EQUILIBRIUM STRATEGIES FOR FINAL-OFFER ARBITRATION

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ABSTRACT

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Final-offer arbitration is a procedure for settling disputes between two parties in which an arbitrator chooses the final offer of the party closest to what he considers a fair settlement. This procedure is modeled as a two-person, zero-sum, infinite game of incomplete information, in which the parties are assumed to know the probability distribution of the abitrator's fair settlements and to make bids that maximize their expected payoffs. Necessary and sufficient conditions for there to be local and global equilibria in pure strategies are derived, and necessary conditions for mixed strategies in a particular case are found. Optimal strategies are shown not always to be convergent, and implications of this finding, and the difficulties that mixed strategies (or no equilibria) pose in seeking settlements, are discussed.

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1. Introduction

A strikingly simple procedure for settling disputes between two parties was proposed fifteen years ago by Carl Stevens [8], though it had been discussed on an ad hoc basis before that [7, p. 113, ftn. 7]. It is called final-offer arbitration (FOA) and involves each party's submitting its final offer for settlement to a (presumably neutral) arbitrator, who must choose one final offer or the other. The offer chosen by the arbitrator determines the settlement. Unlike conventional arbitration, the arbitrator is not permitted to "split the difference," or compromise the offers of each side in any other way. One or the other side "wins" by getting its offer accepted.

Ostensibly we have a game, and one which, it seems, will force the two players to converge, eliminating the need for a settlement imposed by the arbitrator, as in binding arbitration. To one analyst, the reasons for convergence are rather evident:

> The theory which underlies final-offer arbitration is quite simple. If the arbitrator or panel was permitted to select only one or the other of the parties' final offers, with no power to make a choice anywhere in between, it was expected that the logic of the procedure would force negotiating

> parties to continue moving closer together in search of a position that would be most likely to receive neutral sympathy. Ultimately, so the argument went, they would come so close together that they would almost inevitably find their own settlement. In short, final-offer arbitration would obviate its own use thus eliminating the chilling effect of binding arbitration [6, p. 218].

One purpose of our article is to show that while there is a good deal of truth to this theory, it is not quite as simple as is often assumed.

As a practical matter, FOA is now used to settle labor disputes involving certain classes of public employees in six states (Connecticut, Iowa, Massachusetts, Michigan, New Jersey, and Wisconsin), and to resolve salary disputes in major-league baseball. Recent experience with FOA, particularly in baseball, is discussed in [1], and in public-sector disputes in [4].

With the exception of a recent article by Farber [3], there seems to be no analysis of the equilibrium properties of FOA. Farber's model, though developed independently, is similar to ours. He showed, among other things, the sensitivity of final-offer Nash equilibrium strategies to the risk-aversity of parties, the size of the contract zone (defined by the minimum and maximum offers the "low" and "high" parties, respectively, will accept in lieu of using FOA), and uncertainty about the arbitrator's preferences.

The focus of our analysis will be on the degree of convergence--or lack thereof--that different assumptions about the arbitrator's preferences have on the existence and nature of equilibrium strategies of the two parties. We shall provide necessary and sufficient conditions for both local and global equilibria in pure strategies, which Farber does not do, and necessary conditions for two-point mixed strategies, given a symmetry condition, as well.

Specifically, we shall model FOA as an infinite, two-person, zero-sum game of incomplete information between the disputing parties. Because Farber assumed that the parties might have independent utility functions over the possible settlements the arbitrator might choose, the game he modeled was not necessarily zero-sum. Although this permitted him to analyze the effects of such properties as risk aversity on the choice of equilibrium strategies, his model did not allow him to derive the kinds of quantitative results we obtain.

The uncertainty in the game that we assume is played between the two parties stems from their not knowing what the arbitrator considers to be a "fair" settlement. We assume, however, that both parties know, and agree on, the probability distribution over what the arbitrator thinks is fair. The game is for each party to choose an offer that maximizes his expected payoff, given that the arbitrator will choose the offer closest to what he considers fair.

The payoff may be monetary or not, but what we assume is that the greater utility one player has for it, the less the other does, with the sum of their utilities zero for all payoffs. Our first two theorems give (i) necessary and sufficient conditions for there to be local equilibrium strategies, and says what they are, and (ii) necessary and sufficient conditions for these strategies to be global equilibrium strategies, which define a saddlepoint in the FOA game.

Roughly speaking, global equilibrium strateiges are at least as good as, and sometimes better than, any other strategy, regardless of the choice of the other player; by contrast, neither player has an incentive to deviate by a small amount from local equilibrium strategies, though large deviations may be optimal. Thus, it would never be rational for either player to depart from global equilibrium strategies, but a large deviation by one player from a local equilibrium strategy may lead to a larger expected payoff.

Several examples of probability distributions that satisfy the necessary and sufficient conditions are given, as well as examples which admit local but not global equilibria, or which admit no equilibria in pure strategies at all. We then provide necessary conditions for two-point mixed strategies for symmetric distributions and give examples to illustrate this case. Implications of our findings, particularly with respect to the question of convergence of the parties under FOA, are addressed in the concluding section.

2. Equilibrium Solutions Using Pure Strategies: Continuous Distributions

Our main result for continuous distributions consists of two parts (I and II), one giving necessary and sufficient conditions for local equilibria, and the other these conditions for global equilibria, when the solution is in pure strategies. As we show below, for a pure-strategy solution to exist, the probability density function at the assumed median of 0, f(0), must be greater than zero; if this condition and the other conditions are met, then the equilibrium strategies are a reciprocal function of $f(0)$ and equidistant from the median. The reader is referred to [5] for basic definitions and results concerning probability distributions.

THEOREM 1. Suppose the arbitrator's notion of a fair settlement has a continuous distribution with probability density f and distribution function F with $F' = f$ (or at least one-sided derivatives of F exist for each x). Assume without loss of generality, that the median is 0.

I. If $f'(0)$ exists and $f(0) > 0$, then

$$a_0 = -1/[2f(0)] \quad \text{and} \quad b_0 = 1/[2f(0)] \tag{1}$$

are local equilibrium strategies for the low and high bidders, respectively, if

$$|f'(0)| < 4f^2(0) . \tag{2}$$

If the inequality in (2) is reversed, no local equilibrium exists. If equality holds in (2), (a_0, b_0) may or may not be a local equilibrium.

II. If $f(0) > 0$, then the pair (a_0, b_0) defined by (1) is a global equilibrium if and only if the following conditions hold:

$$\int_0^x f(t)dt \leq x/(2b_0 - 2x) \quad \text{for} \quad 0 < x \leq 1/[4f(0)] , \tag{3}$$

$$\int_0^x f(t)dt \geq x/(2b_0 + 2x) \quad \text{for} \quad x > 0 , \tag{4}$$

<u>and the same inequalities hold for</u> $\int_{-x}^{0} f(t)dt$ <u>in place of</u> $\int_{0}^{x} f(t)dt$.

PROOF. The expected payoff to the high bidder (as well as the low bidder since the game is assumed to be zero-sum) is the sum of the bids times the respective probabilities that the arbitrator's choice is closer to each:

$$g(a, b) = aF[(a + b)/2] + b[1 - F(a + b)/2)] \tag{5}$$

$$= b - (b - a)F[(a + b)/2] .$$

To determine local equilibria, we note that g is differentiable (or at least there exist one-sided partial derivatives), and that if a critical point occurs at (a, b), then

$$\frac{\partial g}{\partial a} = F[(a + b)/2] - (1/2)(b - a)f[(a + b)/2] = 0 \tag{6}$$

$$\frac{\partial g}{\partial b} = 1 - F[(a + b)/2] - (1/2)(b - a)f[(a + b)/2] = 0 \tag{7}$$

(where one-sided derivatives of F are used in place of f if necessary). Adding and subtracting (6) and (7), we obtain

$$(b - a)f[(a + b)/2] = 1 \tag{8}$$

$$2F[(a + b)/2] = 1 . \tag{9}$$

From (9), we see that

$$0 = (a + b)/2 \tag{10}$$

by the definition of the median. Thus by (8),

$$b - a = 1/f(0) \,, \tag{11}$$

so that (1) follows.

To determine when this solution for (a_0, b_0) is a local equilibrium, note that

$$\frac{\partial^2 g}{\partial a^2} = f[(a + b)/2] - (1/4)(b - a)f'[(a + b)/2]$$

and

$$\frac{\partial^2 g}{\partial b^2} = - f[(a + b)/2] - (1/4)(b - a)f'[(a + b)/2]$$

when f' is defined. Thus

$$\frac{\partial^2 g}{\partial a^2}(a_0, b_0) = f(0) - \frac{f'(0)}{4f(0)} = \frac{4f^2(0) - f'(0)}{4f(0)} \,,$$

which is > 0 if and only if $f'(0) < 4f^2(0)$. Also

$$\frac{\partial^2 g}{\partial b^2}(a_0, b_0) = - f(0) - \frac{f'(0)}{4f(0)} = \frac{- 4f^2(0) - f'(0)}{4f(0)} \,,$$

which is < 0 if and only if $f'(0) > - 4f^2(0)$. Hence (a_0, b_0) is a local equilibrium point if $|f'(0)| < 4f^2(0)$, and no such point exists if the inequality is reversed. It is easy to construct examples with or without local equilibrium if equality occurs in (2). This completes the proof of part I of the theorem.

In proving part II, we note that f has a local (global) equilibrium at (a_0, b_0) if and only if $h(x) = [1/f(0)]f(x/f(0))$ has a local (global) equilibrium at $(a_0 f(0), b_0 f(0))$. Furthermore, f satisfies (3) and (4) if and only if h is a density satisfying the corresponding conditions for h. Also, $h(0) = 1$. Hence, without loss of generality, we may assume that $f(0) = 1$. Note that h is obtained from f by a linear change of scale.

Under the assumption that $f(0) = 1$, $(a_0, b_0) = (-1/2, 1/2)$. We can demonstrate that $(-1/2, 1/2)$ is a global equilibrium by showing that $g(-1/2, b) \leq g(-1/2, 1/2) = 0$ for all $b \in (-\infty, \infty)$. A similar argument can be used to show that $g(a, 1/2) \geq 0$ for all $a \in (-\infty, \infty)$.

It is clear that $g(-1/2, b) \leq 0$ if $b \leq 0$, so we assume that $b > 0$. Note that the following sequence of statements are equivalent:

$$g(-1/2, b) = b - (b + 1/2)F[(-1/2 + b)/2] \leq 0$$

or

$$F(b/2 - 1/4) \geq b/(b + 1/2) ,$$

or, subtracting 1/2 from both sides,

$$F(b/2 - 1/4) - 1/2 \geq [(b - 1/2)/2]/(b + 1/2) ,$$

or

$$\int_0^y f(t)dt \geq y/(b + 1/2) , \tag{12}$$

where we have set

$$y = b/2 - 1/4 \ . \tag{13}$$

Solving (13) for b, we obtain $b = 2y + 1/2$, so that (12) is in turn equivalent to

$$\int_0^y f(t)dt \geq y/(1 + 2y) \tag{14}$$

for $y \geq 0$ (i.e., $b \geq 1/2$), which is equivalent to the first formula in (4). The other formula in (4) is equivalent to the statement $g(a, 1/2) \geq 0$ for $a \leq -1/2$.

If, on the other hand, $0 < b \leq 1/2$, then $-1/4 \leq y < 0$, and (14) may be replaced by

$$\int_y^0 f(t)dt \leq -y/(1 + 2y) \ .$$

Setting $x = -y$, we have $0 < x \leq 1/4$, and

$$\int_{-x}^0 f(t)dt \leq x/(1 - 2x) \ ,$$

as in (3). The other formula in (3) is equivalent to $g(a, 1/2) \geq 0$ for $-1/2 \leq a < 0$. Q.E.D.

By differentiating the right-hand-side expressions in (3) and (4), we obtain the bounding density functions

$$f_1(x) = b_0/2(b_0 - x)^2 \quad \text{and} \quad f_2(x) = b_0/2(b_0 + x)^2 \ ,$$

which are plotted in Figure 1. According to (3) - (4), the mean value,

$\int_0^x f(t)dt$, between 0 and x of a density f possessing global equilibria must lie between the corresponding mean values of f_1 and f_2. This is not to say that the density f itself must lie between f_1 and f_2 but rather that, for every x, the <u>average</u> value of f between 0 and x must lie between the corresponding average values of f_1 and f_2. An example of such a density, the normal curve, is shown by the dashed line in Figure 1.

Figure 1 about here

Although part II of Theorem 1 provides necessary and sufficient conditions for the existence of a global equilibrium, the following corollary provides a set of sufficient conditions which are more easily applied to many distributions.

COROLLARY. *Suppose that the arbitrator's notion of a fair settlement has a continuous distribution with density* f *and median* 0. *If* $f(0) > 0$, *then the following are sufficient conditions that* (a_0, b_0) *defined by (1) be a global equilibrium:*

$$f(x) \leq f(0) + 4f^2(0)|x| , \quad \text{for} \quad |x| \leq 1/[4f(0)] , \tag{15}$$

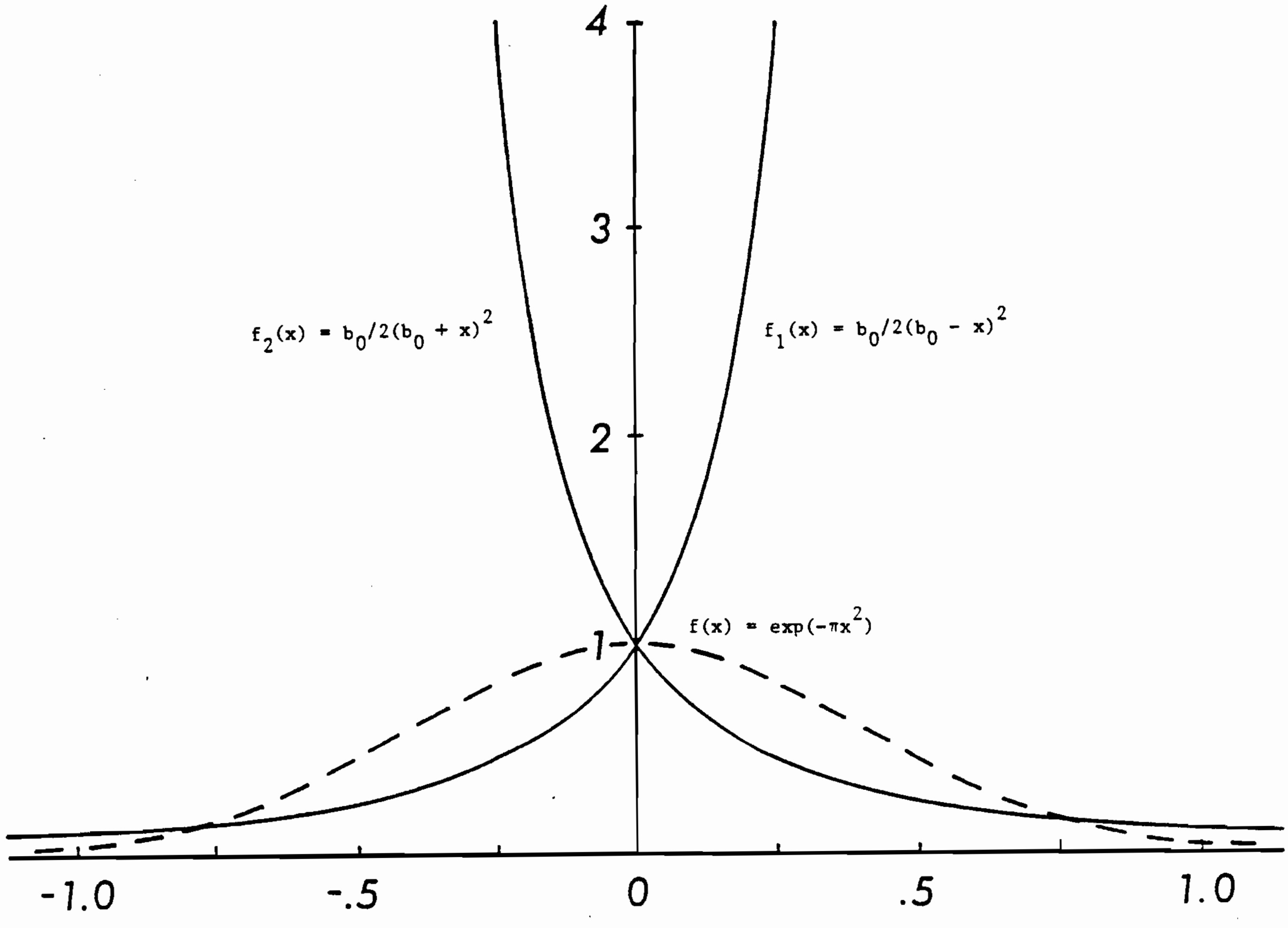
and there exist c_1 *and* c_2 *with* $-\infty \leq c_1 \leq 0 \leq c_2 \leq \infty$ *such that*

$$f(x) \geq f(0)\exp(-2f(0)|x|) , \quad c_1 \leq x \leq c_2 , \quad \text{and} \tag{16a}$$

$$f(x) \leq f(0)\exp(-2f(0)|x|) , \quad x \leq c_1 \quad \text{and} \quad x \geq c_2 . \tag{16b}$$

Figure 1

Bounding Probability Density Functions f_1 and f_2 (Solid Curves) for Continuous Distributions Possessing Global Equilibria for FOA (See Text). Dashed Curve Represents Normal Distribution, Normalized so $f(0) = 1$.



PROOF. As before we may assume that $f(0) = 1$, since conditions (15) and (16) are unaltered by a linear change of scale. By Theorem 1, it suffices to show that (15) implies (3) and (16) implies (4).

Suppose that (15) holds, so that

$$\int_0^x f(t)dt \leq \int_0^x (1 + 4t)dt = x + 2x^2 , \quad \text{for} \quad 0 < x \leq 1/4 .$$

But $x + 2x^2 = x(1 + 2x) \leq x/(1 - 2x)$ for $x > 0$ (the last inequality can be checked by cross-multiplication), so that (3) follows.

Condition (16a) implies that for $0 \leq x \leq c_2$,

$$\int_0^x f(t)d \geq \int_0^x e^{-2t}\, dt = (1/2)[1 - e^{-2x}] .$$

Hence, to show (4), we must prove that

$$1 - e^{-2x} \geq 2x/(1 + 2x) \quad \text{for} \quad 0 \leq x \leq c_2 ,$$

which reduces to

$$e^{2x} \geq 1 + 2x \quad \text{for} \quad 0 \leq x \leq c_2 .$$

But this follows from the power series expansion for e^{2x}.

On the other hand,

$$\int_x^\infty f(t)dt \leq \int_x^\infty e^{-2t}dt = (1/2)e^{-2x} \quad \text{for} \quad x > c_2 ,$$

so that

$$\int_0^x f(t)dt \geq 1/2 - (1/2)e^{-2x} = (1/2)[1 - e^{-2x}] ,$$

so that (4) holds also for $x > c_2$. (The arguments for $x < 0$ are similar.) Q.E.D.

Examples. We first show that several well-known distributions satisfy conditions (15) and (16) and hence by the corollary possess global equilibrium solutions given by (1). Since, as we have seen, the existence of such solutions is independent of scale and location parameters, it is sufficient to consider only the standardized form for each parametric family.

Seven continuous probability distributions possessing global equilibria in pure strategies are plotted in Figure 2, A - G. Four continuous distributions possessing local but not global equilibria in pure strategies are plotted in Figure 2, H - K, together with an example (L) having local and global equilibria in mixed (but not pure) strategies and some discrete distributions, M - O (see sections 3 and 4 for the descriptions of plots L - O). In each plot, local equilibria are designated by heavy dots; global equilibria are designated by circles.

Figure 2 about here

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1. Global Pure

A. Double Exponential. $f(x) = (1/2)\exp(-|x|)$ for all x . First note that f has a maximum at 0 so (15) holds trivially.

Figure 2

Probability Densities, with Local Equilibria (Heavy Dots) and Global Equilibria (Circles) for FOA.

1. Global Pure

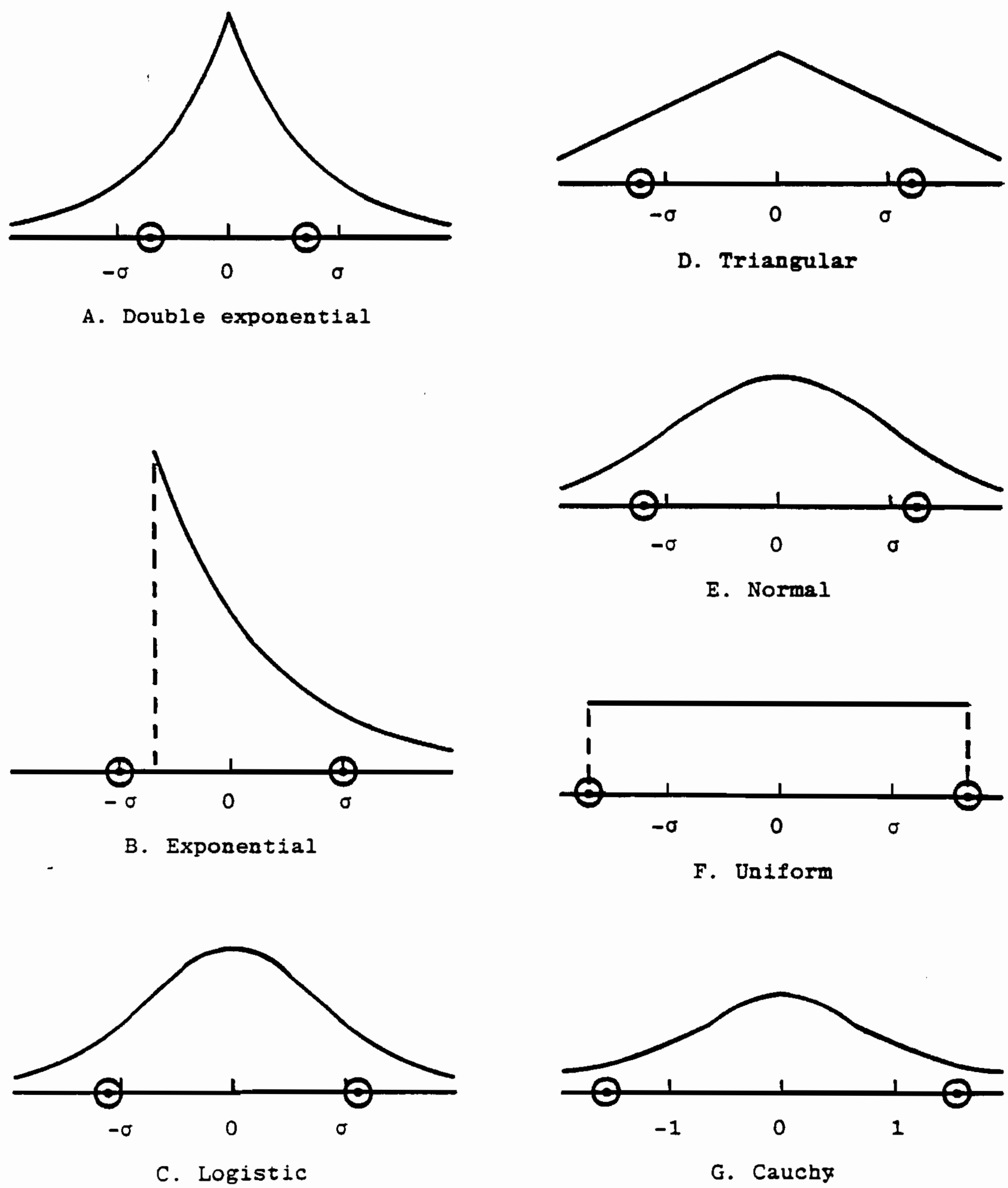
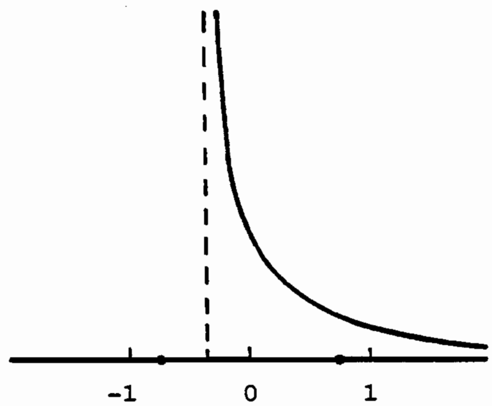
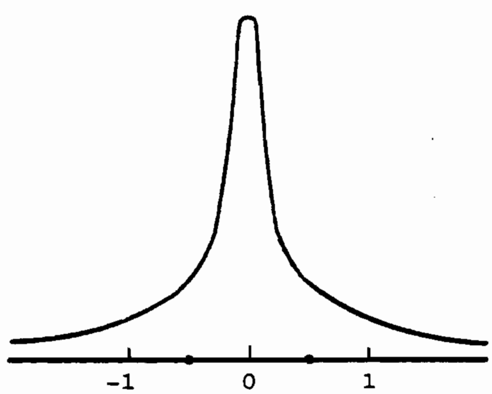


Figure 2 (cont.)

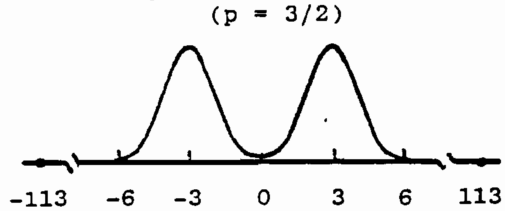
2. Local Pure



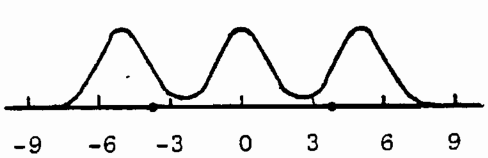
H. Chi-square (1 d.f.)



I. Reciprocal power $1 < p < 2$
($p = 3/2$)

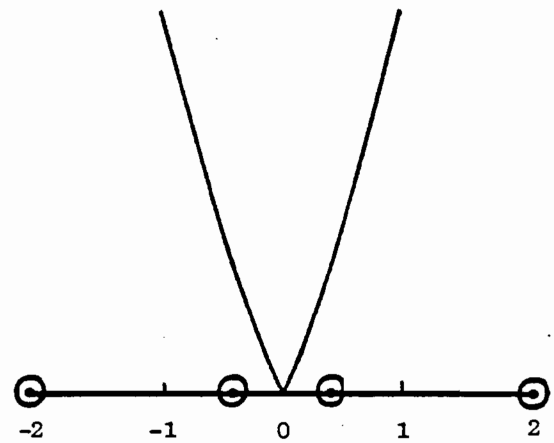


J. Mixture of 2 normals
($\mu = 3$)



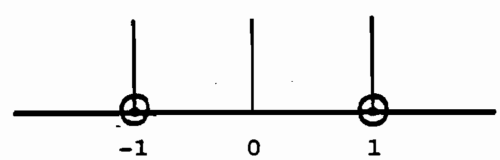
K. Mixture of 3 normals
($\mu = 5$)

3. Global Mixed

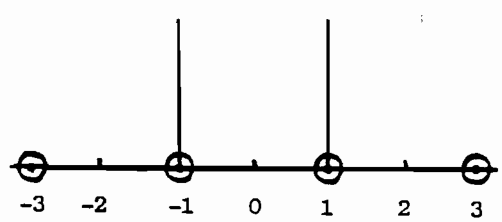


L. Mixed strategies

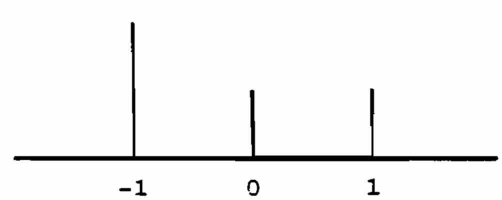
4. Discrete



M. Unique median (global solution)



N. $p_0 = 0$ (solution in mixed strategies)



O. $\bar{p}_0 = 0$ (no solution in pure strategies)

Furthermore, $f(x) = f(0)\exp(-2f(0)|x|)$ for all x, so that (16) is valid for all x (for any choices of c_1 and c_2).

B. Exponential. $f(x) = \begin{cases} (1/2)\exp(-x) , & x \geq -\ln(2) \\ 0 , & x < -\ln(2) . \end{cases}$

(Note that the density has been normalized so that the median is 0 .) Since $f(0) = 1/2$ and $|f'(x)| \leq \exp(0) = 1 = 4f^2(0)$ for all $x > -\ln(2)$, the latter inequality holds *a fortiori* for $x \geq -1/[4f(0)] = -1/2$. Hence (15) is valid by the mean-value theorem. Now set $c_1 = -\ln(2)$ and $c_2 = 0$. For $c_1 \leq x \leq 0$, $f(x) = (1/2)\exp(-x) \geq (1/2)\exp(x) = f(0)\exp(-2f(0)|x|)$, so (16a) is satisfied. If $x < c_1$, $f(x) = 0$, so (16b) is satisfied there. On the other hand, if $x \geq 0$,

$$f(x) = (1/2)\exp(-x) = f(0)\exp(-2f(0)|x|) ,$$

so (16) is satisfied in that range as well.

C. Logistic. $f(x) = e^{-x}/(1 + e^{-x})^2$. Since the maximum of f occurs at 0 and f is symmetric, it suffices to note that $f(x) \leq e^{-x}/4 \leq f(0)\exp(-2f(0)x)$ for $x \geq 0$ and set $-c_1 = c_2 = \infty$.

D. Triangular. $f(x) = 1 - |x|$ for $|x| \leq 1$, 0 otherwise. The verification of conditions (15) - (16) is simple.

E. Normal. $f(x) = (1/\sqrt{2\pi})\exp(-x^2/2)$. Since f has a maximum at 0 , (15) is trivial. Now observe that

$$f(0)\exp(-2f(0)|x|) = (1/\sqrt{2\pi})\exp(-2/(\sqrt{2\pi})\,|x|)$$

$$\geq (1/\sqrt{2\pi})\exp(-x^2/2)$$

if and only if $|x| \geq 4/\sqrt{2\pi} = 1.596$, so that (16b) is satisfied for $-c_1 = c_2 = 4/\sqrt{2\pi}$. If $|x| \leq c_2$,

$$f(x) \geq (1/\sqrt{2\pi}) \exp(-2/\sqrt{2\pi}) |x|) = f(0)\exp(-2f(0)|x|) ,$$

so (16a) is satisfied.

F. Uniform. $f(x) = 1$ for $|x| \leq 1/2$, 0 otherwise. The verification is trivial.

G. Cauchy. $f(x) = 1/(1 + \pi^2 x^2)$. It can be shown that $(-1/2, 1/2)$ is a global equilibrium by appealing to Theorem 1 directly.[1] However, the conditions of the corollary are violated.

Next we provide examples of distributions with local equilibria given by (1) which are not global equilibria.

2. Local Pure

H. Chi-square with one degree of freedom. Condition (3) is violated.[2]

I. Reciprocal power. $1 < p < 2$. $f(x) = k/|x|^p$, for $|x| \geq m$, and defined on $[-m, m]$ so that $f(0) = 1$, $f'(0) = 0$, and f is a density. Condition (4) is violated.[3]

K. Mixture of two normals. $f(x) = [\phi(x + \mu) + \phi(x - \mu)]/2$ where ϕ is standard normal and μ the distance from the median of each of the two modes. It is easy to check that for $\mu \geq 2$, (3) is violated (e.g., take $x = \mu$), and the local equilibrium (at ± 113 for $\mu = 3$) is not global.

L. Mixture of three normals. $f(x) = [\phi(x + \mu) + \phi(x) + \phi(x - \mu)]/3$. For $\mu \geq 4$, the distance from the median of each of the outer modes,

(4) is violated (e.g., take $x = 3$), and the local equilibrium (at ± 3.76 for $\mu = 5$) is not global.

A mixture of n normals provides an appropriate model for an arbitration setting in which there are n likely values for the arbitrator's notion of a fair settlement, each associated with a degree of imprecision. It is noteworthy that for the mixture of three normals (K), the equilibria (± 3.76) stay well within $x = \pm \mu = \pm 5$, but for the mixture of two normals (J), the two equilibria (± 113) lie far outside $x = \pm \mu = \pm 3$. It seems that a mode at the median 0 in the odd case draws in the bidders, whereas no mode at the median in the even case leads to divergence, though it should be emphasized that the equilibria in both cases are only local.

Conditions (3) and (4) and the foregoing examples suggest very roughly that a global equilibrium exists if the probability density (i) does not have a deep dip at the median, and (ii) has tails that are not too heavy. In other words, we can expect global stability when there is neither too much concentration of probability near but not at the median (as in H and J) nor too much probability far from the median (as in I and K). Too deep a dip at the median violates the upper bounding curve (see Figure 1); too heavy tails leaves too little probability near the median, violating the lower bounding curve.

When there are global equilibria in pure strategies, however, they may or may not fall within one standard deviation, i.e., $\pm \sigma$. Moreover, if they lie more than one standard deviation from the median, they may lie at the extremes of the distribution (F), or an equilibrium may even lie outside the domain of f (B).

In the case of the latter distribution (exponential), the low bidder should propose $-\sigma$ even though it has zero probability of being the arbitrator's choice. Intuitively, this is an optimal strategy not because $-\sigma$ will ever be chosen (it won't), but rather because it is sufficiently close to values that will be chosen that it will "beat" the high bidder's strategy often enough to maximize the low bidder's expected payoff.

Because a major objective of FOA is to draw the bids of the two parties closer together, it is useful to measure the efficacy of a family of distributions for FOA in terms of the distance between equilibrium strategies. In order that the measure of efficacy be independent of the unit of measurement, the distance between optimal strategies should be divided by a scale parameter. We adopt the following definition:

$$d = \text{dispersion for FOA} = (b_0 - a_0)/\sigma = 1/\sigma f(0)$$

where σ is the standard deviation. The following table provides values of the dispersion d for all continuous distributions in Figure 1 with global equilibria in pure strategies. It should be noted that a different measure of scale (e.g., the median of the absolute deviations) would give somewhat different values for dispersion.

Distribution	Dispersion d
Double exponential	$\sqrt{2} = 1.41$
Exponential	$2 = 2.00$
Logistic	$4\sqrt{3}/\pi = 2.21$
Triangular	$\sqrt{6} = 2.45$

Normal	$\sqrt{2\pi}$	$= 2.51$
Uniform	$\sqrt{12}$	$= 3.46$
Cauchy	Not defined	

Not surprisingly, the uniform distribution is most dispersive and the double exponential least, illustrating the drawing power of a concentration of probability at the median.

3. Equilibrium Solutions using Pure Strategies: Discrete Distributions

Thus far we have admitted only continuous distributions for the arbitrator's notion of a fair settlement. Suppose instead that both the arbitrator's notion and the bids are restricted to multiples of some quantity $\delta > 0$, which we may assume, without loss of generality, to be 1. We denote the probability function for the arbitrator's distribution by p and write $p_n = p(n)$. An integer m is called a median of p if both Σp_n, $n < m$, and Σp_n, $n > m$, are less than or equal to 1/2. According to this definition, the median m is unique if both of these sums are strictly less than 1/2. In example M below, 0 is the unique median; in example N, - 1, 0, and 1 are all medians; in example O, - 1 and 0 are medians.

If m is a median, we define the partition:

$$p_m = p_m^- + p_m^+ ,$$

where

$$p_m^- = 1/2 - \sum_{n<m} p_n , \quad \text{and} \quad p_m^+ = 1/2 - \sum_{n>m} p_n .$$

Thus, the median is unique if and only if p_m^- and p_m^+ are both different from zero.

Without loss of generality, we may set $m = 0$, where m is chosen as (one of) the medians. In the example M below, where $p_{-1} = p_0 = p_1 = 1/3$, note that $p_0^- = p_0^+ = 1/6$. For $x > 0$, define sums

$$\begin{aligned} S_x &= p_0^+ + \sum_{i=1}^{[x]} p_i \\ S_{-x} &= \sum_{i=[-x]+1}^{-1} p_i + p_0^- , \end{aligned} \tag{17}$$

where $[x]$ is the largest integer $\leq x$.

THEOREM 2. *If* $p_0 > 0$, *define*

$$\begin{aligned} b_0 &= 1/2p_0 + \alpha \\ a_0 &= -b_0 , \end{aligned} \tag{18}$$

where b_0 *is an integer and* $-1/2 < \alpha \leq 1/2$. *Then* (a_0, b_0) *is a global equilibrium if and only if whenever* $2x$ *is an integer*

$$S_x \leq x/(2b_0 - 2x) , \quad 0 < x \leq b_0/2 , \tag{19}$$

$$S_x \geq x/(2b_0 + 2x) , \quad x > 0 , \tag{20}$$

and the same inequalities hold for S_{-x} *in place of* S_x.

PROOF. Since the proof is almost identical with that of Theorem 1, part II, we only provide an outline. Arguing as before, if $b > 0$, the statement

$$g(a_0, b) \leq 0$$

is equivalent to

$$F(y) - 1/2 \geq y/(b - a_0) \ , \tag{21}$$

where we have set

$$y = (a_0 + b)/2 \ .$$

Thus $b = 2y - a_0$, so for $y \geq 0$ (i.e., $b \geq - a_0$),

$$S_y = p_0^+ + \sum_{i=1}^{[y]} p_i \geq y/(2y - 2a_0) = y/(2b_0 + 2y) \ .$$

If, on the other hand, $0 < b \leq - a_0$, then $a_0/2 \leq y < 0$, and setting $x = - y$, (21) may be replaced by

$$S_{-x} = \sum_{i=[-x]+1}^{-1} p_i + p_0^- \leq x/(2b_0 - 2x)$$

for $0 < x \leq b_0/2$. The other inequalities follow from a consideration of $g(a, b_0)$.

Note in particular that if $x = 1/2$, (20) implies that

$$S_{\frac{1}{2}} = p_0^+ \geq (1/2)/(2b_0 + 1) > 0$$

and a similar argument shows that $p_0^- > 0$ as well, so the median must be unique for a global equilibrium to occur. Q.E.D.

Examples (see Figure 2 for plots).

M. p: $p_{-1} = p_0 = p_1 = 1/3$. Then $(-1, 1)$ is a global equilibrium by Theorem 2.

N. p: $p_{-1} = p_1 = 1/2$. Since $p_0 = 0$, no global equilibrium exists in pure strategies. (But see section 4 for a mixed-strategy solution.)

O. p: $p_{-1} = 1/2$; $p_0 = p_1 = 1/4$. Here $p_0^- = 0$ and $p_0^+ = 1/4$. Again by Theorem 2, no global equilibrium exists in pure strategies.

Discrete distribution M seems analogous to continuous distribution F (uniform), with global equilibria at the extremes of the distribution, though an argument can be made that it more resembles continuous distribution K, whose local equilibria are not at the extremes. Distribution N, which has no continuous analogue (except perhaps J), fails to meet the necessary condition for both continuous and discrete distributions that the density be greater than zero at the median.

The closest continuous analogue to distribution 0 is B (exponential), but unlike B it has no unique median. Altogether, these discrete distributions, and the conditions for a global equilibrium of Theorem 2 (note that local equilibria are undefined in the discrete case) suggest that the search for stability may be more difficult for discrete distributions whose arguments can assume only integer values (or multiples of some other positive quantity).

4. Equilibrium Solutions Using Two-Point Mixed Strategies

We now consider the case in which $f(0) = 0$. As we have seen, no equilibrium exists in pure strategies when the density or probability function is zero at the median. However, in this section we obtain necessary (but not sufficient) conditions for an equilibrium to exist for two-point mixed strategies under the assumption that the density or probability function is symmetric. We then provide examples for both the continuous and discrete cases of distributions satisfying these conditions which possess global equilibria.

We treat the continuous case first. Suppose that the low bidder chooses pure strategies a_1 and a_2 with probabilities α and $(1 - \alpha)$; the high bidder chooses pure strategies b_1 and b_2 with probabilities β and $(1 - \beta)$. Assume for convenience that $a_1 < a_2$ and $b_1 > b_2$. Denote these mixed strategies by $\bar{a} = (a_1, a_2, \alpha)$ and $\bar{b} = (b_1, b_2, \beta)$. Then the expected payoff is given by

$$\begin{aligned} g(\bar{a}, \bar{b}) = {} & \alpha\beta F(p_{11})(a_1 - b_1) + \alpha(1 - \beta)F(p_{12})(a_1 - b_2) \\ & + (1 - \alpha)\beta F(p_{21})(a_2 - b_1) + (1 - \alpha)(1 - \beta) \\ & \cdot F(p_{22})(a_2 - b_2) + \beta b_1 + (1 - \beta)b_2 \ , \end{aligned} \tag{22}$$

where $p_{ij} = (a_i + b_j)/2$ and F is the distribution function of the arbitrator's estimate.

If $LE = (\bar{a}, \bar{b})$ constitutes a local equilibrium, then

$$a_1 = -b_1 \ , \quad a_2 = -b_2 \ , \quad \text{and} \quad \alpha = \beta \tag{23}$$

by the symmetry of the distribution. It follows that

$$F(p_{11}) = F(p_{22}) = 1/2 \tag{24}$$

and

$$f(p_{11}) = f(p_{22}) = 0 \ . \tag{25}$$

Furthermore, we can set

$$\begin{aligned} x &= (b_1 + a_2)/2 = -(b_2 + a_1)/2 \\ y &= (b_1 - a_2)/2 = (b_2 - a_1)/2 \ . \end{aligned} \tag{26}$$

Noting that $x = p_{21} = -p_{12}$, we have

$$F(-x) + F(x) = 1 \ . \tag{27}$$

Furthermore, if the partial derivatives are evaluated at LE , we obtain:

$$\frac{\partial g}{\partial a_1} = \alpha^2/2 + \alpha(1 - \alpha)[F(-x) - yf(x)] = 0 \tag{28}$$

$$\frac{\partial g}{\partial a_2} = (1 - \alpha)^2/2 + \alpha(1 - \alpha)[F(x) - yf(x)] = 0 \tag{29}$$

$$\begin{aligned} \frac{\partial g}{\partial \alpha} = \alpha(a_1 - b_1)/2 - 2(1 - \alpha)F(-x)y + 2\alpha F(x)y \\ - (1 - \alpha)(a_2 - b_2)/2 = 0 \ , \end{aligned} \tag{30}$$

where one-sided derivatives of F are used in place of f if necessary. (It follows from the symmetry of f and some algebra that the first partial derivatives with respect to the b_i lead to the same equations.)

Subtracting (29) from (28) yields:

$$\alpha/2 - (1 - \alpha)^2/2 + \alpha(1 - \alpha)[F(-x) - F(x)] = 0 \ ,$$

or

$$2F(x) - 1 = \frac{2\alpha - 1}{2\alpha(1 - \alpha)} = 2\varepsilon/(1 - \varepsilon^2) , \tag{31}$$

where we have set

$$\alpha = (1 + \varepsilon)/2 . \tag{32}$$

Adding equations (28) and (29) and using (27) yields:

$$[\alpha + (1 - \alpha)]^2/2 - 2\alpha(1 - \alpha)yf(x) = 0 ,$$

or

$$yf(x) = 1/[4\alpha(1 - \alpha)] = 1/(1 - \varepsilon^2) . \tag{33}$$

Equation (30) when simplified yields:

$$\alpha(a_1 - b_1)/2 - (1 - \alpha)(a_2 - b_2)/2 - 2y[F(-x) - \alpha] = 0 .$$

Using (23), (27), and (32), this becomes

$$2yF(x) = y + x . \tag{34}$$

Together with (31), this yields

$$x/y = 2\varepsilon/(1 - \varepsilon^2) , \tag{35}$$

which proves the following theorem:

THEOREM 3. _If the distribution function_ F _is continuous and possesses at least one-sided derivatives for each_ x , _and if the corresponding density_ f _is symmetric, then conditions (33) - (35)_

<u>are necessary for the existence of a local equilibrium in two-point mixed strategies</u>.

Thus, any local equilibrium must satisfy equations (33), (34), and (35) and is specified by the quantities x, y, and ε. Since equations (33) - (35) define a differentiable transformation of (x, y, ε) space into itself, Newton's method will provide a convergent solution by iteration if the determinant of the matrix of partial derivatives of this transformation does not vanish at the solution.

Furthermore, note that by our assumptions and (26), x and y are both positive. Thus, it follows from (35), and the fact that $F(x) \leq 1$, that $y \geq x$, i.e., $a_2 \leq 0 \leq b_2$. In particular, $0 < x/y \leq 1$, so by (35), $0 < \varepsilon \leq \sqrt{2} - 1$, i.e., $0.5 < \alpha \leq \sqrt{2}/2 = 0.707$. We conclude that the high bidder should use a mixture of two <u>non-negative</u> pure strategies, weighted slightly toward the higher of these strategies (the low bidder should behave symmetrically).

<u>Example</u>. L. Define $f(t) = [(p + 1)/2]|t|^p$ for $-1 \leq t \leq 1$. For fixed $p > 1$, f is a differentiable density function on $(-1, 1)$ with $f(0) = 0$. If x, y, and ε specify a local equilibrium, equation (34) implies that

$$y = x^{-p} . \tag{36}$$

Combining (33) and (35), we get $yf(x) = x/(2\varepsilon y)$. Using the definition of f and (36), this yields:

$$\varepsilon = x^{p+1}/(p + 1) . \tag{37}$$

Using (37) to substitute in (35) leads after some simplification to:

$$x = (p^2 - 1)^{1/(2p+2)} . \tag{38}$$

Thus (36) - (38) specify a solution for x , y , and ε . Since $x > 0$, we must restrict p so that $1 < p < \sqrt{2} = 1.414$. If we take $p = 1.2$ (so that $f(t) = 1.1t^{1.2}$), numerical solution yields $-a_1 = b_1 = 2.0807$, $-a_2 = b_2 = 0.4212$, and $\alpha = 0.6508$. Computer evaluation of $g(\bar{a}, \bar{b})$ shows that this is a global solution (see Figure 2L for a plot). In fact, a global solution is obtained for p in the vicinity of 1.2; however, for $p = 1.3$, the values obtained from conditions (36) - (38) provide a local but not global equilibrium. For $p = 1.4$ or 1.1, the corresponding values are not even a local equilibrium. Hence conditions (33) - (35), although necessary, are not sufficient for a local equilibrium using two-point mixed strategies.

Clearly, the existence of an equilibrium, and whether it is local or global, is very sensitive to the parameter p for the symmetric density $f(t)$. A mixed-strategy solution is needed when $p = 1.2$ to protect against, on the one hand, the other player's bidding "too high" or "too low," for then one can win by being close to $t = 0$; to exploit, on the other hand, the steep rise in the density by making an "extreme" bid when the other player does and winning half the time (in the symmetric case). Why this subtle blend of protection and exploitation works for some values of p globally, for other values locally, and for still other values not at all raises questions which require further investigation.

Conditions (33) - (35), which are necessary for an equilibrium to exist, apply to continuous distributions. If, on the other hand, the bidders are restricted to discrete and symmetric distributions of the type described in section 3, we note that (34) remains a necessary condition for an equilibrium, for its derivation depends only on differentiation with respect to the weighting factor α . In what follows, we assume that if the two bidders are equidistant from the arbitrator's estimate, one of the bids is chosen by lot as the settlement. This leads us to interpret $F(n)$ in formula (5) for the expected payoff as

$$F(n) = \sum_{i<n} p_i + p_n/2 ,$$

where n is an integer.

We now provide an example of a discrete symmetric distribution possessing a global equilibrium in two-point mixed strategies. For the example, set $p_{-1} = p_1 = 1/2$ (see Figure 2, example N). We first use the necessary condition (34), which we rewrite as

$$b_1/(b_1 + b_2) = F(x) ,$$

to suggest candidate values of b_1 and b_2 for an equilibrium. Since $b_1 > b_2$ by assumption, either $F(x) = 3/4$, in which case $b_1 = 3$, $b_2 = 1$, and $x = 1$; or $F(x) = 1$, so that $b_1 \geq 3$, $b_2 = 0$, with $x = b_1/2$.

The latter does not lead to an equilibrium. However, if $\bar{b} = (3, 1, \alpha)$, direct verification shows that $g(a, \bar{b}) \geq 0$ for any (integral) pure strategy a where $1/2 \leq \alpha \leq 2/3$. Therefore, $\bar{a} = (-3, -1, \alpha)$ and $\bar{b} = (3, 1, \alpha)$ with $1/2 \leq \alpha \leq 2/3$ constitute a global equilibrium in mixed strategies for example N.

Are there probability distributions of the arbitrator that have no equilibrium, even in mixed strategies (without the restriction to two points)? The discrete distribution O, which has no pure-strategy solution, is a candidate, though it would seem that a distribution as simple as O would probably have some mixed-strategy solution. Yet, the necessary condition given by (34) is not applicable to nonsymmetric distributions like O.

We have not been able to find a mixed-strategy solution for O, nor have we been able to prove that one does not exist. The problem seems to be that each player wants to exploit the arbitrator's choice when it is favorable to him (e.g., the high bidder wants to be to the right of 1 when it is chosen) but at the same time protect himself from exploitation by being close to 0 when the arbitrator's choice is unfavorable to him (- 1).

It may not be an unrealistic problem for two bidders, and thus more than of merely mathematical interest. Imagine a situation in which both bidders estimate that the facts of a case will sway the arbitrator toward one position (- 1) half the time; the other half he will either favor the other position (1) or split the difference (0), each with equal probability. Although equilibrium strategies in this nonsymmetric problem are unknown, we do know that if the arbitrator will not or cannot split the difference--and hence will favor one side (- 1) or the other (1) with equal probability--there is a mixed-strategy solution $(\bar{a}, \bar{b})$, given earlier for (symmetric) example N.

This might occur, for example, if the arbitration involves a "lumpy good," which cannot be split and therefore must go to one side or the other. Then, if this is the only thing the dispute is about, the arbitrator cannot choose something "in the middle," which may give rise to a discrete distribution similar to N. But if the conditions for the award of the lumpy good are flexible, the bidders still can choose conditions, defined by points along the real line, which are more or less favorable to them. In this case, though the distribution is discrete, because we do not limit the players to integral values, the previous mixed-strategy solution to example N is not applicable. We believe, however, that limiting the players to bids that are multiples of a common value (e.g., integers) offers a more realistic model of bidding than positing real values if there is lumpiness in the good sought.

Recall that, in example N, $(3, 1, \alpha)$ and $(-3, -1, \alpha)$ are the global equilibria when the arbitrator is limited to discrete (integral) values. Although either bidder could choose the midpoint 0, his mixed strategy does not include this point, so the equilibrium solution, were the arbitrator limited to _odd_ integers, would be exactly the same as that when all integers are feasible. We interpret this to mean that if there is no midpoint that is feasible (e.g., a lumpy good can go to either one side or the other), the previous optimal mixed strategies in the discrete case would _a fortiori_ still obtain.

5. Conclusions

We have shown that final-offer arbitration can be modeled as a two-person, zero-sum, infinite game of incomplete information played between two parties who make "bids," with an arbitrator choosing the bid closest to what he considers a fair settlement. The bidders are assumed to know the probability distribution of the arbitrator's fair settlements and to make bids that maximize their expected payoffs. From these assumptions we derived necessary and sufficient conditions for there to be local and global equilibria in pure strategies (only global equilibria for discrete distributions with equally spaced intervals), which we showed to be reciprocal functions of the density at the median.

Generally speaking, optimal bids are more dispersed for "flat" distributions, such as the uniform, or those with a dip at the median, such as the mixture of two normals. For these, optimal bids violate the stated rationale of FOA--that it will force the two sides together. Indeed, there are distributions, like the exponential, in which one party's globally optimal bid is outside the domain of the arbitrator's probability distribution, and some, like the mixture of two normals, in which both parties' locally optimal bids are far from the median, resulting in divergence rather than convergence.

Necessary conditions for local equilibria sustained by two-point mixed strategies were given for symmetric distributions, but no existence conditions for general mixed strategies are known. One discrete distribution with no pure-strategy equilibria was discussed; it remains an open question whether it has a solution in mixed strategies.

FOA, as modeled herein, neither forces convergence of the parties nor even guarantees that they have optimal strategies. Even if one could show that optimal strategies always exist, the use of mixed strategies by real-life players would seem decidedly problematical and would, by definition, not necessarily lead to convergence.

Is there a solution to the problem of convergence? One analyst has suggested that FOA be modified to allow the arbitrator to make an initial choice (as under binding arbitration)--perhaps the median of his distribution--but if one party found it unacceptable, then FOA would be applied to the parties' two previously submitted bids [2]. But since one party can always guarantee himself at least as much as his expected payoff in FOA, he will challenge the arbitrator's settlement if it is not on his side of the median, given the parties are risk neutral. If they are not, risk as analyzed in [3] would come into play, or perhaps each party might make a Bayesian calculation based on the information revealed by the arbitrator's proposed settlement (which is a calculation we shall not develop here).

This game involving an initial proposal by the arbitrator would allow for the possibility of a settlement that the arbitrator considered fair, such as the median, whereas FOA in its form modeled here would never, for example, lead to the median choice in any single play (though, over a number of plays, it could in an expected-value sense).[4] It highlights the conflict between an imposed outcome, which may be fair (to the arbitrator) but is not one the parties themselves arrived at, and one induced by the play of a game, which may not be fair but is one the parties must accept once they have agreed to the rules of the game.

As FOA has gained acceptance in several states, it seems to have acquired a kind of procedural legitimacy. For many probability distributions of the arbitrators, the parties do have optimal strategies not far (on the order of one standard deviation) from the median. But for other distributions, this is certainly not the case. This finding, we believe, should suggest to proponents of FOA a note of caution on its convergence properties, even in principle.

Suffice it to say that the picture is cloudy--and indeed quite opaque for discrete distributions for which optimal strategies, if they exist, are mixed. Ironically, a clearer recognition of these difficulties may help to vindicate one claim of FOA supporters--"that it increases the pressure on the parties to take realistic bargaining positions and to settle their disputes through direct negotiations without use of arbitration" [7, p. 3]. Thereby the option of using FOA may lead to its own demise once it is understood by the parties that their optimal strategies preclude a median settlement.

Notes

1. For the Cauchy distribution, condition (3) is trivial. To verify (4), we must show that

$$\int_0^x 1/(1 + \pi^2 x^2) = (1/\pi) \text{ arc } \tan(\pi x) \geq x/(1 + 2x)$$

for $x > 0$. To this end define $p(x) = \text{arc } \tan(\pi x) - \pi x/(1 + 2x)$. Setting $p'(x) = 0$, we obtain $x_1 = 0$ and $x_2 = 4/(\pi^2 - 4)$ as the only critical points on $[0, \infty)$. Since $p(x_1) = \lim_{x\to\infty} p(x) = 0$ and $p(x_2) > 0$, we conclude that $p(x) \geq 0$ for $x > 0$. Thus (4) holds, so $(-1/2, 1/2)$ is a global equilibrium by Theorem 1. However, (16b) holds only for $0.268 < |x| < 1.683$ or $x = 0$, so that (16) is not satisfied.

2. The chi-square distribution with one degree of freedom is given by $f(x) = (1/\sqrt{2\pi})x^{-\frac{1}{2}}\exp(-x/2)$. From a standard chi-square table we see that the median $\tilde{\mu}$ is 0.455. Direct calculation shows that $|f'(\tilde{\mu})| = 0.753 < 0.887 = 4f^2(\tilde{\mu})$, so that local equilibrium strategies $a_0 = -0.606$ and $b_0 = 1.516$ are given by (1). To see that this equilibrium is not global, we use a chi-square table again to observe that $F(0.016) = 0.1$. If we choose $x = (a_0 + b)/2 = 0.016$, then $b = 0.638$ and $g(a_0, b) = b - (b - a_0)F(x) = 0.638 - (0.638 + 0.606)(0.1) = 0.514 > 0.455 = g(a_0, b_0)$. Hence, the high bidder may

do better by deviating from the strategy b_0, so the equilibrium is not global. Direct calculation shows that condition (3) is violated.

3. Computing $\int_x^\infty f(t)dt$ shows that (4) fails, so no global equilibrium exists.

4. An exception is if the arbitrator has a (degenerate) probability distribution of choosing the median 0 with probability 1. Then, if all real values are feasible, the unique equilbrium strategy for each party is to bid 0. On the other hand, if feasible bids are restricted to the integers, each party's dominant equilibrium strategy is not to bid 0 but rather +1 (high bidder) or − 1 (low bidder). These bids ensure the expected value of 0 and are better for each party than bidding 0 should the other party's bid exceed 1 in absolute value.

REFERENCES

1. Chelius, James R. and Dworkin, James B., "An Economic Analysis of Final-Offer Arbitration as a Conflict Resolution Device," J. Conflict Resolution, Vol. 24, No. 2 (June 1980), pp. 293-310.

2. Donn, Clifford B., "Games Final-Offer Arbitrators Might Play," Industrial Relations, Vol. 16, No. 3 (October 1977), pp. 306-314.

3. Farber, Henry S., "An Analysis of Final-Offer Arbitration," J. Conflict Resolution, Vol. 24, No. 4 (December 1980), pp. 683-705.

4. Konchon, Thomas A., "Dynamics of Dispute Resolution in the Public Sector," in Public-Sector Bargaining, Benjamin Aaron, Joseph R. Grodin, and James L. Stern, eds. Bureau of National Affairs, Washington, DC, 1979, pp. 150-190.

5. Mood, Alexander M., Graybill, Franklin A., and Boes, Duane C., Introduction to the Theory of Statistics, 3d ed. McGraw-Hill, New York, 1974.

6. Rehmus, Charles M., "Interest Arbitration," in Portrait of a Process: Collective Negotiations in Public Employment, Public Employment Relations Services, ed. Labor Relations Press, Fort Washington, PA, pp. 209-223.

7. Stern, James L., Rehmus, Charles M., Lowenberg, J. Joseph, Kasper, Hirschel, and Dennis, Barbara D., Final-Offer Arbitration. D. C. Heath, Lexington, MA, 1975.

8. Stevens, Carl M., "Is Compulsory Arbitration Compatible with Bargaining?" Industrial Relations, Vol. 5, No. 2 (February 1966), pp. 38-52.