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# CHAOS: SIGNIFICANCE, MECHANISM, AND ECONOMIC APPLICATIONS

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# Chaos: Significance, Mechanism, and Economic Applications

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Chaos theory has all the earmarks of yet another fad in economics. The scarcity of enduring research following up the early postwar spate of construction of dynamic models and more recent work on catastrophe theory may well feed skepticism about the prospects of chaos models, with their deterministic intertemporal structure. There may, however, be substance to the subject. We will show how wide a variety of important economic phenomena can "easily" fall within the domain of chaotic relationships. We will also describe the rich and surprising variety of the time paths that may consequently emerge. We will stress what may be considered the negative implications of the analysis -- that apparently random behavior patterns can in fact prove to be fully deterministic and that these may emerge in unexpected places, that forecasting of particular variables may face enormous difficulties and that such problems may arise in regimes that obey extremely simple relationships.

We will also describe the mechanism that underlies chaotic regimes and the intriguing pattern of their structural response to changes in the parameter values of the generating models. In much of this we will rely on a compelling heuristic explanation that writers in other disciplines have provided for the chaos phenomenon.

## I. Roots in the Earlier Dynamic Models

To understand the sort of deterministic economic model that can generate chaotic behavior we must return to the literature of the three or so decades following 1930. Until fairly recently, economists were very much concerned with business cycles and the literature had accumulated a vast variety of nonmathematical models each undertaking to provide a set of conditions sufficient to generate cyclical behavior in the economy. There was no necessary conflict among these models since one of them might very well have provided a better explanation of the crisis of 18-- while another might more closely approximate the depression of 19--. However, many of these models were vague and their logic difficult to verify.

All this changed in the 1930's with the work of Frisch (e.g., 1933), Lundberg (1937), and Samuelson (1939), who used difference equations, differential equations and mixed models to generate deterministic time paths. These models readily demonstrated (as was already known in ecology and engineering) that it is extremely "easy" for such deterministic dynamic relationships to generate cyclical behavior. For any such model, any parameter values chosen within broad ranges that can easily be predetermined are sure to yield a time path that exhibits cyclical behavior.

For example, the simplest sort of difference equation is the first order (one-period lag) linear equation

$$y_{t+1} = ay_t,$$

where $y_t$ is the value at time t of the variable in which we are interested. It is obvious, given the initial value, $y_0$, that this generates the time path

$$y_1 = ay_0, \quad y_2 = ay_1 = a^2y_0, \quad y_t = a^ty_0.$$

A moment's thought confirms that for _any_ negative value of parameter _a_ the time path will be oscillatory, as $a^ty_0$ successively goes from positive to negative and vice-versa.

Of course, such an equation is much too simple for most plausible economic models (though it is very close to that which emerges from the cobweb theorem). A more suggestive equation is provided by Samuelson's justly famous multiplier-accelerator model which is made up of the three standard relationships

$$Y_t = C_t + I_t$$

$$C_t = cY_{t-1} + k$$

$$I_t = b(Y_{t-1} - Y_{t-2})$$

where Y is national income (output), C is consumption, and I is investment. Here c is the marginal propensity to consume, so that the equation for C is an obvious linear consumption function with a one-period lag. The investment function is a linear lagged accelerator in which investment is assumed to be proportionate to the preceding period's rate of growth of output. Substitution of the two latter equations into the first yields at once

$$Y_t = (c+b)Y_{t-1} - bY_{t-2} + k,$$

which was Samuelson's second-order linear difference equation. It is not difficult to show that for broad ranges of values of c and b this equation, too, generates cycles. It is also not too difficult to provide an intuitive explanation of the economics of the cyclical mechanism, but that is not our concern here.

These models indicated that the presence of cyclical behavior is not hard to explain. Indeed, it almost suggests that what requires explanation is why cycles are not more pervasive in economic reality. The appearance of these models was at first greeted with enthusiasm and generated a considerable body of writings along similar lines, many by leading economists.

Still, it was not long before disappointment seemed to set in and publication in the arena began to dry up. There were two basic reasons. First, it soon became clear that the behavior of the time path generated by such a linear dynamic system can be extremely sensitive to changes in the values of the parameters, as well as the structure of the model. That undercut the prospects for formulation of a model (and econometric estimation of its parameter values) that would constitute a robust and reliable representation of economic reality. Second, it was recognized early that, qualitatively speaking, such linear models were capable of generating only four types of time path: (i) cyclical and stable (i.e., one converging with oscillations of decreasing amplitude toward some fixed equilibrium value); (ii) cyclical and explosive (cycles of ever-increasing amplitude); (iii) noncyclical and stable and (iv) noncyclical and explosive. This is illustrated by the case $Y_t = a^t Y_0$ derived above, whose time path is obviously cyclical and stable for $-1 < a < 0$ (for example, if $a = -0.5$, then $a^2 = +0.25$, $a^3 = -0.125$, etc., thus going successively upward and downward, and converging toward zero). Similarly, the time path is cyclical and explosive for $a < -1$; it is noncyclical and stable for $0 < a < 1$; and it is noncyclical and explosive for $a > 1$. It was soon recognized that linear equations even more complex than Samuelson's would not generate any time paths basically different from these four. This range of possible time path configurations simply was not sufficiently rich for the economists' purposes.

The solution to the problem, brought to our attention by Hicks and Goodwin, was the use of nonlinear models, perhaps of the general form $Y_t = f(Y_{t-1}, \ldots, Y_{t-h})$. These authors, for example, showed that such a nonlinear model can yield a stable limit cycle toward which all possible time paths of the variable $Y_t$ converge. That is, rather than converging to a fixed equilibrium value, $Y^*$, with the cycles dampening out toward zero amplitude, the nonlinear models could instead yield a stable equilibrium cycle, with $Y_t$ forever wandering from peak to trough along the equilibrium cyclical path.

This is pretty much where matters were left, with a little bit of nonlinearity having been injected into the systems. But the work stopped short of introducing explicitly a degree of nonlinearity just sufficiently higher to generate chaotic behavior.

Work in construction of such dynamic models, then, flagged until the new burst of activity stimulated by chaos analysis in the past few years. Yet, even though the earlier product perhaps did not quite live up to its promise, it must not be misjudged to have been arid. On the contrary, it added considerably to our understanding of the pertinent issues. Three brief examples will suffice to illustrate the point.

The first is the observation already cited, which flowed even from the earliest of the formal dynamic models. This is recognition of how easy it is for any deterministic time path to produce oscillation, a fact well recognized by control theory engineers. The analysis simply demonstrated that the construction of a model sufficient to imply the presence of cycles requires neither convoluted reasoning nor premises that are implausible or pathological.

In addition, despite their sensitivity, the dynamic models proved to be effective instruments for disproof of the universal validity of propositions that had previously been accepted too readily, and for corresponding warnings to policy designers. The very flexibility of the dynamic models made it easy to use them to provide the required counterexamples. For instance, such a model was used to disprove the allegation that profitable speculation is always and necessarily stabilizing (see Baumol, 1957). That is, even if speculators buy when price is low and sell when price is high they can conceivably increase the amplitude of any fluctuations in the price of the good in which they are speculating, if its price happens to be rising at the time they buy and declining at the time they sell. Similarly, it was shown that slight lags in response can undermine apparently rational contracyclical policy (Baumol, 1961). A government which spends more whenever the economy's output is below some target level and which reduces its expenditure when output is above target can increase the amplitude of fluctuations even if the lag in its contracyclical measures is unrealistically small. The same can happen if the government uses similar means to offset the trend in income by running a budget deficit (surplus) inversely related to the time derivative of aggregate income.

Chaos theory, as we will indicate, has at least equal power in providing <u>caveats</u> for both the economic analyst and the policy designer. It warns us that apparently random behavior may not be random at all. It demonstrates dramatically the dangers of extrapolation and the difficulties that can beset economic forecasting generally. It shows that negotiation processes may elicit erratic behavior patterns which no one intended and which can occur even if the positions taken by both parties are inherently simple and straightforward. The moral may be that only extreme patience may be able to prevent breakdown of such a negotation process through mutual misunderstanding of the chaotic course the process can take. Thus, it is our belief that chaos analysis does have much to offer economic analysis and that there is a reasonable chance that it will produce a substantial and enduring flow of research.

## II. What is Chaos?

Let us, next, offer an intuitive description of what chaos theory is about. A simple deterministic relationship such as that provided by a first order linear difference equation can yield an extremely complex time path. By superimposition of a large number of cycles that differ in periodicity and other attributes, intertemporal behavior can acquire an appearance of disturbance by random shocks and can undergo violent, abrupt qualitative changes, either with the passage of time or with small changes in the values of the paramters. Among others, chaotic time paths can have the following attributes: a) a trajectory (time path) can sometimes display sharp qualitative changes in behavior of a sort one would expect in a model subject to large random disturbances (for example, very sudden changes from cycles of small amplitude to cycles of much larger ampltiude, and vice-versa), and at least some of the standard tests of randomness are incapable of distinguishing such chaotic patterns of change from "truly random" behavior; b) the time path of a variable in a chaos model is sometimes extremely sensitive to microscopic changes in the values of the parameters -- a change in, say, the fifth decimal place of one parameter can completely transform the qualitative character of the path; c) chaotic or aperiodic trajectories can include superimposed periodic components of every periodicity (i.e., cycles two periods long, plus cycles three periods long, ...) so that the net result is a time path so convoluted that it cannot in practice be distinguished from an aperiodic one.

Where chaos occurs economic forecasting becomes extremely difficult. The two basic forecasting devices -- extrapolation (of various degrees of sophistication) and estimation of a structural forecasting model -- both become extremely questionable. Extrapolation is hardly appropriate for a time path that may, for example, exhibit two period cycles of considerable and steadily increasing amplitude for 50 periods, suddenly to have the cycles all but disappear for the next 20 periods, with still another pattern abruptly emerging thereafter. Forecasting carried out with the aid of estimates of the parameters of a structural model also runs into difficulties in a chaotic regime if an error in calculation of the third decimal place of a parameter can change the qualitative character of the forecast beyond recognition.

## III. How Chaos Arises

Before we can see how chaotic relationships can arise in economics we must first provide a description of their characteristics. Since much of the discussion that follows relates to cyclical and oscillatory behavior it is important to define precisely what we mean by those terms. A time path, $y_t$, will be taken to be characterized by a cycle whose duration is p periods (a p period cycle) if it always replicates itself precisely every p periods from any initial point in its trajectory, and does not always repeat itself precisely in any smaller number of periods.

In contrast, an oscillatory time path is defined more vaguely as one which is not monotonic, involves "frequent" rises and declines in the values of its variables, but in which the time path may rarely or never replicate an earlier portion of its trajectory.[1]

The simplest and most common chaos model involves a nonlinear one-variable difference equation of first order, i.e., one of the form

$$\text{(1a)} \qquad y_{t+1} = f(y_t),$$

where the graph (the phase diagram) of $f(y_t)$ is hill-shaped and "tunable", i.e., the height, steepness and location of the hill can be adjusted as desired by a suitable modification in the values of the parameters of $f(y_t)$. This phase diagram is the geometric instrument used to analyze the time path generated by a difference equation model, and it is employed extensively in chaos analysis.[2]

The function most commonly used to illustrate the chaos phenomenon is the quadratic

$$\text{(1b)} \qquad y_{t+1} = wy_t(1-y_t),$$

with the single parameter, w. The hill-shaped curve in Figure 1 represents this function when w = 3.45. The figure also shows the generation of a time path via the graphic procedure made familiar in the cobweb theorem literature, using the phase curve to find $y_1$ from $y_0$, $y_2$ from $y_1$, etc., and the 45 degree ray to transfer each value of y from the vertical to the horizontal axis.

We see immediately from (1b) that whatever the value of w, the graph (the phase curve) for that equation always must reach its maximum at $y_t = 0.5$, where

$$(2) \qquad dy_{t+1}/dy_t = w(1-2y_t) = 0.$$

At that point its height must be w(0.5)(1-0.5) = w/4, which clearly increases proportionately with w.

Now, for w less than 1 the phase curve will lie entirely below the 45$^{o}$ ray in the positive quadrant (Figure 2a).[3] For values of w that exceed 1 there will be a positive valued ($y_t > 0$) intersection (equilibrium) point, E, satisfying

$$(3) \qquad y_e = (w-1)/w \text{ (see preceding footnote)},$$

between the phase curve and the 45$^{o}$ ray. For $1 < w < 2$, the phase curve's slope at the intersection point will be positive[4] (Figure 2b). For $2 < w < 3$ that slope will be negative but less than unity in absolute value (Figure 2c), while for $w > 3$ the slope will be less than -1 (Figure 2d).

It is this last case, $w > 3$, that is of interest to us here. We know from the elementary theory of difference equations that since the slope of the phase curve is then negative at the equilibrium point the time path must involve oscillations. These cobweb-like oscillations will be two periods in length, with the high point of one period, $y_t$, followed by the low point, $y_{t+1}$, of the next, just as in $y_t = a^t y_0$ when $a < 0$. Moreover, since the slope is greater than unity in absolute value, the oscillations will be explosive (of ever-growing amplitude), moving ever further away from the equilibrium value, $y_e$, in the neighborhood of that value of y.

If the graph were not hill-shaped that would be the end of the story, with the cobweb cycles moving ever further away from the equilibrium point. However, with a hill-shaped phase curve, eventually, as the cobweb expands it will encounter the positively sloping side of the hill and "bounce off it" at a $y_{t+1}$ value closer to the equilibrium level that some earlier $y_{t+1}$ (thus, in Figure 3 the height of point B is closer to that of equilibrium point E than is earlier point A's). This must happen eventually, since as the cobweb expands further to the left during its explosive stage its height in the next move that follows must be reduced because the slope of the pertinent portion of the graph is positive. When this happens, the cycles will begin converging toward E once more, but that can only be temporary since E is an unstable equilibrium that generates an explosive time path, as we have seen.

The analogy with a billiard ball bouncing off the sides of the table in a complicated pattern is suggestive here. It is easy to imagine why, in such circumstances, the time path can turn out to be complex, as chaos requires. What is rather more surprising is that the pattern of chaotic behavior will then follow some very simple and orderly rules.

## IV. The Orderly Structure of Chaotic Behavior[5]

Let us turn now to a more careful description of chaotic behavior. While the discussion that follows will be based almost exclusively on our illustrative chaos equation $y_{t+1} = wy_t(1-y_t)$ it must be emphasized that exactly the same sort of behavior holds for a very wide set of relationships $y_{t+1} = f(y_t)$ whose graph is hill-shaped and "tunable" by adjustment of the parameter values.

Let us first offer a preview of what will be shown in this section and the next, summarizing results, initially without explanation. We will see that as the value of w increases, in addition to the two period cycles which the time path initially contained, it will incorporate successively (and at known values of w) first, cycles of four period length, then it will add 8 period cycles, then it will superimpose 16 period cycles, etc.

Let us begin the story just at the point where the tuning (or controlling) parameter attains a value (w = 3 in our case) at which the basic two period cycle, $y_1$, $y_2$ (with $y_3 = y_1$) becomes unstable because the slope of its graph at equilibrium point E exceeds unity in absolute value. At exactly that value of w we will see that a stable four period cycle, $y_1^*$, $y_2^*$, $y_3^*$, $y_4^*$ (with $y_5^* = y_1^*$) makes its appearance. Two of these four points, $y_1^*$, $y_3^*$, of the four period cycle are generated via a process called "bifurcation" (which will be explained presently) from one of the points, say $y_1$, of the period-two cycle, while the other two points $y_2^*$, $y_4^*$ "bifurcate" from the other point of the period-two cycle. Along such a four period cycle, a trajectory that starts at one of the two high points, say, $y_1^*$, of the four period cycle, first moves to one of the two low points say $y_2^*$), then back to the other high point, $y_3^*$ and finally completes the cycle at the remaining low point, $y_4^*$. Trajectories in the vicinity of this cycle follow a pattern that is very similar, eventually converging to the stable four period cycle. As was stated, the period-doubling bifurcation just described takes place just when the controlling parameter attains a value at which the two period cycle loses its stability, when the parameter w exceeds the value 3 in our illustrative equation.

As the value of the parameter w is increased further, the four period cycle itself becomes unstable in its turn, and from each of the four values $y_1^*$, $y_2^*$, $y_3^*$, $y_4^*$ that constitute the period-four cycle, two additional points emerge via a new bifurcation. These new eight points now constitute a stable period-eight cycle. As w is increased, the period-doubling bifurcation tale will repeat itself, thereby ushering in new stable cycles which any trajectory in its neighborhood will gradually approach.

At first, all of these will involve cycles that only have even periods but, eventually, cycles whose length involves an odd number of periods will appear. The first such odd-period cycles to enter the time path will be very long, but they will be joined by odd period cycles of shorter and shorter duration. Finally, at some value of the controlling parameter, w, even three-period cycles will occur. At such values of the parameter the time path must involve an infinite number of equilibrium points, stable and unstable, and an infinite number of cycle lengths. There will also be an uncountable number of initial values yielding time paths which, while, bounded, will yield a pattern that never repeats iself, no matter how long a set of time periods one permits the calculation to encompass. When this set of conditions holds, true chaos is said to have occurred.

Let us begin to consider, now, why chaos should be approached by such an orderly progression -- from one stable two period cycle, to an unstable two period cycle perturbed by two stable four period oscillations, then (when these all have become unstable) with the addition of four 8 period disturbances, etc. We will use two related procedures to provide an intuitive view of the matter.

To see how the four period disturbances enter, let us, first, quickly review the analagous properties of the two period case. The time path of a two period cycle is studied with the aid of a difference equation (1a) or (1b) relating the values of the endogenous variable, $y_t$ in <u>two successive periods</u>. The equilibrium value can then be calculated from the requirement $y_{t+1} = y_t = y_e$.

Now, in examining whether a four period cycle is present we are not concerned with the relationship between two <u>successive</u> values, $y_t$ and $y_{t+1}$, as we were in the two period case, but rather with that between $y_t$ and $y_{t+2}$. That is, suppose we are given the value of $y_t$. Then, information about the value of $y_{t+1}$ does not permit us to infer anything about the presence or absence of a four period cycle. But if we find that a relatively low $y_t$ is followed by a relatively high value of $y_{t+2}$ (which in turn is followed by a relatively low value of $y_{t+4}$, etc.) we can infer that a four period oscillation is present. For analagous reasons, for such a cycle eventually to converge to an equilibirum point, that equilibrium must satisfy

$$(4) \qquad y_{e2} = y_t = y_{t+2}.$$

Thus, to investigate the genesis of four period cycles we need a relationship between $y_{t+2}$ and $y_t$, not one between $y_{t+1}$ and $y_t$. By (1a) and (1b) such a two period relationship is obtained via a second iteration of the relationship $y_{t+1} = f(y_t)$. That is, we first find $y_{t+1}$ from $f(y_t)$ and then we find $y_{t+2}$, in turn, from $f(y_{t+1})$. Combining them, this process gives us

(5a) $$y_{t+2} = f(y_{t+1}) = f[f(y_t)]$$

(5b) $$y_{t+2} = wy_{t+1}(1-wy_{t+1}) = w[wy_t(1-y_t)(1-wy_t(1-y_t))].$$

(Henceforth, we will use $f^{(2)}$ to represent $f[f(y_t)] = y_{t+2}$, $f^{(3)}$ to represent $f\{f[f(y_t)]\} = y_{t+3}$, etc.).

To tell the story of the introduction of the four period cycles we must first consider some properities of the graph of the general four period relationship (5a). This may be considered the equation of a four period phase curve in the graph which has $y_{t+2}$ rather than $y_{t+1}$ on its vertical axis. In Figure 4a such a phase curve [labelled y(t+2)], with its typical double hump, is superimposed on the hill-shaped two period phase curve [labelled y(t+1)].

Let us examine the relations between the two phase curves, y(t+2) and y(t+1). First we note that the two phase curves in Figure 4a cross the horizontal axis at the same points, that is, at the points $y_t = 0$ and $y_t = 1$. This is generalized in

Proposition 1. If the graph of f goes through the origin, then all roots of f [i.e, points at which $y_{t+1} = f(y_t) = 0$] must also be roots of $f^{(2)}$.

Proof: Let $y^*$ be a root of f. Then, since $f(y_t^*) = 0$, $f[f(y_t^*)] = f(0) = 0$.

Next, we note that the two phase curves cross the $45^o$ ray at a common equilibrium point, E. This is generalized in

Proposition 2. Any equilibrium point of f must also be an equilibrium point of $f^{(2)}$.

Proof: Let $y_e$ be an equilibrium point of f. Then, by definition, $f(y_e) = y_e$, so that $f^{(2)}(y_e) = f[f(y_e)] = f(y_e) = y_e$.

We also have

Proposition 3. The slope of $f^{(2)}$ at an equilibrium point of f must be the square of the slope of f.[6] That is, at any $y_e = y_t = y_{t+1}$ we must have $df^{(2)}/dy_t = (df/dy_t)^2$.

That is why in Figure 4a at point E, where the slope of the $y_{t+1}$ graph is negative but less than -1 in absolute value, the slope of $y_{t+2}$, since it is the square of the other slope, is positive but quite small.

Comment: Proposition 3 must hold, in particular, at the origin, where $y_{t+1} = y_t = 0$. Note also that corresponding propositions also hold for any $f^{(2n)}$ where n is any positive integer.

The final key observation linking the graphs of $f^{(2)}$ and f is that where [as in the case of (2b)] the basic relationship, $f(y_t)$ is quadratic (it includes a term with $y_t^2$ in it), and consequently has one peak, the four period cycle relationship $f^{(2)}(y_t) = f[f(y_t)]$ will be of fourth degree [it has a term involving $y_t^4$, as is illustrated in equation (5b)] and so can be expected to have either two maxima and one minimum, or the reverse. That is why the $f^{(2)}$ graph, as is shown in the figure, typically has a double hump.

Figures 4a-4c illustrate the behavior of the preceding relationships as the value of w increases. Each graph is derived from our basic illustrative equation for a different value of w, as will now be explained, and we will see precisely why two new equilibrium points must appear just at the value of the tuning parameter (w = 3 in our example) where the initial equilibrium point, E, becomes unstable.

In Figure 4a $w = 2.8$ so that at equilibrium point E the slope, $dy_{t+1}/dy_t = 2 - w$ (see footnote 2) must be negative and less than unity (in fact, it must equal -0.8). Therefore, by proposition 3 the derivative of $f^{(2)}(y_t)$ at E must be $(-0.8)^2 = 0.64$, i.e., it must be positive and less than unity. Hence, with $2 < w < 3$ the curve $f^{(2)}(y_t)$ must cut the $45^0$ ray at E <u>from above</u> as we move from left to right.

We now can see just what sort of shape the $f^{(2)}(y_t)$ locus must assume. As we have noted, like f(t), by proposition 1, it must cut the horizontal axis at the same points ($y_t = 0$ and $y_t = 1$). Then, since where $w > 2$ it is easy to show that the slope of $f(y_t)$ is greater than unity near the origin, the slope of $f^{(2)}(y_t)$, by proposition 3, must be greater still and so it must lie above the $45^0$ ray. Ultimately, $f^{(2)}(y_t)$ must meet the $45^0$ ray at E, cutting it from above, as we have just seen. Hence, with w in the range now being considered, $f^{(2)}(y_t)$ need never meet the $45^0$ ray anywhere to the left of E (except at the origin). To the right of E the situation is similar, with $f^{(2)}(y_t)$ leaving that ray from below, finally descending to zero at $y_t = 1$. In short, in this $2 < w < 3$ case $f^{(2)}(y_t)$ need intersect the $45^0$ ray only at the origin and at E, just as $f(y_t)$ does.

Next, in Figure 4b we consider the case $w = 3$ so that at E $df/dy_t = -1$ exactly; and so by proposition 3 $df^{(2)}/dy_t = (df/dy_t)^2 = +1$. Then $f^{(2)}(y_t)$ will be tangent to the $45^0$ ray at E, as the figure shows.

Finally, the full story emerges in Figure 4c where $w = 3.43$ so that $df/dt = -1.43$ at E and therefore $df^{(2)}/dt = (-1.43)^2 = 2$ (approximately). Since that slope exceeds unity, for $w > 3$, $f^{(2)}(y_t)$ must cut the $45^0$ ray <u>from below</u> at E as we move from left to right. This means that as $f^{(2)}(y_t)$ leaves the origin, initially lying above the $45^o$ ray, at some point G, to the left of E, it must cross that ray for it to be possible for that curve to cut the ray at point E from below. Similarly, to the right of E, $f^{(2)}(y_t)$ must first lie above the ray, and so must cross it again at some point, H, in order to reattain the horizontal axis again at $y_t = 1$. We see that just at that value of w at which the slope of the $f(y_t)$ graph begins to exceed unity in absolute value, so that its equilibrium becomes unstable, there appear two new intersection points G and H, which have no counterparts in Figure 4a. These are the two new equilibrium points that constitute the bifurcation of equilibrium E. That is, each new intersection point is the equilibrium point for a four period oscillatory cobweb perturbation generated by any initial point in its vicinity, as illustrated in the graph for initial point y(0) and equilibrium point G. This must be so because for the value of the slope of $f^{(2)}$ at G and H is negative and greater than -1.

That is the basic story of the bifurcation process. We can also use a slightly different way to examine its proliferation of equilibrium points more directly with the aid of equilibrium condition (4) and our illustrative relationship (5b) for $f^{(2)}$. By substituting equilibrium condition (4) into the two period relationship (5a) we obtain the equation $y_{e2} = f^{(2)}(y_{e2})$ from which the two-period equilibrium values, $y_{e2}$, can be deduced. But as we will see now by example this is an equation of higher degree than the analagous two period equation $y_e = f(y_e)$, and so the former will yield more equilibrium points. An illustration is obtained from our specific equation (5b). After we substitute equilibrium requirement (4) into it, we obtain after a bit of manipulation,

$$y_{e2}(wy_{e2} - w+1)[w^2y_{e2}^2 - (w^2 + w)y_{e2} + w + 1] = 0.$$

Obviously, this is an equation of fourth degree, in contrast to the quadratic form of the analagous equilibrium equation $y_e = wy_e\ (1-y_e)$ for the two period case. The fourth degree equation clearly has the trivial root $y_{e2} = 0$, as before. The first parenthetic expression indicates that a root is also given by

$$wy_{e2} - w + 1 = 0 \qquad \text{or } y_{e2} = (1 - w)/w,$$

which, by (3) is our two period cycle equilibrium point. This shows that (3) (point E in Figures 4a-4c) continues to be an equilibrium value even after w exceeds 3, when this equilibrium becomes unstable.

Finally, the expression in the square brackets, being quadratic, yields two roots. If we use the standard formula to find those two roots it is easy (but somewhat tedious) to show that for any $w < 3$ those roots will be complex numbers (i.e., they will not take real values). However, for any $w > 3$ these roots become real and, consequently, constitute two additional equilibrium values. That is the second way of seeing how the first bifurcation arises.

The key point is that, as seen from (4), (5a) and (5b), an increase in the length, s, of the cycle under study automatically brings an increase in the degree of the equilibrium value equation $y_e = f^{(s)}(y_e)$, yielding a correspondingly increased number of roots.

## V. How Eight Period Oscillations and Their Equilibrium Points Arise

We can now quickly indicate by analogy how the four period cobweb oscillations at G and H in Figure 4c become unstable and how this leads to the appearance of four new equilibrium points. The story is a precise replication of what was described in the previous section, with a second bifurcation step succeeeding the first. As the value of w increases, the absolute slopes of $f^{(2)}(y_t)$ at G and H will increase monotonically.[7] Ultimately, these slopes must exceed unity in absolute value and then their surrounding cobwebs must become unstable, as before. In addition, if we form the function $y_{t+4} = f^{(4)}(y_t) \equiv f(f(f(f(y_t))))$, for precisely the same reasons as before the curve representing $f^{(4)}(\cdot)$ will become tangent to the $45^0$ ray at G just when the slope of $f^{(2)}(\cdot) = -1$, and the same will be true at H. Thus, for w slightly larger than this, G and H will each be surrounded by two new equilibrium points for 8 period cobweb oscillations. Each such oscillatory disturbance will initially be stable but will grow unstable as w increases still further.

The process obviously can repeat itself ad infinitum thus giving rise to an infinite set of superimposed oscillations, each of an even number of periods in duration.

Later, we will pause briefly to see how cycles of odd periods of length can arise. But first, we show some actual superimposed oscillations of periods 2 and 4 to give concreteness to the abstractions discussed so far.

## VI. Superimposition of Two Sets of Oscillations: Example

While it is clear from the preceding discussion that the initial two period cycles will at some point be joined by a pair of four period perturbing oscillations, the tangible form their conjunction assumes still needs to be made clear. Figure 1, in addition to showing $f(y_t)$, also includes the time path of the first 14 periods, starting with $y_0 = 0.999$. The resulting cobweb obviously involves cyclical behavior about point E, but the pattern is not obvious. However,[8] Figure 5a represents the time path for periods 47 to 50. We see here that the time path has settled down into an (approximately) recurrent pattern (w has not yet entered into the region of true chaotic behavior). We seem to have two nested cycles with the time path alternating between them. The cyclical path ABCD does not return to starting point A but instead goes to neighboring point K; then it follows the cycle KFGH and then, apparently as something of a miracle, returns to starting point A of the other cycle. However, this is no great coincidence, but a normal part of the process, for reasons the analysis of the preceding sections has indicated. It is just in this way that oscillations of four period length are superimposed on cycles of two period length. Both types of oscillation are shown in Figure 5b which is the time path generated by the phase diagram in Figure 5a.

At first glance we see only a persistent (but imperfectly replicated) oscillation exactly two periods in length, which clearly dominates behavior. Where are the superimposed four period oscillations hidden? They are concealed here by the larger two period oscillations. To see them one must first look exclusively at the upper horizontal segments a, c, e, g and i of the time path, and then by looking in turn only at the lower segments, b, d, f, h and j.

The upper segments describe the first of the four period oscillations. Starting from a, <u>and skipping one period</u>, $y_t$ falls to c. Then, after a gap of another period, $y_t$ rises again to e. Continuing in this way we see that one has an oscillatory disturbance whose high points are a, e, i, ... and whose low points are c, g, ..., with four periods elapsing between, say, one high point and the next. This oscillation corresponds to the difference in height between horizontal segments GH and CD in the time path of the phase diagram 5a. The reader will now readily recognize the other four period oscillatory disturbance in Figure 5b by looking at the lower horizontal segments b, d, f, h, and j.

We can see now where we are left by the superimposition of four period oscillatory disturbances upon our initial two period oscillation. The net result, in the limit, is a single four period cycle, that is, a cycle that repeats itself precisely every four periods, as is confirmed by careful examination of the right hand end of the time path in Figure 5b.

We can also usefully return to Figure 5a to tie together the various strands of our explanatory discussion.[8] In Figure 5a suppose we were to superimpose the four period phase graph given by $y_{t+4} = f^{(4)}(y_t)$, drawing it in just as $f^{(2)}$ was drawn into Figures 4a-4c. Then, it is not difficult to confirm that $f^{(4)}$ contains four points of negatively sloping intersection with the $45^0$ line. While $f^{(4)}$ is not actually drawn into Figure 5a it turns out, for reasons about to be suggested, that these intersection (equilibrium) points are B, F, D and H -- the intersection points of the $45^0$ line with the path ABCDKFGH. Why must the equilibrium path go through those intersection points? Consider the case where the negative slopes of $f^{(4)}$ at its four intersection points, B, F, D and H, with the $45^0$ line are all very small. Then each of these points, e.g., B, will be surrounded by cobweb paths each of which converges rapidly to the intersection point in its interior (point B). Very rapidly all movement of $y_t$ will therefore take it to points very close to one or another of the four points B, F, D or H. Moreover, should $y_t$ attain one of these four points, say B, then $y_t$ cannot return to B until precisely four periods later since B is an equilibrium point of $f^{(4)}$ but not of $f^{(2)}$ or f. That is, if we let $y_b$ represent the value of y at B, then if for some $t = t^*$ $y_{t^*} = y_b$ we must also have $y_{t^*+4} = y_b$ but not $y_{t^*+1} = y_b$ or $y_{t^*+2} = y_b$. A similar observation holds for the other three intersection points, F, D and H. It follows (in the limit) since $y_t$ has only four places where it can go, and can go to any particular one of them only once in four periods, that $y_t$ must divide its time equally among all four points, and since $y_{t+1}$ is determined uniquely by $y_t$, it must repeat its circuit every four periods. Thus, for any value of w at which $f^{(4)}$ has four stable intersection points with the $45^0$ line, those four intersection points will constitute a single, rather messy, limit cycle four periods in length, like that in Figure 5a. Similarly, at values of w where $f^{(8)}$ has eight stable intersections, the time path will be characterized by a single even messier limit cycle eight periods in length, and so on, *ad infinitum*.

## VII. How Three Period Cycles Arise

So far we have dealt only with cycles whose duration is an even number of periods. However, it is now easy to show how odd period cycles arise, using essentially the same approach as before. To find the three period cycles, for example, we plot the phase diagram for

$$y_{t+3} = f^{(3)}(y_t) = f(f(f(y_t))).$$

If $f(y_t)$ has a single hill, $f^{(3)}(y_t)$ will normally exhibit four hills when the tuning parameters have made the $f(y_t)$ hill sufficiently steep. The graph is shown in Figures 6a and 6b. In 6a, with w relatively small, the phase curve only crosses the $45^0$ ray once, at a location, E, which is also the nontrivial equilibrium point of $f(y_t)$. However, as w increases in value the hilltops of $f^{(3)}$ will rise and the valleys will deepen, and eventually the $45^0$ ray will be crossed seven times. Initially the corresponding six new crossings will correspond to points on two cycles of three periods, one of which is stable and the other unstable. Unlike the bifurcations of cycles of even order which derive from cycles of lower period as their equilibrium points undergo a loss of stability, odd period cycles do not bifurcate from lower order cycles, but emerge or disappear in pairs, with a stable and unstable cycle constituting each pair.

## VIII. Chaos and Strange Attractors

Despite its aura of erotic kinkiness, "strange attractor" is a technical term which offers yet another insight into the workings of the chaos phenomenon. An attractor is what most of us might describe as the equilibrium or limit time path of a stable dynamic system, whether or not that system is chaotic. For example, the difference equation $y_{t+1} = 0.5y_t$ clearly converges toward the equilibrium value $y_e = 0$ so that any time path of the equation, whatever the initial point, will converge in the limit to the origin in the phase diagram. The origin is then said to be the attractor for this relationship; and in this case the attractor is clearly a single point.

In other cases, the attractor is more complex. For example, all time paths of the system may be cobwebs which converge toward a simple rectangle in the phase diagram. This means that the time path will settle down in the limit to a two period oscillation -- a repeated traversing of that rectangle, going endlessly back and forth from its upper to its lower edge and then back up again. Here the attractor is the rectangle, that is, it is a two period limit cycle, toward which all time paths of the system converge.

Attractors can grow more complex still, as illustrated in Figures 1 and 5a. In the former we see a complicated cobweb path which converges to the attractor shown in Figure 5a, an attractor which can perhaps be described as a pair of intertwined rectangles. The result is an equilibrium time path involving somewhat messy oscillations approximating those in Figure 5b (that have already been discussed).

Now, intuition suggests, correctly, that in the stable case, as the attractor of the system is made increasingly complex by changes in the pertinent parameter values, the time path will increasingly take on chaotic attributes. Indeed, when the parameter values of a stable system enter the true chaotic region the limit path will achieve the degree of complexity that leads it to be referred to as a "strange attractor."

It is possible to provide pictures of strange attractors, but they are sufficiently convoluted that it is fairly difficult to do so without recourse to three dimensional colored diagrams (for nice examples, see Crutchfield, Farmer, Packard and Shaw, 1987, pp. 50-51).[9]

## IX. Sensitivity of the Time Path

Next, let us deal with the extreme sensitivity of a chaotic time path and its qualitative properties to very small changes in initial conditions. This property has attracted a great deal of attention in disciplines outside of economics which have also utilized chaos theory. For example, meteorologists have dubbed this sensitivity the "butterfly's wing phenomenon." They refer to the possibility that a butterfly fortuitously flapping its wings in Hong Kong can cause tornados in Oklahoma if weather is controlled by chaotic relationships.

A few graphs will illustrate this degree of sensitivity. In Figures 7a, 7b and 7c there is no difference in initial conditions or anything else, except that in 7a $w = 3.935$, in 7b $w = 3.94$, while in 7c $w = 3.945$. We see that changes in the third decimal place in the parameter value can transform the entire picture unrecognizably. It is also easy to demonstrate that far smaller changes in the value of the parameter can cause very similar upheavals. Not only that. If we hold the parameter value constant and change the initial condition by microscopic amounts in a chaotic regime equally startling qualitative changes in the time path will follow. (We provide no graphic example because when one changes the initial conditions, leaving w constant, the resulting graphs differ from one another very much like figures 7a, 7b and 7c do.)

The sensitivity of the time path of a variable governed by a chaotic time path can be brought out in another way. In a calculation by Richard Quandt, the time path of our illustrative equation $y_{t+1} = wy_t(1-y_t)$ was determined twice, each time for 640 periods. The first calculation was carried out by a process that rounded after 7 decimal places while, in the second, rounding occurred after 14 decimal places. With w sufficiently low so that the equation had not yet actually entered the chaotic region the two calculated time paths remained virtually identical even after 600 iterations. In contrast, with a w value sufficiently large to produce chaos, after only 30 iterations the two series lost virtually any resemblance to one another.

To be specific, letting, e.g., $y(14)_{640}$ represent the 640th observation with a calcuation accurate to 14 decimal places, for a nonchaotic $w = 3.5$ Quandt obtained

$$y(14)_{640} = 0.3828196830, \qquad y(7)_{640} = 0.3828207254.$$

But for a chaotic $w = 3.9$, the two series lost all resemblance after only a few periods, and gave, for example

$$y(14)_{31} = 0.8823060155, \qquad y(7)_{31} = 0.4794570208.$$

These figures dramatize the extreme sensitivity of trajectories to initial conditions. Sensitive dependence to initial conditions will not be observed if there exist stable periodic time paths that attract trajectories from almost all initial points. For the quadratic case there will be many values of w between 3 and 4 for which stable orbits exist but there will also be a large set of w's for which trajectories will be sensitive to initial conditions. The studies of Shaw [1981] also confirm that for the quadratic case sensitive dependence will be prevalent for values of w between 3.5 and 4.

These figures indicate the difficulties that are apt to beset forecasting in the presence of chaos. Even a forecasting procedure of unprecedented accuracy is likely in such a case to yield results that differ vastly from the actual course of future developments.

## X. Sudden Qualitative Breaks in the Time Path

Figure 7b also dramatizes another of the characterizing attributes of chaotic trajectories -- their propensity to introduce sharp and unheralded qualitative breaks in time path. From the initial point, A, of the time path until point B, some 25 periods later, there is a fairly homogeneous regime of (somewhat lopsided) cycles which seem to exhibit no clear trend in amplitude. Then, suddenly, the time path becomes almost horizontal, and for 10 periods (from B to C) cyclical behavior all but disappears. At that point, just as unexpectably, several fairly sharp oscillations arise, apparently out of nowhere, abruptly becoming very moderate again to the right of point D. It is difficult to imagine how any forecasting technique that relies upon extrapolation, direct or indirect, could have correctly predicted events during the period encompassed between points B and C from even the most accurate and fullest set of data about the 25 period interval that preceded it.

This graph suggests that chaotic behavior does not generally mimic pure randomness in the performance of its basic variable. Rather, the time path can resemble one that might be expected of a moderately orderly deterministic model, but which is at the same time subject to very large random disturbances occurring at randomly determined intervals.[10]

## XI. Some Basic Mathematical Results

Modern methods of qualitative analysis of dynamical systems go back to Poincaré [1880], [1899]. Since the classic work of Smale [1967], it has become clear that very complicated trajectories (time paths) can easily arise in certain dynamical systems and that such complicated trajectories can persist when small perturbations of the underlying system occur. (For a clear exposition see Guckenheimer and Holmes [1983].) The papers of Li and Yorke [1975] and others and the work of Šarkovskii [1964] which has recently been rediscovered (see Stephan [1977]) have greatly facilitated exploration of the pertinence of such complicated dynamics, arising in simple first order dynamic systems, to a variety of fields, such as physics, biology or economics.

Let us now describe two of the basic mathematical theorems of chaos analysis, translating them into terms that economists can follow more easily. The theorems describe the superimposition of cycles of periods of different length and the resulting chaotic behavior of the time path when cycles of every integer periodicity are included. We will also note a few pitfalls besetting interpetation of the basic theorems, pitfalls that have led to some debate in the literature.

We begin, chronologically with the theorem of Šarkovskii [1964] which has recently reemerged.

Consider the sequence of all odd integers, followed by twice that sequence, then three times that sequence, etc., finally followed by the descending sequence $2^n$, where n is positive and integer (e.g., ... $2^6 = 64$, $2^5 = 32$, $2^4 = 16$, $2^3 = 8$, ...). Let aPb mean "a precedes b"; then we are considering the ordering

(6) 3P5P7...P(2·3)P(2·5)P(2·7)...P(3·3)P(3·5)P(3·7)...64P32P8P4P2.

Also, let f be a continuous difference equation. Then we have

The Šarkovskii Theorem. If the function f gives rise to a time path with cycles of period m, then this time path generated by f will also contain cycles of each and every period $m^*$ such that $mPm^*$ in sequence (6).

For example, since the numbers 8, 4 and 2 follow the number 16 in sequence (6), the theorem tells us that if a difference equation's time path happens to contain cycles of period 16, that time path must, in addition, contain cycles of period 8, of period 4, and of period 2. It follows as a corollary of the preceding theorem, since 3 is the first number in sequence (6), that when a time path of the sort under discussion contains cycles three periods in length it must also contain cycles of every other possible (integer) length! This is, essentially, what is meant by "chaos" and underlies the expression in the literature that "period three implies chaos."

We now turn to the Li-Yorke Theorem which contains some of the results of Šarkokvskii.

The Li-Yorke Theorem. Let f be a difference equation that is continuous, and for which there exist two numbers, a and b, such that if $a \leq y_t \leq b$, then $a \leq y_{t+1} \leq b$. Now, if one can find a $y_t$ such that when $y_t$ rises for two successive periods it will fall back to below its initial value in the next period, i.e.,

(7) $$y_{t+1} = f(y_t) > y_t \text{ and } y_{t+2} = f^{(2)}(y_t) > y_t \text{ but } y_{t+3} = f^{(3)}(y_t) \leq y_t,$$

then two major consequences follow:

(a) For any integer $k > 1$ there is at least one initial point $y_0$ between a and b such that the subsequent time path, $y_t$, is characterized by cycles of period k.

(b) There exists an uncountable set, S, of initial points[11] in the interval between a and b such that if initial points $x_0$ and $y_0$ both lie in S, then (i) ultimately the difference $(x_t - y_t)$ between their respective time paths will approach zero, that is, the two paths will (temporarily) move as close to one another as may be desired; (ii) however, after some interval of close proximity the two time paths must always diverge again; (iii) moreover, no such time path will ever converge asymptotically to any stable periodic time path; (iv) indeed, if $w_0$ is any initial point either outside or inside of S and $w_t$ is its subsequent time path (which may or may not be periodic) $y_t$ (for any $y_0$ in S) will never converge asymptotically to $w_t$.

All of this means that if the Li-Yorke conditions (7) are satisfied for a given difference equation there will exist an uncountable set of initial values which generate a time path that is sensitive to the choice of initial value and which never approximates any simple and regular path for an indefinite period. It is these two features that are the formal attributes of chaotic regimes.

There are a number of issues that require care in use of the preceding theorems. As has been noted already, the aperiodic (chaotic) trajectories, whose existence is shown in the Li-Yorke theorem, will normally be generated by an uncountable infinity of different initial conditions. Speaking very roughly, this may make it appear that the chaotic region in the realm of initial condition is "very large." Yet, the Lebesque measure[12] (a standard measure of the area or volume occupied by a set of points) of these intitial points may be zero, that is, for some chaotic models their behavior may be nonchaotic "almost everywhere." Here it should be noted that the Lebesque measure is defined so that the measure of any interval is equal to its length, while a set of "isolated" points, even if there is a nondenumerable infinity of them, has Lebesque measure zero (that is, these points can be covered with a countable set of intervals whose total length is arbitrarily small). Under certain conditions it has been shown that[13] the time paths will have at most one stable periodic orbit which will attract all intitial points except for a set of Lebesque measure zero. On the other hand, there may be no stable orbits at all and almost all initial conditions will then lead to some aperiodic trajectory. This, for instance, will be true of first order difference equation systems whose phase graphs look roughly like inverted V's, and the derivatives of whose phase graphs are everywhere larger than unity in absolute value except at the peak of the graph, where the map will not be differentiable[14] (see for example Li and Yorke [1975], Theorem 3, or Lasota and Yorke [1977]). It is observations such as this that constitute the core of discussions about *a priori* grounds for belief in the prevalence and significance of chaotic relationships.

An important matter, especially for forecasting purposes, is whether intial points that are close together give rise to trajectories that stay close together or diverge (move apart by a distance larger than a preselected $\epsilon > 0$ for periods beyond some date, $t^*$). Such divergence is called "sensitive dependence on initial conditions." Obviously, if a map (difference equation) has a periodic time path attracting almost all initial points, sensitive dependence cannot arise. For certain classes of parametrized families of equations, it has been shown (Jacobson [1981], Collet and Eckman [1980]) that sensitive dependence arises for a set of parameter values that is large, that is, it is of positive Lebesque measure. Jacobson [1981] has also shown that for the quadratic family of difference equations $y_{t+1} = wy_t(1-y_t)$ where $0 \leq y_0 \leq 1$, $0 \leq w \leq 4$, the time paths of $y_t$ constitute a set of points whose asymptotic distribution is independent of the initial value of y, for a large set of values of w (a set whose Lebesque measure is not zero). The behavior of the determinate time path, then, can have this in common with the paths generated by certain stochastic systems: the limiting behavior of these paths can be described with the aid of a frequency distribution. However, while the distributions for the deterministic and the stochastic systems may seem similiar, they differ critically because in the deterministic system the location of the current point by itself must obviously indicate completely where the next point will lie.

## XII. Chaos in Higher Order and Multivariate Systems

A number of economic models use n simultaneous difference equations of first order to relate a vector of n dated variables $(x_{1t}, x_{2t}, \ldots, x_{nt})$ to their values $(x_{1t+1}, x_{2t+1}, \ldots, x_{nt+1})$ in the subsequent period.

Other models employ an nth order difference equation in a single variable $y_t = f(y_{t-1}, \ldots, y_{t-n})$ (see e.g., the justly famed Samuelson [1939] model of the accelerator-multiplier cycle). Any theorem about chaotic behavior in n simultaneous first order systems (i.e., about systems whose variables are n dimensional) must also apply to a single nth order equation. This is so since, as is well-known, such an nth order equation can easily be rewritten as the simultaneous first order system in n variables

$$x_{1t} = y_{t-1}; \; x_{2t} = y_{t-2}, \ldots, x_{nt} = y_{t-n}, \quad y_t = f(x_{1t}, \ldots, x_{nt}).$$

All of the chaotic economic models referred to in this paper employ one-dimensional (that is, single variable first-order) difference equations. One exception is the paper by Benhabib and Day [1981] which studies the dynamics of endogenous choice and provides conditions on preferences under which chaotic choice sequences of n-commodity vectors arise under stationary conditions. They use the results of P. Diamond [1976] which generalize the Li and Yorke [1975] propositions to the n dimensional (n variable or n period lag) case. A further generalization is also reported by Llibre [1981]. (See also Marotto [1978].) However, so far the available mathematical theorems that give sufficient conditions for the presence of chaotic behavior in higher dimensions are not easy to employ and their use up to now has been limited. They only suggest the conjecture that in higher order systems sufficient conditions for chaos to arise are "easier" to satisfy than in the case of first order systems; that is, chaos is "more likely" to occur in higher order systems.

## XIII. Economic Models with Chaotic Properties

In economics, the possibility of cyclical and chaotic dynamic behavior was perhaps first suggested by May and Beddington [1975], and has been shown to arise in simple ad hoc macroeconomic models (Stutzer [1980], Day and Shafer [1983]), in duopoly models (Rand [1978]), in models of growth cycles (Day [1983], R.A. Dana and P. Malgrange [1984]), in cobweb models of demand and supply (R.V. Jensen and R. Urban [1982]), in models of the firm subject to borrowing constraints (Day [1982]), in dynamic models of choice with enodgenous tastes (Benhabib and Day [1981]), in models of productivity growth (Baumol and Wolff [1983]), in dynamic models of advertising expenditures (Baumol and Quandt [1985]), in models analyzing military arms races and disarmament and negotiations (Baumol [1986]), in overlapping generations models (Benhabib and Day [1982], [1980a], Benhabib [1980], Grandmont [1985] as well as growth models with infinitely lived representative agents (Boldrin and Montrucchio [1985], Deneckre and Pelikan [1984]). This is only part of a growing list.

Cyclic and chaotic dynamics have been shown to arise in a number of competitive models of intertemporal general equilibrium. These results are of particular interest since they demonstrate that prices and outputs can oscillate even under standard competitive assumptions such as market clearing, perfect information and perfect foresight. For overlapping generations models of exchange, Benhabib and Day [1981] have provided sufficent conditions for cyclic and chaotic dynamics under perfect foresight when the young are net borrowers (the classical case; see Gale [1973]). Grandmont studied the case where young are net savers (the Samuelsonian case; see Gale [1973]) and correctly learn to forecast periodic equilibria. In equilibrium models with infinitely lived agents Benhabib and Nishimura [1979], provided sufficent conditions for cyclic equilibrium and recently, albeit in a more abstract setting, Boldrin and Montracchio [1985] and Deneckre and Pelikan [1984] have shown that chaotic trajectories can occur in such models.

## XIV. How Hill-Shaped Phase Diagrams Arise in Economics

The key to construction of a model in which chaotic behavior may arise, as we have seen, is the generation of a hill-shaped phase graph, at least if the model is built upon a difference equation of first order. We must, then, indicate how such hill-shaped dynamic relationships can arise in economics. Let us provide brief intuitive discussions of several models which have this property. We begin with an example that is an oversimplification, to say the least, but in which there is a very clear connection with the shape of phase curve in which we are interested.

Consider the relationship between a firm's profits and its advertising budget decision. Suppose that without any expenditure on advertising the firm cannot sell anything. As advertising outlay rises, total net profit first increases, then gradually levels off and finally begins to decline, yielding the traditional hill-shaped profit curve. If $P_t$ represents total profit in period t and $y_t$ is total advertising outlay, $p_t$ can, for illustration, be taken to follow the expression

(8) $p_t = ay_t(1 - y_t)$.

If, in addition, the firm devotes a fixed proportion, b, of its current profit to advertising outlays in the following period so that

(9) $y_{t+1} = bp_t$,

equation (8) is immediately transformed into our basic chaotic equation (1b), with $w = ab$.

The reason the slope of the phase graph turns from positive to negative in this case is clear and widely recognized. Even if an increase in advertising outlay always raises total revenue, after a point its marginal net profit yield becomes negative and, hence, the phase diagram exhibits a hill-shaped curve.

A moment's thought also indicates why the time path of $y_t$ can be expected to be oscillatory. Suppose the initial level of advertising, $y_0$, is an intermediate one that yields a high profit figure, $P_0$. That will lead to a large (excessive) advertising outlay, $y_1$, in the next period, thereby bringing down the value of profit, $P_1$. That, in its turn, will reduce advertising again and raise profit and so on ad infinitum.

The thing to be noted about this process is that it gives us good reason to expect the time paths of profit and advertising expenditure to be oscillatory. But it does not give us any reason to expect that these time paths need either be convergent or perfectly replicatory. Exactly the same logical structure is consistent with "sloppiness" in the cycles, so that past behavior is reproduced only imperfectly in the future. That, then, is how chaotic behavior patterns can arise.

Another example has been provided in the theory of productivity growth (Baumol and Wolff [1983]). It involves the relationship between the rate of productivity growth, $(\Pi_{t+1} - \Pi_t)/\Pi_t$, (which we can write as $\Pi^*_t$) and the level, $r_t$, of R and D expenditures by private industry. Obviously, a rise in $r_t$ can be expected to increase $\Pi_t^*$. However, because research can be interpreted as a service activity with a more or less fixed labor component, its cost will be raised by productivity growth in the remainder of the economy and the resulting stimulus to real wages. This, in turn, will cut back the quantity of R and D demanded. The result, as a formal model easily confirms, will be analagous to the corn-hog (cobweb) cycle with high productivity growth rates leading to high R and D prices which restrict the next period's productivity growth, and so reduce R and D prices, etc. If R and D costs ultimately increase disproportionately with increases in productivity growth it is clear that the relation $\Pi_{t+1}^* = f(\Pi_t^*)$ can generate the sort of hill shaped phase graph that is consistent with a chaotic regime.

To make this a bit more explicit, let $r_t^*$ represent the growth rate of $r_t$. Then if $P_t$ is the price level of R and D in period t and $P_t^*$ is its growth rate, we expect by the negative slope of the demand function

$$r_t^* = F\ (P_t^*), \qquad F' < 0$$

and since $P^*$ is a rising function of $\Pi_t^*$ which is, in turn, a rising function of $r_t$, this becomes

$$r_t^* \equiv (r_{t+1} - r_t)/r_t = G(r_t),\ G' < 0,$$

that is,

$$r_{t+1} = r_t + r_t G(r_t), = r_t(1+G),$$

so that where both $r_{t+1}$ and $r_t$ are positive we must have $G > -1$. We see also that $r_{t+1}$ will be zero if $r_t = 0$ or if $r_t$ can assume a value sufficiently large to reduce $G(r_t)$ to $-1$. Even if the value of G never falls that low, the graph of the last difference equation is apt to have a negatively sloping segment and hence assume the hill shape that interests us here.

Another model that can generate cyclic or chaotic dynamics is a standard growth model of Solow type in which the propensity to save out of wages is lower than that for profits (for a more complex version of this model see Akerlof and Stiglitz [1969]). Suppose that at low levels of capital stock, K, one obtains increasing marginal returns to increased capital and the elasticity of substitution of labor for capital is initially low; but that diminishing returns eventually set in and the elasticity of substitution moves the other way. Then total profits can rise at first, relative to total wages, but later profits may fall both relative to wages, and even absolutely. This can immediately generate a hill shaped relationship between $K_{t+1}$ and $K_t$ as rising $K_t$ at first elicits rising savings and then eventually depresses them as profits fall.

Similar results can be obtained for a model in which the propensity to save out of profits and wages is the same but where this propensity declines as the society grows progressively richer. (For a formulation in terms of an overlapping generations model in which the discount factor increases with wealth see Benhabib and Day [1980].)

Questions have been raised about the possibility of constructing simple chaotic macromodels that are consistent with the presence of long lived agents who optimize intertemporally, have perfect foresight and in which market clearing occurs. For it has been suggested that previous models of this type may generate chaos only because they involved agents whose life spans were less than the duration of many of the cycles, and who were thereby prevented from eliminating the cycles through acts of arbitrage.

A model that overcomes this problem can in fact be constructed.[15] Full explanation of the model requires a careful formal analysis. However, we will try to describe it briefly and intuitively. To do so, let us aggregate all commodities into a single good whose price then constitutes the price level. The objective of the model is to determine how much money M (real balance) will be held by an agent with infinite life, given the expected rate of inflation, or what is equivalent here, the money stock expected to be held in the next period. There are assumed to be two reasons for holding money. First, there is the obvious fact that it offers its proprietor future purchasing power (that can be increased by deflation). Second, money balances facilitate transactions and, hence, reduce their real costs. Thus the amount of money that people are willing to hold today can be expressed as

$$M_t = f\left[M^e_{t+1}, T'\left(M^e_{t+1}\right)\right]$$

where $M^e_{t+1}$ is the expected cash balance in period t+1, $T(M^e_{t+1})$ is the expected value of the resources that must be used up in carrying out transactions in that future period, and $T' = dT/dM^e_{t+1}$ is the expected <u>marginal</u> transactions cost saving resulting from an increase in money holdings. This relationship is, as a matter of fact, the first order condition for utility maximization by the agent. If expectations turn out to be correct then $M^e_{t+1} = M_{t+1}$ and the preceding equation clearly becomes a dynamic model, (implicitly) relating $M_{t+1}$ and $M_t$.

To see intuitively why the preceding model generates cycles with chaotic properties we consider the intertemporal equilibrium behavior of its variables. Along such a time path for money balances in the model the quantity of money held will increase only if the return on money increases. The return on money has two components: (i) appreciation (positive or negative) in its purchasing power, (ii) money's <u>marginal</u> yield in reduced transactions costs, given by $dT/dM^e_{t+1}$.

While deflation, which must be associated with an increase in M in the phase diagram, then contributes a positive amount to the return on money holding and, hence, serves to increase balances, the enlarged balances in turn lead to a decline in marginal liquidity yields (diminishing returns) so that the transactions savings return to increased balances will also fall. This eventually can begin to dominate the first component. Along an equilibrium path the magnitude of money balances can then reverse direction and start to fall. (Along an equilibrium path in which expectations may be defined to be correct, we can, of course, drop the expectation symbol e.) It is easy to show that all of this is consistent with intertemporal optimization and market clearing.

We can avoid the initial deflation element in the story if we permit money, M, to pay a suitable nominal interest rate. Then part of the return on the holding of money will derive from the interest payment rather than from deflation and the same argument will hold.

## XV. Empirical Evidence on the Presence of Chaos

The evidence on whether chaos does or does not occur in economic phenomena so far is only suggestive.

Brock [1986] has used some new techniques, (see also Brock and Dechert [1986]) to test whether a particular time series is most likely to have been generated by a stochastic system or, instead, by a regime that is (predominantly) chaotic, i.e., by a deterministic system giving rise to complicated dynamics (perhaps with minor random influences). Brock and Sayers [1985] have used these techniques to study a number of macroeconomic series. While the evidence is weak and somewhat inconclusive there seem to be grounds for the tentative conclusion that the use in econometric analysis of simple linear systems with stochastic disturbances may in some particular cases be inadequate and misleading and that non-linear systems may be more appropriate in some cases.

On the other hand, macro variables may not be the most promising place to look for chaos. Rather, there is reason to expect, from the very nature of its logic, that chaotic dynamics is more likely to affect disaggregated variables (such as the production of pig iron) rather than an aggregate series such as GNP, particularly when the micro variables are inherently subject to resource constraints that interconnect future values of the variables with their current levels (as in the case of resource depletion) All in all, then, the evidence for the existence of chaotic behavior in real economic time series is far from compelling so far, though what there is does suggest the value of further research in that direction.

## XVI. Empirical Determinability of the Underlying Structural Model

How does one test empirically for the presence of chaos? One approach, based on a very simple and rather naive procedure that seeks to discover the underlying dynamic system generating a "chaotic" time series will next be described. To dramatize the simplicity of the procedure, we use the data for a highly "disorderly" time path, the one shown in Figure 7b, and employ it to reconstruct the hill-shaped graph of our generating equation $y_{t+1} = 3.94\ y_t(1-y_t)$. Here we assume only (or rather, we allow ourselves to act only on the knowledge) that the underlying relationship is a single difference equation of first order. This enables us to proceed in the most direct manner possible, taking pairs of adjacent observations of y in Figure 7b and then merely plotting each $y_{t+1}$ against the corresponding $y_t$. The result is shown in Figure 8, which indicates that despite the complicated pattern of the data, we obtain in this way a virtually perfect reconstruction of our underlying phase curve.

The success of this procedure relies, of course, on the fact that there is absolutely nothing random in the process. Each generated point in the time path slavishly follows the dictates of the underlying model and so must correspond unambiguously to a point on the graph of an equation of the model. This immediately suggests a naive test to determine whether a time series involves random or chaotic influences: if a reconstruction of the underlying model of the sort just described can be carried out and yields a highly regular relationship, the time path can be presumed to be chaotic rather than random.

However, the calculation that has just been described suffers from a variety of pitfalls. First, the underlying system may have many variables and/or may have a complicated lag structure involving an unknown number of periods. Then this underlying mechanism is no longer "simple", and the kind of "simple" structure to look for is never obvious. Of course, this problem is no different from that besetting the choice of structure of the model to be used in an econometric estimation process. Second, the time series or observations at our disposal may not provide information on the variables of the underlying system but on some function of those variables. For example, if the system is described by a difference equation which is a function of $n$ variables which make up the vector $x_t$, then instead of observing the states of the system given by the vector $x_t$ we may only be in a position to observe a function of $x_t$, $y_t = h(x_t)$ where $y_t$ is a scalar (a single variable). This is often so when we deal with aggregated time series such as GNP.

Finally, the underlying system may also involve some small amount of random noise, although the deterministic part of the system by itself generates chaotic behavior. The problem of distinguishing essentially deterministic dynamics from dynamics primarily governed by stochastic elements as a result becomes difficult if not ambiguous. Attempts have been made to devise more sophisticated methods to deal with such problems. First, means have been sought to determine whether a given time series is generated by a stable and stochastic dynamical (difference equation) system or a chaotic but deterministic system. Here, the "dimension" of the set of points toward which the time path tends in the limit has proved a helpful criterion. To explain the concept and its relevance, we note that in the stable stochastic case, given initial conditions (or an initial distribution), the state of the system at a future date, as perceived today, is a random variable. Under appropriate assumptions this random variable converges to a frequency distribution as the pertinent date moves toward infinity. Consequently, in the case of randomness only a continuum may be sufficient to contain all possible limit points of the time path. This is so because the random variables themselves can assume any of the set of values in the continuum corresponding to the pertinent frequency distribution. In contrast, a trajectory of a deterministic dynamical system may, as some of the preceding illustrations show, converge to a finite number of points (e.g., a stationary point or a cycle). In addition, it may either follow a chaotic path or converge to a chaotic set (a "strange attractor"). In the last two cases, the trajectory, while not constituted by a finite number of points, can nevertheless be distinguished from a continuum because, roughly speaking, the former contains "fewer points". (This will be explained presently in somewhat greater detail).

Methods that seek to distinguish empirically whether the underlying mechanism is deterministic or not on the basis of finite (but large) and possibly aggregated sets of time series data are based on this distinction between the "dimension" of a "strange attractor" which, under an appropriate definition is finite, and the "dimension" of the support of a stationary distribution generated by a stable stochastic dynamical system driven by shocks that have continuous density functions. The "dimension" of the latter, suitably defined, can be shown to be infinite. Several definitions of "dimension", which are appropriate for use in testing for a finite but large data set, have been provided by Takens [1985], Proccacia and Grassberger

[1985] and others. For a description and some applications of these methods see Brock [1986], Brock and Dechert [1986], Scheinkman and LeBaron [1986], Brock and Sayers [1985], and Brock, Dechert and Scheinkman [1986].

The logic of the dimension approach to empirical testing of whether a time path is chaotic or random is perhaps more readily explained intuitively with the aid of Figure 8. We have seen a few paragraphs ago how plotting of successive values of $y_{t+1}$ against the corresponding $y_t$ from the chaotic time path in Figure 7b gives us a series of points all lying on the parabolic phase curve of the system, as shown in Figure 8. Now, even if in the limit these points were to fill in the entire segment of the parabola in the positive octant, they would still only constitute a one dimensional set -- a curve in two dimensional space.

In contrast, had the generation of the time path involved substantial random influences, then the same exercise as that we have just carried out would obviously have yielded a set of points scattered about the parabola: at best, a grey area, an area that can be covered only by a continuous region in the diagram and only if that covering region is at least of two dimensions. This, then, is the sense in which chaotic behavior is associated with a set of points lower in dimension than is randomness, and this indicates, consequently, how dimension can be used, at least in principle, to distinguish the one case from the other.

However, this leaves us with the difficult problem of distinguishing empirically any mixed case in which chaotic and random influences are <u>both</u> present. Thus, one complication that may well beset many economic time series is that random noise may well be present in a time series generated by a system with a clear "deterministic" structure, e.g., a structure encompassed in a difference equation. The problem, as we have seen, is that such a time series falls into an intermediate category. The deterministic structure of $y_t$ may involve a time path with a limit of low dimension (possibly corresponding to a "strange attractor") and the deterministic part can be large relative to the magnitude of the variation in the series of independently distributed random shocks. The possibility of a "strange attractor" arising out of the deterministic part of the system, with some noise superimposed upon it has been called "noisy chaos". Methods to detect such cases in which the noise component of the time path is "small" relative to the autoregressive part are given by Ben-Mizrachi [1984] and also Brock and Dechert [1986].

These approaches are complicated by the fact that the data sets used in reality are necessarily finite. With finite data, a linear stochastic difference equation system can appear to generate a "finite dimensional" attractor if its stochastic component is small enough, and can therefore suggest the conclusion that the underlying dynamic system is strictly deterministic. (For a discussion of the difficulties that arise in this context as a result of finiteness of the data samples, see Ramsey [1987].)

An additional check that is promising tests whether the underlying system is, on the average, "stable". Such a test involves methods designed to ascertain whether trajectories arising out of a given relationship but with different initial conditions, that are initially close together, remain close, as they would not do if they were chaotic.[16] Some methods have been designed to estimate the mean rate of divergence of such trajectories. A positive divergence rate is then taken as further evidence indicating the presence of a strange attractor rather than a stable but stochastic dynamical system, whose stability prevents marked divergence of the trajectories of its variables.

Though necessarily incomplete, this discussion should offer the reader some impression of the methods now being used in empirical studies of chaotic phenomena in economics.

## XVII. Concluding Comment

This paper has sought to introduce the reader to the logic of the chaos phenomenon, to its implications for economics and to some of the pertinent economic literature. While not pretending to constitute a systematic survey of writings in the field we do hope we have provided the reader with some sense of the range and significance of the work it encompasses. At best, the analysis may offer an expanded view of possible qualitative properties of intertemporal economic processes. At the very least, it is a warning against simplistic interpretations of complex intertemporal interrelationships.

## Footnotes

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**New York University. The authors are grateful to the C.V. Starr Center for Applied Economics for its assistance in preparation of this paper. We are also deeply indebted to William Brock, Dermot Gately, Allanah Orrison, Richard Quandt and Joseph Stiglitz for their very substantial help.

1. In particular, if f(t) is a cyclical time path, then for $0 < r < 1$, $(r)^t f(t)$ is oscillatory in our sense, since the exponential dampening of its underlying cycles prevents its trajectory from constantly repeating itself precisely.

2. Figure 1 can also be used to remind the reader how one uses a phase diagram to calculate the time path. Suppose we start our calculation with an initial value of $y_t$ indicated by the point labelled y(0) on the horizontal axis of the graph. Then the next period's value of y, that is, y(1), is given by the height of point A on the phase graph of $y_{t+1} = f(y_t)$ directly above y(0). Next, we want to repeat the process, this time starting from y(1), in order to find y(2), the next value of $y_t$. For this purpose we first move horizontally from point A to point B on the 45 degree line. We do this to move y(1), i.e., the height of point A, to the horizontal axis. This is the point directly below B because the two coordinates of any point on the $45^o$ line must be equal. Having

found y(1) on the horizontal axis we now move directly upward to point C on the phase graph. Continuing in this way we trace out the time path of $y_t$.

We notice that in the leftward region of the diagram, where the phase graph is upward sloping, the time path ABCDH... does not change direction (i.e., in this case it goes steadily upward). Thus, as in our earlier example, $y_t = a^t y_0$ with $a > 0$, the time path has no oscillations. However, toward the right hand end of the diagram, where the phase graph has a negative slope (as where $a < 0$ in $y_t = a^t y_0$) the time path starts to oscillate. It goes up and down in a cobweb pattern (such as HJKLM) around the equilibrium point E. (E is the equilibrium point since that is where the phase curve cuts the 45 degree line, so that there $y_{t+1} = y_t$, as equilibrium requires.

More generally, the graph confirms why difference equation models tend to generate the four basic time path patterns already described. If near the equilibrium point the slope of the phase graph is positive but less than unity the time path near that point will be nonoscillatory but stable (e.g., the case $y_t = a^t y_0$ with $0 < a < 1$); if that slope is less than unity then the nearby time path will be nonoscillatory and unstable ("explosive"). If the slope is negative but greater than -1 the time path near the equilibrium point will be oscillatory but stable (oscillations of ever declining amplitude); while if the slope of the phase graph near equilibrium point E is less than -1 the time path will be oscillatory and unstable (oscillations of ever greater amplitude).

3. For at a crossing of the phase line with the $45^0$ ray $y_{t+1} = y_t = y_e$ so that $y_e = wy_e(1-y_e)$ or $wy_e^2 = (w-1)y_e$. Hence, $y_e = (w-1)/w$ will not have a positive value for $w < 1$.

4. Substituting (3) into (2), we obtain $dy_{t+1}/dy_t = 2-w$ at $y_t = y_e$, from which the results of this paragraph follow at once.

5. The bulk of the following discussion is based on the beautiful analysis in May [1976].

6. Proof: $df^{(2)}/dy_t = (dy_{t+2}/dy_{t+1})(dy_{t+1}/dy_t)$, but at $y_t = y_{t+1} = y_e$ we must have $dy_{t+1}/dy_t = dy_{t+2}/dy_{t+1} = df/dy_t$.

7. It can be proved that the slope of the phase curve at equilibrium points G and H must be the same, and the analagous result holds at any equilibrium points that emerge at any subsequent bifurcation.

8. The value of w has been changed from 3.45 in Figure 1 to 3.5 in Figure 5a, to make the patterns clearer.

9. We should be cautious not to confuse chaotic sets with attractors. As has been pointed out, the chaotic set may be small so that the trajectories emanating from most initial points may converge to a periodic orbit. Only those chaotic sets toward which all nearby trajectories converge qualify as strange attractors.

While "strange attractors" exist (as they do, for example, for the quadratic difference equation $y_{t+1} = wy_t(1-y_t)$), when they persist under small perturbations of the equation defining the dynamics is still an open question. It has, however, been shown (Misiurewicz [1980] that some such relationships possess strange attractors that do not disappear when the relation is perturbed.

10. Professor Quandt has carried out a simulation exercise in which the behavior of a chaotic time path generated by our basic illustrative difference equation was contrasted with one that followed an uncomplicated deterministic regime that was subject to substantial random disturbance of moderately low probability. Spectral analysis then yielded very similar results for the chaotic series and the series subject to random disturbances, properties very different from those that held for the time paths of series generated by our equation with w values not far from the chaotic region. The implication is that standard statistical procedures may fail to determine correctly in any particular case whether a set of observations has been subject to random disturbances or whether it has been generated by a model that is perfectly deterministic but chaotic. For details of the simulations see Baumol and Quandt [1985].

11. Thus the number of points in S will, in one sense, be "very large," i.e., its cardinal number will exceed the number of integers. Nevertheless,

the number of those points <u>may</u> be "very small" in the sense of constituting a set of measure zero (roughly speaking, S occupies zero percent of any continuous region that contains S). More will be said about this later.

12. The Lebesque measure of the size of a set S on the real line is technically defined as the greatest lower bound of the sum of the lengths of a denumerable set of intervals coverings. When one calculates the Lebesque measure of a subset of a continuum, if that measure turns out to be zero, it is taken to mean that "almost all" of the points in the continuum lie outside the subset in question.

13. These conditions are that the map be unimodal, continuous and have a negative Schwartzian derivative. The Schwartzian derivative of $f(x)$ is given by

$$\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right].$$

See Singer [1978].

14. Almost all time paths in any such model are chaotic.

15. We should note at this point that similar models can be derived with money in the utility function, as in Brock [1974] or Calvo [1979]. The reduced forms of these models are similar to ours. One can also interpret the transaction cost function as a generalization of a Clower

constraint which allows trade without costs up to the level permitted by the available money balances and prohibits trade beyond that.

16. The requisite analysis can not be conducted by studying roots at "the steady state" since the underlying system is not even known. However, a procedure that does test whether, on the average, the system is expansive or contractive can be carried out by studying "Lyapunov exponents" which generalize the idea of checking roots at a "steady state." "Lyapunov exponents" for an underlying system can be estimated from time series data and give an indication of whether a system is on the average expansive. If the system is expansive (positive Lyapunov exponents) and the time series generates low dimensional surfaces, then there is a presumption that what is involved is a non-linear deterministic system which has given rise to chaotic dynamics.

Figure 1. Phase Diagram, Periods 0-10
$y(t+1)=3.45y(t)[1-y(t)]$, $y(0)=0.999$

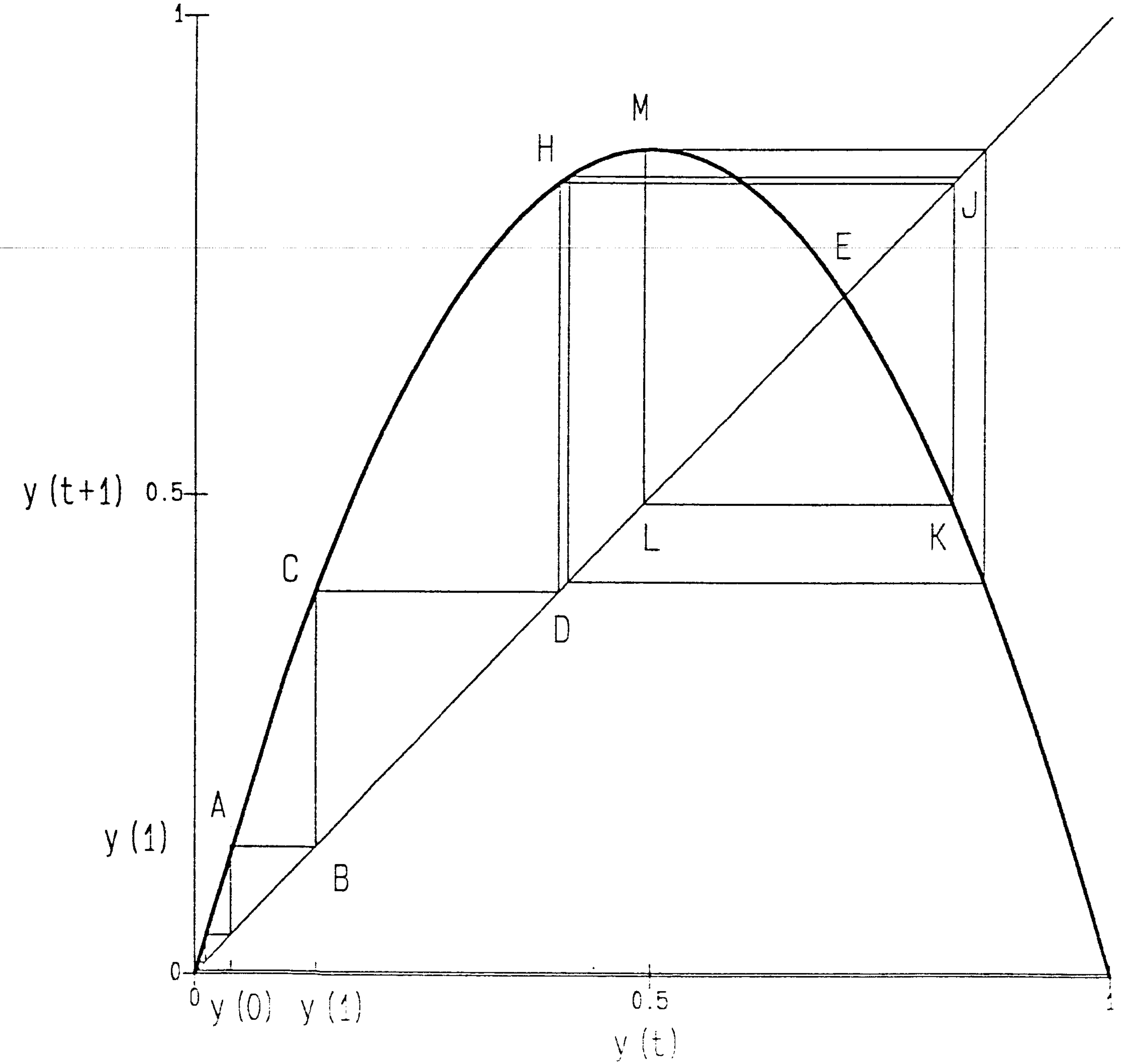


Figure 2a. $y(t+1)=f[y(t)]$

$y(t+1)=wy(t)[1-y(t)]$, $w=0.8$

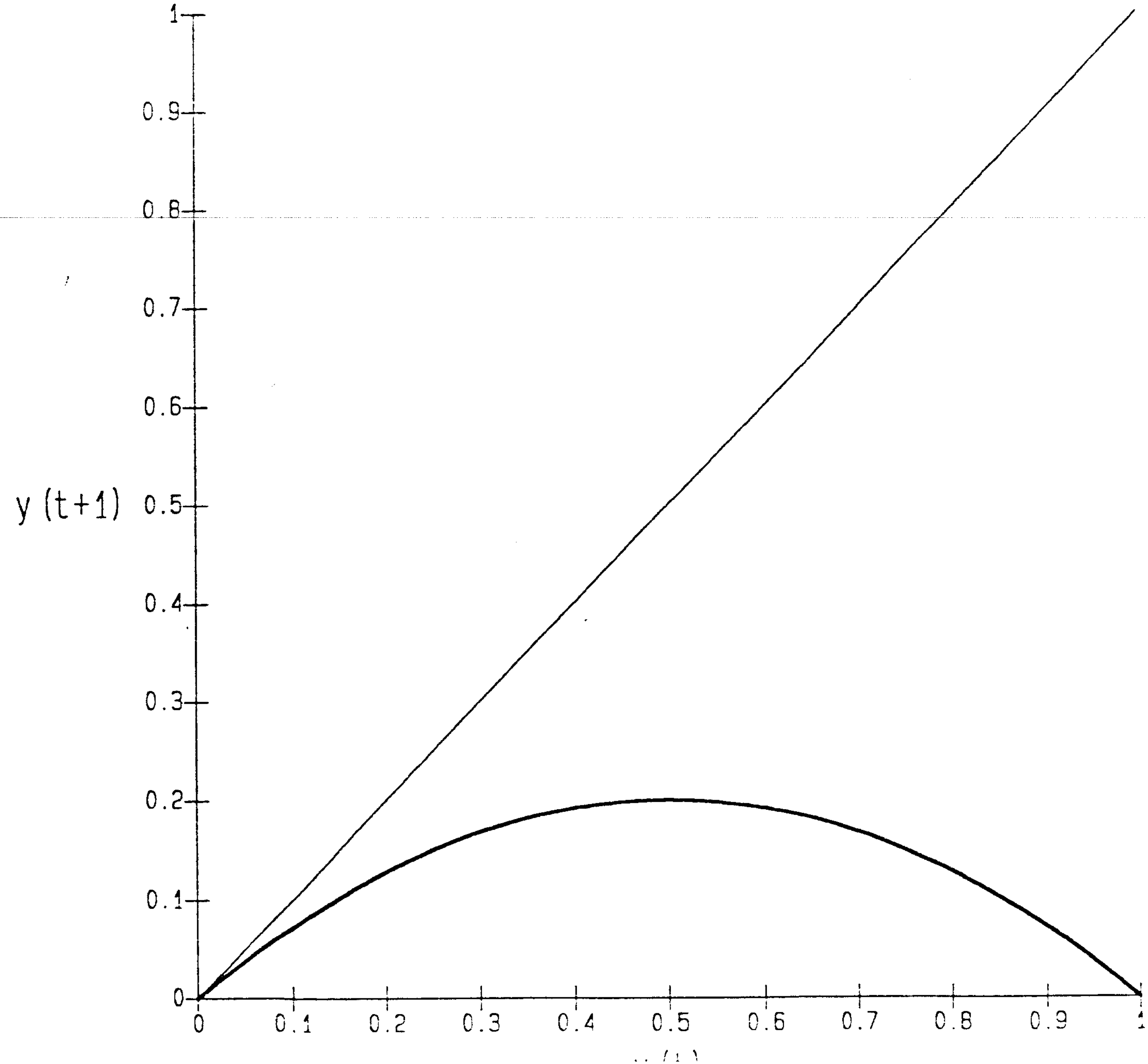


Figure 2b. $y(t+1) = f[y(t)]$
$y(t+1) = wy(t)[1-y(t)]$, $w=1.7$

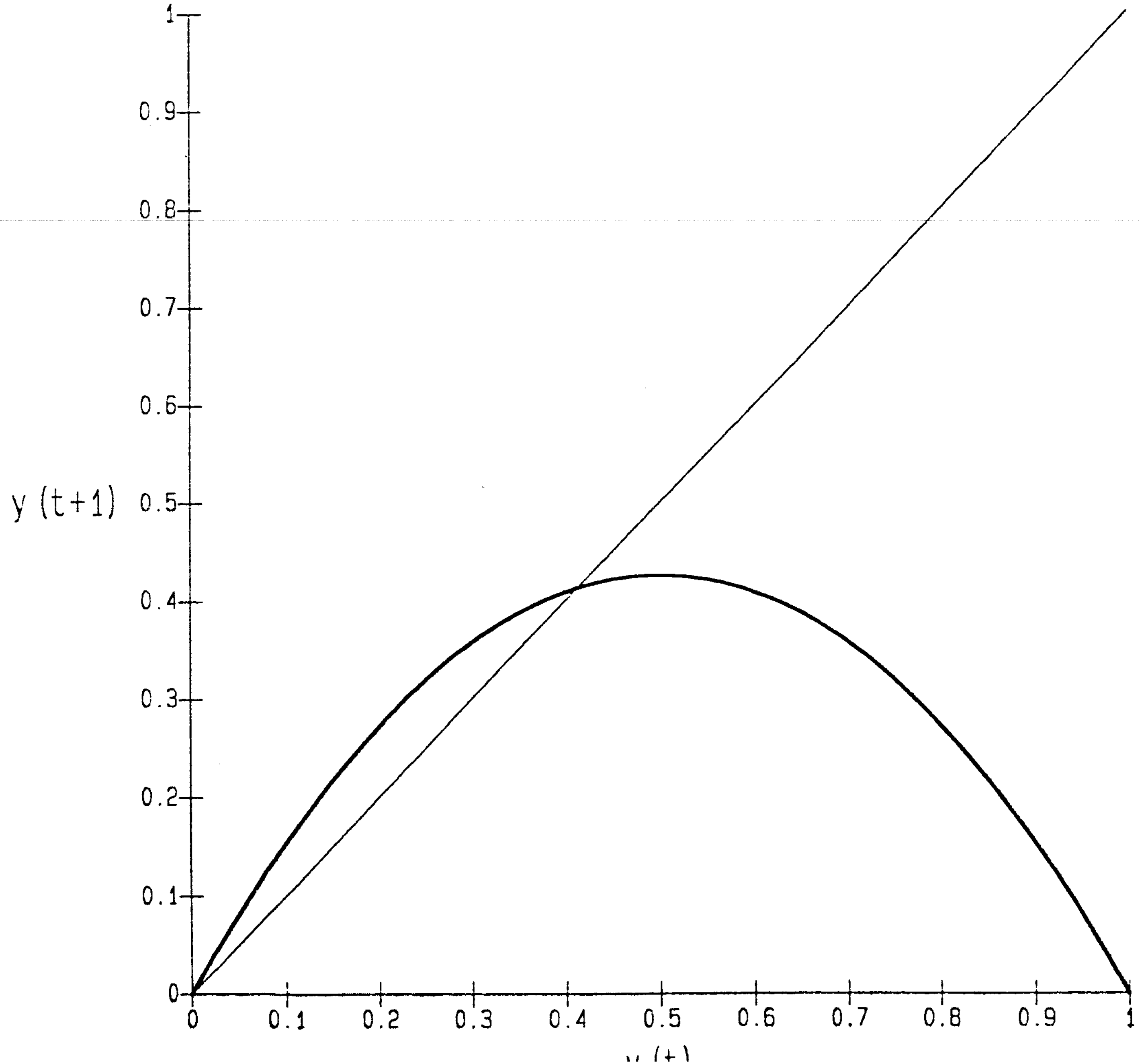


Figure 2c. $y(t+1) = f[y(t)]$

$y(t+1) = wy(t)[1-y(t)]$, $w=2.5$

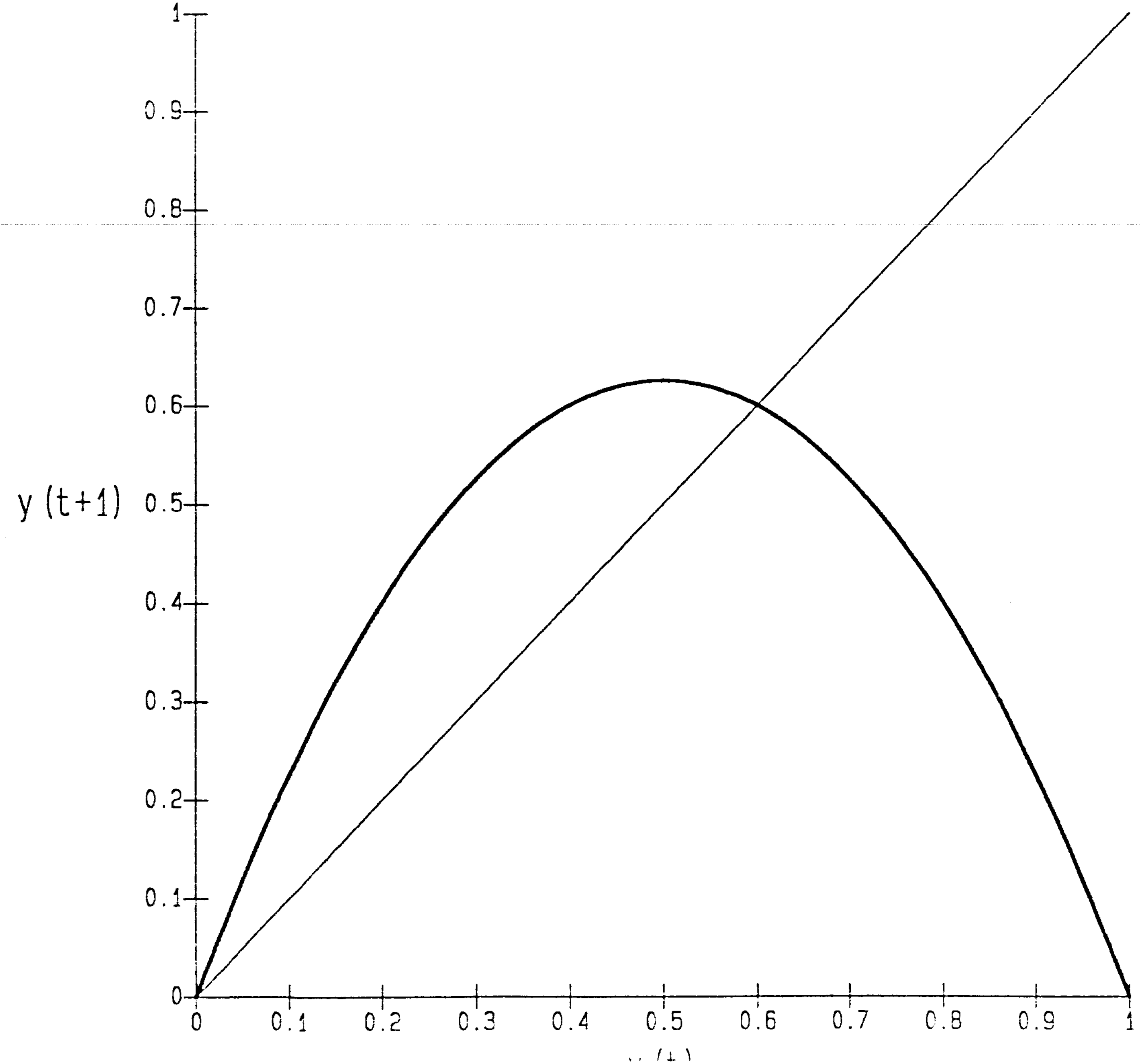


Figure 2d. y (t+1) = f [y (t)]
y (t+1) = wy (t) [1−y (t)], w=3.95

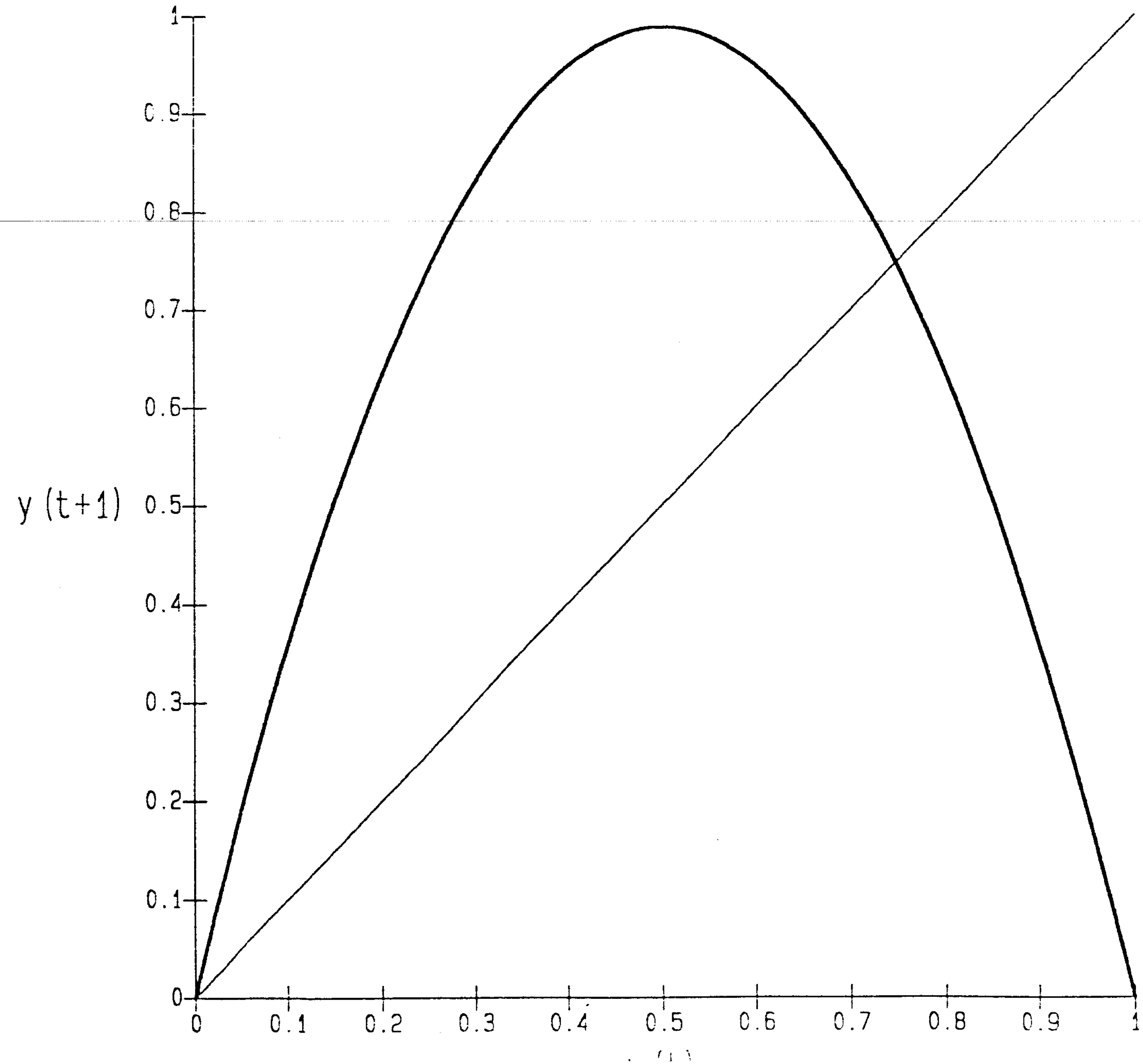


Figure 3. Phase Diagram, Periods 0-8
y (t+1) =3.94y (t) [1-y (t) ], y (0) =0.76

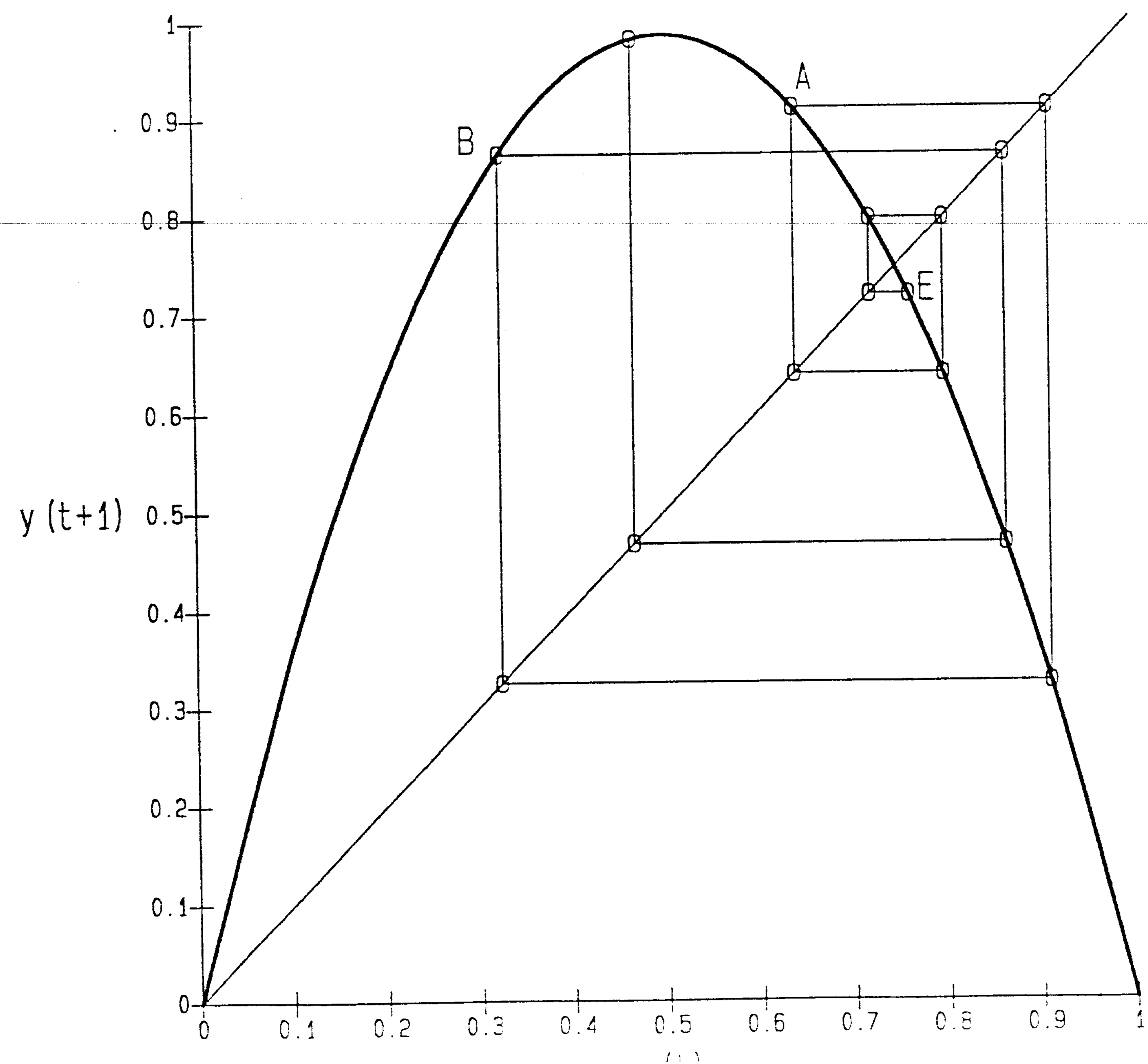


Figure 4a. f [y (t) ], f {f [y (t) ] }
y (t+1) = f [y (t) ] = 2.8y (t) [1-y (t) ]

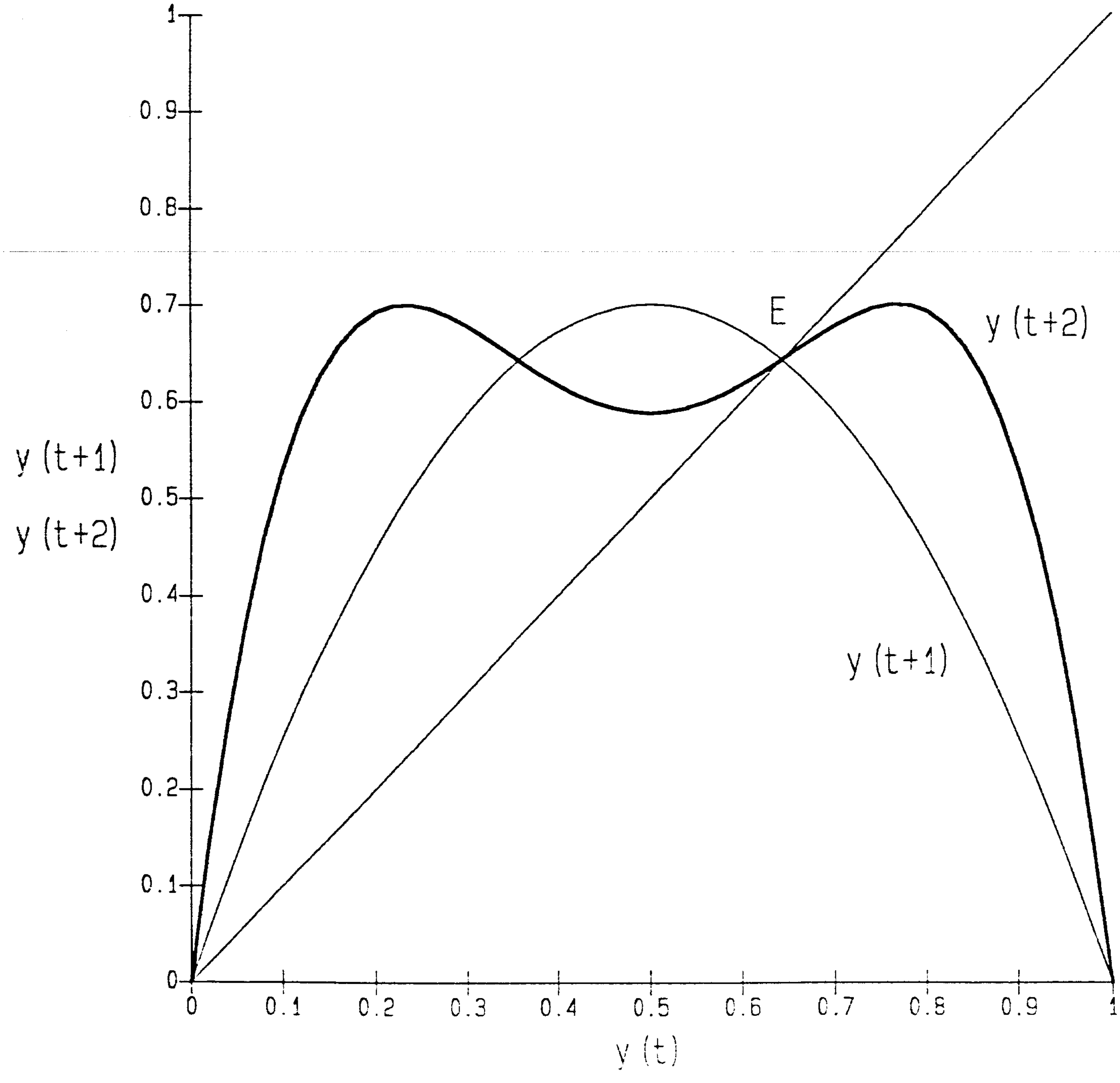


Figure 4b. $f[y(t)]$, $f\{f[y(t)]\}$
$y(t+1)=f[y(t)]=3y(t)[1-y(t)]$

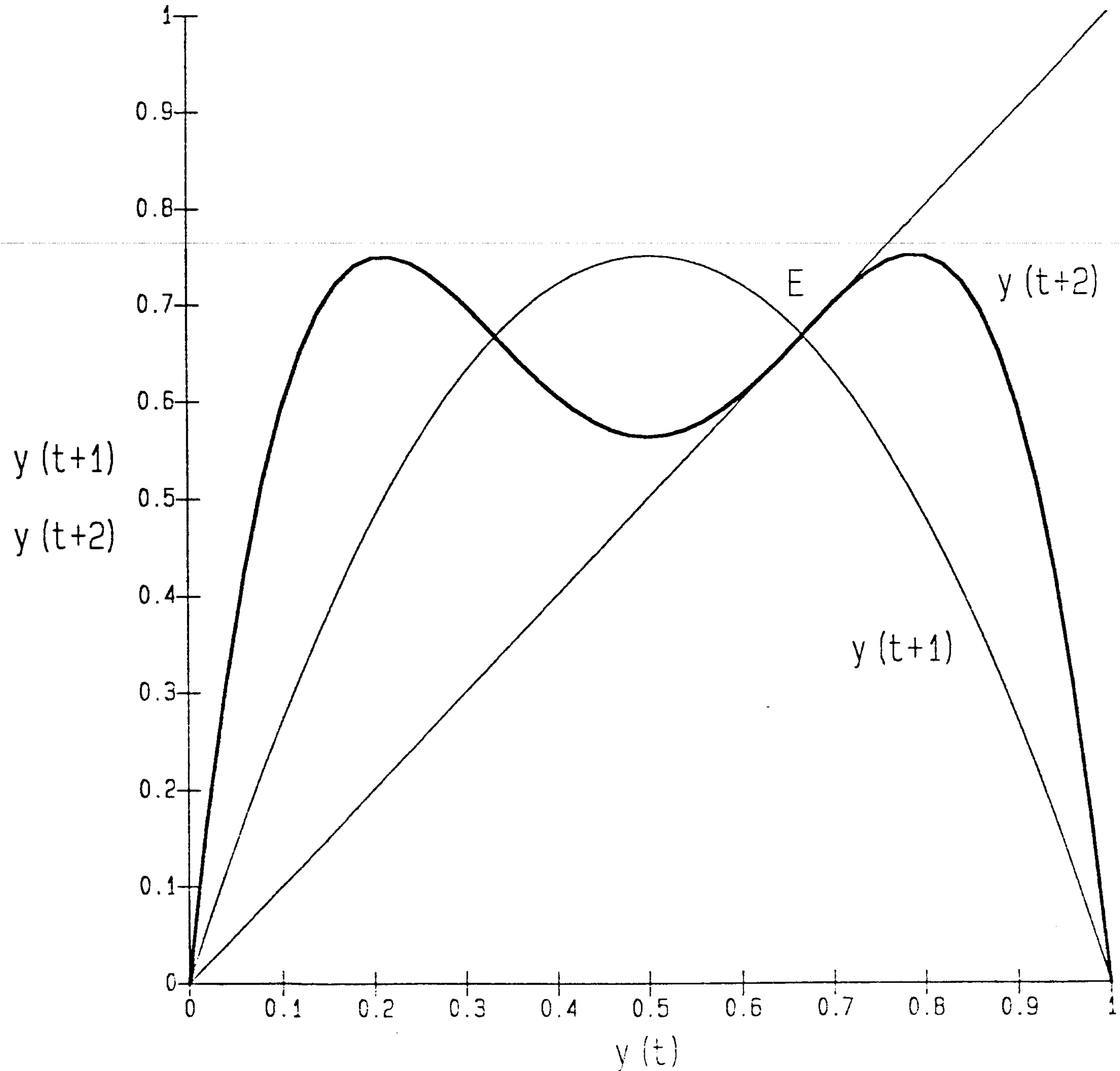


Figure 4c. f [y (t)], f {f [y (t)]}

y (t+1) = f [y (t)] = 3.43y (t) [1-y (t)]

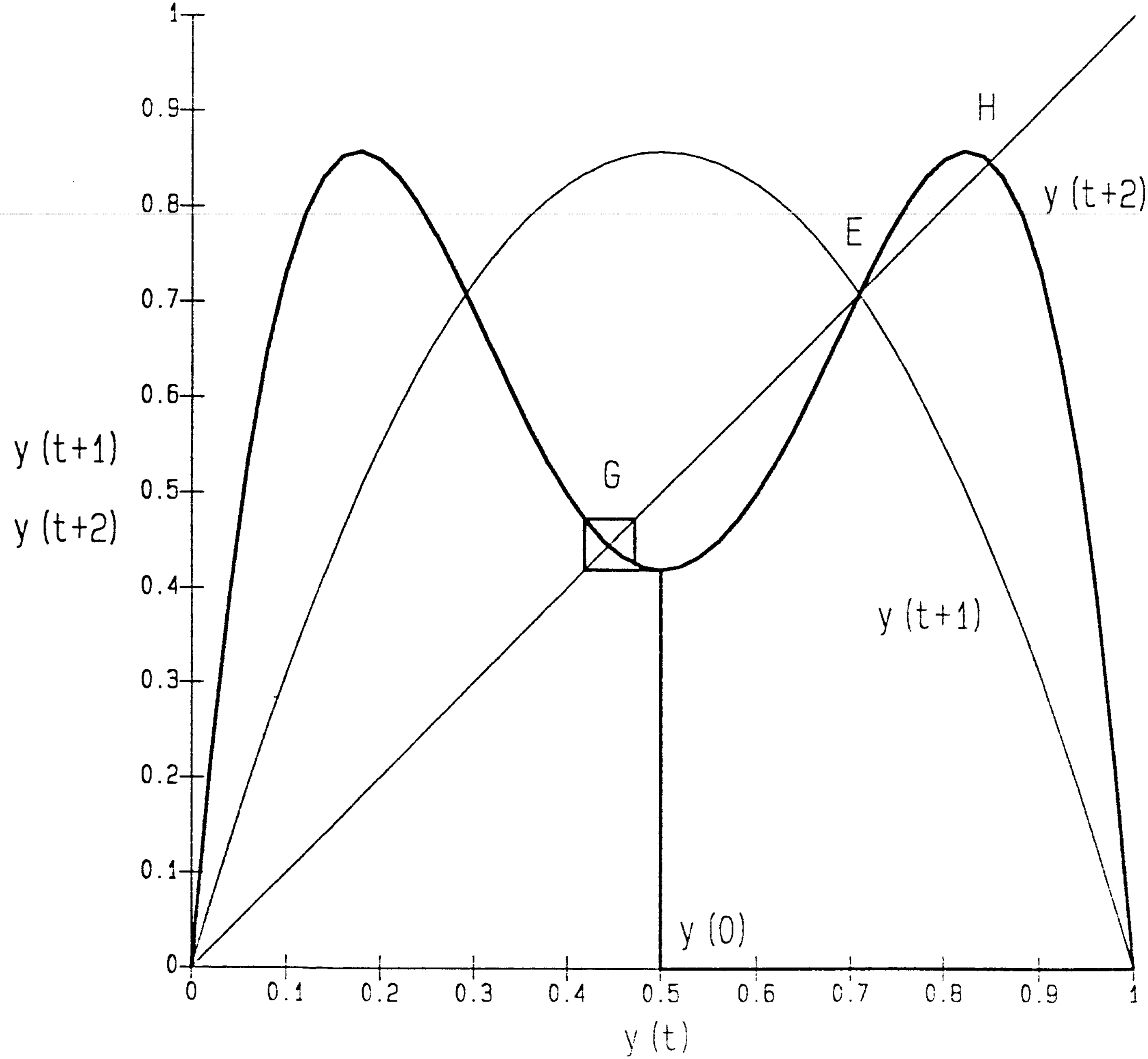


Figure 5a. Phase Diagram, Periods 47–50

$y(t+1) = 3.5y(t)[1-y(t)],\ y(0) = 0.999$

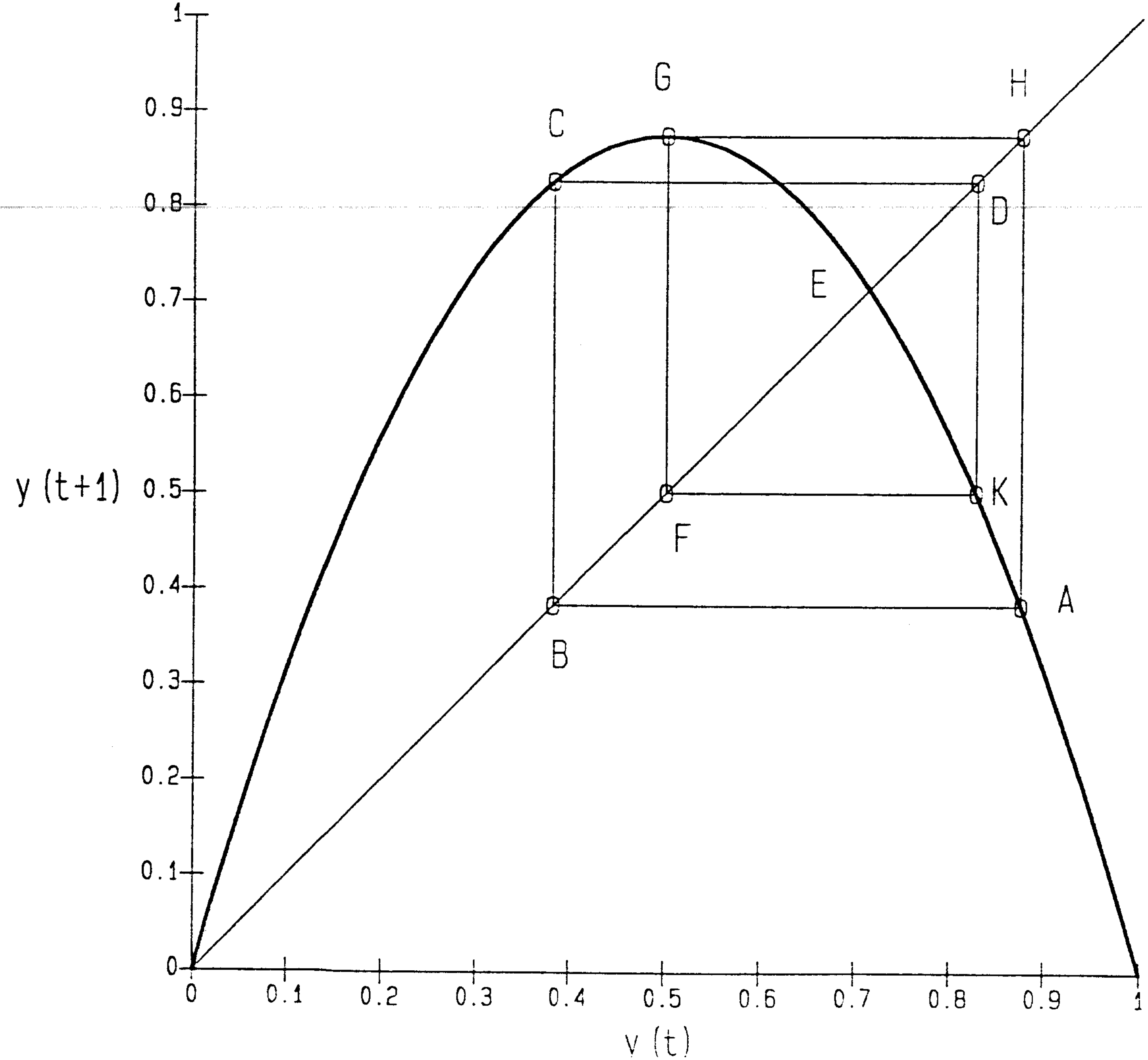


Figure 5b. Time Path, Periods 0-50
$y(t+1)=3.45y(t)[1-y(t)]$, $y(0)=0.999$

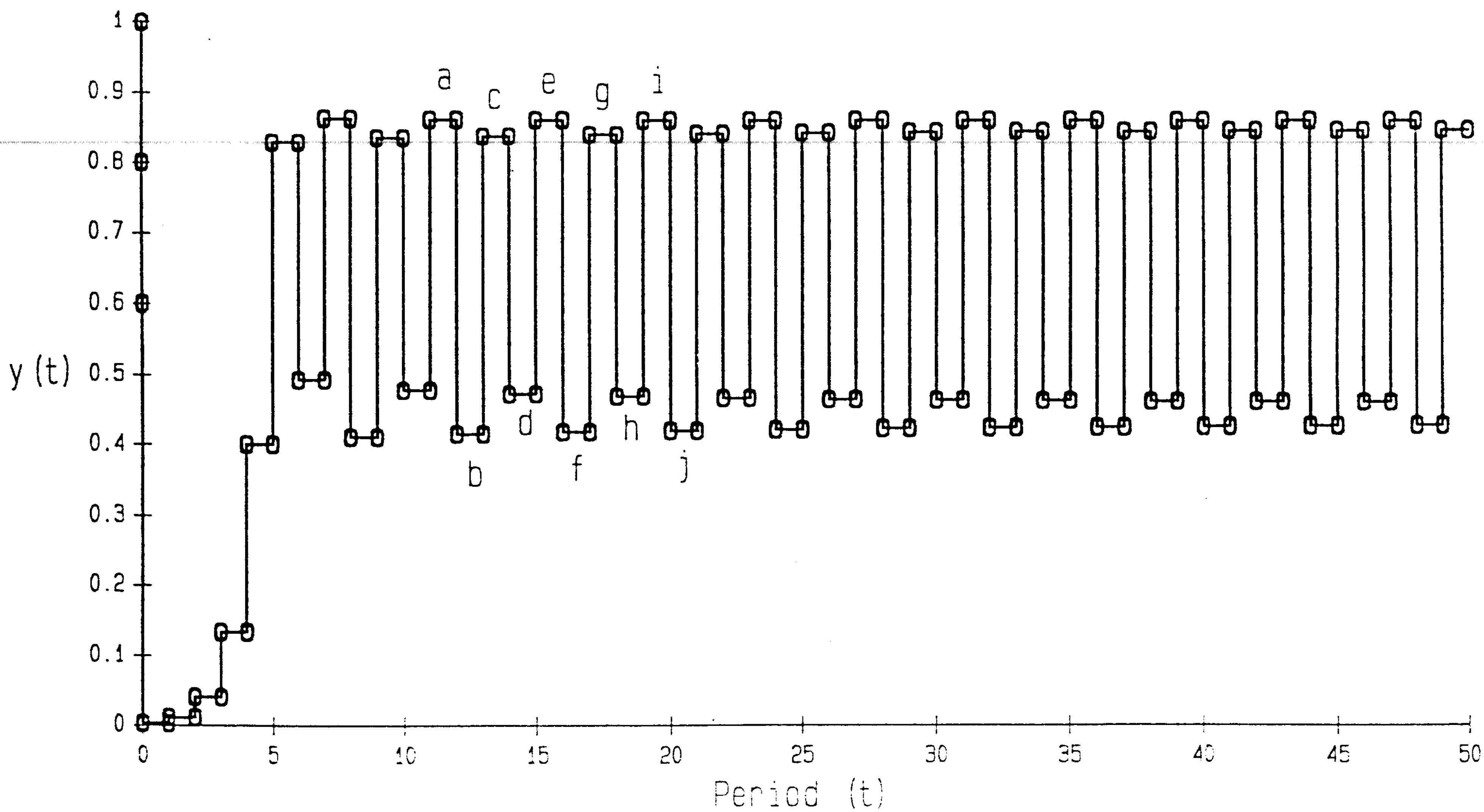


Figure 6a. $y(t+3) = f(f\{f[y(t)]\})$
$y(t+1) = f[y(t)] = 3.6y(t)[1-y(t)]$

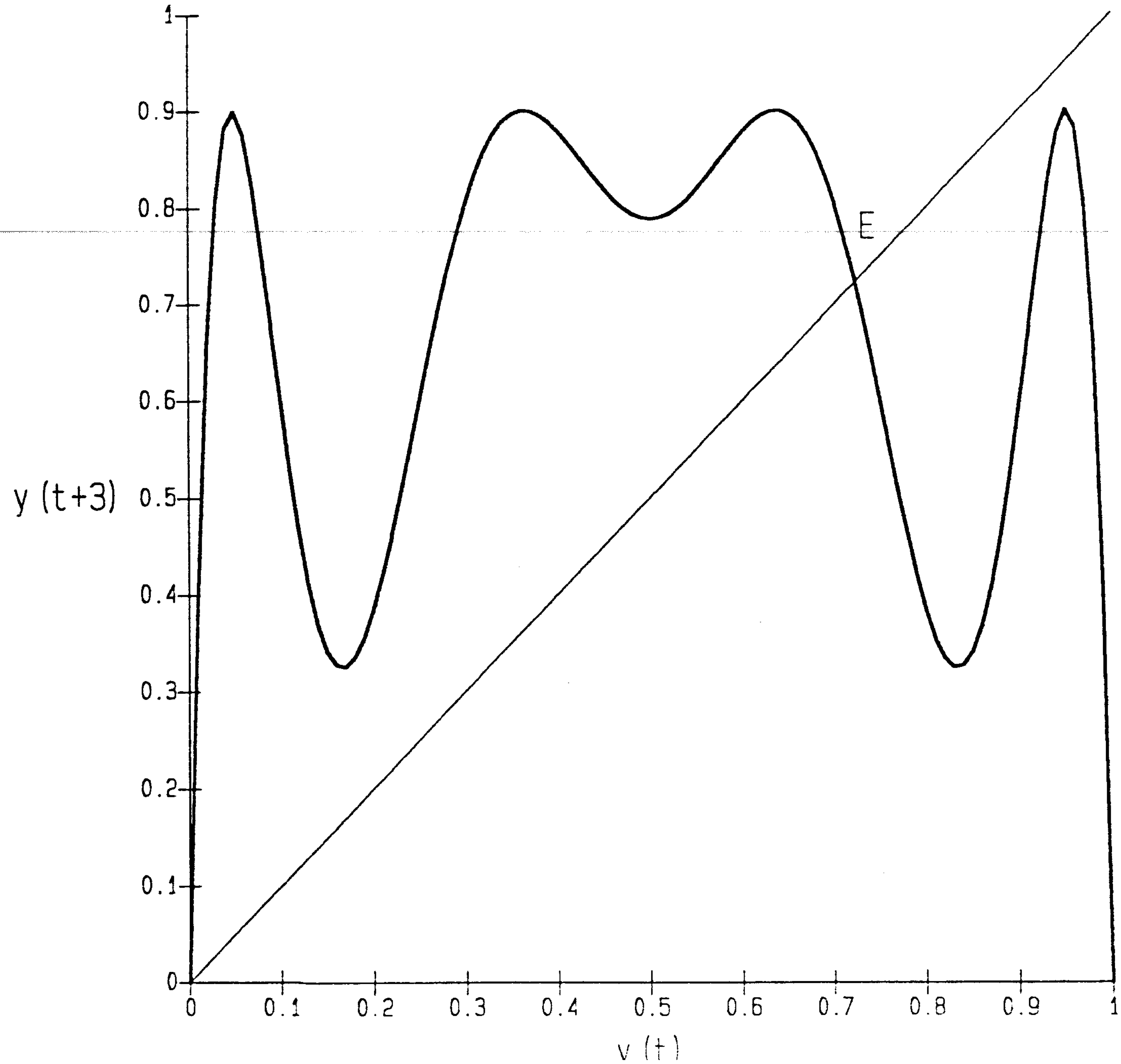


Figure 6b. $y(t+3) = f(f\{f[y(t)]\})$
$y(t+1) = f[y(t)] = 3.95y(t)[1-y(t)]$

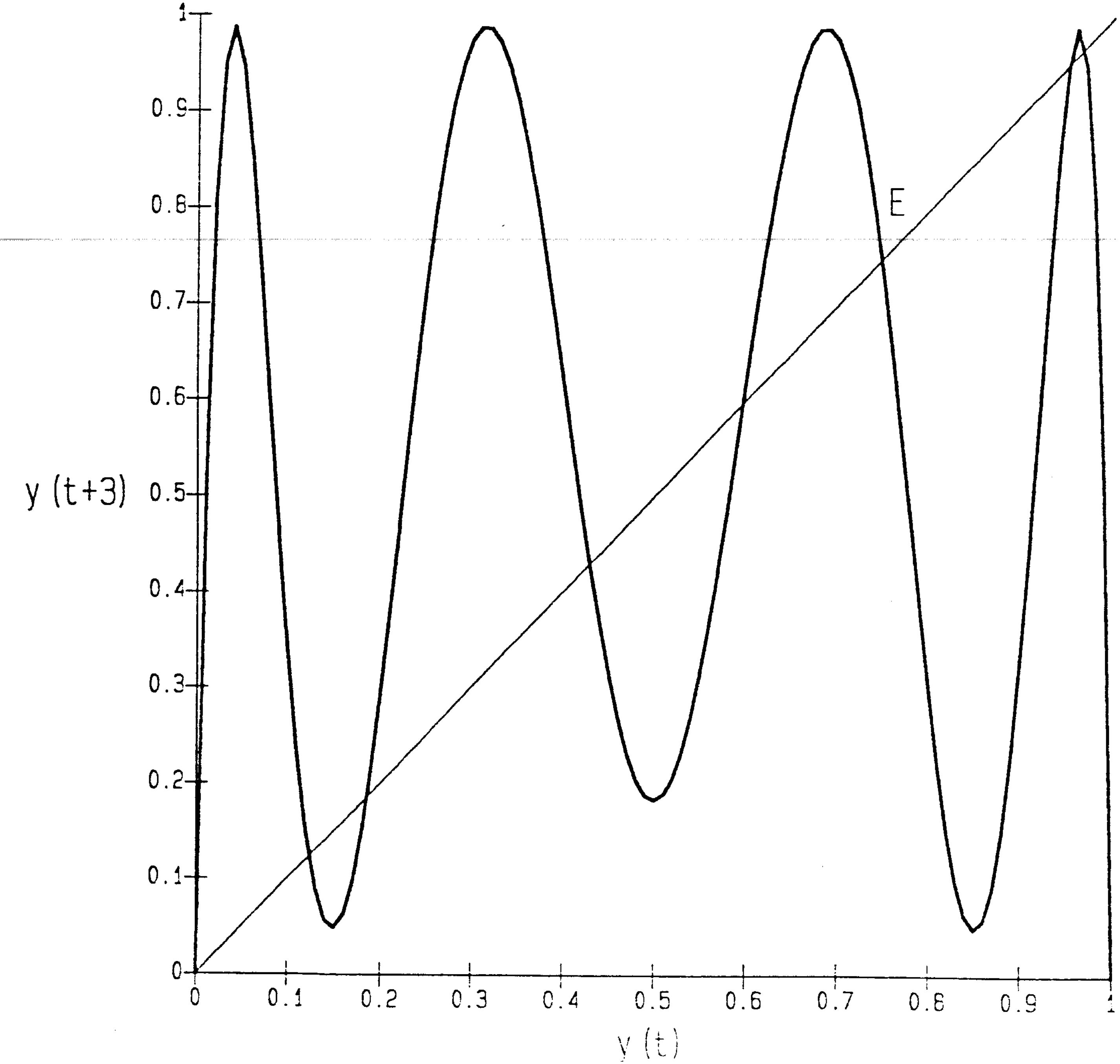


Figure 7a. Time Path, Periods 0-50
$y(t+1) = 3.935y(t)[1-y(t)], \; y(0) = 0.99$

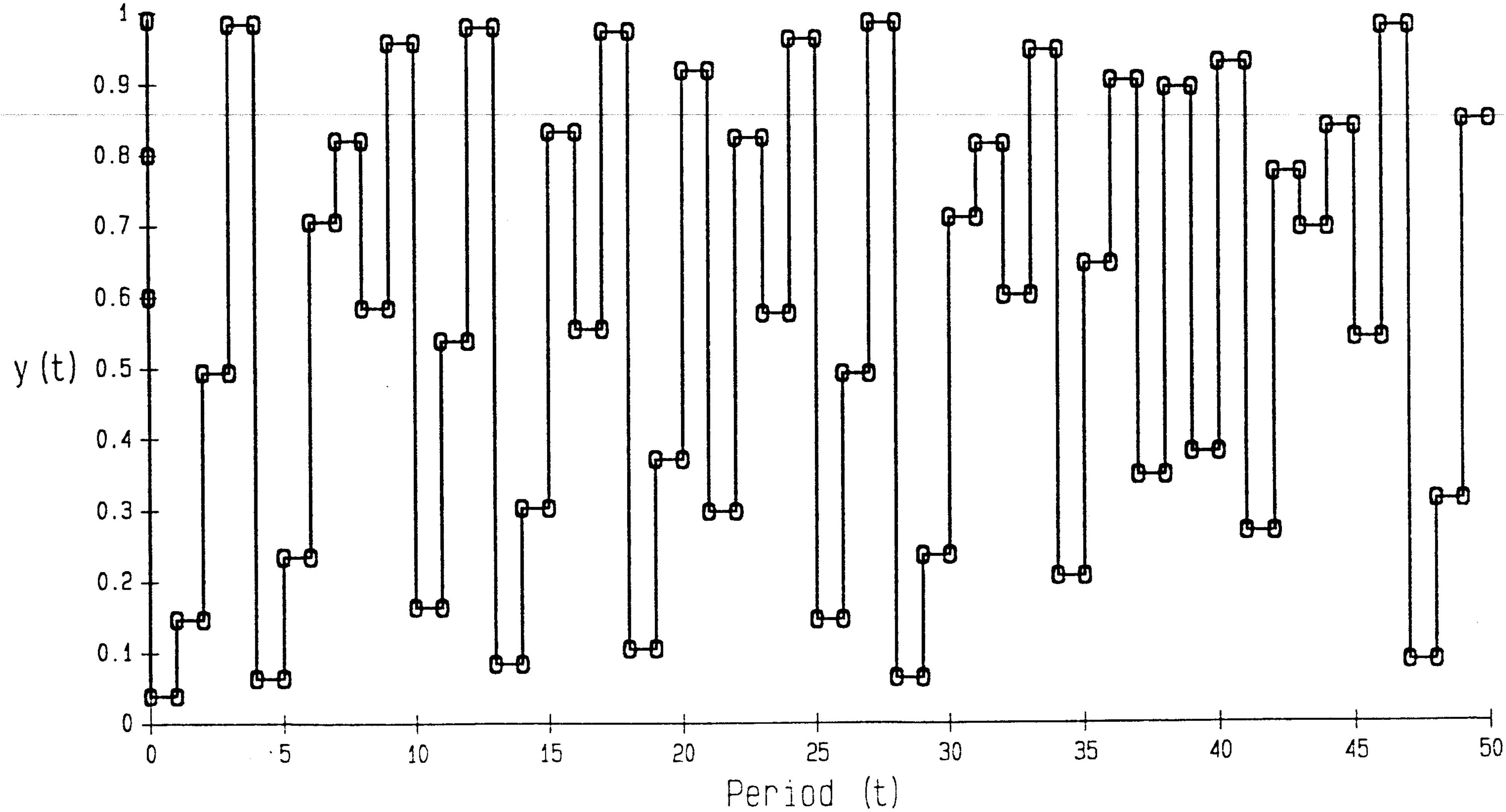


Figure 7b. Time Path, Periods 0-50
$y(t+1)=3.94y(t)[1-y(t)], \quad y(0)=0.99$

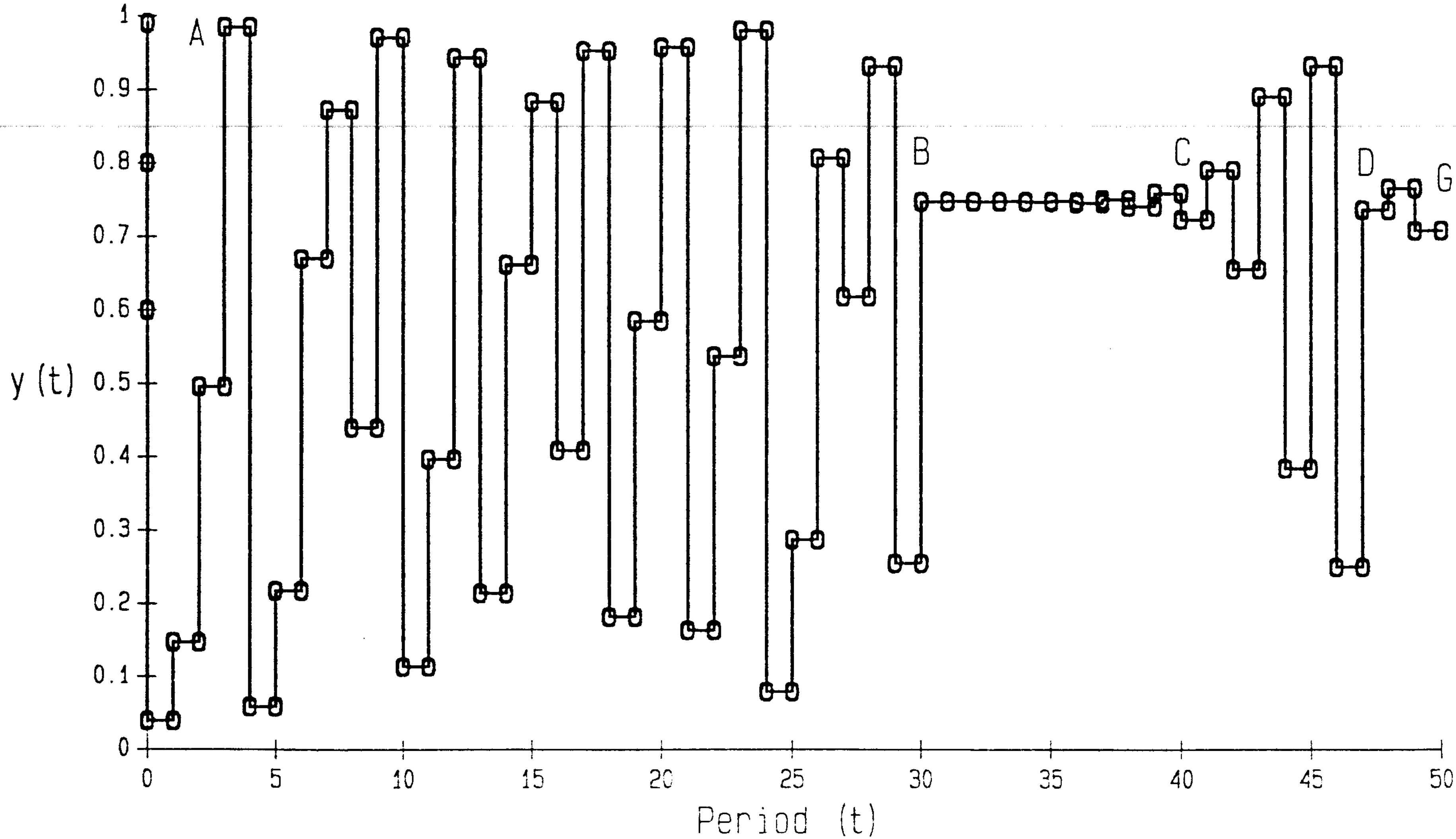


Figure 7c. Time Path, Periods 0-50
$y(t+1) = 3.945y(t)[1-y(t)], \quad y(0) = 0.99$

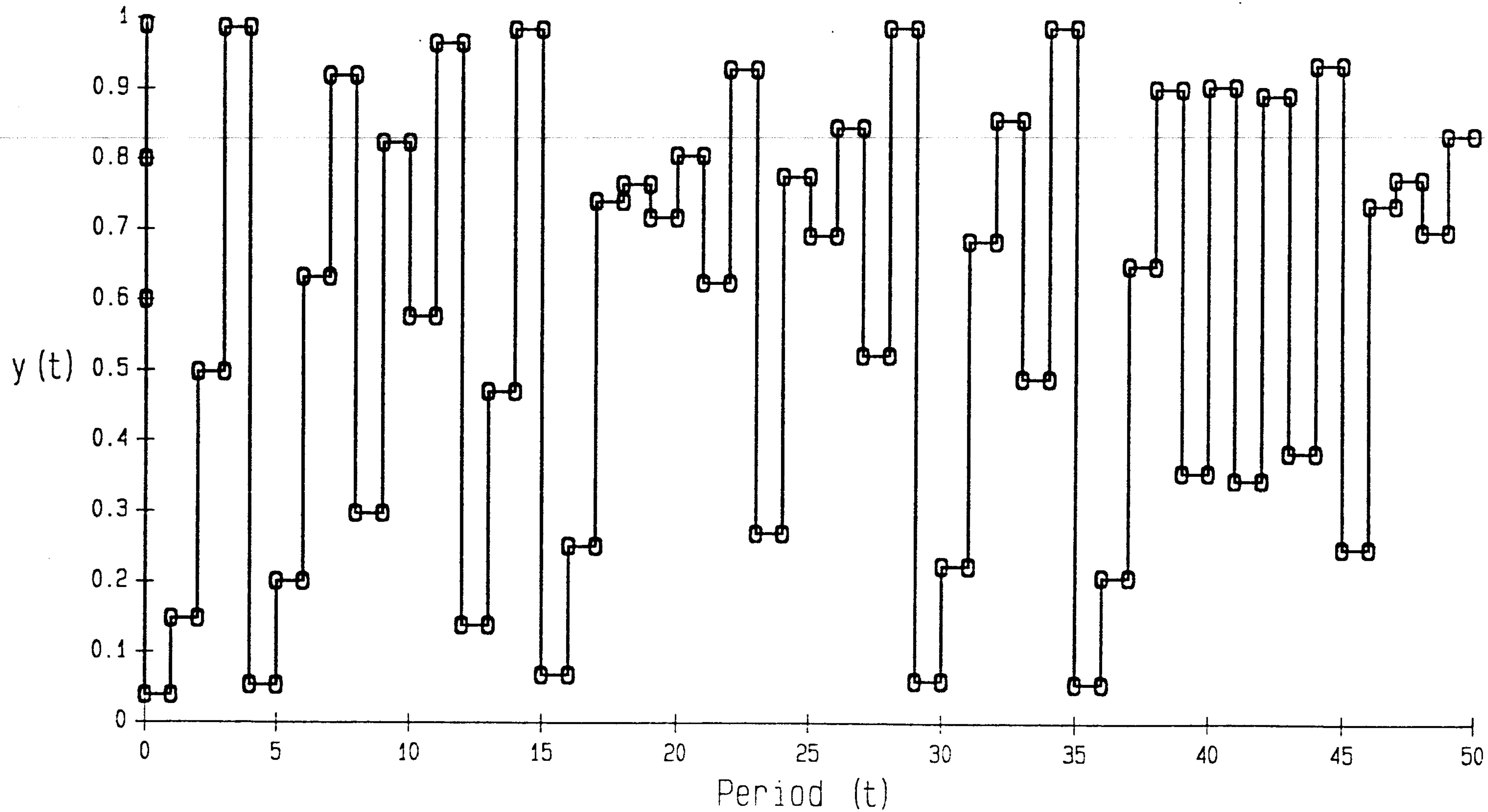
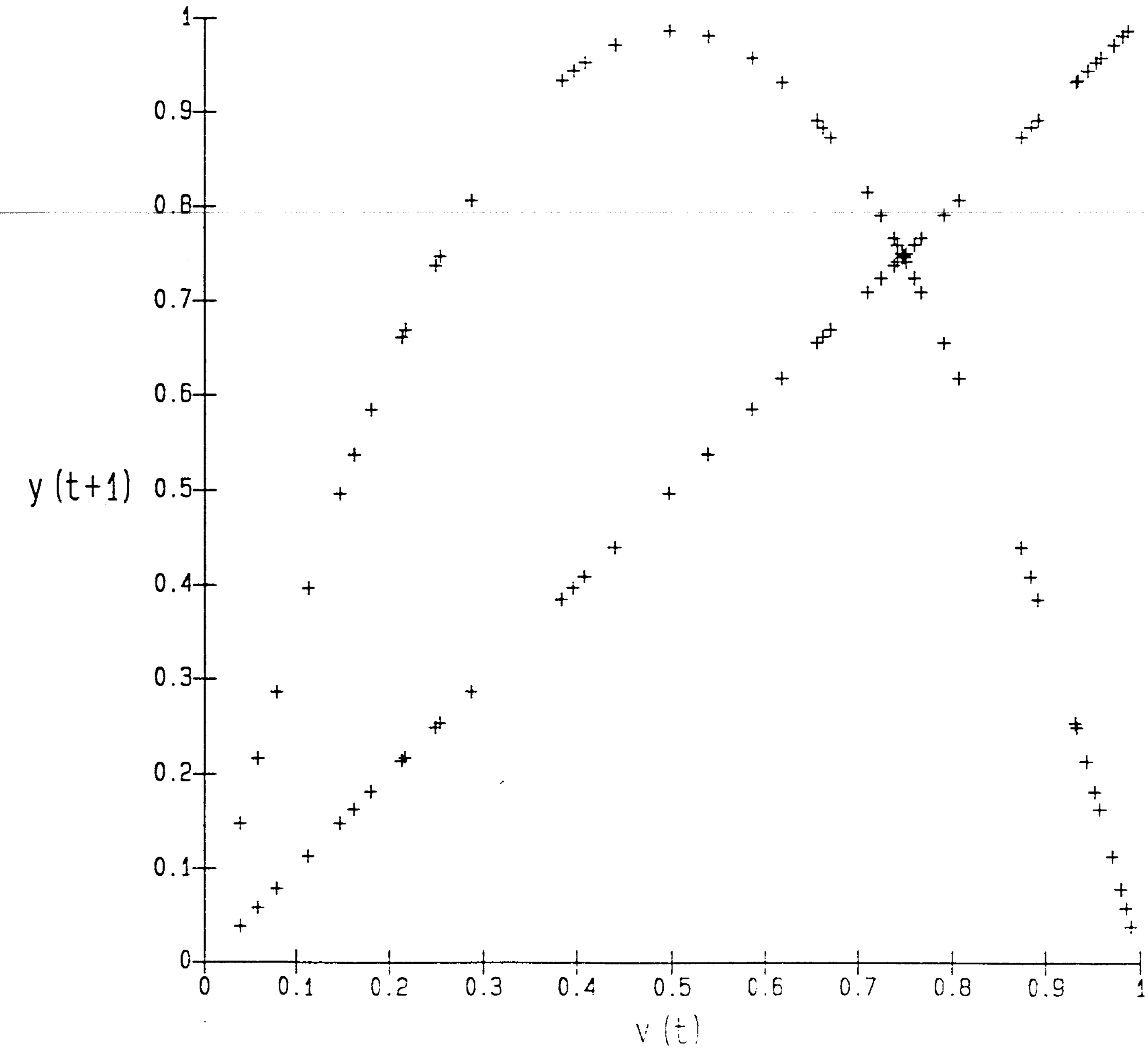


Figure 8. Phase Graph
Simulated From Chaotic Data
$y(t+1) = 3.94y(t)[1-y(t)], \quad y(0) = 0.99$



## References

Akerlof, George and Joseph Stiglitz, "Capital Wages and Structural Unemployment," The Economic Journal, June 1969, pp. 269-281.

Baumol, William, "Speculation, Profitability, and Stability," Review of Economics and Statistics, August 1957, 39, pp. 263-71.

Baumol, William, "Pitfalls on Contracyclical Policies: Some Tools and Results," Review of Economics and Statistics, February 1961, 43, pp. 21-26.

Baumol, William and Edward Wolff, "Feedback from Productivity Growth to R&D," Scandinavian Journal of Economics, 1983, 85, No. 2. pp. 147-57.

Baumol, William and Richard E. Quandt, "Chaos Models and their Implications for Forecasting," Eastern Economic Journal, January - March 1985, 11, pp. 3-15.

Benhabib, Jess, "Adaptive Monetary Policy and Rational Expectations," Journal of Economic Theory, October 1980, 23, pp. 261-266.

Benhabib, Jess and Richard H. Day, "Erratic Accumulation," Economics Letters, 1980, 6, pp. 113-117.

Benhabib, Jess and Richard H. Day, "Rational Choice and Erratic Behavior," Review of Economic Studies, July 1981, 48, pp. 459-472.

Benhabib, Jess and Richard H. Day, "A Characterization of Erratic Dynamics in the Overlapping Generations Model," Journal of Economic Dynamics and Control, 1982, 4, pp. 37-55.

Benhabib, Jess and Kazuo Nishimura, "The Hopf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimum Economic Growth," Journal of Economic Theory, December 1979, 21, pp. 421-444.

Benhabib, Jess and Kazuo Nishimura, "Competitive Equilibrium Cycles," Journal of Economic Theory, April 1985, 35, pp. 284-306.

Ben-Mizrachi, A. et all., "Characterization of Experimental (noisy) Strange Attractors," Phys. Rev. A(3), 1984, 29, pp. 975-977.

Boldrin, Michele and Luigi Montrucchio, "The Emergence of Dynamic Complexities in Models of Optimal Growth: The Role of Impatience," Journal of Economic Theory, October 1986, 40, pp. 26-39.

Brock, William A., "Money and Growth: The Case of Long-Run Perfect Foresight," International Economic Review, October 1974, 15, pp. 750-777.

Brock, William A., "Distinguishing Random and Deterministic Systems: Abridged Version," Journal of Economic Theory, J.M. Grandmont, ed., October 1986, 40, pp. 168-195.

Brock, William A., and William D. Dechert, "Theorems on Distinguishing Deterministic from Random Systems, preliminary draft, 1986.

Brock, William and Chera L. Sayers, "Is the Business Cycle Characterized by Deterministic Chaos?" Working Paper, Department, of Economics, University of Wisconsin, Madison, WI, Sept. 1985.

Brock, William A. and William D. Dechert and Jose Scheinkman, "A Test for Independence Based on Correlation Dimension," mimeo.

Calvo, Guillermo, "On the Time Consistency of Optimal Policy in a Monetary Economy," International Economic Review, February 1979, 20, pp. 83-103.

Collet, Pierre and Jo Eckman, Iterated Maps on the Interval as Dynamical Systems, Basel: Birkhauser, 1980.

Dana, Rose-Anne and Pierre Malgrange, "The Dynamics of a Discrete Version of a Growth Cycle Model," in J.P. Ancot (ed.). Analyzing the Structure of Econometric Models, Amsterdam: M. Nijoff, 1984.

Day, Richard, "Irregular Growth Cycles," American Economic Review, June 1982, 72, pp. 406-414.

Day, Richard, "The Emergence of Chaos from Classical Economic Growth," Quarterly Journal of Economics, May 1985, 98, pp. 201-213.

Day, Richard and James W. Shafer, "Keynesian Chaos," Working Paper, Department of Economics, University of Southern California, 1983.

Deneckere, Raymond and Steve Pelikan, "Competitive Chaos," Journal of Economic Theory, October 1986, 40, pp. 13-25.

Diamond, Phil, "Chaotic Behavior of Systems of Difference Equations," International Journal of Systems Science, 1976, 7, pp. 953-956.

Frisch, Ragner, "Propagation Problems and Impulse Problems in Dynamic Economics," Economic Essays in Honor of Gustav Cassel, London, 1983.

Gale, David, "Pure Exchange Equilibrium of Dynamic Economic Model," _Journal of Economic Theory_, February 1973, _6_, pp. 12-36.

Grassberger, P. and I. Procaccia, "Measuring the Strangeness of Strange Attractors," _Phys. D_, 1983, pp. 189-208.

Guckenheimer, John and Philip Holmes, _Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields_, New York, 1983.

Grandmont, Jean Michel, "On Endogenous Competitive Business Cycles," _Econometrica_, September 1985, _5_, pp. 995-1045.

Jacobson, M., "Absolutely Continuous Invariant Measures for One-Parameter Families of One-Dimensional Maps," _Comm. Math Phys._, September 1981, _81_, pp. 39-88.

Jensen, R.U. and R. Urban, "Chaotic Price Behavior in a Nonlinear Cobweb Model," mimeo, Yale University, 1982.

Lasota, A. and James A. Yorke, "On the Existence of Invariant Measure of Transformations with Strictly Turbulant Trajectories," _Bulletin de L'Academie Polonaise des Sciences_, Serie des Sciences Mathematiques, Astronomiques et Physique, 1977, _25_, pp. 233-238.

Li, Tien-Yien and James A. Yorke, "Period Three Implies Chaos," *American Mathematical Monthly*, November 1975, *82*, pp. 985-922.

Llibre, J., "Sufficient Conditions for a Continuous Map of $R^n$ into Itself to Have n-periodic Points for all m > 0," in M. Atteia, D. Dancel and I. Cernowski (eds.), *Nonlinear Problems of Analysis in Geometry and Mechanics*, New York: Pitman, 1981.

Lunberk, Erik, *Studies in the Theory of Economic Expansion*, London: P.S. King & Son Ltd., 1937.

Marotto, Frederic R., "Snapback Repellers Imply Chaos in $R^n$," *Journal of Mathematical Analysis and Applications*, 1978, *63*, pp. 199-223.

May, Robert, "Single Mathematical Models with very Complicated Dynamics," *Nature*, June 1976, *261*, pp. 459-467.

May, Robert M. and John R. Beddington, "Nonlinear Difference Equations: Stable Points, Stable Cycles, Chaos" (unpublished), 1975.

Misiurewicz, M., "The Loci Mapping has a Strange Attractor," in R.H.A. Helleman (ed.), *Nonlinear Dynamics*, New York: Academy of Sciences, 1980, pp. 398-358.

Ramsey, James and Hsiao-Jane Yun, "The Statistical Properties of Dimension Calculations Using Small Data Sets," mimeo, New York University.

Rand, David, "Exotic Phenomena in Games and Duopoly Models," Journal of Mathematical Economics, September 1978, 5, pp. 173 - 184.

Samuelson, Paul, "Interactions Between the Multiplier Analysis and the Principle of Acceleration," Review of Economics and Statistics, May 1939, 21, pp. 75-78.

Sarkovskii, A.N., "Coexistence of Cycles of a Continuous Map of a Line into Itself," Ukr. Math. Z., 1964, 16, pp. 61-71.

Scheinkman, Jose and Blake Le Baron, "Nonlinear Dynamics and Stock Returns," Department of Economics, University of Chicago, 1986.

Shaw, Robert, "Strange Attractors, Chaotic Behavior, and Information Flow," Z. Naturforsch., 1981, 36A, 80. , pp. 80-112.

Stefan, P., "A Theorem of Sarkovskii on the Existence of Periodic Orbits of Continuous Endomorphism of the Real Line," Comm. Math. Phys., June 1977, 54, pp. 237-248.

Stiglitz, Joseph, "Recurrence of Techniques in a Dynamic Economy," in J.A. Mirlees and N.H. Stern (eds.), Models of Economic Growth, London: MacMillan Press, 1973.

Stutzer, Michael, "Chaotic Dynamics and Bifurcations in a Macro Model," Journal of Economic Dynamics and Control, 1980, 2, pp. 353-376.

Takens, Floris, "Distinguishing Deterministic and Random Systems," in G. Rosenblatt, G. Iooss, and D. Joseph (eds.), Nonlinear Dynamics and Turbulence, Boston: Pitman, 1985, pp. 315-333.