$\lambda_D = \frac{1}{1 + (\beta\alpha_1)^{1/\epsilon}(1/2)^{(1-\epsilon)/\epsilon}}$$

Of course in equilibrium it is a dominant strategy for both players not to withold any of their labor endowments because there is no disutility of labor.

The value function for each player in this case is given by

$$V_D(k) = s_D k^{1-\epsilon} + I_D$$

where

$$s_D = \left[\alpha_1/(1-\epsilon)\right]\left\{\lambda_D^{1-\epsilon}(1-\lambda)^{1-\epsilon} + \beta\alpha_1\left[(1-\lambda)(1-\lambda_D)(1/2)\right]^{1-\epsilon}\right.$$

$$I_D = \left\{(\alpha_0/\alpha_1)s_D + \alpha_0\left[\beta/(1-\beta)\right]\left[1/(1-\epsilon)\right]^{1-\epsilon}\right\}\ell^1$$

While we have not formally proved the analogue of Lemma 1 for this case, simulations on the computer suggest that an analogous result holds for this case as well Benhabib and Radner 1988] have shown that if agents have a HARA utility function (of the type $(c+\eta)^\alpha$ where η is a positive constant) and face a simple linear technology without labor inputs, then it is possible to construct examples where cooperation cannot be enforced from low stocks but can be enforced by high stocks (see also Benhabib and Ferri 1987]) Furthermore, in their example, a symmetric non-cooperative consumption strategy that starts in the region where cooperation is unenforcable and grows into the cooperative region after which players switch to cooperative strategies is enforcable as an equilibrium under the threat of reverting to the extreme equilibrium Benhabib and Radner 1988] termed the strategies followed along such an equilibrium path switching strategies Of course since they constitute an equilibrium, the payoffs associated with switching strategies must dominate the value of defecting from them

We can offer a conjecture as to why equilibria with switching strategies are possible in the HARA case with a linear technology and not in the case of a constant relative risk aversion utility coupled with CES production. Both formulations require sufficient curvature of the utility of consumption to dampen the value of defection with a large amount of consumption when the stock is large. Similarly, both formulations require some positive utility when the stock levels have been depleted so that the value of defection is high enough for low stocks This is acheived by the η term in the HARA case so that players can continue to accumulate utility after stocks are depleted and consumption is zero. In the CES case positive consumption can continue after stocks are depleted because labor alone produces output However in the latter case the curvature of utility is tied to the curvature of production A strongly curved utility also implies a low elasticity of substitution in production so that labor is not effective in producing output on its own Thus it may be possible to dampen the value of a defection from high stocks with a large amount of consumption by imposing a strong curvature on the utility function, while simultaneously allowing for a sufficiently high level of output after the stock has been depleted, so that with low stocks the value of defection can be maintained above that of cooperation

When cooperation cannot be immediately enforced as an equilibrium from the initial stock by using some appropriate trigger strategy, the above discussion raises the issue of finding strategies that constitute a second best equilibrium. Characterizing such strategies remains an open problem

4. Some Final Remarks

We have shown in section 2 that for the simple dynamic programming problem that arises in growth theory, setting appropriate elasticities of the utility and production functions equal to each other results in savings policies that are independent of capital stocks. In terms of Figure 1 however, not only points on the diagonal but also those on the vertical axis results in such policies, although for a given utility function the policies will be different. First consider the CES production function $\left[ak^{1-\epsilon}+(1-a)\ell^{1-\epsilon}\right]^{1/(1-\epsilon)}$ where ℓ denotes the fixed quantity of labor. For $\ell = 0$, the function becomes linear and is given by $a^{1/(1-\epsilon)}k$ For purposes of comparison we specify a linear production function $bk + (1-b)\ell$, where $b = a^{1/(1-\epsilon)}$ We set the utility function $(c^{1-\epsilon})/(1-\epsilon)$. As is already known, the consumption policy for the linear technology can be computed as $c = \lambda(y-\ell)$, where $\lambda = 1-(\beta a)^{1/\epsilon}$ For the CES production function, consumption is given by $c = \lambda y$, again with $\lambda = 1-(\beta a)^{1/\epsilon}$ Therefore the two consumption functions only differ by the amount $\lambda\ell$; consumption is always higher with the CES production function. Therefore we can conjecture that, in terms of Figure 1, moving from the diagonal towards the vertical axis where technology is linear tends to increase the propensity to save out of income. The problem of obtaining exact solutions for parameters not on the diagonal or on the axes remains open

References

[1 . Benhabib, J. and G. Ferri, "Bargaining and the Evolution of Cooperation in a Dynamic Game," Economic Letters 24 (1987), 107-111.

[2]. Benhabib, J. and R. Radner, "Joint Exploitation of a Productive Asset: A Game Theoretic Approach," C.V. Starr Center Working Paper Series 88-17 (1988), New York University.

[3]. Benhabib, J. and A. Rustichini, "A Vinatge Model of Investment and Growth: Theory and Evidence," C.V. Starr Center Working Paper Series 89-26 (1989), New York University.

[4]. Levhari, D. and L. Mirman, "The Great-Fish War: An Example of Using the Cournot-Nash Solution," Bell Journal of Economics 11 (1980), 322-344.

[5]. Long, J.B. and C.I. Plosser, "Real Business Cycles," Journal of Political Economy, 91 (1983), 39-69.

[6]. Radner, R., "Optimal Growth in a Linear-Logarithmic Economy," International Economic Review 7 (1986), 1-33.