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ABSTRACT. An algorithm is given that provides an envy-free division of a cake among n people, each of whom may have a different way of measuring the value of fractional parts of the cake. Previous results were (1) non-constructive, (2) produced a division based on the weaker notion of fairness in which each person believed he or she received at least $1/n^{th}$ of the cake, but not necessarily the largest piece, or (3) worked only for the case n = 3.

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1. INTRODUCTION

The general cake division problem dates back almost fifty years to Knaster (1946) and Steinhaus (1948, 1949) and can be roughly phrased as follows: Suppose we have a cake C and n people who may value different parts of the cake differently. Can one find a way to divide C among the n people so that each is satisfied with his share? Over the years, a number of expository treatments of this problem have appeared, including those of Gamow and Stern (1958), Dubins and Spanier (1961), Kuhn (1967), Gardner (1978), Rebman (1979), Bennet et al (1987), Hill (1991), and Olivastro (1992a).

A consideration of the cake division problem requires addressing the following three issues: (1) a choice of the mathematical framework in which to formalize the problem, (2) a definition of "satisfaction," and (3) a decision as to whether a "division process" is to be an algorithm or an existence proof. We briefly discuss each of these in turn.

The mathematical framework we choose to work with is quite general. We assume that C is a set and $\mu_1, \ldots, \mu_n$ are finitely additive measures defined on some common algebra A of subsets of C. We assume that $\mu_i(C) = 1$ for every i and that, if $X \in A$ and k is finite, then X can be partitioned into k pairwise disjoint sets of equal μ_i-measure. Finally, we assume that the following Trimming Postulate (TP) is satisfied for every i:

TP: If $X, Y \in A$ and $\mu_i(X) < \mu_i(Y)$, then there exists a set $Z \subset Y$ so that $Z \in A$ and $\mu_i(X) = \mu_i(Z)$.

Intuitively, TP corresponds to the ability of each person ("player") to trim a piece of cake Y to a size determined by X.

Another framework for the cake-division problem is to assume the measures are non-atomic and countably additive, although non-measure theoretic interpretations have also been considered (e.g., Stromquist, 1980). Our formulation is quite close to the original one of Steinhaus (1948).

In discussing whether or not a player will be satisfied with the share of cake he receives, there are two fundamental notions that have been the center of investigation. Let us call an ordered partition $X_1, \ldots, X_n$ of C _fair_ if $\mu_i(X_i) \geq 1/n$ for every i, and _envy-free_ if $\mu_i(X_i) \geq \mu_i(X_j)$ for every i and j. Thus, a partition is fair if it corresponds to a division of the cake into pieces such that each player thinks he received at least $1/n^{th}$ of the cake. A partition is envy-free if it corresponds to a division of the cake in which no player would desire to swap the piece he received for the piece another player received. It is easy to see that every envy-free partition is fair, but not every fair partition is envy-free.

There is a stronger version of fairness found in the literature. Steinhaus (1948) attributed the following observation to Knaster: if the n measures are not all identical, then C can be divided so that each player thinks he is receiving _more_ than $1/n^{th}$ of the cake. Proofs were independently provided by Urbanik (1955) and Dubins and Spanier (1961). These proofs are non-constructive, but an algorithmic proof was later found by Woodall (1986), based on an idea of Fink (1964).

We strengthen Woodall's result in the present paper by producing a division of the cake that is not only strongly fair, but also strongly envy-free. That is, not only does each player think his piece is _as large_ as everyone else's piece, but each player thinks his piece is _strictly larger_ than the pieces of those players who have a measure that is different from his own. It is again easy to see that a partition that is envy-free in this stronger sense is also fair in the stronger sense considered by Knaster, Urbanik, Dubins and Spanier, and Woodall.

The primary distinction in the cake-division literature that has been made over the past half century is between algorithmic procedures (like those of Banach and Knaster, Conway, Selfridge, Guy, Fink, and Woodall discussed in section 2) and existence proofs (like those of Neyman, Steinhaus, Urbanik, Dubins and Spanier, and Stromquist, also discussed in section 2). Typically, algorithms provide divisions that are fair, whereas existence proofs provide divisions with the stronger property of envy-freeness. "Moving knife" solutions - some of which are described in section 2 - seem to lie somewhere between the two, being on the one hand quite constructive, at least as long as one sticks with a framework quite true to the cake metaphor but, on the other hand, not really the kind of finite, discrete process one associates with the term algorithm.

In what follows we provide a finite and discrete algorithm for obtaining an envy-free partition of a cake among n players. The lack of such an algorithm for $n > 3$ has been explicitly mentioned by Gamow and Stern (1958), Gardner (1978), Rebman (1979), Woodall (1980), Austin (1982), Woodall (1986), Bennett et al. (1987), Webb (1992), Hill (1991, 1992), and Olivastro (1992a). More precisely, we prove the following:

THEOREM. Suppose C is a set and $\mu_1, \ldots, \mu_n$ are finitely additive measures defined on some common algebra A of subsets of C so that $\mu_i(C) = 1$ for every i and so that TP is satisfied. Then there exists a finite algorithm for producing an ordered partition $X_1, \ldots, X_n$ of C that is envy-free. Moreover, if we have available triples $(X_{ij}, \mu_i(X_{ij}), \mu_j(X_{ij}))$ so that if $\mu_i \neq \mu_j$, then $\mu_i(X_{ij}) \neq \mu_j(X_{ij})$, then there exists a finite algorithm for producing an ordered partition $X_1, \ldots, X_n$ of C that is strongly envy-free.

The rest of the paper is organized as follows. In section 2 we present a brief history of the mathematical problem of fair division organized around five main themes: fairness, strong fairness, envy-freeness, Pareto-optimality, and entitlements.

In section 3 we present an infinite constructive procedure for obtaining an envy-free division of a cake. This serves as an easy warm-up for the results in the next three sections. However, it also does more than this. The infinite scheme can be truncated to give a rather simple finite algorithm for obtaining a division that is "within ε" of being envy-free. For real-world applications, of course, this is often enough. We will say more about such applications elsewhere.

In section 4 we present the finite algorithm for obtaining an envy-free division of a cake.

In section 5, we show how the algorithm from section 4 can be cast as a game with "rules" and "methods" of the kind mentioned in Steinhaus (1948, p. 102). The importance of this game-theoretic framework is that one can then see that each player has a strategy that will, in the words of Austin (1962, p. 212), "ensure justice for an honest person even when there is dishonest collusion by the other people."

Section 6 contains the generalization of Woodall's strongly fair algorithm to the context of strongly envy-free divisions. Section 7 contains an entitlement result in the context of no envy. In section 8 we show how the methods of the present paper can be adapted to the situation where one wants a small piece of cake rather than a large piece. Gardner (1978) discussed this in the context of dividing up "chores." Finally, section 9 offers a framework for relating present results to a number of open questions.

We owe thanks to many people. Our interest in fair division was sparked by Olivastro (1992a). Valuable mathematical contributions were made by Fred Galvin and William Zwicker, and will be pointed out at the appropriate places. Specific observations by Sergiu Hart, Douglas Woodall, and William Webb also proved helpful. In addition, we have benefitted from conversations and correspondence with the following people: Ethan Akins, Julius Barbanel, Morton Davis, Karl Dunz, David Gale, Martin Gardner, Theodore Hill, D. Marc Kilgour, Jerzy Legut, Herve Moulin, Barry O'Neill, Philip Reynolds, William Thomson, Hal Varian, and Charles Wilson.

2. SOME HISTORY OF THE PROBLEM

In this section we identify five themes in the mathematical history of fair division that are relevant to the considerations of the present paper. For each of these five themes, we will briefly trace the sequence of results that unfolded over the past half century.

Theme 1. Fairness

Within this theme we are concerned only with the basic issue of obtaining an ordered partition $A_1, \ldots, A_n$ so that $\mu_i(A_i) \geq 1/n$ for $i = 1, \ldots, n$. The first result here, of course, is the "one cuts and the other chooses" algorithm for $n = 2$. Its origin is apparently unknown. The modern era of cake cutting began with Steinhaus' observation "during the war" (Steinhaus, 1948, P. 102) that the cut-and-choose scheme easily extends to $n = 3$. This is discussed in Knaster (1946). The following description of Steinhaus' procedure is based on the presentation in Kuhn (1967).

Let us call a piece of cake <u>acceptable</u> to a player if that player thinks the piece is of size at least 1/3. One player (say

A) divides the cake into 3 pieces that he considers acceptable. If either of the other two players (say, B) thinks that two or more of the pieces are acceptable, then we let player C take any one of the three pieces he considers acceptable. Player B then has available at least one of the pieces he considered to be acceptable, and of course player A considers the remaining piece to be acceptable.

On the other hand, if player B thinks at most one piece is acceptable, and player C thinks at most one piece is acceptable, then there is a single piece that players B and C agree is not acceptable. But now we can give this piece to player A and let players B and C redivide what remains according to cut-and-choose. This yields a division that each player finds acceptable.

Having found this solution for $n = 3$, Steinhaus asked whether a procedure could be found for $n > 3$. This was answered by Banach and Knaster and reported in Steinhaus (1949, pp. 315-316), and again in Steinhaus (1948, p. 102) as follows:

> The partners being ranged A, B, C, . . . , N, A cuts from the cake an arbitrary part. B has now the right, but is not obliged, to diminish the slice cut off. Whatever he does, C has the right (without obligation) to diminish still the already diminished (or not diminished) slice, and so on up to N. The rule obliges the "last diminisher" to take as his part the slice he was the last to touch. This partner thus disposed of, the remaining n-1 persons start the same game with the remainder of the cake. After the number of participants has been reduced to two, they apply the classical rule for halving the remainder.

A more well-known version of the Banach-Knaster solution is the "moving-knife" scheme, which is apparently due to Dubins and Spanier (1961, p. 2). They present it as follows:

> A knife is slowly moved at constant speed parallel to itself over the top of the cake. At each instant the knife is poised so that it could cut a unique slice of the cake. As time goes by the potential slice increases monotonically from nothing until it becomes the whole cake. The first person to indicate satisfaction with the slice then determined by the position of the knife receives that slice and is eliminated from further distribution of the cake. (If two or more participants simultaneously indicate satisfaction with the slice, it is given to any one of them.) The process is repeated with the other n-1 participants and with what remains of the cake.

They also point out that it is an easy matter to verify that if each of the players adheres to the strategy of calling "cut" any time such an action would yield him a piece of size 1/n, then "independently of the strategies of the other participants, even allowing for coalitions and duplicity, it is assured that . . . [each] possesses a strategy which ultimately yields him at least 1/nth of the original cake according to his own evaluation" (Dubins and Spanier, 1961, p. 2).

Two other algorithms were presented in the 1960s, one by Fink (1964) and one by Kuhn (1967). The one by Fink, in particular, led to later results of some importance (Even and Paz, 1984; Woodall, 1986). Hill (1991, p. 4) describes Fink's algorithm as follows.

> In Fink's algorithm, the first person bisects the cake C according to his own measure. The second person arriving chooses between the two pieces cut by the first player, and if a third person arrives then each of these first two players trisects his own portion, and the third person selects one portion from each. The algorithm continues in this manner (e.g., quadrisection at the next stage) until no new arrivals appear and the algorithm terminates.

Theme 2. Strong Fairness

One of the most influential works on fair division is Dubins and Spanier (1961), and one of the most oft-quoted results from that paper concerns the question of whether everyone can get more than his fair share. Dubins and Spanier give an existence proof that, except in the obviously prohibitive case where all the measures are the same, there exists an ordered partition $A_1, \ldots, A_n$ of C such that $\mu_i(A_i) > 1/n$ for each i. We will call such an ordered partition strongly fair (although "superfair" is used by Hill, 1991 and, in a different sense, by Baumal, 1985). However, this result had already been discovered and rediscovered. Thus, for example, Steinhaus (1948, pp. 102-103) reports:

> It may be stated incidentally that if there are two (or more) partners with different estimations, there exists a division giving to everybody more than his due part (Knaster); this fact disproves the common opinion that differences in estimations make fair division difficult.

Moreover, Urbanik (1955) first provided a proof of the strong fair-division result, but in the more narrow context of measures that are absolutely continuous with respect to Lebesgue measure. (Dubins and Spanier assume only countable additivity of the measures.) What exactly Knaster assumed can not be gleaned from Steinhaus (1948).

The next contribution to strong fair division seems to be the observation of Rebman (1979) that one can, in fact, algorithmically get a strongly fair division if one postulates there to be no (proper) slice of the cake on which any two people agree. This assumption, however, is highly restrictive and leaves open the question of whether one can algorithmically get a strongly fair division assuming only that two of the measures are distinct. After showing the existence of such a strongly fair division, Rebman (1979, p. 33) wistfully remarks that it provides "no clue as to how to accomplish such a wonderful partition."

This problem was solved by Woodall (1986) in the following sense. Assuming there is a set X to which two players, say A and B, assign different measures $\alpha > \beta$, Woodall provides a finite algorithm, based on that of Fink (1964), which produces a strongly fair division of the cake.

Finally, more recent investigations have been concerned with the problem of obtaining quantified versions of strongly fair divisions. For example, Elton, Hill, and Kertz (1986) showed (via an existence proof) that one can always get a division in which $\mu(A_i) \geq 1/(n - M + 1)$, where M is the total mass of $\vee\mu_i$. Hill (1987) obtained a similar result using $\wedge\mu_i$. The Elton-Hill-Kertz result was strengthened by Legut (1988).

Theme 3. Envy-Freeness

The earliest existence proofs for envy-free division go back to the 1940's. The first appears to be Neyman's (1946) proof that if one has n measures $\mu_1, \ldots, \mu_n$, then for every k there exists a partition $A_1, \ldots, A_k$ so that $\mu_i(A_j) = 1/k$ for every i and j. The fact that his result allows $k \neq n$ yields a solution to Fisher's (1938) "problem of the Nile," which is discussed in Dubins and Spanier (1961). Steinhaus, himself, announced in his 1949 paper that, using methods similar to Stone and Tukey (1942), it is possible to prove the following: if $\alpha_i > 0$ for every i and $\Sigma\alpha_i = 1$, then there exists a partition $A_1, \ldots, A_n$ of the cake so that $\mu_i(A_j) = \alpha_j$ for every i and j.

Existence proofs for both these results can be found in Dubins and Spanier (1961). An additional existence result of some significance is the one obtained independently by Stromquist (1980) and Woodall (1980). In the case where the cake is an interval on the real line, they showed the existence of a partition that is both envy-free and connected. The results of Weller (1985) and Berliant, Dunz, and Thomson (1992) are also relevant here, but they will be discussed under theme 4.

The first algorithmic results for envy-free division all appeared around 1980 and worked only for the case n = 3. Selfridge, Conway, and Guy have schemes attributed to them by Gardner (1978), Woodall (1980), Stromquist (1980), and Austin (1982), but the problem seems to have first been posed in Gamow and Stern (1958, pp. 117-119). Austin (1982, p. 214) describes the scheme he attributes to Conway (and others) as follows:

> Person 1 cuts the cake into 3 pieces.
> Person 2 cuts a bit off at most 1 of the 3 pieces.
> Person 3 takes one of the 3 pieces.
> Person 2 takes one of the 2 remaining pieces but he must not leave the piece he cut a bit off, if in fact he did cut a bit off.
> Person 1 takes the remaining piece.
> If person 2 did not cut any off then the procedure is finished.
> Suppose person 2 did cut a bit off.
> The one of persons 2 and 3 who did not get the cut piece above, takes the bit cut off and divides it into 3 pieces.
> The other of persons 2 and 3 takes one of the three pieces.
> Person 1 takes one of the 2 remaining pieces.
> The person who divided the bit into 3 takes the remaining piece.
>
> The strategy for each person is as follows. Person 1 divides the cake into 3 equal pieces. Person 2 cuts the largest piece so as to make it equal to the larger of the other 2 pieces (he may not actually have to make a cut if there is the appropriate equality already). The divider of the bit cuts that bit into 3 equal pieces. The reader is left to show that the strategy works. He should let himself be persons 1, 2, 3 in turn and show that in each case he

> can ensure the required aim for himself, whatever the other 2 persons get up to.

Woodall (1980) presents the same scheme and attributes it to Selfridge.

Stromquist (1980, p. 641) has a moving-knife version of an envy-free algorithm for n = 3. He describes it as follows.

> A referee moves a sword from left to right over the cake, hypothetically dividing it into a small left piece and a large right piece. Each player holds a knife over what he considers to be the midpoint of the right piece. As the referee moves his sword, the players continually adjust their knives, always keeping them parallel to the sword. When any player shouts "cut," the cake is cut by the sword and by whichever of the players' knives happens to be in the middle of the three.
>
> The player who shouted "cut" receives the left piece. He must be satisfied, because he knew what all three pieces would be when he said the word. Then the player whose knife ended nearest to the sword, if he didn't shout "cut," takes the center piece. The player whose knife was farthest from the sword, if he didn't shout "cut," takes the right piece. The player whose knife was used to cut the cake, if he hasn't already taken the left piece, will be satisfied with whichever piece is left over. If ties must be broken - either because two or three players shout simultaneously or because two or three knives coincide - they may be broken arbitrarily.

Notice that the correct strategy is not to call "cut" when you think the referee's knife is at the one-third point, but only when you will be non-envious by the resulting distribution. This point has caused confusion and required clarification at different times (Stromquist, 1981; Austin and Stromquist, 1983; Olivastro, 1992a, 1992b).

Finally, there is another procedure for producing an envy-free division among three people. It is due to Levmore and Cook (1982) and seems to have been largely overlooked. It is essentially a moving knife algorithm (although they describe it as a process with "infinitely small shavings"). It can be described as follows:

Player 1 divides the cake into three pieces P, Q, R that he considers equal. Each of the other two players selects the pieces he considers largest. If they choose different pieces, we are done. Otherwise, we can assume they both choose P. Now player 1 starts a moving knife as in the Banach-Knaster scheme, but at the same time he places a second knife perpendicular to the first and over the portion of the cake that the first knife has already swept over. Notice that if the knives were to make a cut from such a positioning, the cake would be cut into three pieces, exactly two of which would involve both knives. Let S denote the leftmost of these two and let T denote the rightmost. The second knife is moved back and forth in such a manner that player 1 thinks $Q \cup S$ is the same size as $R \cup T$. In Figure 1 below, the pieces are drawn from player 2's point of view.

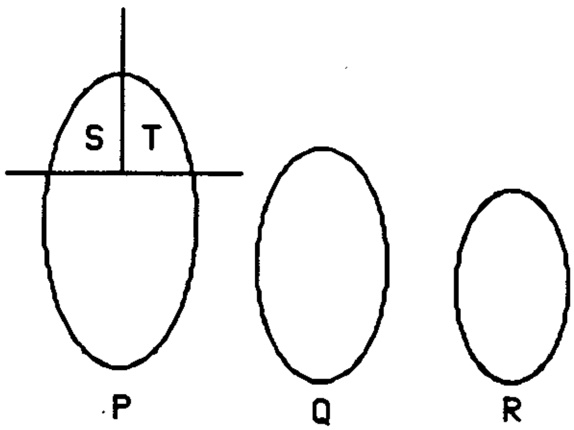

Figure 1

When the process begins, both S and T are the empty set, so player 2 and player 3 both think P-(SUT) is larger than both QUS and RUT. Now let either player 2 or player 3 yell when the first of these four inequalities reverses. Without loss of generality, assume that player 2 is the one to yell and that it is because he now thinks P-(SUT) is no longer larger than RUT. At this point we can obtain an envy-free division by giving to player 2 the piece RUT, player 3 the piece P-(SUT), and player 1 the piece QUS.

Levmore and Cook (1981, p. 53) also assert that:

> "The most interesting aspect of shaving is that it does the trick for three *and* four people. In fact, it looks useful for *any* number of claimants, subject to the "objection" that we might need infinitely small shavings until we can declare with confidence that people are equally happy (indifferent, in economics) with any two or more packages.

There description for n = 4 is phrased in terms of four daughters dividing up land. It goes as follows.

> The four daughters are A, B, C, and D. Have A divide the land into four plots such that she would be happy with any of them. Call the plots 1, 2, 3, and 4. The other three daughters indicate their preferences. Needless to say we are stuck with somewhat similar tastes for the purpose of this problem. Perhaps B and C settle into plot 1 and D chooses 2. Now A must shave from 1 into 2, 3, and 4. (We might have had the shaving done into 3 and 4, figuring that we can leave well enough alone in plot 2. However, as we increase 3 and 4 we can count on D's increasing jealousy and her reentry, so we take this into account at the outset.) A continues the shaving until B or C moves into a new plot. Whenever two daughters are in the same plot (or three in the same plot), as when B moves into 2 in competition with D, A

turns her attention to that plot and shaves from it into all the others (from 2 into 1, 3, and 4).

This, however, does not seem to work. For example, suppose that daughter C thinks that plot 1 has measure one and that daughter D thinks plot 2 has measure one. Assume daughter B thinks plots 1 and 2 have measures 5/8 and 3/8 respectively. Suppose that on the first trimming, daughter A chooses subsets X, Y, and Z of plot 1 that A, C, and D think are of measure zero and transfers these to plots 2, 3, and 4. Assume daughter B thinks X has measure 1/4 and Y and Z have measure zero. The result is that daughter B will jump to plot 2. Now daughter A still thinks all four plots are the same size, daughter C still thinks plot 1 has measure one, and daughter D still thinks plot 2 has measure one. Daughter B now thinks plots 2 and 1 have measures 5/8 and 3/8 respectively. Thus, the situation is exactly as when we started (except that the roles of plots 1 and 2 have been reversed). Thus, the Levmore-Cook scheme (at least as described above) could have daughter B jumping back and forth between plot 1 and plot 2 with no sense of convergence except the transfer of sets of measure zero from plots 1 and 2 to plots 3 and 4. Whether or not a moving knife version of this procedure can be made to work remains to be seen. At any rate, the Levmore-Cook ideas deserve more attention than they have received.

Theme 4. Pareto-Optimality

Given measures $\mu_1, \ldots, \mu_n$, an ordered partition $A_1, \ldots, A_n$ is said to be Pareto optimal if there is no other ordered partition $B_1, \ldots, B_n$ such that $\mu_i(B_i) \geq \mu_i(A_i)$ for every i with strict inequality holding for at least one i. Economists consider Pareto optimality to be of considerable importance, as pointed out in Berliant-Dunz-Thomson (1992).

The first result having implications for the question of Pareto optimality seems to be Urbanik's 1955 existence proof of a partition that maximizes (among all partitions) the fraction of the cake that the person who fares the worst will get. That is, it maximizes $\min(\mu_i(A_i)\colon i=1, \ldots n)$ over all partitions (Legut, 1988). This immediately implies the division is fair, since the existence of a fair division shows this minimum must be at least 1/n. Moreover, it is Pareto optimal in the context of measures that are absolutely continuous with respect to each other. (If you could make one person slightly better off without making anyone else worse off, you could transfer a bit of that person's gain to whomever fared the worst.) Dubins and Spanier (1961) also obtained partitions that are fair and Pareto-optimal.

The existence of partitions that are envy-free and Pareto-optimal was first established by Weller (1985) in the context of measures that are countably additive. A similar result, yielding a stronger notion of no envy but requiring absolute continuity of the measures involved, was found by Berliant, Dunz and Thomson (1992). As far as we know, there are no algorithmic results concerning Pareto optimality in the context of either fair or envy-free divisions.

Theme 5. Entitlements

In many real-world situations, one seeks a mutually satisfactory distribution in which some people - by law, mutual agreement, or otherwise - are entitled to more of a good than others. This immediately suggests a version of fair division where one starts with a sequence $\alpha_1, \ldots, \alpha_n$ of positive reals (or rationals) whose sum is 1, and then seeks an ordered partition $A_1, \ldots, A_n$ so that $\mu_i(A_i) \geq \alpha_i$. Dubins and Spanier (1961) point out that that Neyman's (1946) result immediately gives such a division in the case where each α_i is rational.

However, the earliest explicit reference to an entitlement version of fair division again goes back to Steinhaus (1948, p. 103). Referring to the Banach-Knaster scheme, he says: "The procedure described above applies also, under slight modifications, to the case of different (but rational) ideal shares." Steinhaus (1948, p. 103) actually states his envy-free existence theorem in the context of entitlements. Legut (1985) goes a step further and shows that if we have an infinite sequence $\langle \alpha_i \rangle$ of positive reals whose sum is one, then there exists an (infinite) ordered partition $\langle X_i \rangle$ so that $\mu_i(X_i) = \alpha_i$ for all i. A final paper of interest in this context is that of McAvaney, Robertson, and Webb (1992), in which an entitlement question is nicely settled in the context of what they call Ramsey partitions.

We have restricted ourselves to the above five themes because they are the ones most relevant to the considerations of the present paper. Nevertheless, there are at least two others that should not go unmentioned.

The first concerns the minimum number of cuts required for fair and envy-free divisions. Once again, Steinhaus (1948, p. 103) first raised the issue: "Interesting mathematical problems arise if we are to determine the minimal number of 'cuts' necessary for fair division." The two most direct responses to Steinhaus' question seem to be the $\Theta(n\log n)$-cut versions of Fink's algorithm found by Even and Paz (1984) and Webb (1991). Robertson and Webb (1992) also consider the problem of approximating fair division with a limited number of cuts.

The second theme concerns partitioning a piece of land. Early work here included Hill's (1983) existence proof of a suitable partition in this context. Beck (1987) provided a moving-knife solution for the same problem, and Webb (1990) provided a finite algorithm.

3. THE INFINITE ALGORITHM

In the proofs that follow, one may think of the players actually going through a step-by-step process of dividing a cake among themselves. Thus, if we use a phrase like, "player i thinks X_i is at least as large as any X_j for $j \neq i$," it may be interpreted as: "For all $j \neq i$, $\mu_i(X_i) \geq \mu_i(X_j)$." Whenever we use the phrase "piece of cake," we shall mean a set X in the common algebra A. Finally, a collection $X_1, \ldots, X_n$ of pairwise disjoint pieces of cake will be called an envy-free partition if each player thinks his piece (X_i always goes to player i) is at least as large as everyone else's. (The union of the X_i's after any finite number of steps in the procedure described next will typically not be all of C.)

Our starting point is the following:

LEMMA 3.1. Suppose P is a piece of cake. Then there exists a pairwise disjoint partition $X_1, \ldots, X_n$, L of P into n+1 pieces so that:

(i) $X_1, \ldots, X_n$ is an envy-free partition; and

(ii) Player n thinks the size of the "leftover" piece L is at most $(2^{n-1}-1)/2^{n-1}$ that of P.

PROOF. Player n begins the process by dividing P into 2^{n-1} pieces that he considers equal and among which, therefore, he is therefore indifferent. Players n-1, n-2, . . . , 2 now go in order (n-1 next, then n-2, etc.), with player k proceeding as follows. Given the collection of 2^{n-1} pieces (now perhaps altered, as discussed next), he first arranges them according to size, with ties grouped together, and then identifies the 2^{k-1st} piece in the list, counting from largest to smallest. He then trims all the strictly larger pieces down to the size of this 2^{k-1st} piece; the trimmings are set aside. The result, minus the trimmings, is a collection of size at least 2^{k-1} that he considers tied for largest. Player k now passes this collection of 2^{n-1} pieces on to player k-1. This continues until there is only one player remaining (i.e., player 1).

Once all the trimming by players n-1, . . . , 2 is done, the players proceed in the order 1, 2, . . . , n to choose one of the (perhaps trimmed) pieces. Player 1, who gets to make the first choice, obviously will not envy any other piece. Notice that for k = 2, . . . , n, player k will have, among the 2^{k-1} or more pieces he considers tied for largest, at least one that has neither been trimmed nor chosen because

$$1 + 2 + 4 + \ldots + 2^{k-2} = 2^{k-1}-1 < 2^{k-1}.$$

Let X_i be the piece chosen by player i and let

$$L = P - (X_1 \cup \cdots \cup X_n).$$

Notice that player i thinks X_i is large as any X_j since player i is choosing one of the pieces he considered tied for largest, while everyone else is choosing - from his point of view - a piece that was the same or smaller (possibly because it was later trimmed). This shows that (i) holds. For (ii), simply notice that $\mu_n(X_n) = \mu_n(P)/(2^{n-1})$ and $L \subset P-X_n$. This completes the proof of Lemma 3.1.

A variant of the above process allows one to begin with fewer than the 2^{n-1} pieces used in the above proof. For example, if one demands that a player take a piece he actually trimmed if one is available, then the number of pieces that player n must cut initially is $2^{n-2} + 1$. This essentially halves the number of initial pieces and could be significant in applications that will be described in subsequent work.

For future reference, it is worth noting that if player n-1 picks a piece he trimmed, then it is possible to ensure that player n thinks the (untrimmed) piece he receives is strictly larger than the (trimmed) piece received by player n-1. This is possible because player n-1 can trim a portion of piece p so that each player who regards P as being of positive measure also views the trimmed portion to be of positive measure. Thus, suppose player i wants to trim a portion with μ_i-measure β from P. Then he can simply partition P into enough equal-size subpieces (according to his measure) so that he thinks each subpiece is of size at most β/n. Each player j, with $\mu_j(P) > 0$, can now choose and delete one of the subpieces that he (player j) thinks is of positive measure. Because a set of size at most β has been deleted from P, we get a trimming with the desired property.

The next lemma simply says that if we iterate the scheme from Lemma 3.1, we arrive at an envy-free partition of part of the cake, with a leftover piece L that player n thinks is arbitrarily small.

LEMMA 3.2. Suppose P is a piece of cake and $\varepsilon > 0$. Then there exists a pairwise disjoint partition $X_1, \ldots, X_n$, L of P into n+1 pieces so that:

(i) $X_1, \ldots, X_n$ is an envy free-partition; and

(ii) Player n thinks the size of the "leftover" piece L is less than ε.

PROOF. Let $X_{11}, \ldots, X_{1n}$, L_1 be the partition of P guaranteed to exist by Lemma 3.1, and let $\beta = (2^{n-1}-1)/2^{n-1}$. Now, apply Lemma 3.1 again, but with L_1 playing the role of P. This yields a partition $X_{21}, \ldots, X_{2n}$, L_2 of L_1 that is envy-free and has the property that $\mu_n(L_2) \leq \beta\mu_n(L_1) \leq \beta^2\mu_n(P)$. Continue this until k is large enough so that $\beta^k < \varepsilon$, and let $X_i = X_{1i} \cup \cdots \cup X_{ki}$. (That is, collect together player i's pieces from each step in the iteration.) Let $L = L_k$. Since each player is non-envious at each stage of the iteration, he is non-envious when the pieces are collected together. Moreover,

$$\mu_n(L) = \mu_n(L_k) \leq \beta^k\mu_n(P) \leq \beta^k < \varepsilon.$$

This completes the proof of Lemma 3.2

If we now iterate the previous lemma n times (with the k^{th} application being applied to the piece leftover after the k-1^{st} application), and give each player a chance to play the role of player n for exactly one of the iterations, then we immediately obtain the following:

THEOREM 3.3. Suppose P is a piece of cake and $\varepsilon > 0$. Then there exists a pairwise disjoint partition $X_1, \ldots, X_n$, L of P into n+1 pieces so that:

(i) $X_1, \ldots, X_n$ is an envy-free partition; and
(ii) <u>everyone</u> thinks the size of the leftover piece L is less than ε.

4. THE FINITE ALGORITHM

In this section we show how the infinite procedure can be made finite. The basic idea is the following. Recall that the infinite procedure began with player n's dividing the cake into 2^{n-1} pieces that he considered to be the same size; then player n-1 (and the others) trim some of the pieces, if necessary, to create ties among the ones each considers to be largest.

Now suppose that we can arrange the division so that player n-1 definitely trims some piece and definitely chooses one of the pieces he has trimmed. Then player n will think that his piece is strictly larger (say by ε) than the piece received by player n-1. Theorem 3.3 can now be applied to extend this division to an envy-free partition of all of the cake, except for a leftover piece L that everyone thinks is of size less than ε. But then player n will be perfectly content to have the leftover piece L given to player n-1. Of course, others might object, so we have to proceed a little more carefully than this. Nevertheless, this is roughly the plan of attack.

The proof itself requires the following four lemmas. Our original proof of Lemma 4.3 below was based on an idea suggested to us by William Zwicker (1992). The version of that proof given below is somewhat simpler and is due to Fred Galvin (1992). The proofs of the other lemmas have also benefitted from a reworking of our original proof by Galvin (1992).

LEMMA 4.1. Let $k = 2^{n-2}$ and let P be a piece of cake. Suppose we have disjoint subsets $Y_1, \ldots, Y_k, Z_1, \ldots, Z_k$ of P such that $\mu_n(Y_i) > \mu_n(Z_j)$ and $\mu_{n-1}(Y_i) \le \mu_n(Z_j)$ for all i, j = 1, . . . k. Then there exists a pairwise disjoint partition $X_1, \ldots, X_n$, L of P into n+1 pieces so that:

(i) $X_1, \ldots, X_n$ is an envy-free partition; and
(ii) $\mu_n(X_n) > \mu_n(X_{n-1})$.

PROOF. First, player n trims the sets $Y_1, \ldots, Y_n$ to the size of the smallest one, and player n-1 does the same with $Z_1, \ldots, Z_k$. Players n-2, n-3, . . . , 2 now go in order (n-2 next, then n-3, etc.) and trim the sets down exactly as in the proof of Lemma 3.1. The selection then begins with player 1, as in the proof of Lemma 3.1 again, except that player n-1 now takes an untouched piece from $\{Z_1, \ldots, Z_k\}$ and player n takes an untouched piece from $\{Y_1, \ldots, Y_k\}$.

LEMMA 4.2. Let $k = 2^{n-2}$, and let P be a piece of cake. Suppose we have disjoint subsets $Y_1, \ldots, Y_k, Z_1, \ldots, Z_k$ of P such that $\mu_n(Y_i) > \mu_n(Z_j)$ and $\mu_{n-1}(Y_i) \leq \mu_n(Z_j)$ for all i, j = 1, . . . k. Then there exists a pairwise disjoint partition $X_1, \ldots, X_n$, L of P into n+1 pieces so that:

(i) $X_1, \ldots, X_n$ is an envy-free partition; and
(ii) $\mu_n(X_n) > \mu_n(X_{n-1} \cup L)$.

PROOF. Use Lemma 4.1 and then use Theorem 3.3 with $\varepsilon = \mu_n(X_n) - \mu_n(X_{n-1})$.

LEMMA 4.3. Suppose k is a positive integer, and assume we have sets A and B so that $\mu_n(A) > \mu_n(B)$ and $\mu_{n-1}(A) = \mu_{n-1}(B)$. Then we can find disjoint sets $Y_1, \ldots, Y_k, Z_1, \ldots, Z_k \subset A \cup B$ such that $\mu_n(Y_i) > \mu_n(Z_j)$ and $\mu_{n-1}(Y_i) \leq \mu_n(Z_j)$ for all i, j = 1, . . . k.

PROOF. We can assume A and B are disjoint. Let m be a positive integer such that

(1) $\mu_n(A) > (1 + k^2/m)\mu_n(B)$.

Let player n-1 divide each of A and B into m+k equal pieces, which we index so that $\mu_n(A_1) \geq \ldots \geq \mu_n(A_{m+k})$ and $\mu_n(B_1) \geq \ldots \geq \mu_n(B_{m+k})$. Let $\{Z_1, \ldots, Z_k\} = \{B_{m+1}, \ldots, B_{m+k}\}$. Notice that:

(2) $\mu_n(A) \leq m\mu_n(A_{k+1}) + k\mu_n(A_1)$; and

(3) $\mu_n(B) \geq m\mu_n(B_m)$.

Substituting (2) and (3) into (1), we see that

(4) $m\mu_n(A_{k+1}) + k\mu_n(A_1) > m\mu_n(B_m) + k^2\mu_n(B_m)$.

Thus, either $\mu_n(A_{k+1}) > \mu_n(B_m)$, in which case we let $\{Y_1, \ldots, Y_k\} = \{A_1, \ldots, A_k\}$, or else $\mu_n(A_1) > k\mu_n(B_m)$, in which case we let player n divide A_1 into k equal pieces and call these $Y_1, \ldots, Y_k$.

LEMMA 4.4. Suppose P is a piece of cake, and assume we have sets $A, B \subset P$ so that $\mu_n(A) > \mu_n(B)$ while $\mu_{n-1}(A) = \mu_{n-1}(B)$. Then there exists a pairwise disjoint partition $X_1, \ldots, X_n$, L of P into n+1 pieces so that:

(i) $X_1, \ldots, X_n$ is an envy-free partition; and
(ii) $\mu_n(X_n) > \mu_n(X_{n-1} \cup L)$.

PROOF. Use Lemma 4.3 with $k = 2^{n-2}$ and then use Lemma 4.2.

THEOREM 4.5. Suppose C is a set and $\mu_1, \ldots, \mu_n$ are finitely additive measures defined on some common algebra A of subsets of C so that $\mu_i(C) = 1$ for every i and so that TP is satisfied. Then there exists a finite algorithm for producing an ordered partition $X_1, \ldots, X_n$ of C that is envy-free.

PROOF. Player 1 splits C into n! sets he considers to be the same size. Let A be the collection of players who agree these sets are the same size, and let D be the collection of players who disagree. If $D = \emptyset$, we are done, since player 1 can simply give n!/n pieces to each of the other players. Otherwise, choose

$i \in D$ and $j \in A$ and apply Lemma 4.4 to get a partition $X_1, \ldots, X_n, L$ so that $X_1, \ldots, X_n$ is envy-free and $\mu_i(X_i) \geq \mu_i(X_j \cup L)$. In this case we say that "player i has an irrevocable advantage over player j."

We now turn to the "crumb" L and repeat the procedure, again with player 1 splitting L into n! sets that he considers to be the same size. Let A and D be as before. If there is some $i \in D$ and $j \in A$ so that player i does not yet have an irrevocable advantage over player j, then we apply Lemma 4.4 as above. Since there are only finitely many pairs i, j, we eventually reach a point where there are no such pairs i, j. We can now give $n!/|A|$ pieces to each player in A. This preserves envy-freeness as far as everyone in A is concerned, and it cannot make anyone in D envious, because each player in D already had an advantage (in his view) over everyone in A that exceeded the size of the crumb being distributed. This completes the proof.

Fred Galvin (1992) William Webb (1992b) and Douglas Woodall (1992) have pointed out to us that one can complete the proof using partitions into n pieces instead of the n! we used. Galvin (1992) describes the procedure as follows:

> Suppose we've reached a state $(X_1, \ldots, X_n, L)$ where X_i has been allocated to Player i and L is the unallocated portion; we assume that $\mu_i(X_i) \geq \mu_i(X_j)$ for all i and j. Construct a digraph on $\{1, \ldots, n\}$ by drawing an arc from j to i if Player i "envies" Player j, meaning that $\mu_i(X_i) < \mu_i(X_j \cup L)$. Let $\mathcal{E}$ be the set of nodes (i.e. players) accessible from 1, and let $m = \#(\mathcal{E})$. Let Player 1 divide L into m equal pieces. If all the players in $\mathcal{E}$ agree that the m pieces are equal, we finish the job by giving one of them to each player in $\mathcal{E}$, since no player outside $\mathcal{E}$ envies anybody in $\mathcal{E}$. Otherwise, we have two players $i, j \in \mathcal{E}$ such that j agrees with the division, i disagrees, and i envies j. In this case we use

[Lemma 4.4 with i and j replacing n and n-1] to allocate part of L so that i no longer envies j, and our digraph loses an arc. The procedure can't go on forever, since we will run out of arcs.

5. GAME-THEORETIC CONSIDERATIONS

A game entails the notion of free and independent choices by players who are constrained by rules that define legal play, but whose specific choices at every stage are not prescribed. A variety of solution concepts in game theory, including that of Nash equilibrium and the core, have been applied to cake-cutting by Legut (1986, 1987, 1988, 1990).

Our interest in a game-theoretic treatment of cake-cutting is inspired by Steinhaus (1948, 1969), Austin (1962), Kuhn (1967), and the above-cited work of Legut. Indeed, cake-cutting algorithms naturally correspond to, and are often explicitly described as, n-person non-cooperative games in which each player has a strategy that will guarantee him - regardless of what the other players collectively or individually know or do - an outcome that is satisfactory in some prescribed sense (e.g., ensures a piece of cake that is of size at least 1/n, or a piece that is at least as large as that of anyone else).

Such a guarantee does not rule out the possibility that a player could do even better by (unilaterally) abandoning this strategy in favor of another, and so we are not talking about a Nash equilibrium (Nash, 1951). On the other hand, even if "satisfaction" means only "fair," any such strategy will be what is called a maximin strategy: it maximizes the minimum one can get in the worse possible case. That is, it guarantees a player a piece of size 1/n, even when all n measures are the same and the game is therefore zero-sum.

In this section we indicate how the arguments in section 4 can be put in a game-theoretic framework. Specifically, we show that each player has a strategy, based on knowledge of his measure alone, that will guarantee him a piece of cake that he considers to be as large as that received by anyone else - even if the others know his measure and conspire to deny him this level of satisfaction. The presentation of such a game-theoretic result requires:

(i) a specification of the rules of the game; and

(ii) a description of the strategies with the desired properties.

The rules we propose require that players at various times make a verbal statement (although the rules cannot compel a player to tell the truth). The reader uncomfortable with this framework should realize that such verbal announcements can be avoided by letting a large fraction of moves be partitions that serve no purpose except to convey, via the number of sets in the partitions, such information using any standard coding of the English language (with names for all integers) as positive integers.

In fact, the proof of Theorem 4.5 can readily be translated into a game-theoretic form, given that players can make declarations as to whether they want to be considered to be in class A (agreement) or class D (disagreement) and that we have rules that label players as having an irrevocable advantage over some other players at certain times. What requires some elaboration is the subgame corresponding to the lemmas leading up to this. For example, in the proof of Lemma 4.3, we have player n-1 dividing a set into a number of equal pieces based on player n's measure of a set. In the game-theoretic framework, we cannot assume that one player knows another's measure. Issues such as this arise and are dealt with in the

Irrevocable Advantage Subgame

Suppose player 1 and player 2 have at their disposal disjoint sets A and B so that player 1 has declared A to be larger than B while player 2 has declared them to be the same size. Let $k = 2^{n-2}$.

Move 1. Player 1 announces an integer m.

Strategy: Player 1 chooses m so that $\mu_1(A) > (1 + k^2/m)\ \mu_1(B)$.

Move 2. Player 2 divides each of A and B into m + k pieces.

Strategy: He makes all the pieces the same size.

Move 3. Player 1 orders the pieces as $A_1, \ldots, A_{m+k}$ and $B_1, \ldots, B_{m+k}$ and sets $\{Z_1, \ldots, Z_k\} = \{B_{m+1}, \ldots, B_{m+k}\}$. He then either sets $\{Y_1, \ldots, Y_k\} = \{A_1, \ldots, A_k\}$, or he divides A_1 into k pieces and calls these $Y_1, \ldots, Y_k$.

Strategy: He orders the pieces so that $\mu(A_1) \geq \ldots \geq \mu(A_{m+k})$, and $\mu(B_1) \geq \ldots \geq \mu(B_{m+k})$. If $\mu_1(A_{k+1}) > \mu_1(B_m)$, he then sets $\{Y_1, \ldots, Y_k\} = \{A_1, \ldots, A_k\}$. Otherwise, he makes the pieces he is dividing A_1 into all the same size.

The proof of Lemma 4.3 shows that the result of this subgame is a collection of disjoint sets $Y_1, \ldots, Y_k, Z_1, \ldots, Z_k$ so that if player 1 has followed the aforementioned strategy, then $\mu_1(Y_i) > \mu_1(Z_j)$ for all i and j, and if player 2 has followed the aforementioned strategy, then $\mu_2(Y_i) \leq \mu_2(Z_j)$ for all i and j.

The next sequence of moves in the subgame corresponds in the obvious way to the trimming and choosing procedure described in Lemma 4.1 (and based on the infinite scheme from section 3). The rules will require player n-1 to take one of the pieces he trimmed. We leave it to the reader to convince himself that the obvious strategies are impervious to any hostile efforts of the others.

There is, however, an issue to be dealt with in going from the conclusion of Lemma 4.1 [that is, $\mu_1(X_1) > \mu_1(X_2)$] to the conclusion of Lemma 4.2 [that $\mu_1(X_1) > \mu_1(X_2 \cup L)$]. The first move here is the declaration by player 1 of a number p. The correct strategy is for him to choose p large enough that if the players iterate the trimming and choosing scheme p times, then he (at least) will think the size of the leftover piece is less than $\mu_1(X_1) - \mu_1(X_2)$.

Notice that we cannot dispense with the declaration of p in favor of simply letting player 1 say "stop" (strategically at the point when he thinks the leftover piece L is sufficiently small), because player 1 might keep the game going forever in an effort to deprive the other players of their due. (Recall that we are looking for strategies that protect one player from another even in the event that the other player is willing to inflict harm upon himself for the sake of harming the first.) The reader should also note that the convention about trimming presented in section 3 can be handled easily in a game-theoretic context.

6. STRONGLY ENVY-FREE DIVISIONS

Recall that a division of the cake among n people is said to be strongly fair if each player thinks he received strictly more than $1/n^{th}$ of the cake; it is said to be strongly envy-free if it is envy-free and, whenever two people have different measures, each thinks he received strictly more than the other. Our goal in this section is to generalize Woodall's 1986 strongly fair division result by proving the following.

THEOREM 6.1. Suppose C is a set and $\mu_1, \ldots, \mu_n$ are finitely additive measures defined on some common algebra A of subsets of C, so that $\mu_i(C) = 1$ for every i and so that TP is satisfied. Suppose we also have available triples

$$(X_{ij}, \mu_i(X_{ij}), \mu_j(X_{ij}))$$

so that if $\mu_i \neq \mu_j$, then $\mu_i(X_{ij}) \neq \mu_j(X_{ij})$. Then there exists a finite algorithm for producing an ordered partition $X_1, \ldots, X_n$ of C that is strongly envy-free.

PROOF. Our starting point in the proof is the following observation of Woodall (1986): Given a set X_{ij} as postulated to exist, one can algorithmically find disjoint sets $A, B \subset X_{ij}$ so that $\mu_i(A) > \mu_i(B)$ while $\mu_j(A) < \mu_j(B)$. It is now easy to see that Theorem 6.1 would be a trivial consequence of the basic envy-free algorithm (Theorem 4.5) if the sets X_{ij} postulated to exist were pairwise disjoint. That is, if the sets X_{ij} were pairwise disjoint, we could apply Lemma 4.4 to each one and obtain a partition $X_1, \ldots, X_n$, L of X_{ij} so that $X_1, \ldots, X_n$ is envy-free and $\mu_i(X_i) > \mu_i(X_j \cup L)$. We could then partition the rest of the cake (including the leftover pieces L in each X_{ij}) in an envy-free way by Theorem 4.5 to arrive at the desired strongly envy-free division.

We begin with some terminology. Suppose that X is a piece of cake and $1 \leq i,j \leq n$. Then we shall say that <u>X is ε-good for (i,j)</u> iff there exist $A, B \subset X$ such that:

(i) $A \cap B = \emptyset$,
(ii) $\mu_i(A) = \mu_i(B) > 0$.

(iii) $\mu_j(A) + \varepsilon = \mu_j(B)$.

Thus, X is ε-good for (i,j) iff there exist $A,B \subset X$ so that player i thinks A and B are the same size, but player j thinks B is ε larger than A.

We shall say that X is good for (i,j) iff X is ε-good for (i,j) for some $\varepsilon > 0$.

Finally, we shall say that X is very good for (i,j) iff, for every positive integer k, there exist pairwise disjoint sets

$$Y_1, \ldots, Y_k, Z_1, \ldots, Z_k \subset X$$

such that player i thinks all 2k sets are the same size, but player j thinks all the Ys are larger than all the Zs. Notice that, in this case, each set $Z_r \cup Y_s$ is good for (i,j).

Trivial modifications in the proof of Lemma 4.3 suffice to establish the following.

LEMMA 6.2. If X is good for (i,j), then X is very good for (i,j).

The next three lemmas produce the pairwise disjoint collection of sets referred to above and thus suffice to complete the proof.

LEMMA 6.3. If X is very good for (i,j) and $\beta > 0$, then there exists a set $Y \subset X$ so that Y is good for (i,j) and $\mu_s(Y) < \beta$ for every $s = 1, \ldots, n$.

PROOF. Suppose X is very good for (i,j), and suppose $\beta > 0$ is given. We want to produce a set $Y \subset X$ such that Y is good for (i,j) and $\mu_s(Y) < \beta$ for $s = 1, \ldots, n$. Suppose $Y_1, \ldots, Y_k$,

$Z_1, \ldots, Z_k$ are the sets guaranteed to exist by the fact that X is very good for (i,j), and assume $k > 1/\beta$. Choose r so that if $X_1 = Y_r \cup Z_r$, then $\mu_1(X_1) < \beta$. Then X_1 is good for (i,j) since player i thinks Y_r and Z_r are the same size and player j does not.

Since X_1 is good for (i,j), Lemma 6.2 guarantees that X_1 is, in fact, very good for (i,j). Thus, we can repeat the above construction to arrive at a set $X_2 \subset X_1$ so that X_2 is good for (i,j) and $\mu_2(X_2) < \beta$. Continuing this for $s = 3, \ldots, n$ yields the desired set X_n.

LEMMA 6.4. If X is good for (i,j), then there exists a $\gamma > 0$ such that for every $T \subset X$, if $\mu_i(T) < \gamma$ and $\mu_j(T) < \gamma$, then X-T is also good for (i,j).

PROOF. We want to show that if X is good for (i,j), then X remains good for (i,j) when a set that both consider sufficiently small is deleted. So suppose that X is ε-good for (i,j), and let A and B be witnesses to this fact. Choose p so large that

$$\mu_j(B)/(p-1) < \varepsilon,$$

and let γ be given by

$$\gamma = \min(\varepsilon - \mu_j(B)/(p-1), \mu_i(B)/p).$$

Now suppose that $T \subset X$ and $\mu_i(T), \mu_j(T) < \gamma$. Let $A' = A - T$ and let $B' = B - T$.

The basic idea behind the rest of the proof is as follows. Player j still thinks B' is larger than A'. However, if most (or all) of T came out of A, then p layer i may now also think that B' is larger than A'. We want to show that there exists a set $S \subset B'$ such that

(i) Player i thinks S is as large as T; and

(ii) Player j thinks S is small enough so that when it is deleted from B', the result is still larger than A.

Let player i partition B into p many pieces of size $\mu_i(B)/p$ so that a single one of these pieces contains the set $T \cap B$. This is possible since $\mu_i(T) < \gamma < \mu_i(B)/p$. Consider the p-1 pieces that do not contain $T \cap B$. Choose one of these - call it S - so that

$$\mu_j(S) \leq \mu_j(B)/(p-1).$$

Let B'' = B' - S.

We first claim that $\mu_j(B'') > \mu_j(A')$. To see this, notice that because

$$\mu_j(T) < \gamma \leq \varepsilon - \mu_j(B)/(p-1),$$

we have

$$\mu_j(T) + \mu_j(B)/(p-1) < \varepsilon,$$

and so

$$\mu_j(B'') > \mu_j(A) \geq \mu_j(A').$$

On the other hand, we claim that player i thinks A' is at least as large as B''. To see this, notice that

$$\begin{aligned} \mu_i(A') &= \mu_i(A-T) \\ &\geq \mu_i(A) - \mu_i(T) \\ &\geq \mu_i(A) - \gamma \end{aligned}$$

$$\geq \mu_j(A) - \mu_i(B)/p$$

$$= \mu_i(B) - \mu_i(B)/p$$

$$= \mu_i(B'').$$

Thus, player i can now trim A' to a set A'' that he considers to be the same size as B''. Of course, player j thinks B'' is strictly larger than A''. Thus A'' and B'' show that X-T is good for (i,j).

LEMMA 6.5. There exists a collection $\{S(i,j): 1 \leq i,j \leq n\}$ of pairwise disjoint sets such that if $\mu_i \neq \mu_j$, then S(i,j) is very good for (i,j).

PROOF. We want to produce a collection

$$\{S(i,j): 1 \leq i,j < n\}$$

of pairwise disjoint sets such that if $\mu_i \neq \mu_j$, then S(i,j) is good for (i,j).

Enumerate n x n as $\langle p_1, \ldots p_k \rangle$. We shall construct a sequence $\langle X_1, \ldots, X_k \rangle$ of sets and a sequence of positive numbers $\langle \gamma_1, \ldots, \gamma_k \rangle$ so that

(i) If $p_r = (i,j)$ and $\mu_i \neq \mu_j$, then $X_r - T$ is good for (i,j) whenever $\mu_s(T) < \gamma_r$ for every s = 1, . . . , n, and

(ii) $\mu_s(X_r) < \min\{\gamma_1/k, \ldots, \gamma_{r-1}/k\}$ for all s = 1, . . . , n and r = 2, . . . , k.

We start with $X_1 = C$ and note that (i) is satisfied by Lemma 6.4. Suppose that we have $X_1, \ldots, X_{r-1}$ and $\gamma_1, \ldots, \gamma_{r-1}$ for some r with $2 \leq r \leq k$. Suppose $p_r = (i,j)$. If $\mu_i = \mu_j$, then we take $X_r = \emptyset$ and $\gamma_r = 1$. If and $\mu_i \neq \mu_j$. Let $\beta = \min\{\gamma_1/k, \ldots, \gamma_{r-1}/k\}$. Since $\mu_i \neq \mu_j$, C is good for (i,j). Lemma 6.3 now guarantees there exists a set X_r so that X_r is good for (i,j) and $\mu_s(X_r) < \beta$ for every $s = 1, \ldots, n$. Lemma 6.4 now provides a γ_r so that $X_r - T$ is good for (i,j) whenever $\mu_s(T) < \gamma_r$ for every $s = 1, \ldots, n$. This completes the construction.

Now, to get the collection $\{S(i,j): 1 \leq i,j < n\}$, suppose $(i,j) = p_r$. Let $S(i,j) = X_r - (X_{r+1} \cup \ldots \cup X_k)$. Because the sets are clearly pairwise disjoint, it suffices to show that if $\mu_i \neq \mu_j$, then S(i,j) is good for (i,j). Let $T = X_{r+1} \cup \ldots \cup X_k$ and let $s \in \{1, \ldots, n\}$. Then

$$\mu_s(T) \leq \mu_s(X_{r+1}) + \ldots + \mu_s(X_k) < k(\gamma_r/k) = \gamma_r.$$

Thus, it follows from (i) above that S(i,j) is good for (i,j) as desired.

7. ENTITLEMENTS

Recall that there is a natural generalization of fair division based on the idea of entitlements. That is, if $r_1, \ldots, r_n$ are rational numbers which sum to 1, we can ask if a cake can be partitioned so that each person thinks he received at least the corresponding fraction of the cake. Dubins and Spanier (1961) point out that this entitlement version of fair division easily follows from the special case where $r_i = 1/n$ for each i. That is, if we get a common denominator q so that $r_i = p_i/q$, then we can simply let player i be replaced in any scheme for the special case with p_i copies of himself.

Of course, the same can be done in the envy-free context, and it leads to a rather pleasing answer to the following non-obvious question: Suppose we fix positive rationals summing to 1 as above, and suppose everyone agrees that these rationals represent the fractions of cake to which the corresponding individuals are entitled. What would it then mean to say that a partition is envy-free? One answer that immediately suggests itself is to demand that everyone agree that the piece given to player i has measure exactly r_i. This, however, is easily seen to be sufficient, but it is not necessary. For example, when $r_i = 1/n$ for each i, we can surely have partitions that one wants to call envy-free, but are such that some players think others (not themselves) received much less than $1/n^{th}$ of the cake.

The answer suggested by the idea of splitting a player into several copies of himself is the following: Given a sequence $p_1/q, \ldots, p_n/q$ of positive rationals summing to 1, and represented so that all have a common denominator, we will say that a partition $X_1, \ldots, X_n$ is envy-free if for each i there is a partition of X_i into p_i pairwise disjoint subpieces so that no player would desire to trade any one of his subpieces for a subpiece of any other person.

Thus if $p_i = 2$, this definition says that player i thinks that each of the two subpieces he received is at least as good as each of the other subpieces - whatever their number - that all the other players received; in our opinion, this notion captures the intuitive idea of envy-freeness in the context of entitlements. The correspondingly stronger version of our envy-free algorithm (Theorem 4.5) now follows as did its analogue for Dubins and Spanier (1961).

Woodall (1992) has pointed out to us that one can also handle entitlements $r_1, \ldots, r_n$ in the envy-free context by demanding that $\mu_i(X_i) \geq (r_i/r_j)\mu_i(Xj)$. This has the advantage of applying to irrational r_is as well as to rational ones, but it has the disadvantage of being so tied to the real numbers that it does not work in the context of preference relations over subsets.

8. CHORES

Gardner (1978) suggests the following variant of the fair division problem. Suppose one is dividing up chores to be done. (Economists speak of distributing "bads" instead of "goods.") One would then like to obtain a division so that each player is satisfied that the fraction allocated to him is sufficiently small. "Sufficiently small" might mean "at most $1/n$" or it might mean "no larger than anyone else's share." Our interest, of course, is in the latter notion.

It turns out that the same ideas that led to the finite envy-free division algorithm in section 4 can be used here, but there are some slight complications. The obvious approach is to simply replace "trimming" with "adding on." The problem is the following: where do players get the extra cake to add on, and how to we guarantee that each has a large enough supply at his disposal? One way of handling this is illustrated in the following variant of Lemma 3.1:

THEOREM 8.1. Suppose P is a piece of cake. Then there exists a pairwise disjoint partition $X_1, \ldots, X_n$, L of P into n+1 pieces so that:

(i) $\mu_i(X_i) \leq \mu_i(X_j)$ for every i and j; and

(ii) Player n thinks the size of the leftover piece L is at most ε times the size of P for some $\varepsilon < 1$.

PROOF. Consider the sequence $\langle a_2, a_3, \ldots, a_n \rangle$ where

$$a_2 = 1,$$
$$a_k = a_1 + \ldots + a_{k-1} + 2^{k-1} \text{ for } 3 \leq k \leq n.$$

For $k = 2, \ldots, n-1$, we give player k a total of a_k colored tags (with different players getting different colored tags). The process begins with player n dividing the cake into a_n pieces that he considers equal. Player n-1 now places one of his tags on each of the a_{n-1} pieces that he considers largest, and passes the collection of a_n pieces (many of which are now tagged) to player n-2. Player n-2 now selects from the a_n pieces the a_{n-2} pieces he considers largest, and places one of his tags on each of these, removing tags left by player n-1 if necessary. This continues for player n-3, . . . , player 2. Notice that for $k = n-1, \ldots, 2$, player k still has his tags on at least 2^{k-1} pieces. These tagged pieces will constitute the surplus for the "adding-on" that will replace the "trimming" of Lemma 3.1.

With the tags in place, we return to player n-1 for the adding-on process. Notice that of player n's original a_n pieces, we can choose a subcollection of size 2^{n-1} with no tags. We use this collection to start the process. Player n-1 now takes this collection of 2^{n-1} pieces and identifies the 2^{n-2} smallest pieces. The largest of these is smaller than each of the 2^{n-2} pieces that still have one of player n-1's tags. Hence, player n-1 can use his tagged pieces to bring all 2^{n-2} pieces in this collection up to the size of the largest. This creates a collection of at least 2^{n-2} pieces tied for smallest in player n-1's view. The process continues with player n-2, . . . , player n-1 each taking his turn. Notice that when player k identifies the 2^{k-1} smallest pieces in the altered collection he is handed, the largest piece in this subcollection is still no larger than all the tagged pieces he has at his disposal. Hence, player k can add-on to create a collection of at least 2^{k-1} that he considers tied for smallest.

At this point, the choosing takes place with player 1 going first, then player 2, and so on up to player n. At each stage, the player choosing has available one of the pieces he considered tied for smallest. Notice that player n thinks the size of the leftover piece is at most $(a_n-1)/a_n$ the size of P. This proves the lemma.

We leave it to the reader to check that the remaining arguments in sections 3 and 4 go through to yield:

COROLLARY 8.2. Suppose C is a set and $\mu_1, \ldots, \mu_n$ are finitely additive measures defined on some common algebra A of subsets of C so that $\mu_i(C) = 1$ for every i and so that TP is satisfied. Then there exists a finite algorithm for producing an ordered partition $X_1, \ldots, X_n$ of C so that $\mu_i(X_i) \le \mu_i(X_j)$ for every i and j.

The problem of building up a reserve of cake to be used in the adding on process can be handled in ways other than what we did in Theorem 8.1 above. For example, William Webb (1992) has pointed out to us that a recent result due to Jack Robertson and himself can be used to facilitate this part of the proof. Sergiu Hart (1992), on the other hand, noticed that the basic envy-free algorithm itself can be used to provide such stockpiles of cake for the adding-on process. And for the case n = 3, Oskui (1992) gives three algorithms, including two of the moving-knife variety.

9. A FRAMEWORK FOR QUESTIONS

The variety of theorems that have appeared over the past half century in the context of fair division suggests that it might be of some use to a have a framework that would both serve to display the known results and their relationships to each other and to suggest new questions based on these results.

Our attempt at such a framework involves twelve issues and three choices for each issue, all displayed in the following chart. The idea is that a typical theorem should correspond to a sequence of length twelve. For example, Theorem 4.5 corresponds to the sequence <1a, 2c, 3a, 4c, 5b, 6a, 7b, 8a, 9b, 10c, 11c, 12c>. Notice that a's are better than b's, and b's are better than c's.

Thus, one way of asking if a theorem can be improved is to ask if any c can be replaced by b or any b by an a (leaving everything else the same). Notice, for example, that Theorem 4.5 is not sharp in this sense since the results of section 6 show that 5b can be replaced by 5a.

MATHEMATICAL FRAMEWORK

1a. Heterogeneous	1b. Homogeneous	1c. Discrete
2a. Ordinal Prefs	2b. Utility Function	2c. Additive Measure
3a. Finitely Additive	3b. Countably Add.	3c. Abs. Continuity
4a. No Assumptions	4b. Splittable	4c. Trimmable

DIVISION METHOD

5a. Unilateral Game	5b. Algorithm	5c. Existence Proof
6a. No Information	6b. Information $\mu \neq \nu$	6c. Witnesses for $\mu \neq \nu$
7a. Finite Bounded	7b. Finite Unbounded	7c. Infinite Process
8a. Works for all n	8b. Works only if $n=3$	8c. Works only if $n=2$

SATISFACTION CRITERION

9a $\mu_i(A_j) = 1/n$	9b. Envy-free (EF)	9c. Fair: $\mu_i(Ai) \geq 1/n$
10a All Measure 1	10b. Strongly F/EF	10c. Weakly F/EF
11a. Parallel Slices	11b. Connected Sets	11c. Disconnected
12a. Maximize Sum	12b. Pareto Optimal	12c. Not Optimal

Some of the cryptic terms in the chart require explanation. In line 1, for example, the distinction between the heterogeneous context and the homogeneous context is the major point separating virtually all results in the economics literature from those in the mathematics literature on fair division. Basically, the homogeneous context corresponds to the metaphor in which the cake is made up of, say, three layers of ice cream: chocolate, strawberry, and vanilla. Differences in perceived value are due to different preferences for the three flavors. In this context, an envy free allocation is trivial to achieve since one can simply divide each layer evenly among the n people. This is the sense in which "homogeneous" lies between "discrete" and "heterogeneous."

In line 2 of the chart, the "additive measure" context is the one in which mathematicians have primarily worked. Economists, however, assume that preferences in the real world are often not additive and thus tend to work either with utility functions (measures which satisfy a monotonicity property in place of additivity) or ordinal preference relations (preorders on some collection of subsets of the cake).

Line 3 really only applies in the measure theoretic context and asks if one is assuming only finite additivity, countable additivity, or some kind of absolute continuity either with respect to Lebesgue measure or among the measures themselves. Line 4 identifies two relevant properties measures might have. Call a measure trimmable if it satisfies the condition TP used in the main theorems of this paper, and call it splittable if every piece can be split into an arbitrary finite number of equal size pieces.

The distinction made in line 5 is the one illustrated in sections 4 and 6. Unilateral game refers to the existence of a strategy for each player that will guarantee him a fair or envy-free share regardless of what the others collectively know or do, whereas an algorithm specifies what all the players must do.

The distinction in line 6 is the one illustrated in sections 4 and 5, in which the stronger conclusion obtained in section 6 required "witnesses" (i.e. sets upon which measures disagreed).

Line 7 raises a question suggested independently by Douglas Woodall (1992) and Fred Galvin (1992). The algorithm in section 4 is finite but unbounded. That is, the required number of steps depends on the measures as opposed to just depending on the number of people. (Some of the algorithms for $n=3$ are finite and bounded.) Woodall and Galvin both asked if there exists a bounded algorithm for producing an envy-free division of a cake.

Line 8 needs no explanation. In line 9, the condition "$\mu_i(A_j) = 1/n$" could be replaced by the notion of group envy-freeness used in Berliant-Dunz-Thomson (1992). The distinction between strongly F/EF (fair/envy-free) and weakly F/EF made in line 10 is, for example, the question of whether one gets a piece strictly larger than that received by anyone else (as in section 6) or simply at least as large (as in section 4). Again, lines 11 should be clear. Finally, in line 12, the sum we are talking about maximizing is $\mu_1(X_1) + \cdots + \mu_n(X_n)$.

Of particular interest would be results saying that certain combinations are impossible. We plan to say more on this elsewhere, as well as discuss various applications of the different procedures to real-life situations.

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