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SHARES OF WORLD OUTPUT, ECONOMIES OF SCALE, AND REGIONS FILLED WITH EQUILIBRIA

by

Ralph E. Gomory

and

William J. Baumol

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C. V. STARR CENTER FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

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Ralph E. Gomory* and William J. Baumol**

I. Introduction

Ricardian models of international trade traditionally employ production functions that exhibit decreasing returns to scale. This paper will use a Ricardian model with scale *economies* and substantial *startup costs*. Their inherent non-convexity makes these models technically difficult to analyze. Models of the same type as ours have already been the basis of an important, revealing, and varied literature. The progress in Gomory [1994] now enables us to analyze an important class of problems in a new way. Our model has much in common with the earlier literature, but it reveals important features that are different from and complementary to the earlier work, yielding significant consequences for the theory of international trade and for its policy implications.[1]

It will be shown that in models with increasing returns or, more generally, with high start up costs for an industry new to a country there are always *vast numbers of locally stable equilibria* arranged not haphazardly but rather in dense groupings or *regions* of equilibrium points whose surprisingly simple shape has significant implications for theory and policy. Regions of equilibria were introduced in Gomory [1991,1994], where their characteristic shape was first described and some of the economic consequences of that shape were first described. However, in those papers the characteristic shape, on which so much of the economic and policy conclusions depends, was an observation based largely on numerous model simulations. Here we provide for the first time a rigorous derivation for that shape, and examine its implications much further.

The study of equilibrium regions reveals that, in contrast with the classical cases of diminishing or constant returns to scale: (1) A country will do better for itself by taking more than its "natural" share of world markets; (2) It will usually damage its trading partner by so doing; (3) It is possible for a country to take too large a share of world markets and damage itself as well as others. (4) In contrast to the classical case, a country can gain at the expense of others by acquiring some of their industries.

Gains from trade for both countries remain a dominant feature of the model. Yet, if one country, either through historical accident or through an aggressive and successful policy of industry promotion, should succeed in obtaining something close to the share of markets that maximizes its own utility, it will certainly obtain a very high level of prosperity and very large gains from trade. But it is likely by so doing to deprive its trading partner of most, and sometimes all of the gains from trade. In fact, groups of equilibria that are worse than autarky for one of the countries will be present in any reasonably large model. Such equilibria represent outcomes where one of the countries has only a small share of world markets. The key role of market *share* in all these conclusions shows the significant possibility of direct conflict in the interests of trading partners with all the policy implications that entails.

The results we obtain here, together with the important characteristics of scale economies equilibria already well known in the literature[2] -- the tendency toward specialization and multiple equilibria, many Pareto dominated by others -- depict a world very different from the Ricardian world with its unique and mutually beneficial trading equilibrium. The world we describe is one with an enormous range of possible outcomes differing widely in the benefits they offer. This. suggests that a policy of extreme *laissez-faire*, which recommends itself in a Ricardian world, at least requires reexamination in the presence of scale economies.

The discussion in this paper is expressed, because of its familiarity, in terms of the traditional concept of scale economies In fact, our main results hold for *any* industry of a type we have elsewhere called *retainable* (Gomory and Baumol [1992b]). A retainable industry is one that, because of high startup costs, can be held on to by a country despite lower wage costs in other countries. The industry can be retained because its success requires a large amount and variety of non-tradable support services, or because knowledge gained by experience in the operation of such an industry offers the country a strong competitive edge, or because other sorts of substantial sunk costs must be incurred in order for the industry to be replicated in another country. Because of this, a retainable industry will generally *not* be contestable. Retainable industries may well produce a very considerable share of the world's traded goods.

II. The Basic Diagram and The Equilibrium Region.

We plot equilibria as points in a diagram whose horizontal axis measures income share and whose vertical axis measures utility. The horizontal axis measures $Z_1=Y_1/(Y_1+Y_2)$ where Y_j is any measure of national income in Country 1. Z_1 is the share of the total income of the two countries that accrues to Country 1. Country 2's share, $Z_2=1-Z_1$, is measured by the distance from Z_1 to 1. As one moves rightward from $Z_1=0$ toward $Z_1=1$, Z_1 increases, Z_2 decreases and

Country 1 acquires an ever-larger share of the world's income. The vertical axes indicate the utility[3] offered to each country by an equilibrium. The utility U_1 of Country 1 is plotted on the right vertical axis and the utility U_2 of Country 2 is plotted on the left vertical axis. Each each equilibrium is represented by *two* points $S_1=(Z_1, U_1)$ which show the utility of the equilibrium for Country 1, and $S_2=(Z_2,U_2)$which show the utility of the same utility point for Country 2. Both points lie on the same vertical line which represents the partition of world income at that equilibrium (Figure 1).

The equilibria for each country lie between two curves that are the upper and lower boundaries of the equilibrium region for that country. As the number of goods increases the equilibrium points tend to "fill up" these regions. In Figure 2 we show the regions for a model with 13 traded goods. For Country 1 we show the S_1 points and the boundary curves B_1 and BL_1. For Country 2 the S_2 points (not shown) would lie between the curves B_2 and BL_2. The horizontal bars labelled AUT_1 and AUT_2 represent the utility that each country achieves when in a state of autarky. The vertical bar marked Z_C marks the "Classical Level" of relative income which we will describe in Section VI.

In Figure 2 we have plotted only the *perfectly specialized equilibria,* that is, the equilibria in which no commodity is produced at the same time by more than one country. Later we will in fact see (Theorem 9.6) that the specialized equilibria determine the shape of the region of equilibria to a very considerable extent even *for non-specialized equilibria*.

Before proceeding with a more detailed analysis we will summarize the main features of that shape and their economic consequences.

III. Properties of the Regional Boundaries and their Economic Implications

Country 1's upper utility frontier is described by three properties which we will establish rigorously in Sections VII and VIII, and which are critical for the economic interpretation of the model: The three properties are: **(P1)** Country 1's upper boundary rises steadily from a height of zero as Country 1's world income share Z_1 increases from 0. **(P2)** Somewhere to the right of the Classical Level, Z_C, the boundary peaks and then turns and descends. Then, as Z_1 approaches 1, the height of the boundary approaches U^1_A, the utility level achieved by Country 1 when it is in a state of autarky. **(P3)** At all points between Z_C and $Z_1=1$ the boundary lies above U^1_A.

The lower boundary also rises to the left of Z_C **(P1)** and then peaks and descends to the autarky level as Z_1 approaches 1 **(P2)**, so that both curves come together at $Z_1=0$ and at $Z_1=1$. Both curves are convex except for some barely discernable wobbles, and therefore, except for the wobbles, each is *hill shaped*, with a single peak. The equilibrium region has roughly the shape of a crescent, as shown in Figure 2. All these properties hold for Country 2 with $Z_2=1\text{-}Z_1$ replacing Z_1.

<u>Economic Implications:</u> P1 indicates that there is always a range of values of Z_1, around Z_C, over which Country 1's upper utility frontier is rising, while Country 2's is falling. This means, roughly, that in this range Country 1 can gain utility as its share Z_1 is rising, but it does so only at Country 2's expense. P2 indicates that this gain for Country 1 with increasing share will continue until it reaches the utility peak. This is the type of conflict in the interests of the trading countries, mentioned in Conclusions 1 and 2 of the introduction, that has no counterpart in a world of diminishing returns. Near $Z_1=1$ both countries' frontiers slope down. This means

that close to $Z_1=1$, where one country is much richer than the other, further increase in share of wealth by the wealthier country is harmful to both countries (Conclusion 3 of the introduction), thus providing the opportunity for mutually beneficial change. Finally, since both boundaries of Country 1's region start at zero utility at $Z_1=0$, there is inevitably a section of the region that lies below Country 1's autarky level. The equilibria in this part of the region will give results worse than autarky for Country 1.

We now describe the formal model that permits rigorous proofs of these conclusions.

IV. Equilibria in the Formal Model[4]

We start the rigorous development of the ideas of region, boundary, and regional shape with a discussion of equilibria.

We will assume that each country has a single wage w_j and a well defined utility. The utility is of Cobb-Douglas form, $U_j=\cap_i y_{i,j} d_{i,j}$ with $y_{i,j}$, the quantity of good i consumed in Country j and $\Sigma_i d_{i,j}=1$. As is well known, the demand-determined expenditure for the ith good in Country j is independent of price and equals $d_{i,j}Y_j$ for each of the n goods.

The production functions $f_{i,j}(l_{i,j})$ in the single input, labor, are assumed to have economies of scale in the sense of declining average costs, i.e., $l_{i,j} > l'_{i,j}$ implies $f_{i,j}(l_{i,j})/l_{i,j} > f_{i,j}(l'_{ij})/l'_{i,j}$. We also assume a zero derivative at the origin, meaning intuitively that some minimum level of activity is required before any output can be produced. It is this important *startup cost* assumption that stabilizes our many specialized equilibria.

The Equilibrium Conditions: We now introduce the key variables x which we will call the "market share" variables, that we use to describe the various equilibria. $x_{i,j}$ is defined to be the fraction of world outlay on the ith good that is spent for ith goods made in Country j. In a specialized equilibrium $x_{i,j}=1$ if Country j is the producing country; and $x_{i,j}=0$ if Country j is the non producing country.

Definition: By a zero-profit equilibrium point we will, as usual, mean an assignment of market shares between the two countries (that is, a set of non-negative $x_{i,j}$, $x_{i,1} + x_{i,2}=1$), a price vector p_i, a set of wage rates w_j, and an allocation $l_{i,j}$ of each country's labor supply L_j among the industries in which that country is a producer, in which (E1) the supply of each good from each country equals the demand for it, (E2) each industry earns zero profit, (E3) the demand for labor in each country equals L_j, the total quantity of labor supplied.[5]

Theorem 4.1 (Specialized Equilibrium Existence). Any set of *integer* (i.e., 0,1) market share variables $x_{i,j}$ must yield an equilibrium. That is, *any* such perfectly specialized assignment of market share (or, equivalently, of production) between countries will satisfy the three requirements for an equilibrium at suitable prices, and wage rates. We except the two assignments that give the entire world market for *all* products to one of the two countries.

This means, perhaps somewhat surprisingly, that there always exist equilibria in which a country is exclusive producer of only a few goods, say, 10 percent of the world's goods, and others where it produces 50 or even 90 percent of the total.

Proof-Preliminaries: Since Country j's income is given by the sum of world expenditures $x_{i,j}(d_{i,1}Y_1+d_{i,2}Y_2)$ for each of its products we have:

(4.1),(4.2) $$Y_1=\sum_i x_{i,1}(d_{i,1}Y_1+\ d_{i,2}Y_2) \quad and \quad Y_2=\sum_i x_{i,2}(d_{i,1}Y_1+\ d_{i,2}Y_2).$$

It is a consequence of Walras' law that either (4.1) or (4.2) alone implies both[6]. If we put *any* market share values $x_{i,j}$ into (4.1) or (4.2) we can find the corresponding national incomes by solving the resulting simple linear equation in Y_1 and Y_2. Since there is only one equation and two unknowns Y_1 and Y_2, (we have excluded the degenerate case by not allowing one country to produce everything) there is not one solution, but rather a ray of solutions. If (Y_1,Y_2) is a solution so is (kY_1,kY_2), which, as we will see below, changes the scale of wages and prices but represents the same equilibrium point.

Proving (E2) - Zero Profit: Having determined an (Y_1,Y_2) from $X_{i,j}$, we next find wage rates w_j from $w_jL_j=Y_j$. Then we determine the labor quantities $l_{i,j}$ in each industry in each country by equating the expenditure on good i from country j with the wage bill:

$$w_j l_{i,j} = (d_{i,1}Y_1 + d_{i,2}Y_2)x_{i,j}. \tag{4.3}$$

These wage rates and labor quantities certainly fulfill the zero profit equilibrium requirement (E2).

Proving (E3) - Zero Excess Demand for Labor: Next we show that the $l_{i,j}$ use up no more and no less than the quantity of labor supplied in each country. Using (4.3) we substitute $w_j l_{i,j}$ for $(d_{i,1}Y_1+d_{i,2}Y_2)\ x_{i,j}$ in (4.1) and (4.2) and $w_j\ L_j$ for Y_j to obtain:

$$\sum_i w_1 l_{i,1} = w_1 L_1 \qquad \sum_i w_2 l_{i,2} = w_2 L_2.$$

On cancelling out the wage rates we have (E3).

Proving (E1) - Supply Demand Equality: Since the $l_{i,j}$, and Y_j, are now determined, the only possible prices that can equilibrate supply and demand for the ith industry in each country, condition (E1), are the *two* prices $p_{i,1}$ and $p_{i,2}$ determined from:

$$p_{i,1}f_{i,1}(l_{i,1})=(d_{i,1}Y_1+d_{i,2}Y_2)x_{i,1} \quad and \quad p_{i,2}f_{i,2}(l_{i,2})=(d_{i,1}Y_1+d_{i,2}Y_2)x_{i,2}.$$

Since equilibrium requires a *single* world price p_i, we will only have equilibrium if $p_{i,1}= p_{i,2}$, i.e., if both countries produce goods at the same price. For *integer* x this condition *is always met because there is only one producer.* For example if $x_{i,1}=1$ and $x_{i,2}=0$, then choosing $p_i=p_{i,1}$ satisfies the first equation and with $x_{i,2}=0$, $l_{i,2}=0$, $f_{i,2}(l_{i,2})=0$ it also satisfies the second. So condition (E1) holds and this establishes the theorem.

Normalization: If we had chosen (kY_1,kY_2) instead of (Y_1,Y_2) we would have had essentially the same equilibrium point. We would have had a wage kw_j and a price kp_i, but the wage ratio w_1/w_2, the ratios of prices to wages p_i/w_j, the labor quantities $l_{i,j}$ and the quantities produced $y_{i,j}$ would all be the same. We will normalize by always choosing $k=1/(Y_1+Y_2)$. This means that we use relative national incomes (or share of world income) $Z_1=Y_1/(Y_1+Y_2)$ and $Z_2=Y_2/(Y_1+Y_2)$.

Using $Z=(Z_1,Z_2)$ in place of (Y_1,Y_2) in (4.1,2) gives.

(4.4),(4.5) $$\sum_i x_{i,1}(d_{i,1}Z_1+ d_{i,2}Z_2)=Z_1 \quad and \quad \sum_i x_{i,2}(d_{i,1}Z_1+ d_{i,2}Z_2)=Z_2.$$

(4.4) (or equivalently (4.5)) now determines, by the steps given above, for any market shares x, first a *unique* relative national income or world income share Z(x) and then the uniquely determined wage rates w_j, labor quantities $l_{i,j}$, and quantities produced $f_{i,j}(l_{i,j})$. For integer x we also have a unique price p_i for each good. With this normalization we have well determined wages and prices at all the equilibria of Theorem 4.1.

We can easily count these equilibria:

Theorem 4.2 Number of Specialized Equilibria. With two countries and n goods there are 2^n-2 specialized equilibria.

Proof. By the specialized equilibrium Theorem 4.1, each and every specialized assignment of outputs among countries yields an equilibrium. Because in each specialized assignment there are two choices of producer country for each of the n goods, this yields 2^n possible assignments. Since we have excluded the two extreme cases in which no commodity production at all is assigned to one or the other country, this yields 2^n-2 equilibria, the vast number of equilibria mentioned earlier.

Stability: The properties of the production functions $f_{i,j}$ were not used in this proof of the Specialized Equilibrium Theorem so the equilibrium existence result, surprisingly, holds for a diseconomies model as well as for an economies one. However with scale *diseconomies*, any perfectly specialized assignment will be extremely *unstable* unless the producing country is the cheaper producer *at all output levels*. The reason is easy to see. For if, say, only Country 1 produces good i and Country 2 is the cheaper producer at some output level then, *a fortiori,* Country 2 can enter successfully on a small scale because of the even lower costs this small size entails.

In contrast, every specialized equilibrium will be *stable* locally in a world of substantial *startup costs* or scale economies with $f_{i,j}'(0)=0$.[7] From the assumption that the function has a zero derivative at the origin, it follows that a non-producer of a commodity (quantity=zero), who attempts its production, will always have, for sufficiently low levels of output, an arbitrarily large average cost. Therefore the non-producer would have a higher average cost than the producer with a positive output of that good. Consequently, the former non-producer must earn a negative profit at current prices if it undertakes sufficiently low quantities of production. On the Marshallian dynamics premise, this will lead to a fall in output of i in that country. More formally, we have:

Theorem 4.3 (Stability Theorem): Under scale economies with production functions $f_{i,j}(l)$ having $f'(0)=0$, perfectly specialized assignments will be locally stable equilibria in the Marshallian sense.[8]

Of course, none of these equilibria will be stable globally since a suitable change in the values of the variables will move the economy from one locally stable equilibrium to another.

Acquiring Industries: If one country acquires an industry from its trading partner it simply moves from one locally stable equilibrium to another without any inherent need to lose any of the industries it had before. There is no mechanism leading to an offsetting loss of some or all of another industry as in the linear or diseconomies case. There is no requirement that the assignment of industries to countries at any of these equilibria satisfy comparative advantage.[9] This permits a country to gain by acquiring more industries.[10]

V. Obtaining the Regional Boundaries by Linear Programming:

We next determine the boundaries of the region in which all these equilibria lie by using linear programming to find the largest and smallest possible values of the utility for each value of Z_1, To do this we need an explicit expression for utility.

Utility: Utility, of course, depends on the amounts consumed in the country. From Section IV, given an equilibrium, x, the amount of the ith good produced is $q_{i,j}(x,Z)$, where $q_{i,j}(x,Z)$ is defined by:

(5.1) $$q_{i,j}(x,Z) = f_{i,j}(l_{i,j}) \quad with \quad l_{i,j}w_j = (d_{i,1}Z_1 + d_{i,2}Z_2)x_{i,j} \quad and \quad w_jL_j = Z_j.$$

Since x is an equilibrium point, there is a single world price p_i for each good. Country 1, which spends $d_{i,1}Z_1$, on good i, will obtain a fraction $F_{i,1}(Z) = d_{i,1}Z_1/(d_{i,1}Z_1 + d_{i,2}Z_2)$ of world production of that good. Therefore $u_1(x,Z)$, the natural log of its Cobb-Douglas utility is:

$$u_1(x,Z) = \ln U_1(x,Z) = \sum_i d_{i,1}\ln\, y_{i,1} = \sum_i d_{i,1}\ln F_{i,1}(Z)\{q_{i,1}(x_{i,1},Z) + q_{i,2}(x_{i,2},Z)\}.$$

This expression is extremely complicated in its dependence on the $x_{i,j}$, even for fixed Z. Fortunately we are able to simplify it in a way that decisively facilitates both theory and calculation.

__The Linearized Utility:__ For *integer x only*, and therefore *for perfectly specialized equilibria*, the Cobb-Douglas utility always has the same values as the linearized utility $Lu_1(x,Z)$ which we define by:

(5.2) $$Lu_1(x,Z) = \sum_i \{x_{i,1}d_{i,1}\ln F_{i,1}(Z)q_{i,1}(1,Z_1) + x_{i,2}d_{i,1}\ln F_{i,1}(Z)q_{i,2}(1,Z_2)\}.$$

This extremely useful result can be proved immediately by comparing the ith term in the sum for $u_1(x,Z)$ with the ith term in $Lu_1(x,Z)$ for the perfect specialization values $x_{i,1}=1$, $x_{i,2}=0$, and $x_{i,1}=0$, $x_{i,2}=1$. We call Lu_1 the *linearized utility* because for a fixed Z it is linear in x.

__**Linear Programs for the Upper and Lower Boundary Curves**__: We can now define two curves in the Z-U plane which will turn out to be the upper and lower boundaries of the region of equilibria. We first define $B_1(Z)$, which will be the upper boundary for Country 1, by the maximizing linear program using equation (4.4) as its constraint:

(5.3)
$$\ln B_1(Z) = Max_x \; Lu_1(x,Z)$$
$$subject\ to \quad \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}x_{i,1} = Z_1 \quad and \quad 0 \le x_{i,1} \le 1$$

and then define a lower boundary $BL_1(Z)$ by minimizing the same objective function:

These programs lead to:

Theorem 5.1 (The Boundaries): All specialized equilibria lie between the curves $U= B_1(Z)$ and $U= BL_1(Z)$.

Proof: Let x be any specialized equilibrium. From (4.4) we compute its income share $Z(x)=Z'$. This means that in the (Z,U) plane x lies on a vertical line running through Z'. For $Z=Z'$, x is a feasible solution to both the maximizing and minimizing variants of (5.3). Therefore its *linearized* utility $Lu_1(x,Z')$ lies between $\ln B_1(Z')$ and $\ln BL_1(Z')$. Since x is a specialized equilibrium, utility $u_1(x,Z')$ equals linearized utility $Lu_1(x,Z')$, so $BL_1(Z') \le U_1(x,Z') \le B_1(Z')$, and the equilibrium point corresponding to x lies between the two curves. (QED).

<u>**Properties of the Linear Programs:**</u> The linear program (5.3) is remarkably simple. It is a one equation linear programs with upper bounds.[11] We will exploit this simplicity both for actual numerical computation, and to provide a theory of the regional shape.[12]

<u>Form of the Solution - at most one non-integer variable:</u> In a one equation linear program with no upper bounds it is well known that there is always an optimizing solution with no more than one non-zero variable. In a one equation linear program *with* upper bounds there is at most

one variable that is both non-zero and *not* at its upper bound. Since our upper bounds are all 1's, this means that the *optimizing solution to (5.3) has at most one non-integer variable.* All the others are 0's or 1's.

Numerical Computation: Both $B_1(Z)$ and $BL_1(Z)$ can be computed by any standard linear programming method for each fixed Z_1 value. The calculations that gave us the boundaries of the various figures used a grid of Z_1 values extending from $Z_1=.05$ to $Z_1=.95$ at intervals of .02. The complete boundary for a 27-industry model required about one minute of calculation on a personal computer, and the required computations grow rather slowly with problem size. We can contrast this ease of calculating the boundaries with the enormous difficulty of trying to deal individually with, e.g., the more than 100,000,000 specialized equilibria in such a 27-good model.

VI. The Classical Level

The Classical Assignment and the Classical Level are, roughly, our counterpart to the unique Ricardian solution, and play a key role in the economic interpretation of the shape of the equilibrium region.

Definition: *The classical assignment* is the assignment $x^C(Z)$ that allocates, at each Z, the market for the ith good entirely to Country 1 if Country 1 is the larger potential producer of the ith good, $(q_{i,1}(1,Z) > q_{i,2}(1,Z))$, otherwise it allocates the market entirely to Country 2.

Country 1 being the larger potential producer of good i means that the amount of the ith good County 1 produces as sole producer is larger than the amount Country 2's would produce as sole producer for that Z. Since with Cobb-Douglas utility the amount spent worldwide is the same no matter which country produces, it follows that Country 1 is also the cheaper (lower average cost) producer.

For each Z the classical assignment produces a total excess demand $E_1(Z)$ defined by

(6.1) $$E_1(Z)=\sum_i (d_{i,1}Z_1+d_{i,2}Z_2)x^{C}_{i,1}-Z_1.$$

$E_1(Z)$ is monotone decreasing as Z_1 increases This decrease has two sources. (1) If as Z_1 increases the classical assignment does not change, then, using $Z_2=1-Z_1$, the derivative dE_1/dZ_1 is $d_{i,1}-d_{i,2}-1$ which is negative. (2) If x^C does change, the change is always that Country 1 loses an industry from its assignment. This is because as Z_1 increases the Country 1 wage, $w_1=Z_1/L_1$, rises and by (5.1), with w_1 rising and $(d_{i,1}Z_1+d_{i,2}Z_2)$ falling, $l_{i,1}$ and $q_{i,1}$ must fall. Similarly $q_{i,2}$ must rise. Thus as Z_1 increases Country 1 changes from being the larger potential producer to being the smaller producer in industry after industry. We refer to these points of change (where both countries are equally large potential producers) as *switching points* and we will make use of them in Section VIII.

$E_1(Z)$ is clearly positive near $Z_1=0$, where Country 1 is the larger potential producer of everything, and negative near $Z_1=1$. Therefore there is a unique transition value of Z_1 that separates the Z_1 for which $E_1(Z)$ is positive from the Z_1 where $E_1(Z)$ is negative. We call this transition value the Classical Level, Z_c.[13]

The Classical Level separates two rather different regimes. To the left of Z_C Country 1 is a relatively low wage country and is the larger potential producer in more industries than match its national income. To the right of Z_C industries in which it is the larger producer are insufficient to provide its high national income and high wage.

<u>**The Classical Level and the Maximization Problem:**</u> We can now prove the theorem that underlies the analysis of the regional shape. The theorem asserts that, *to the left of* Z_C, all Country 1 industries that have positive market share in the maximizing solution $x^*(Z)$ to (5.3) are ones in which Country 1 is the larger potential producer. Thus the maximizing solution assigns Country 1 only industries in which it contributes more than would Country 2 to world output. More precisely:

Theorem 6.1 (Maximization). If x^* is the maximizing solution to (5.3), then for $Z_1 < Z_C$, $x^*_{i,1} > 0$ implies $q_{i,1}(1,Z) \geq q_{i,2}(1,Z)$. Similarly, if $Z_1 > Z_C$, then $x^*_{i,2} > 0$ implies $q_{i,2}(1,Z) \geq q_{i,1}(1,Z)$.

<u>Proof</u>: If we are below the Classical Level, in accord with (6.1), x^C does not satisfy the equation in (5.3). There are more industries in which Country 1 is the larger producer than can appear with $x_{i,1}=1$ in x^*. Therefore, at least one of these industries, say the pth, has $x^*_{p,1} < 1$ in x^*. Now suppose x^* gave positive market share to some industry, say the kth, in which Country 1 is *not* the larger producer. Then using a small parameter λ we can form a new feasible market share vector x' that will increase Country 1 utility by decreasing $x'_{k,1}$ and increasing $x'_{p,1}$ as follows:

$$x'_{k,1} = x^*_{k,1} - \lambda\left(\frac{1}{d_{k,1}Z_1 + d_{k,2}Z_2}\right), \quad x'_{k,2} = 1 - x'_{k,1}, \quad x'_{p,1} = x^*_{p,1} + \lambda\left(\frac{1}{d_{p,1}Z_1 + d_{p,2}Z_2}\right), \quad x'_{p,2} = 1 - x'_{p,1}.$$

All other components of x' are the same as those of x^*. (1) Checking Feasibility: Clearly, $0 \leq x'_{k,1} \leq 1$ and $0 \leq x'_{p,1} \leq 1$ for sufficiently small λ. We can also immediately verify by substituting these expressions into the equation that x' satisfies the equation in (5.3). (2) Checking Increased Utility: The effect of increasing λ from zero is to decrease $x'_{k,1}$ and increase $x'_{k,2}$ thus increasing the coefficients of the larger output, $q_{k,2}$, at the expense of the smaller output, $q_{k,1}$, and doing the same in the p^{th} term of (5.2). This increases both the kth and pth termsof the linearized utility function (5.2) Therefore x′ has larger linearized utility than x*. This contradicts the hypothesis that x* is the maximizing solution and proves the first part of the theorem. The proof of the second half is identical.

We will use this theorem in next two sections to provide for the first time rigorous proofs of the three properties that characterize the critical hill-shape of the utility frontiers.

VII. Proof of Property P1-The Monotone Rise of the Upper Boundary Below Z_C.

To simplify the proof of P1 as much as possible we will assume symmetric demand, i.e., $d_{i,1}=d_{i,2}$. A proof for a much wider class of problems is available[14], but the concepts are the same, while the complexity of the terms tends to obscure what is going on. This analysis will also enable us, at the end of this section, to give an intuitive explanation of the hill shape of the entire region.

The Zero Starting Point: Both the upper and lower boundaries start at $(Z_1,U_1)=(0,0)$ since the only way to satisfy (5.3) at $Z_1=0$ is to have all $x_{i,1}=0$. To prove the monotonicity of the rise of the upper boundary from there, we must show that *the slope of the upper boundary is positive to the left of* Z_c.

The Slope of $B_1(z)$: The slope of the boundary curve, $B_1(Z)$ defined in (5.3), will be positive if the slope of $b_1(Z)=\ln B_1(Z)$ is positive. Since $b_1(Z)=u_1(x^*(Z),Z)$ where $x^*(Z)$ is the x that solves the maximizing problem for each Z, we differentiate this expression (see Appendix). The result is that the derivative b_1' is the sum of three terms $(b_1'=T_1+T_2+T_3)$ each calculated in the

Appendix. The three terms are:

$$
(7.1)\quad
\begin{aligned}
&T_1=\sum_i \frac{\partial b_1}{\partial F_{i,1}}\frac{dF_{i,1}}{dZ_1}=\frac{1}{Z_1} \qquad T_2=\sum_{i,j}\frac{\partial b_1}{\partial x*_{i,j}}\frac{dx*_{i,j}}{dZ_1}=\ln q_{k,1}(1,Z)-\ln q_{k,2}(1,Z)\\
&T_3=\sum_{i,j}\frac{\partial b_1}{\partial q_{i,j}}\frac{dq_{i,j}}{dZ_1}=-\sum_i\left(\frac{f'_{i,1}(l_{i,1})l_{i,1}}{f_{i,1}(l_{i,1})}\right)\frac{x*_{i,1}d_{i,1}}{Z_1}+\sum_i\left(\frac{f'_{i,2}(l_{i,2})l_{i,2}}{f_{i,2}(l_{i,2})}\right)\frac{x*_{i,2}d_{i,2}}{Z_2}.
\end{aligned}
$$

Each term has a distinct economic meaning which will be explained below. We will also show that the first term is positive everywhere, and that the second is positive to the left of the classical level Z_C and negative to the right of it, and that under relatively weak assumptions the third term does not have a very large effect.

T_1, The Consumption Share Term: This first term in the derivative shows the effect on utility of the change in the Country 1 consumption shares $F_{i,1}$ as Z_1 changes. Since the consumption share $F_{i,1}=d_{i,1}Z_1/(d_{i,1}Z_1+d_{i,2}Z_2)$ of Country 1 increases with increasing Z_1, T_1, which is $1/Z_1$, should be and is *always positive*.

T_2, The Market Shares Term: This term results from the change in the $x*_{i,j}$, the share of Country j in the world market for good i. For sufficiently small changes in Z only $x*_{k,1}$ the non-integer variable, will be changed in value, the other $x*_{i,1}$ will remain 0 or 1 as before.[15] Therefore as Z_1 rises, part of exactly *one* industry, the kth, shifts from Country 2 to Country 1. Since from (7.1) $T_2=\ln q_{k,1}-\ln q_{k,2}$, the effect of this shift will be positive if Country 1 is the larger potential producer of the kth good ($q_{k,1}>q_{k,2}$), and negative if Country 2 is. By Theorem 7.1, any industry assigned to Country 1 by x*(Z) to the left of Z_C *is* an industry in which Country 1 is the larger producer. Therefore this term will be *positive to the left of* Z_C. From the same theorem, T_2 will be *negative to the right of* Z_C.

T_3, The Labor Shift Term: The two sums in term T_3 reflect (1) the loss in County 1's utility as some of Country 1's labor moves from other Country 1 industries into the industry that is being acquired, thus reducing the quantities produced in those other industries, and (2) the gain in Country 1's utility as Country 2's labor is released from the industry being lost, and is shifted into increased output of Country 2's remaining industries. This term can be either positive or negative, as intuition suggests. We will, however, offer very plausible assumptions ensuring that T_3 is close to zero.

Each term within the sums consists of two parts. The first part, $f'_{i,j}(l_{i,j})l_{i,j}/f(l_{i,j})$, is the ratio of marginal to average productivity in the ith industry when Country j is the sole producer of the world's supply. With scale economies the marginal product of labor, the sole input, exceeds average product, so these fractions all exceed unity. The second part of a term in T_3 is $x^*_{i,j}\, d_{i,j}/Z_j$. Since we have symmetric demands, $Z_j=\Sigma(d_{i,1}Z_1+d_{i,2}Z_2)x^*_{i,j}=\Sigma d_{i,j}\ x^*_{i,j}$ so these expressions add up to 1. Consequently, the $x^*_{i,j}\, d_{i,j}/Z_j$ are weights adding up to 1, and each sum in T_3 produces a *weighted average* ratio of marginal to average productivity β_j for each country.

$$\textbf{(7.2)}\qquad -\sum_i \Big(\frac{f'_{i,1}(l_{i,1})l_{i,1}}{f_{i,1}(l_{i,1})}\Big)\frac{x*_{i,1}d_{i,1}}{Z_1} = -\ \beta_1(Z) \quad \textit{and} \quad \sum_i \Big(\frac{f'_{i,2}(l_{i,2})l_{i,2}}{f_{i,2}(l_{i,2})}\Big)\frac{x*_{i,2}d_{i,2}}{Z_2} = \ \beta_2(Z).$$

Both β_1 and β_2 are greater than or equal to unity because each is a weighted sum of ratios of marginal over average products, each of which is greater than unity because of scale economies. *If, on average, scale economies are about equally strong in the two countries, these two averages will approximately cancel out.*[16] This occurred in our many numerical examples. However, by restricting the β_j we will have a more definitive theorem.

Adding the Three Terms: If we add up the three terms of the boundary's derivative we obtain:

$$b_1'(Z)=T_1+T_2+T_3=\frac{1}{Z_1}+\ln q_{i,1}-\ln q_{i,2}+(-\beta_1+\beta_2)\geq\frac{1}{Z_1}+\beta_2-\beta_1.$$

since $\ln q_{i,1}-\ln q_{i,2}>0$ for $Z<Z_C$. If we restrict β_2-β_1 by $|\beta_2(Z)-\beta_1(Z)|<1/Z_1$, meaning that, on the average, scale economies in the two countries are not very dissimilar, this sum will be positive to the left of Z_C, and the upper boundary will have positive slope. So we have derived the following theorem:

Theorem 7.1 (Monotone Rise). If $d_{i,1}=d_{i,2}$, $B_1(Z_1)$ will be monotone increasing for $0\leq Z_1\leq Z_c$ provided that $\beta_2(Z)$-$\beta_1(Z)$, the difference in average ratio of marginal to average productivity of the two countries' industries as sole suppliers always satisfies $|\beta_2(Z)-\beta_1(Z)|<1/Z$ for $Z\leq Z_C$.

Note that the bound $1/Z_1$ is very large when Z_1 is near 0, and is always greater than $1/Z_C$ which in turn is always >1. The reasoning that led to the Monotone Rise Theorem can easily be shown to apply equally well to the lower boundary, so it too is monotone increasing below Z_C.

Intuitive Explanation of the Regional Shape: The reasoning that led to the Monotone Rise Theorem 7.1 also suggests an economic explanation of the characteristic hill shape of the *entire* upper boundary.

Let us simply ignore the effect of the labor shift term, T_3 since it is likely to be small. Then we have:

(7.3) $$b_1'(Z) = T_1 + T_2 = \frac{1}{Z_1} + \ln q_{k,1} - \ln q_{k,2}.$$

As we remarked above, the share term T_1 indicates that as Country 1 produces a larger and larger percent of the world's goods, it gets a larger and larger share of what is produced both for $Z_1 < Z_C$ and for $Z_1 \geq Z_C$. However the market shares term T_2 is positive for $Z_1 < Z_C$ and then becomes negative to the right of Z_C. Country 1's acquisition of industries increases their output at first, when these are industries in which Country 1 is the larger potential producer while, later, to the right of Z_C, Country 1's further acquisition of industries from the low wage Country 2 results in a loss in production in the acquired industry. That loss is produced by the excess of $q_{k,2}$ over $q_{k,1}$, and as the wage in Country 2 sinks toward 0 this quantity becomes unboundedly large so b_1' becomes negative. So, while at first Country 1 gets more and more of a growing world output pie, and its utility rises sharply, it eventually gets more and more of a rapidly shrinking pie, so that eventually its utility turns down.

This reasoning can be put in a more direct mathematical form. Recalling that $b_1(Z) = \ln B_1(Z)$, if we integrate (7.3) we get an expression for the boundary of the form:

$$B_1(Z) = K_1 Z_1 e^{\int_0^{Z_1} (\ln q_{k,1} - \ln q_{k,2}) dZ_1} = K_1 Z_1 Q(Z).$$

Z_1 represents Country 1's output share, and Q(Z) (which can be regarded as a measure of output) clearly increases with Z_1 up to Z_C and then decreases. [17]

VIII. The Upper Boundary to the Right of Z_C - Properties P3 and P2.

To prove that the boundary is above the autarky level to the right of Z_C (Property P3), we need two theorems about the circumstances in which trade in a commodity must increase its consumption in a particular country. The first of these somewhat surprising theorems asserts that if Country 1 is the sole producer of the ith good, for both countries, whether or not it is the larger potential producer, then it always gets more of that good to consume domestically than the amount, $f_{i,1}(l_{i,1}^a)$, it would obtain in autarky. More precisely:

Theorem 8.1 (Producer's Trade Gains). The amount Country 1 receives as sole producer of the ith good equals $\rho_{i,1}(l_{i,1})f_{i,1}(l_{i,1}^a)$ where $\rho_{i,1}(l_{i,1})=(f_{i,1}(l_{i,1})/l_{ij})/(f_{i,1}(l_{i,1}^a)/l_{i,1}^a)$ and $\rho_{i,1}(l_{i,1})\geq 1$.

Remarks:(1) Note that ρ is the ratio of labor productivity at the labor quantity $l_{i,1}$ to the labor productivity in autarky, and (2) the labor quantity $l_{i,1}$ used by Country 1 to produce the world's supply is greater than the amount used by Country 1 in autarky since $l_{i,1}=(1/w_1)(d_{i,1}Z_1+d_{i,2}Z_2)$ $=d_{i,1}L_1+(w_2/w_1)d_{i,2}L_2\geq d_{i,1}L_1=l_{i,1}^a$.

Proof: When Country 1 is the producer, the amount Country 1 retains for consumption is $F_{i,1}\, q_{i,1}(1,Z_1)=F_{i,1}f_{i,1}(l_{i,1})$. So:

$$F_{i1}f_{i1}(l_{i1})=\frac{d_{i,1}Z_1}{d_{i,1}Z_1+d_{i,2}Z_2}f_{i,1}(l_{i,1})=\frac{w_1 l^a_{i,1}}{w_1 l_{i,1}}f_{i,1}(l_{i,1})=\{\frac{f_{i,1}(l_{i,1})}{l_{i,1}}\frac{l^a_{i,1}}{f_{i,1}(l^a_{i,1})}\}f_{i,1}(l^a_{i,1}).$$

since we have economies of scale and $l_{i,1}\geq l^a_{i,1}$, the bracket, which is ρ, is ≥ 1, QED.

Note that the gains from trade in Theorem 8.1 are due to economies of scale above the autarky labor quantities.

Next we find that where Country 1 is *not* the producer of good i it *still* gains from trade in good i provided Country 2 is the larger potential producer of i.

Theorem 8.2 (Non-Producer Trade Gains). The amount Country 1 receives as non-producer of the ith good is $\rho'_{i,1}(l_{i,1})\ f_{i,1}(l_{i,1}{}^{a})$ where $\rho'_{i,1}(l_{i,1})=\{f_{i,2}(l_{i,2})/f_{i,1}(l_{i,1})\}\rho_{i,j}(l_{i,j})$. If Country 2 is the larger producer $\rho'_{i,1}(l_{i,1})\geq 1$.

Proof: Since Country 2 is the producer, an amount $q_{i,2}(1,Z)=f_{i,2}(l_{i,2})=\{f_{i,2}(l_{i,2})/(f_{i,1}(l_{i,1})\}f_{i,1}(l_{i,1})$ is produced. Country 1 obtains $\{f_{i,2}(l_{i,2})/(f_{i,1}(l_{i,1})\}F_{i,1}f_{i,1}(l_{i,1})$, which from Theorem 8.1 is $\{f_{i,2}(l_{i,2})/f_{i,1}(l_{i,1})\}\rho_{i,j}(l_{i,j})\ f_{i,1}(l_{i,1}{}^{a})$. If Country 2 is the larger producer, then $f_{i,2}/f_{i,1}>1$, consequently $\rho'_{i,1}>\rho_{i,1}\geq 1$ so County 1 obtains more than it would have obtained in autarky and also more than it would have obtained if it were the producer of that good. QED.

We can now easily prove P3, the next important property of the upper boundary. Using $U_1{}^A$ for Country 1's utility in autarky, we assert:

Theorem 8.3 (Gains From Trade). $B_1(Z)\geq U_1{}^A$ for all $Z_1>Z_c$. That is, the upper boundary is always above autarky to the right of Z_C.

Proof. By Theorem 6.1, to the right of Z_C, any positive $x_{i,2}$ chosen by the algorithm will represent a good for which Country 2 is the larger producer. So if $x_{i,2}$ is positive, the Non-Producer Trade Gains Theorem 8.2 applies to the ith term of the linearized utility (5.2) with a $\rho'_{i,1}\geq 1$, while if $x_{i,2}=0$, $x_{i,1}$ is positive so the Producer Trade Gains Theorem 8.1 applies. In either case, Country 1 consumes more of every good, i, and the ith term in the linearized utility (5.2) is larger than it would be in autarky; so the total linearized utility must also exceed autarky utility. Q.E.D.

The Lower Boundary to the Right of Z_C: The lower boundary need not lie entirely above autarky to the right of Z_c but *the last part of it* always does. Recalling the definition of switching point from Section VI as a point where production quantity advantage shifts from Country 1 to Country 2 we have:

Theorem 8.4: The lower boundary is always above autarky to the right of the last switching point.

Proof: This can be shown by applying Theorems 8.1 and 8.2 to the terms of the lower boundary's utility function. Theorem 8.2 applies to every term in which Country 2 is a producer, because to the right of the last switching point Country 2 (by virtue of its low relative wage $w_2/w_1 = (Z_2/Z_1)(L_1/L_2)$) has become the larger producer of everything. QED.

It is now easy to prove the third and last property of the shapes of the utility frontiers- - their convergence to autarky as Z_1 approaches unity. (Property P2).

Theorem 8.5 (Autarky Neighborhood). Both the upper and lower utility boundaries of the region of specialized equilibria slope down to Country 1's autarky value at $Z_1=1$.

Proof: For $Z_1=1$ the only possible solution to the equation in (5.3) clearly is $x_{i,1}=1$ for all i. So for Z_1 near 1 both the minimizing and maximizing x must be near $x_{i,1}=1$ for all i . Since Z_1 is near 1 and Z_2 is near 0, $F_{i,1}=d_{i,1}Z_1/(d_{i,1}Z_1+d_{i,2}Z_2)$ is near 1. Also since $w_1 l_{i1}=(d_{i1}Z_1+d_{i2}Z_2)$, $l_{i,1}$ is near $d_{i,1}L_1$ so the quantity of the ith good produced is near $f_{i,1}(d_{i,1}L_1)$. Therefore, by (5.2) the linearized utility for both the minimizing and maximizing x is near $\Sigma d_{i,1}\ln(d_{i,1}L_1)$ which is Country 1's utility in autarky. Since by Theorems 8.3 and 8.4 both the upper and lower boundaries are above autarky near $Z_1=1$, we must have a downward sloping region near Z_1. QED. So we have shown the last of our three major properties.

Near $Z_1=1$ a rise in Z_1 *reduces utility for both countries.* While generally speaking Z_C separates a region which is rewarding (relative to autarky) for Country 1 from one which is generally rewarding for Country 2, it is possible to go too far to the right, i.e., $Z_1 \rightarrow 1$ and wind up in a region where increasing Country 1's share generally makes thing worse for both countries.

IX. Equilibria in the Region.

Now that we have the regional boundaries, we can discuss the rather illuminating connection between them and actual equilibria. The most fundamental relation is that the entire region tends to fill with equilibria. We see this in Figure 2, our 13-good model, where the equilibrium points nearly blacken the region.

Filling In: We offer a rough statement of a fill-in theorem and describe the idea of the proof. Gomory [1994] provides a full statement and proof.

Let $\{P_n\}$ be a sequence of n commodity models P_n with bounded parameters.[18] Let B_1^n and BL_1^n be the upper and lower boundary curves in the nth model. Let distance in the (Z,U) plane mean $|Z-Z'|+|U-U'|$. Then:

Theorem 9.1 (Equilibrium Region Fill-In). For any fixed Z, and any ε, for all n sufficiently large any point p lying on the vertical line segment between $B_1^n(Z)$ and $BL_1^n(Z)$ has a specialized equilibrium point x within distance ε.

Idea of the Proof. In Section V we showed that any optimizing solution x to (5.3) determines a point on the upper boundary curve, and will have at most one non-integer variable. If we round that non-integer variable either up or down it will produce a specialized equilibrium point (Theorem 4.1). That point should be close to x in large models, and therefore close to the boundary. A similar observation can be made about the lower boundary, $BL_1(Z)$. This suggests that there are always equilibria close to both boundaries. A somewhat less obvious extension of this reasoning shows that there are equilibria near any point within the region as well.

Equilibria and the Regional Shape. Since in the limit the region fills in solidly with equilibria, all the properties of the region itself are reflected in the equilibria. There are equilibria near the top of Country 1's hill that give Country 1 very large gains from trade, there are equilibria in the low parts of Country 1's region that give Country 1 little or no benefit, etc.

Sources of Gains from Trade. Equilibria differ in the sources of any gains from trade they offer. In our model a country's gains from trade derive from three sources: (1) Appropriate patterns of specialization, for example if the larger potential producer of i produces i. (2) Share of world income -- if Country j gets a large share of world income it gets a large share of whatever is produced. (3) Economies of scale -- if there are further scale economies from producing for the world market rather than a single domestic market. These further economies are measured by the $\rho_{i,j}$ of Theorem 8.1. Next, we examine how different groups of equilibria differ in their reliance on these sources of trade gains.

Appropriately Specialized Equilibria: We say that an equilibrium is *appropriately specialized for Country 1* if Country 2 only produces goods of which it is the larger potential producer. For appropriately specialized equilibria we have:

Theorem 9.2 (Appropriate Specialization--Autarky Dominance). At an appropriately specialized equilibrium point Country 1 receive *more of every single good* than it does in autarky.

Proof: Theorem 8.1 gives the result for the goods that Country 1 produces, and Theorem 8.2 does so for the goods that Country 2 produces.

Corollary 9.2.1. The appropriately specialized equilibria for Country 1 all lie to the right of Z_C.

Proof: At an appropriately specialized x, Country 2's whole labor supply produces goods of which it is the larger potential producer, so $Z_1(x)$ must be to the right of the Classical Level by Theorem 6.1.

Two Remarks about Theorem 9.2: (1) These gains from trade derive from specialization and share. They occur even when the production functions do not provide further economies at employment exceeding the autarky level, or equivalently even when the $\rho_{i,j}$ of Theorem 8.1 is 1. (2) Since Country 1 gets more of *every single good* than in autarky, everyone in Country 1 benefits from the resulting lower real prices, even those with preferences very different from those of the country overall.

There are many Appropriately Specialized Equilibria in almost every model. They are easily constructed, as we see from:

Theorem 9.3 (Construction of Appropriately Specialized Equilibria). Choose any Z_1 on or to the right of Z_C, let $x^*(Z)$ be the optimizing solution to (5.3) for that Z. Construct an equilibrium point by (1) rounding the one non-integer component (if any) $x^*_{i,1}$ of x^* up. (2) If the rounding has not made Country 1 the sole producer of every good, move *any* proper subset of the remaining Country 2 industries to Country 1. The resulting x' will always be an appropriately specialized equilibrium point.

Proof: Since Z_1 is to the right of Z_C and x^* is optimizing, Country 2 is the larger potential producer in every industry in which it produces (Theorem 6.1). Since Country 2 only loses industries to get from x^* to x', its share of world income (and its wage) goes down so it is still the larger producer in its remaining industries.[19] So the new x' is an appropriately specialized equilibrium.

The construction is unworkable only if $x^*(Z)$ assigns Country 2 market share in only part of one industry and no market share in any other. Therefore:

Corollary 9.3.1 If Country 2 is the larger producer in *any* industry at Z_C, the region must contain appropriately specialized equilibria for Country 1.

Proof: $x^*(Z_C)$, by definition of Z_C will assign that industry to Country 2, so the procedure of the Construction Theorem can be carried out directly at Z_C.

The Construction Theorem produces many different x' for a given Z_1, so appropriately specialized equilibria are usually numerous. In Figure 3, which is an 11 good model, the appropriately specialized equilibria are the small dots to the right of Z_C in the Country 1 region. In this model 512 of the 2046 specialized equilibria are appropriately specialized for Country 1.

Inappropriately Specialized Equilibria We define an *inappropriately specialized equilibrium* for Country 1 to be one where Country 2 is the smaller potential producer of everything it produces.

An analysis like that for appropriately specialized equilibria can be carried out for inappropriately specialized equilibria. By Theorem 6.1, inappropriately specialized equilibria can exist only to the left of Z_C. At all inappropriately specialized equilibria Country 1 gets less of each good produced by Country 2 than it would if it were the producer at that same Z. Inappropriately specialized equilibria are present in almost all models and are numerous. The small dots to the left of Z_C in Figure 3 are the 690 inappropriately specialized equilibria for Country 1.

However, unlike appropriately specialized equilibria, the outcome relative to autarky at inappropriately specialized equilibria depends on the properties of the production functions. Inappropriately specialized equilibria suffer both from poor specialization patterns and a low income share for Country 1, since $Z_1 < Z_C$. Their gains from trade, if any, derive only from scale economies. They can be good (poor) relative to autarky if economies of scale are strong (weak) beyond the autarky labor quantity.

When there are *no* economies of scale above the autarky labor quantities, it is plausible that inappropriately specialized equilibria will yield no gains at all from trade.[20] This is confirmed by:

Theorem 9.4 (Inappropriate Specialization). If there are no further economies of scale above the autarky labor quantities, then all inappropriately specialized equilibria for Country 1 give Country 1 less utility than autarky.

Proof: At each inappropriately specialized equilibrium Country 1 will receive exactly the autarky quantities of goods of which it is the producer, (Theorem 8.1 with $\rho_{i,j}=1$ because of the assumed absence of scale economies) and less in all others (Non-Producer Theorem 8.2 with $\rho_{i,j}=1$ and $f_{i,2}(l_{i,2})/f_{i,1}(l_{i,1})<1$). Its total utility is then less than in autarky.

Almost Classical Equilibria: It is natural to look for equilibria that are appropriately specialized from the point of view of *both* countries. This would require each producing industry in each country to be the larger potential producer, and by definition this can occur only at Z_C. As we mentioned in Section VI, footnote 13, there may or may not be an equilibrium point at Z_C. However it is plausible that there are equilibria near Z_C that are *almost* appropriately specialized from the point of view of both countries. These could make up for their slight deficiency in specialization with gains from economies of scale above the autarky labor quantity. This suggests the following autarky-dominance theorem:

Theorem 9.5 (Almost Classical Equilibria--Universal Trade Gains). Given any $k>1$, there is an n such that any model with more than n industries, and with $\rho_{i,j}(Z_C) \geq k > 1$ for all i and j, contains a specialized equilibrium point near Z_C at which *both countries get more of every good* than they do in autarky.

While the thought behind this theorem is relatively simple, the rigorous proof is long and is available on request. However there is generally more than one Almost Classical Equilibrium. In Figure 3 the dark dots show the 32 Almost Classical Equilibria of our 11 industry model. These equilibria are good for both countries. They do not, however, provide as much utility to any one country as do the equilibria near the peak of that county's upper utility frontier.

Non-Specialized Equilibria: Our utility frontiers have been obtained from the perfectly specialized equilibria alone. But where do the nonspecialized equilibria lie? Here we merely report the following theorem from Gomory [1994].

Theorem 9.6 (Non-Specialized Equilibria). Let x be any equilibrium solution, *whether specialized or not*. Let Z(x) be the corresponding Z, and $U_1(x,Z)$ the utility of x to Country 1, then $U_1(x,Z) \leq B_1(Z)$.

So *all equilibria*, not just the specialized ones, *lie under the upper boundary curve*. Non-specialized equilibria can, however, lie below the lower boundary curve. In the example plotted in Figure 4, the light gray dots are the non-specialized equilibria, the darker dots are the specialized equilibria.[21]

X. Mixed Scale Economies and Diseconomies

The two ends of Figure 2 depict extreme situations. Near either vertical axis one of the countries has been shut out of most trade activities, with the other nation having coopted almost every product for itself. At least two features of reality modify this outcome. (1) Nontraded goods: each country produces many commodities such as personal services and housing that are rarely exported. When a country has lost all of its traded products it can and will continue to produce nontraded goods excluded from the model. This is easily dealt with by considering the zero level of utility in our diagrams actually to represent the utility obtained by each country from non-traded goods; (2) Scale diseconomies industries: there are a number of industries that reach the stage of diminishing returns at a relatively low level of output. Because such products invite the simultaneous activity of a multiplicity of small suppliers, the country that is driven out of all activities with scale economies will continue to find a market for its export of goods produced under conditions of diseconomies of scale. Those countries will be driven to specialize in the export of agricultural products, primary materials, textile manufactures and the like -- all the products that the LDCs offer in reality. Because of the nature of diseconomies of scale, this will not drive the industrialized economies out of such fields altogether. Instead, countries of both types may continue indefinitely to be sources of both sorts of products.

We have extended our analysis to the case where some industries have economies of scale while others have diseconomies. Very roughly our procedure is to separate out the scale economies and scale diseconomies industries. The former are studied as before. However, the scale diseconomies industries, since they generally entail production by both countries, are constrained by the requirement that in equilibrium the marginal cost of producing any good must be the same in Country 1 as in Country 2.

The general results of this analysis[22] are illustrated in Figures 5a-5d. These figures show a steady contraction from a large region of equilibria toward a single point. While in Figure 2, which we have already discussed, *all* products are characterized by scale economies, in Figure

5a the share of diminishing returns products is 25 percent. In the following graphs this proportion rises successively to 50, 75 and, finally, 95 percent of the total. In every figure but the last the utility frontiers retain their characteristic hill shape, and in every figure, in the central areas, the gains of one country tend to come at the expense of the other. But as scale diseconomies dominate the world economy increasingly, the range over which the frontiers extend grows increasingly narrow[23] and the range around Z_C over which utility values extend also diminishes. Finally, when the world is exclusively devoted to the production of items with scale diseconomies the frontiers of the two economies degenerate into a single and common point. This is the familiar and classical single Ricardian equilibrium point toward which market forces drive all producers.

XI. Relation to Other Scale Economies Models

The previous trade literature has emphasized three different variants of industry scale economies. While each corresponds to real and probably significant phenomena, they require markedly different analytic methods and yield very different conclusions (see, e.g., Krugman [1984, 109-110]).

One set of widely-used models of scale economies assume them to be internal to the firm. As is well known, this leads us to expect markets to be monopolistic or subject to monopolistic competition, and unless the markets are perfectly contestable, it is likely to entail non-zero profits.[24] Helpman and Krugman [1985] have been the leading users of this approach, and have produced extremely valuable results with its aid (see also, e.g., Krugman [1979], Helpman [1984]), and Grossman and Helpman [1991]).

The second of the previously studied scale economies models entails world-wide scale economies. Though investigated by eminent scholars including Viner [194], Ethier [1979], and Helpman and Krugman [1985] no more need be said about it here because it goes in a direction so very different from ours.

The third group of models with which the analysis in this paper can be associated, assumes that *firms* are perfectly competitive, and operate under constant or diminishing returns to scale,[25] but that industry scale economies are produced by *externalities* that depend on the proximity of the firms in question, and that they therefore benefit the firms within an industry only in a given country. Then, competition will, of course, drive profits to zero. Examples of the many writings using this approach include Kemp [1969] and Ethier [1982]. The concept goes back to Marshall's *Principles*. (For a good review of the history see Chipman [1965 p.740ff]). Our model is associated with this third group because we, too, assume that profits are zero despite the presence of scale economies.

Though this is probably the most widely used of the scale economies constructs, it has always aroused controversy. It is sometimes suggested that this case rarely arises except where specialized labor is most effectively trained by experience on the job and the labor force is immobile internationally. There are many more cases, however, in which proximity generates economies external to the firm because the activities of one firm lend support to those of others. The modern semiconductor industry or, indeed, any complex manufacturing industry, is dependent on a host of specialized and experienced suppliers, especially of services, whose absence greatly complicates the start-up of an industry and whose presence contributes greatly to efficiency. In all such cases, we can indeed have a range of scale economies for the industry, yet constant returns for the firm, perfect competition and, hence, zero profits.

XII. Conclusions

Implications For Policy: We have shown that there is a very wide range of outcomes in a free trade environment. These outcomes depend on market share and range from those that are worse than autarky for one country, to outcomes for that country that are far better than any equilibrium based on the classical market share Our analysis suggests that a policy designed to increase share by acquisition of industries, when applied to trade between two developed countries, can benefit a country that can make it work. The industries to acquire are not necessarily those that seem to promise high growth, but rather those that are retainable, because high startup costs impede competitive entry by other nations. Here it should be noted that a retainable industry is far easier to identify than an industry with a brilliant growth future.

The analysis also suggests that under some circumstances it can serve the interests of a very dominant country to lose industries to an less developed trading partner.

We do not take any of this as an argument for protectionism. We believe that gains from trade for both trading partners are the most likely outcome in a free trade environment. However within that framework there is considerable scope for actions enabling one country to improve its position markedly at the expense of the other. In our view what this indicates is the enormous value of efficiency and high product quality in industries that engage substantially in international trade. This will then almost certainly spill over and benefit the rest of the economy as well.

Implications For Technique: The analysis also indicates that it is useful to employ integer variables in analyses entailing scale economies. Economies of scale are likely to produce local maxima, but the traditional economic variables such as price and marginal cost are tools for the analysis of local behavior only. While these tools work effectively in the presence of convexity, where a local maximum is always a global one, they cannot cope with global problems in the presence of non-convexity. Integer variables, on the other hand, encapsulate the non-local nature of things, $x_{i,j}$ is either 0 or 1, either Country 1 or Country 2 is the sole producer. These are two different and widely separated solutions. Use of integer variables does not make these fundamental difficulties go away, but it does enable us to use what is known about integer programming or asymptotic linear programming, and start to separate what are inherent mathematical problems of maximization from the description of economic models. For that reason we think that the techniques introduced in Gomory [1994] and described here are likely to be useful in analysis of other issues entailing economies of scale.

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Appendix A - The Derivative of the Upper Utility Frontier

Symmetric Demand: Assuming $d_{i,1}=d_{i,2}$ offers the useful simplification $(d_{i,1}Z_1+d_{i,2}Z_2)=d_{i,1}$. This leads to:

(A.1) $$\Sigma_i\ (d_{i,1}Z_1+d_{i,2}Z_2)x_{ij}=\Sigma_i\ d_{i,j}x_{ij}=Z_j \text{ and } F_{i,j}=d_{i,1}Z_1/(d_{i,1}Z_1+d_{i,2}Z_2)=Z_1.$$

The Expression for the Derivative of $b_1(Z)$--The Slope of the Upper Utility Frontier: $b_1(Z)=u_1(x^*(Z),Z)$, so using the expression for the linearized utility from (5.2):

(A.2) $$b_1(Z)=\sum_i d_{i,1}\{x*_{i,1}(Z)lnF_{i,1}(Z)q_{i,1}(1,Z)+x*_{i,2}(Z)lnF_{i,1}(Z)q_{i,2}(1,Z)\}.$$

To get the slope we differentiate this expression.The derivative $b_1'(Z)$ will consist of three terms:

(A.3) $$b_1'(Z)=\sum_i \frac{\partial b_1}{\partial F_{i,1}}\frac{dF_{i,1}}{dZ_1}+\sum_{i,j}\frac{\partial b_1}{\partial x*_{i,j}}\frac{dx*_{i,j}}{dZ_1}+\sum_{i,j}\frac{\partial b_1}{\partial q_{i,j}}\frac{dq_{i,j}}{dZ_1}=T_1+T_2+T_3.$$

These, then, are the three terms in this derivative: T_1 related to $F_{i,1}$, Country 1's changing *share* of the world output of i, T_2 , related to $x^*_{i,j}$ indicating the changing *market shares*, and T_3 related to the changing output of industry i in Country j in response to the changing quantity of *labor* allocated to that industry.

T_1 - The Share Term: If we differentiate the expression for $b_1(Z)$ with respect to Z_1 and use

$F_{i,1}=Z_1$ from (A.1) we obtain:

$$\frac{\partial b_1(Z)}{\partial F_{i,1}}=d_{i,1}(x*_{i,1}\frac{1}{F_{i,1}}+x_{i,2}\frac{1}{F_{i,1}})=\frac{d_{i,1}}{F_{i,1}}=\frac{d_{i,1}}{Z_1}.\quad So\quad T_1=\sum_i\frac{\partial b_1(Z)}{\partial F_{i,1}}\frac{dF_{i,1}}{dZ_1}=\sum_i\frac{d_{i,1}}{Z_1}\frac{dZ_1}{dZ_1}=\frac{1}{Z_1}.$$

T_2 The Market Share Term: As noted in Section VI, with the exception of a finite number of transition points, as Z_1 increases, $x^*(Z)$ will have only one term that changes. All variables will remain 1 or 0 as before except $x^*_{k,1}$ which increases steadily. At these general points $dx^*_{i,j}/dZ_1=0$ for $i\neq k$. For $i=k$, since $x^*(Z_1)$ satisfies $\Sigma_i d_{i,1}x^*_{i,1}=Z_1$ by (A.1), we have on differentiating:

(A.4)

$$\frac{dx*_{k,1}}{dZ_1}=\frac{1}{d_{k,1}},\ and\ using\ x*_{i,1}+x*_{i,2}=1,\frac{dx*_{k,2}}{dZ_1}=-\frac{1}{d_{k,1}}.\ Since\ \frac{\partial b_1(Z)}{\partial x*_{i,j}}=d_{i,j}\{\ln F_{i,1}q_{i,j}\}$$

we can now sum over all i and j to get:

(A.5)
$$T_2=\sum_{i,j}\frac{\partial b_1(Z)}{\partial x*_{i,j}}\frac{dx*_{i,j}}{dZ_1}=\ln F_{k,1}q_{k,1}-\ln F_{k,1}q_{k,2}=\ln q_{k,1}-ln q_{k,2}.$$

Which is the desired result.

There are also a finite number of transition points. For example after reaching a Z_1 value Z'_1 where $x_{k,1}=1$, we start to increase the next variable. Thus, at Z'_1 the k in (A.4) changes so that the boundary has a left hand derivative from one value of k and a right hand derivative with another. Since both the derivatives are positive this produces a kink in the curve but does not affect the monotonicity argument. The same thing occurs when some other variable overtakes the current $x_{k,1}$ in utility density; there is just a transition to a different choice of the non-integer variable $x^*_{k,1}$.[26]

<u>T_3-The Labor Shift Term:</u> To find T_3 we need $dq_{i,j}/dZ_1$. Since $q_{i,j}(1,Z)=f_{i,j}(l_{i,j})$, $dq_{i,j}/dZ_1=f'_{i,j}(l_{i,j})dl_{i,j}/dZ_1$. We obtain $dl_{i,j}/dZ_1$ by using $w_1 l_{i,j}=d_{i,1}Z_1+d_{i,2}Z_2=d_{i,1}$ or equivalently, by $w_jL_j=Z_j$, $l_{i,j}=(L_j/Z_j)d_{i,1}$.

$$\text{From } l_{i,j}=\frac{L_j d_{i,1}}{Z_j} \text{ we get } \frac{dl_{i,j}}{dZ_1}=\frac{-L_j d_{i,1}}{Z_j^2}\frac{dZ_j}{dZ_1}=-\frac{l_{i,j}}{Z_j}\frac{dZ_j}{dZ_1}. \quad \text{So } \frac{dq_{i,j}}{dZ_1}=\frac{-f'_{i,j}(l_{i,j})l_{i,j}}{Z_j}\frac{dZ_j}{dZ_1}.$$

$$\text{Since } \frac{\partial b_1(Z)}{\partial q_{i,j}}=d_{i,1}x^*_{i,1}\frac{1}{q_{i,j}}=\frac{d_{i,1}x^*_{i,1}}{f_{i,j}(l_{i,j})}, \text{ we have } \frac{\partial b_1(Z)}{\partial q_{i,j}}\frac{dq_{i,j}}{dZ_1}=\frac{-f'_{i,j}(l_{i,j})l_{i,j}}{f_{i,j}(l_{i,j})}\frac{d_{i,1}x^*_{i,1}}{Z_j}\frac{dZ_j}{dZ_1}.$$

For j=1 $dZj/dZ_1=1$ and for j=2 $dZj/dZ_1=-1$. Therefore

$$T_3=\sum_{i,j}\frac{\partial b_1}{\partial q_{i,j}}\frac{dq_{i,j}}{dZ_1}=-\sum_i\left(\frac{f'_{i,1}(l_{i,1})l_{i,1}}{f_{i,1}(l_{i,1})}\right)\frac{x^*_{i,1}d_{i,1}}{Z_1}+\sum_i\left(\frac{f'_{i,2}(l_{i,2})l_{i,2}}{f_{i,2}(l_{i,2})}\right)\frac{x^*_{i,2}d_{i,2}}{Z_2}.$$

Thus, we have obtained all three terms used in Section VII.

[*] President, Alfred P. Sloan Foundation, 630 Fifth Avenue, Suite 2550, New York, New York 10111. This author mentions with pleasure the decisive help he received from Herbert E. Scarf in the early stages of this work.

[**] Director, C. V. Starr Center for Applied Economics, New York University, 269 Mercer Street, Room 307, New York, New York 10003. Also, Professor emeritus, Princeton University. We are extremely grateful for helpful comments to Jagdish Bhagwati, Alan Blinder, Avinash Dixit, David Dollar, Gene Grossman, Peter Kenen, Paul Krugman, Robert Willig and Edward Wolff.

[1] In three invaluable papers [1983, 1984 and 1987] Professor Krugman has explored the policy implications of the scale economies literature. Other recent writings, e.g., Chichilnisky and Heal [1986, especially chapters 3, 11 and 12] contribute valuable insights about such applications. The foci of most of these contributions are, however, very different from ours.

[2] Section X offers some remarks on the relation of our model and our substantial debt, to the very valuable earlier discussions of trade equilibrium under scale economies.

[3] We assume that for each country it is possible to combine the preferences of its inhabitants into an (ordinal) social utility function that does not, however, permit comparison with the magnitude of total utility in the other country. For this reason we can normalize these utilities so that at their maxima their value is unity. Our social utility premise takes us only a small step beyond the common use of community indifference curves in international trade models.

[4] Sections IV-VII are a clearer and simpler version of those results of Gomory [1994] that are needed to prove the new results on regional shape, on the various types of equilibrium points,

and on their economic meaning.

5. Usually in an international trade model there is also a fourth equilibrium condition -- the requirement that trade between the two countries be in balance. In our model, it is easy to show, this is necessarily always satisfied as a result of the assumption that the utility function is such that all income is always spent. This is certainly true with our Cobb-Douglas utility functions.

6. The dependence of the two equations can also be shown by simply adding them up.

7. Stability could be further enhanced by requiring an interval of zero output. A more significant and quite feasible extension is to allow the production functions to depend on the relative national income Z as well as on the labor input. See Appendix B.

8. A similar dynamic argument has been employed by others to show this sort of result, e.g., Ethier [1979 p.14]. The argument in the text can be formalized via a differential or difference equation yielding for any equilibrium vector x* and any nonequilibrium x sufficiently close to x* a trajectory for x that converges to x*.

It is necessary to show also that as x approaches x* the quantity of labor assigned to each industry approaches its magnitude at x* and that the same is true for the wage rate given an x that has approached x*. It is, indeed, possible to demonstrate that the wage and labor quantities will behave in this way. For this we employ the following three very plausible assumptions: (A1) If at point p, w_1L_1, the total wage income in Country j, w_jL_j, is greater than (less than)(equal to) the total income from the sale of the goods it produces in all industries, then w_j will be strictly decreasing (increasing) (stationary), i.e., $dw_j/dt < 0$ ($dw_j/dt > 0$), ($dw_j/dt = 0$). (A2) If at point p the amount of labor $l_{i,1}$ in the ith industry is such that the wage bill in the industry $l_{i,1}w_j$ is greater than (less than)(equal to) the income into that industry from the sale of goods, then $l_{i,1}$ will be strictly decreasing $dl_{i,1}/dt < 0$ ($dl_{i,1}/dt > 0$)($dl_{i,1}/dt=0$). (A3) If at point p with market share $x_{i,j}$ Country j makes the ith good at lower unit cost than Country 2 its market share $x_{i,j}$ will increase (or if it equals 1 it will stay at 1) and if makes the goods at larger unit cost $x_{i,j}$ will decrease (or if it is 0 it will stay at 0).

When put explicitly in the form of a dynamic model satisfying these three premises we can show that with w_1* the equilibrium wage and $l^*_{i,1}$ the equilibrium quantity of labor in industry i in Country 1, any vector $w_{i,}$ $l_{i\,1},\ldots,l_{n,1}$ will approach the Country 1 equilibrium vector w_1^*, $l_{i1}^*,\ldots,l_{n1}^*$. Of course the same holds for Country 2. A full formal proof is available from the authors on request.

9. Most of the equilibria will *not* satisfy comparative advantage. This and their efficiency is discussed in Baumol and Gomory[1994].

10. In common with much of the literature we do not deal here with foreign *ownership* of a another county's industry. Tthis could be developed as an extension of this model.

11. Exactly as written both $x_{i,1}$ and $x_{i,2}$ appear in the expression for $Lu_1(x,Z)$ in (5.2), so we also have the n additional equations $x_{i,1}+x_{i,2}=1$. But if we use $x_{i,2}=x_{i,1}-1$ to eliminate the $x_{i,2}$ we do then have a linear program in the $x_{i,1}$ with only one equation and upper bounds.

12. Here we describe only one of the two boundary approaches methods introduced in Gomory [1993]. The advantages of the *linear* programming approach described here is the ease with which it produces a relatively smooth boundary, and one that seems to be theoretically tractable. The *integer* programming approach, not described here, produces a boundary that is slightly tighter but is harder to analyze. The integer approach, however, is much better at finding the equilibria near the boundary. In fact integer programming of some sort (even if it is only rounding of a linear programming solution) is needed to deal with actual equilibrium points,

while both methods will produce a boundary. Both approaches have provided distinct and useful ways of thinking about the problem.

13. In a linear or diseconomies model the classical assignment would be the Z_1 of the unique equilibrium point. It is also true (see Section X) that in a model in which some industries have scale economies and others have diseconomies, as the share of the latter approaches 100 percent x^c will approach that of the classical trade equilibrium. In our model there may or not be an equilibrium point at Z_C. For a discussion of $x^C(Z_C)$ and the concept of Classical Point see Gomory [1992].

14. In Gomory and Baumol [1994] .

15. This is a property of linear programming. For almost all changes in the right hand side the linear programming basis does not change, the non-basic variables remain zero, and the basic variables change value to satisfy the slightly changed equations. Here we have only one basic variable, the non-integer one. See Appendix for a discussion of the exceptional cases.

16. If the values of the $f'l/f$ were drawn from a random distribution the law of large numbers would produce this effect.

17. Since we will see in the next section that $B_1(1)=U_1^A$ so $K_1=U_1^A/Q(1)$.

18. The key bounding assumption is that the individual industry sizes $d_{i,j}L_j$ are kept between some fixed upper and lower bounds as n increases.

19. This is the same reasoning as in the beginning of Section VI.

20. For an early example of recognition that under scale economies it is possible to experience losses from trade see Graham [1923].

21. There is also an argument in Gomory [1992] suggesting that the least unstable non-specialized equilibria either lie in or very close to the region defined by the specialized equilibria.

[22]. The analysis is given in some detail in Gomory and Baumol [1992a].

[23]. This is because at some Z the diseconomies industries alone consume the entire labor force of one country or the other, and for Z beyond that no solution is possible.

[24]. This was not so in earlier models of trade in differentiated products because they assumed entry to be unrestricted and firms to be symmetrical.

[25]. It is also adequate to assume that the production functions of the firms entail economies up to some intermediate scale of operation, after which average costs either remain approximately constant or gradually rise. Empirical evidence suggests that this may be a common case.

[26]. For more details see Gomory [ ] Appendix.

Appendix B - Production Functions $f_{i,j}(l,Z)$

Introduction: Production functions $f_{i,j}(l,Z)$ allow production to reflect not only the "labor" input but also other factors such as the general state of development of the country. Since the dependence on Z could be result in functions of almost any form, these functions will have to be restricted in some way to get the results we have already obtained for production functions $f_{i,j}(l)$ with economies of scale. An appropriate restriction will be given below.

Theorems

The formula $l_{i,j}=(L_j/Z_j)(d_{i,j}Z_1+d_{i,j}Z_2)$ sets up a one to one correspondence between Z values (Z_1,Z_2) and $l_{i,j}$ values for $l_{i,j}$ values $l_{i,j}\geq d_{i,j}L_j$. We will refer to this correspondence as $\phi_{i,j}$ so $l_{i,j}=\phi_{i,j}(Z)$ and $Z=\phi^{-1}_{i,j}(l)$.

If we have production functions $f_{i,j}(l,Z)$ we define the ***related production function*** $f^*_{i,j}$, which is a function of l only, by $f^*_{i,j}(l)=f_{i,j}(l,\phi^{-1}_{i,j}(l))$.

If we have a model M with demands $d_{i,j}$. country sizes L_j, and production functions $f_{i,j}(l,Z)$ we define the ***related model*** M* as the model with the same demands and country sizes but with the related production functions $f^*_{i,j}(l)$.

Theorem 1: M and M* have (1) the same specialized equilibria and (2) the same regional boundaries.

Proof of (1): The specialized equilibrium x of M produces a Z(x) and an $l_{i,j}$, and the same Z(x) and an $l_{i,j}$ are produced by x in M*. The production amounts are given by:

$$f_{i,j}(l,Z)=f_{i,j}(\varphi_{i,j}(Z),Z) \quad \text{in M, while in M* they are}$$

$$f*_{i,j}(l)=f*_{i,j}(\varphi_{i,j}(Z))=f_{i,j}(\varphi_{i,j}(Z),\varphi^{-1}_{i,j}(\varphi_{i,j}(Z))=f_{i,j}(\varphi_{i,j}(Z),Z),$$

so these are the same. Since x, Z, the $l_{i,j}$ and the amounts produced are the same, these are the same equilibria.

Proof of (2): The boundaries of the two models are produced from equations (5.3) which differ (if at all) only in the $q_{i,j}(1,Z)$ that appear in the linearized utility. For M these are:

$$q_{i,j}(1,Z)=f_{i,j}(\varphi_{i,j},Z) \quad \text{and for M* they are} \quad q*_{i,j}(1,Z)=f*(\varphi_{i,j}(Z))=f_{i,j}(\varphi_{i,j}(Z),Z)$$

so the equations are the same and therefore produce the same boundaries. QED.

However we have said nothing so far about restricting the $f_{i,j}(l,Z)$, and therefore we know nothing about the regional shape that would be produced by the specialized equilibria produced by the $f_{i,j}$ (l,Z). However , now that we have Theorem 1, a plausible restriction suggests itself.

Theorem 2. The region of equilibria resulting from productions functions $f_{i,j}(l,Z)$ will have the properties P1, P2, and P3 provided that the related production functions $f^*_{i,j}$ have economies of scale.

Because of Theorem 1, the proof is immediate. Since the model M* has these properties, so does the model M. However the degree of stability of each equilibrium point, as well as the properties of non-specialized equilibria, can be very different in M and M*.

An Example. The intuitive concept behind this example is that an industry has to pay (in labor)) a fraction of the world market in order to have non-zero production. We obtain this effect by taking for the $f_{i,j}(l,Z)$, $f_{i,j}(l,Z)=h_{i,j}((1+a_{i,j}\}l-(a_{i,j}L_j/Z_j)(d_{i,j1}+d_{i,j2}))$ where the $h_{i,j}(l)$ have economies of scale, zero derivatives at the origin, and $h_{i,j}(l)=0$ for $l<0$. With these $f_{i,j}(l,Z)$, for any fixed Z, there is no positive production until the wage bill exceeds the fraction $a_{i,j}/(1+a_{i,j})$ of the world market for the ith good. Since under the mapping ϕ_j $(L_j/Z_j)(d_{i,j1}+d_{i,j2})$ corresponds to l, the related production function $f^*_{i,j}(l)$ is $h_{i,j}(l)$.

In figure B1 we show one $h_{i,j}$ which we have chosen to be $h_{i,j}=l^{1.4}$. The horizontal axis measures the input l, the vertical axis measures output. $h_{i,j}(l)$ is the dashed line in the figure. The four solid lines are, from left to right, the related $f_{i,1}(l,Z)$ for $Z_1=.8,.6,.3$, and .2.

Figure B1