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Abstract:
We discuss the circumstances under which improvements in productivity in a trading partner are beneficial to the home country and when they are detrimental. We show that there are gains from trade that a country can capture from a partly developed trading partner that strongly exceed the gains it can obtain by trading with either a fully developed trading partner or one that is relatively undeveloped. Once a trading partner passes this partly developed state, further improvements in the trading partner's productivity usually *decrease* the welfare of the home country so that we have an inherent conflict in the interests of the two countries.

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National Trade Conflicts Caused By Productivity Changes:
The Analysis With Full Proofs

by Ralph E. Gomory[1] and William J. Baumol[2]

Many practical issues require an understanding of the circumstances in which a country gains and in which it loses through improvements in productivity of its trading partner. Does the U.S. gain or lose when U.S. companies make major investments in plants in South East Asia? Does the U.S. gain or lose when it trains students in the latest technology and they return to their home countries. Does it matter whether that home country is relatively undeveloped or is a fully developed industrial country like Japan?

This paper will deal with these and related subjects, and in the process will make contributions in two areas, first theory, much of it with policy implications, and second the introduction of new and powerful analytic tools.

First, the *theory pertinent to policy* includes the conclusion that the gains from trade that a country can capture from a partly developed trading partner can substantially exceed the gains it can obtain by trading either with a fully developed country or one that is extremely underdeveloped. We also determine what characteristics of a nation's trading partner allow it to serve the interests of the home country most effectively. To be such an "ideal trading partner" the partner's economy must have low productivity in most industries so that it cannot compete effectively with the home country in those industries, even at its relatively low wage. But it should be very productive in a small proportion of industries so that in these industries it can produce more cheaply than the home country can.

We will show that if the trading partner is below this state of development, for example if it is an almost entirely undeveloped country trading with a well developed one, there is a natural symbiosis: improvements in the undeveloped trading partner usually benefit both countries. But once the ideal trading partner stage of development is reached, further improvements in the trading partner's productivity usually *decrease* the welfare of the home country. From that point on we no longer have symbiosis, but rather an inherent conflict in the interests of the two countries. The nature of this conflict is fundamentally different from conflicts that result from the trade restrictions and protectionist policies more commonly noted as sources of conflicting national interests in international trade.

The second contribution of this paper is to provide *analytic tools* that are new to the study

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of linear trade models.[3] These tools enable us to display as points of a graph each of the equilibria that can emerge from a linear trade model as a result of different values of the productivity parameters in the various industries of two trading countries. We can obtain these equilibria and this graph not only for two industry models but for models with n industries as well. We show that the set of these possible equilibria has robust attributes with the clear economic implications described above. We can obtain these equilibria and this graph not only for two industry models but for models with n industries as well, anda we find that models with large numbers of industries behave fundamentally differently from the more familiar two industry models.

These methods also allow us to quantify the qualitative conclusions we reach. We can compute the magnitude of the gains and losses to the two countries when a country goes from being underdeveloped to being the ideal trading partner, or from the ideal trading partner to being fully developed. This enables us to assert for example, that, for the home country, the difference between having an ideal trading partner and having a fully developed one is often as large as the difference between trading with a fully developed partner and being in a state of autarky.

Many of the comparative statics results of general equilibrium trade theory have been inherently local in character, providing partial derivatives of an endogenous variable with respect to an infinitesimal change in the value of one of the parameters. Our analysis is quite different in that it yields results for *all* possible parameter values. There is no distinguished starting point or infinitesimal variation. Consequently for any arbitrary starting point our analysis yields comparative statics results for changes in the relevant parameters of any magnitude within the ranges consistent with the constraints of the model. These larger changes in parameter values can, of course, have effects upon the endogenous variables that can be very different from the results of local changes in the parameter values. The non-local character of our results explains why they also enable us to connect linear models with those having economies of scale.

Roots in the Previous Literature: In conducting our analysis we touch on a subject long discussed by specialists in international trade, the circumstances under which improvements in productivity in a trading partner are beneficial to the home country. A significant part of the economic literature on this subject has been based on the analysis of Ricardian models whose trade equilibrium is shifted by improvements in productivity in one country or the other. In his noted inaugural lecture Professor Hicks [1953] sketched out an intuitive Ricardian model of the effect of improved productivity in Country A on its own welfare and that of its trading partner Country B. He concluded first that *uniform* improvements in productivity in a trading partner benefitted both countries, and then went on to distinguish two other cases. In the first the improvements in Country B are concentrated in its export industries, and he concluded that this improvement is beneficial to both Countries. In the second case the improvements are concentrated in Country B's import industries and he showed that although this is good for Country A, Country B is worse off.

This fruitful line of thought was taken up again by Dornbush, Fischer, and Samuelson

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[3]One of us, Gomory [1994], has used similar methods for the study of trade models with economies of scale.

[1977] in an explicit Ricardian model using a ground breaking approach to deal with an infinity of goods. Their conclusion was, like that of Hicks, that technological change spread uniformly among the products of the improving country was good for both countries. They also pointed out, however, that the international transfer of technology from a high-wage country to a less advanced low-wage country can be harmful to the welfare of the transferring country. In an illuminating paper that builds on the ideas of both these articles, Krugman [1986] considered the subject of trade between a technologically advanced country and its less advanced partner trading partner using some new and realistic premises. He assumed that the technologically advanced country was likely to make progress more rapidly in its more technologically advanced sectors and traced out the effect of this progress on both countries using a method of analysis similar to that of Dornbush, Fischer, and Samuelson. He found an interesting asymmetry. Progress in the advanced country was always beneficial to both countries, while progress in the less advanced country, while always beneficial to it, could, depending on circumstances, either be harmful or beneficial to the more advanced country. He pointed out that these results can be interpreted in terms of the tendency of the advanced country to make export biased improvements and of the less advanced country to make improvements that were more import biased.

More recently Johnson, Hyman and Stafford [1993,1995] have analyzed the effect of the improvement in a single industry in one of the countries. They found, consistent with the earlier work, that if the industry starts from a very low level of productivity and improves to a competitive level, the initial effects benefit the improving country but harm the trading partner, but later, when the industry is entirely shifted to the improving country, further improvements (which can now be regarded as export biased) are beneficial to both. Whether this is a net gain or net loss for the other country depends on the balance of the two phases.

All this work indicates that productivity improvements in one country are always good for it, but that the effect on its trading partner depends on the balance between the damaging effect on importing industries and the beneficial effect on exporting industries.

Approach of this Paper: In this paper we discuss a different but related issue. Instead of looking at the effect of changes in productivity around an existing equilibrium, we investigate what characteristics of a country best serve the interests of its trading partner. Is Country 1 better off if Country 2, the Country with which it trades, has a high income and its productivity levels in many of its industries are high? Or is Country 1 better off if Country 2 is relatively poor and in many of its industries has low productivity?

To answer these questions we consider the full range of productivity parameters that do not exceed some natural or technological limit. Specifically, for each industry i in each Country j we consider *all* levels of productivities $e_{i,j} \leq e^{max}_{i,j}$. We then ask which values of these productivity parameters yield the best results for one country or the other, or possibly simultaneously for both. Using integer and linear programming methods we give sharp answers to these questions.

Limits on productivity play an important role in our analysis as they do in real economic activity. We know that plants will shift to a low labor cost country when it is no longer possible to increase productivity in the home country to compensate for the cheapness of labor abroad. Or, at the analytical level, we know that in the last phase of the Johnson, Hyman, Stafford analysis, limits on productivity will determine the overall balance of beneficial and detrimental effects. Our methods allow us to deal systematically with the effects of limits on productivity.

Analysis of the full range of possible parameters, and explicit inclusion of the limits of productivity, reveals an inherent conflict in international trade. Generally, productivity parameter sets that are good for one country are poor for the other. The productivity parameters sets that produce the most beneficial outcomes for one country usually require its trading partner to be relatively undeveloped, meaning that the trading partner has not attained its maximal productivity in many of its industries. These productivity parameter values, while good for Country 1, are usually poor for Country 2. In fact we show that the very best trading partner for a developed Country 1, is *always* one that is only developed in a small proportion of the traded industries. This state of development in Country 2, while good for Country 1, is *always* a poor outcome for Country 2 itself.

Relative national income plays a large role in our analysis. We find that there is a range of relative national incomes in which the countries' incomes are linked in a way that permits mutual gains. In this range productivity parameter changes that increase utility in one country usually increase utility in the other as well. In this range *even import biased improvement in the trading partner is usually good for the home country.* But there is another range of relative national incomes where the utilities of the two countries move in opposite directions as productivity parameters change. Here there is inherent conflict in the interests of the countries.

We distinguish in our work between models with only a few industries and those with many. The number of industries matters. The results we have just described are valid for models with a larger number of industries, usually six or more. These models are different from small models, such as the familiar England-Portugal wine-textile example, which we discuss in detail below. In this famous example the outcome at which Portugal specializes in wine and attains its maximal productivity, and England specializes in textiles and attains its maximal productivity, is best for both countries. It remains the best even when we consider, as we do, all models with lower productivities, i.e. $e_{i,j} \leq e^{max}_{i,j}$. However, we will see that this single best outcome is a property of some small models, with two, three, or perhaps four industries, that does not carry over to large ones.

Our methods work well for large models. However, it is possible to go a step further and follow Dornbush, Fischer, and Samuelson [1977] in assuming symmetric demands and developing a continuous model. This is a model in which individual industries make up only an infinitesimal part of the national income. Use of a continuous model enables us to analyze the effect that both different country sizes and different profiles of maximal productivities, $e^{max}_{i,j}$ have on the economic outcomes. In fact we are able to obtain explicit formulas linking a county's utility to the production parameter sets of the two countries. We find that (1) in many cases knowledge of country size alone enables us to predict the exchange rate that gives a country its best possible outcome (2) the modern tendency of industries to depend more on acquirable skills and less on natural resource advantages tends to exacerbate the conflict between countries and generally lessen the gains from trade.

Finally we show that the use of regions of equilibria allows us to connect linear models with those having economies of scale such as those in Gomory [1994]. We introduce what we call the *correspondence principle* to explain this remarkably close connection. The use of the correspondence principle explains the similarity of many of the results that we obtain here for linear models with those that can be obtained with economies of scale.

call the *correspondence principle* to explain this remarkably close connection. The use of the correspondence principle explains the similarity of many of the results that we obtain here for linear models with those that can be obtained with economies of scale.

I. Bounded Productivity Families, Equilibria, and the Basic Graph

Families of Linear Models In our linear model, the quantities $q_{i,j}$ produced of each good i in Country j are determined by linear production functions $e_{i,j}l_{i,j}$. Each of the two countries participating in trade has a given utility function of Cobb-Douglas form[4] with demand parameters $d_{i,j}$. We fix the labor-force sizes L_j of the two countries as well as n, the number of industries. A single model is then completely specified by the vector of productivity coefficients $\epsilon=\{e_{i,j}\}$. However, instead of dealing with just one model we will discuss the equilibrium outcomes of the *family of models* obtained by considering *all productivity coefficients* ϵ *subject only to a maximal productivity condition* $e_{i,j} \leq e^{max}_{i,j}$ and holding everything else constant. This will enable us to analyze the effect of different productivity levels on the welfare of the two countries. We will refer to these as bounded productivity families or "BP families."

Each equilibrium of a BP family of linear models is represented as a point in a graph of utility versus relative national income, described below. The region of that graph that contains *all* the stable equilibrium points for a BP family of linear models has a definite and characteristic shape for large models. It is this shape that leads to the economic implications of this article.

The Basic Graph: For any given vector of productivity parameters $\epsilon=\{e_{i,j}\}$ of a BP family, satisfying $e_{i,j}\leq e_{i,j}^{max}$, there is a stable equilibrium giving a national income Y_j and a utility U_j for each country. From the Y_j we can compute *relative* national income $Z_j=Y_j/(Y_1+Y_2)$. We can then plot this equilibrium as a point $r_1(\epsilon)$ in a (Z_1,U_1) diagram, which displays Country 1's utility, or as a point $r_2(\epsilon)$ in a (Z_1,U_2) diagram which displays Country 2's utility.

Each ϵ gives us one point in each diagram. The 14 dots in Figure 1.1a represent 14 such $r_1(\epsilon)$ from one of our models. Z_1 is measured horizontally from 0 to 1. Utility is measured vertically with the scale chosen so that unity represents Country 1's utility in autarky using the maximal productivities $e^{max}_{i,1}$. In Figure 1.1b we have the $r_2(\epsilon)$. They have the same Z_1 values as the $r_1(\epsilon)$ but describe Country 2's utility. The unit value of utility now represents Country 2's utility in autarky using the $e^{max}_{i,2}$.

By combining the two diagrams we can see when equilibria that are good for one country are, or are not, good for the other. We do this in Figure 1.1c. The equilibrium of each ϵ is now represented by both $r_1(\epsilon)$ and $r_2(\epsilon)$. The black r_1 points represent Country 1's utility in Country 1 autarky units, and the gray r_2 points represent Country 2's utility in Country 2 autarky units. In the (randomly chosen) example in Figure 1.1c, we see that the equilibria that yield the most utility for Country 1 tend to yield a low utility for Country 2 and vice versa.

Stable Equilibria: Next we describe the equilibrium conditions that yield these equilibria. For this we need some notation. Z_j just defined, is Country j' s (relative) national income (Country j's *share*). We normalize analogously all our pecuniary expressions, so p_i the price of good I, and w_j, the wage in Country j, are also divided by total income $Y_1 + Y_2$. Country j's

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[4]Using the Cobb-Douglas demand function enables us to carry out explicit computations and therefore provide *quantitative* results about the effects we describe. However, as we will see in Section 7, many of our *qualitative* results do not require restrictive assumptions of this type.

consumption of good i is denoted by $y_{i,j}$ and its *production* of good i by $q_{i,j}$. Country j's *production share* or *market share* of world output of good i is represented by $x_{i,j}=q_{i,j} / (q_{i,i}+q_{i,2})$, so that the vector $x=\{x_{i,j}\}$ describes the pattern of production.

We can now describe our equilibrium conditions, noting that, henceforth, the term "equilibrium" will mean *stable* equilibrium. First, (relative) national income of Country j must equal the total revenue from domestic and foreign sales of that country's products. Since with a Cobb-Douglas utility function Country 1's expenditure on good i will be $d_{i,j}Z_j$, this condition is:

(1.1)
$$\sum_i x_{i,j}(d_{i,1}Z_1+d_{i,2}Z_2) = Z_j$$

for each country. However only one of the two equations (1.1) is needed.[5] Second, we have a zero-profit condition. World expenditure on Country j's output of good i all goes into the wages of the labor $l_{i,j}$ employed in that industry, so:

(1.2)
$$w_j\ l_{i,j} = x_{i,j}(d_{i,1}Z_1+d_{i,2}Z_2).$$

Third, is the full-employment requirement for each country. This is expressed as the condition that the wage rate times the country's total labor force equals national income:

(1.3)
$$w_j\ L_j = Z_j.$$

Fourth, we have the requirement that, for each good, quantity supplied equals quantity demanded, or equivalently, that the value of the output of good i at the equilibrium price equals the amount consumers are willing to spend on it

(1.4)
$$p_i(q_{i,1}+q_{i,2}) = d_{i,1}Z_1+d_{i,2}Z_2 \quad or \quad p_i q_{i,j} = w_j\ l_{i,j}$$

where the second form of (1.4) follows directly from the first by multiplying through by $x_{i,j}$ and using (1.2). Finally we have the stability conditions that make entry by non-producers unprofitable. These require producers not to have higher unit costs than non-producers For example if Country 1 is the producer in industry i and Country 2 is a non-producer, we must have $e_{i,1}/w_1 \geq e_{i,2}/w_2$. More generally:

(1.5)
$$\begin{array}{l} if\ x_{i,1}>0 \quad and \quad x_{i,2}=0 \quad then \quad e_{i,1}/w_1 \geq e_{i,2}/w_2 \\ if\ x_{i,2}>0 \quad and \quad x_{i,1}=0 \quad then \quad e_{i,1}/w_1 \leq e_{i,2}/w_2 \\ if\ x_{i,2}>0 \quad and \quad x_{i,1}>0 \quad then \quad e_{i,1}/w_1 = e_{i,2}/w_2. \end{array}$$

The conditions (1.5) are, of course, a form of the familiar comparative-advantage criterion.

In our model then, equilibrium is determined by the relative national income relation, supply-demand equality for each good, zero profit in each industry, full employment in each country, and the stability conditions.

Two Remarks: (1) It is easily shown that at equilibrium trade must also be in balance and that the exchange rate is w_1/w_2. (2) If an x and Z_1 satisfy the revenue balance condition (1.1), and the

[5] Since $Z_1+Z_2=1$ and $x_{i,1}+x_{i,2}=1$ the two equations (1.1) are dependent.

stability conditions (1.5) using the wage $w_j=Z_j/L_j$ from (1.3), then that x,Z_1 is already an equilibrium. This is because for any x,Z_1 (1.2) uniquely determines the labor quantities $l_{i,j}$, and with these known, prices are uniquely determined by (1.4) so the two remaining equilibrium conditions are automatically satisfied.

II. Preliminaries: Utility and Linearized Utility

We adopt the notation (x,Z_1,ϵ) for the equilibrium determined by the productivity parameters $\epsilon=\{e_{i,j}\}$ and having market shares $x=\{x_{i,j}\}$, and relative national income Z_1 for Country 1.

Generally a large ϵ means high productivity and high utility, and therefore yields points high up in either diagram. When the $e_{i,1}$ are large relative to the $e_{i,2}$, Country 1 is the producer in most industries so Country 1's income share is large. The equilibrium (x,Z_1,ϵ) will have a large Z_1, and the equilibrium point $p_1(\epsilon)$ will be near the right edge in both diagrams. Similarly, a large $e_{i,2}$ relative to the $e_{i,1}$ yields points near the left edges.[6]

Cobb-Douglas Utility: Cobb-Douglas Utility $U_j=\Pi_i\, y_{i,1}^{\,d_{i,j}}$ is a function of $y_{i,j}$, the consumption of good i in Country j. At any equilibrium (x,Z_1,ϵ) the consumption $y_{i,j}$ can be found by multiplying world output of good i, which is $q_{i,1}(x,Z_1,\epsilon)+q_{i,2}(x,Z_1,\epsilon)$, by the fraction $F_{i,j}$ that County j obtains, so the (log) utility is:

$$\text{(2.1)}\qquad \ln U_j(x,Z_1,\varepsilon)=u_j(x,Z_1,\varepsilon)=\sum_i d_{i,j}\ln\, y_{i,j}=\sum_i d_{i,j}\ln\, F_{i,j}\,\{q_{i,1}(x_{i,1},Z_1,\epsilon)+q_{i,2}(x_{i,2},Z_2,\epsilon)\}.$$

With Cobb-Douglas utility, the fraction Country j obtains is $F_{i,j}=d_{i,1}Z_1/(d_{i,1}Z_1+d_{i,2}Z_2)$,.while the quantities produced in each country are can be expressed in terms of x, Z_1, and ϵ by:

$$\text{(2.2)}\qquad q_{i,j}(x_{i,j},Z_1,\epsilon)=e_{i,j}l_{i,j}=e_{i,j}\frac{x_{i,j}(d_{i,1}Z_1+d_{i,2}Z_2)}{w_j}=e_{i,j}\frac{x_{i,j}(d_{i,1}Z_1+d_{i,2}Z_2)L_j}{Z_j}.$$

(2.2) and (2.1) then give us the utility value for any equilibrium (x,Z_1,ϵ), and in fact assign a utility value for any triple (x,Z_1,ϵ) whether or not it is an equilibrium.

We will often need to compare the utilities of different non-equilibrium production assignments x, and (2.1) with (2.2) enables us to do that. The simplest comparison is when we simply shift production in one industry from one country to another keeping Z_1 and ϵ fixed. As one would expect, such a shift increases utility when the shift is to the country with lower unit cost. More precisely:

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[6]One point in a (Z_1,U_j) plane can correspond to many quite different equilibria. The following definition reduces this duplication somewhat. Two equilibria (x,Z_1,ϵ) and (x,Z,ϵ') are *equivalent* if they differ only in the productivities of industries in which the country is *not* a producer. Equivalent equilibria have the same quantities of labor employed in each industry and have the same outputs, the same Z, the same x, and the same utility. They correspond to the same point in the (Z_1,U_j) plane.

Lemma 2.1 - Production Shift Lemma: If in a triple (x,Z_1,ϵ) County 1 has higher (lower) unit cost than Country 2 in industry i, then increasing $x_{i,1}$ and correspondingly decreasing $x_{i,2}$ increases (decreases) utility.

Proof: The marginal effect of such a shift on the $(q_{i,1}+q_{i,2})$ term in (2.1) is, from (2.2), $(e_{i,1}/w_1 - e_{i,2}/w_1)(d_{i,1}Z_1 + d_{i,2}Z_2)$. This also shows, as one would expect, that when unit costs are equal, shifting production between countries has no effect on utility.

Shifting production also leads us to the concept of *linearized utility* whose use will simplify both computation and theory.

Linearized Utility: To obtain the *linearized utility* we first compute the utility that would be contributed to Country j by the ith industry if all production were shifted to Country 1,$(x_{i,1} = 1)$, and then compute the utility as if all production were in Country 2 $(x_{i,2}=1)$. We then weight both contributions by the actual shares of world production $x_{i,j}$. So, as in Gomory [1994], we have defined linearized utility $Lu_j(x,Z_1,\epsilon)$ by:

$$\textbf{(2.3)} \qquad Lu_j(x,Z_1,\epsilon) = \sum_i x_{i,1} \{d_{i,j}\ln F_{i,j}(Z_1)q_{i.1}(1,Z_1,\epsilon)\} + x_{i,2} \{d_{i,j}\ln F_{i,j}(Z_1)q_{i.2}(1,Z_1,\epsilon)\} .$$

Maximizing the linearized utility (2.3) for different assignments x , while keeping relative income Z_1 and productivities ϵ fixed, is maximizing a *linear* expression in x with constant coefficients; a very simple thing to do. In what follows we will want to maximize in just this way using Cobb-Douglas utility. Fortunately, linearized utility and Cobb-Douglas utility are the same where it matters, as is shown by the Linearized Utility Theorem:

Theorem 2.1 - Linearized Utility Theorem: Utility $u_j(x,Z_1,\epsilon)$ and linearized utility $Lu_j(x,Z_1,\epsilon)$ are equal (a) at every equilibrium (x,Z_1,ϵ), and (b) whenever all the $x_{i,j}$ are all either 0 or 1, whether or not (x,Z_1,ϵ) is an equilibrium.[7]

World Utility: In addition to country utility, we will often refer to *world utility*, the total utility of all goods produced. This is most useful when both countries have the same utility functions but we will use it to some extent when they are different as well. We define world utility as

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[7] Proof: If at an equilibrium the ith industry is shared, then both countries must produce at equal unit cost in that industry. It then follows from Lemma 2.1 that the contribution to utility is the same whether Country 1 produces everything, in which case the contribution to utility is $d_{i,j}\ln\, F_{i,j}(Z_1)\; q_{i,1}(1,Z_1,\epsilon)$, or Country 2 produces everything , in which case the contribution to utility is $d_{i,j}\ln\, F_{i,j}(Z_1)\; q_{i,2}(1,Z_2,\epsilon)$, or if both produce. In every case the contribution to utility is $d_{i,j}x_{i,1}\ln\, F_{i,j}(Z_1)\; q_{i,1}(1,Z_1,\epsilon) + d_{i,j}x_{i,2}\ln\, F_{i,j}(Z_1)\; q_{i,2}(1,Z_2,\epsilon)$. Similarly if we look at an industry that is assigned entirely to one country, the contribution to utility is given by this same expression whether we look at $x_{i,1}=1$ and $x_{i,2}=0$ or $x_{i,1}=0$ and $x_{i,2}=1$.

$U_w = \Pi_i\ (q_{i,1}+q_{i,2})^{d_{i,1}}$. This means we are measuring the utility of total world utility using Country 1's preferences. The expression for world utility in terms of (x,Z_1,ϵ) is obtained immediately from (2.1)and (2.2) by putting j=1 and all the fractions $F_{i,j}=1$.

III. Basic Structure of the Region

The regions we plot are composed of *all* the equilibrium points of a BP family. For a given BP family we can plot R_1, the region of all points $r_1(\epsilon)$ representing Country 1 utility, and R_2 the region of all points $r_2(\epsilon)$ representing Country 2 utility. We will also plot the world utility region R_w , which is composed of the points representing world utility. The general regional structure is the same for all three regions and for all families. Each region always consists of an upper boundary curve $C_j(Z_1)$ and *all* the points below it. Figures 3.1a,b,c, 3.2, and 3.3 show some regions for small models.

The boundary curve $C_j(Z_1)$ completely determines the shape of each region, and it is from that shape that the economic consequences flow. We will eventually see that, remarkably enough, $C_j(Z_1)$ can be determined by solving a very simple one equation (linear) integer programming problem. But before we show that we will discuss the general structure of the region and of its boundary.

Theorem 3.1 - General Regional Structure Theorem: R_j consists of a boundary curve $C_1(Z_1)$ and all the points on or below it.

The assertion of the theorem is certainly plausible. Given any equilibrium (x,Z_1,ϵ), varying all its productivity parameters up and down around ϵ should give us equilibria all around the point corresponding to (x,Z_1,ϵ) in our diagram. This should happen unless at (x,Z_1,ϵ) most of the productivity parameters are $e_{i,j} = e^{max}_{i,j}$ so that the parameter variation is restricted. Since the larger $e_{i,j}$ give higher points in the diagram, a completely filled out region of equilibria, topped by a boundary curve involving equilibria with many $e_{i,j}=e^{max}_{i,j}$ seems at least intuitively plausible. That this is actually the case is shown in Appendix A.

We also make some plausibility remarks about the shape of the *world* boundary region. At the extreme left of the region $Z_1=0$, so Country 1 contributes nothing to world output. The largest world output there is obtained when Country 2 is at its maximal productivity in every industry and that largest world utility is simply Country 2's largest possible autarky value. Similarly at the right hand edge of the diagram, the largest world utility is simply Country 1's largest utility in autarky. However in between, when both countries can contribute we would expect to obtain greater world utility than in autarky (for a simple proof see Dixit and Norman 1980 p71ff). This means that we should expect a world utility boundary curve that is high in the middle of the diagram and lower at the two ends.

We will see that this is so, we will describe the boundary shape of the individual countries in greater detail, we will show how to calculate the boundaries exactly and analyze the equilibria that lie on them, and we will draw its economic consequences.

The Boundary Curve and the Classical Assignment: Because of the general structure of the region, it is the boundary curve that shapes the region and is critical for our analysis. So we now discuss the structure of the boundary curve itself. The equilibria on the boundary curve at Z_1 have the highest utility possible for each given Z_1. The natural way to get the highest possible utility is to assign each industry to the country that has the lowest unit cost at the wage implied by Z_1. This leads to a consideration of what we will call the classical assignment.

The Classical Assignment: If we choose any value of Z_1, we can determine the wages $w_1=Z_1/L_1$ and $w_1=Z_2/L_2$ at $Z_{,1}$ and therefore, for each industry, we can find which country would be the cheaper producer *if both were at their maximal productivities.* As in Gomory [1994] we define the *classical assignment* at Z_1, $x^C(Z_1)$, to be the assignment of industries to countries at Z_1 that assigns production entirely to the country that would be the cheaper producer *at maximal productivity.* This assignment is usually not an equilibrium. However there is one important exception.

Consider the equilibrium that results when $\epsilon=\epsilon^{max}=\{e^{max}_{i,j}\}$, i.e. when each country *is* at its maximal productivity in every industry. We call this equilibrium the *classical equilibrium.* The level of relative income Z_1 at this equilibrium we call the *classical level* Z_C. Since (1.5) tells us that at any equilibrium only the cheaper producer produces in each industry, at the classical equilibrium, each country produces in the industries in which it is the cheaper producer using its full productivity potential. Therefore at Z_C the classical assignment produces the classical equilibrium.

However, the classical assignment is never an equilibrium except at $Z_1=Z_C$. For example, for relative income Z_1 sufficiently small, Country 1's wage is extremely low. Therefore the classical assignment would give Country 1 the entire production of almost all industries. But the large resulting fraction of world revenue Country 1 would obtain would not match the small relative income Z_1, and (1.1) would not hold. Generalizing this thought gives us a theorem we will use repeatedly.

Theorem 3.2 - Classical Assignment Theorem: The classical assignment gives Country 1 a revenue greater than Z_1 for $Z_1<Z_C$ and a revenue less than Z_1 for $Z_1>Z_C$. The classical assignment gives Country 2 a revenue greater than Z_2 for $Z_1>Z_C$, and a revenue less than Z_2 for $Z_1<Z_C$.[8]

Strategy of theAnalysis: Now we are going to work out the structure of boundary equilibria. We will see that boundary equilibria are made up from an initial non-equilibrium assignment of industries to countries, closely related to the classical assignment. This assignment is then

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[8]Given the intuitive background we have just discussed, this theorem may hardly seem to require a proof. It would seem that in the $Z_1<Z_C$ case the lower wage must lead to larger relative national income. The only point not covered by that reasoning is the possible effect of the change in demand with different Z_1. However, that change does not affect the outcome. Proof:

$\sum_i x^C_{i,1}(d_{i,1}+d_{i,2}Z_2/Z_1)$ represents the revenue to Country 1 from the classical assignment divided by Z_1. Each parenthesis increases as Z_1 decreases from Z_C. The $x^C_{i,1}$ also only increase because, as Z_1 and Country 1 wage decrease, Country 1 can only gain industries in the classical assignment. Since the revenue to Z_1 ratio is 1 at Z_C, it is >1, for $Z_1<Z_C$. The other parts of the theorem are proved similarly.

modified, without changing its utility, to satisfy the equilibrium requirements (1.1)-(1.5). Knowing this structure will enable us to understand small models and to find their boundary curves.

Next the boundary structure will lead us to a simple integer programming problem which enables us to obtain the boundary shape for any BP family, large or small, rapidly and automatically.

The Assignments x^S: Let us assume $Z_1<Z_C$ and work to obtain the Country 1 boundary at Z_1. Since we want as much output and utility as is possible we initially choose $e_{i,j}= e^{max}_{i,j}$ for all industries. Also, as much as possible, we want to assign to Country 1 the industries in which it is the cheaper producer. So we will assign to Country 1 a *subset* S of the industries S^C it would get from the classical assignment, assigning all other industries to Country 2. We will denote this assignment by x^S and we will choose S small enough that Country 1 has *too little* revenue, i.e.: that we have:

(3.1) $$\sum_i (d_{i,1}Z_1+d_{i,2}Z_2)x^S_{i,1} \leq Z_1.$$

Next we make the assignment x^S into an equilibrium. In the industries in S^C-S Country 1 is the cheaper producer but x^S has nevertheless assigned their production to Country 2. We now revise *downward* the productivity of Country 1 in all the S^C-S industries until their unit costs match those of Country 2. This change does not affect utility as we have only changed the productivity of non-producers. Then, if we have strict inequality in (3.1), we shift entire industries or parts of the S^C-S industries from Country 2 to Country 1 to produce equality in (3.1). With equal unit costs in these countries the Production Shift Lemma tells us this change does not affect utility, and the Classical Assignment Theorem tells us that the there is more than enough revenue in S-S^C to produce equality. In fact there is often a choice of what industries or parts of industries to shift. But whatever the choice we end up with equilibrium conditions (1.1), and (1.5) satisfied. Then Remark 2 of Section 1 assures us that we have created an equilibrium.

Describing the Equilibria : We will call the set of equilibria resulting from the set S, E(S). Since none of the changes we made changed utility from the original assignment x^S, each equilibrium has the utility value that (2.1)-(2.2) would compute for x^S, i.e. for the triple (x^S,Z_1,e^{max}). Each equilibrium also has $e_{i,j} = e^{max}_{i,j}$ for all industries not in S^C-S. For the industries in S^C-S productivity is still $e^{max}_{i,2}$ in Country 2, but Country 1's productivity has been lowered to $e_{i,1}=(w_1/w_2)e^{max}_{i,2}$. The market shares are $x_{i,1}=1$ for industries in S, $x_{i,2}=1$ for those not in S^C, and for the industries S^C-S any choices of $x_{i,1}$ and $x_{i,2}$ that produce equality in (3.1).

The Maximizing Equilibrium: Since we are looking for the boundary equilibrium, we are particularly interested in the subset S of S^C that satisfies (3.1), *and whose assignment x^S has the greatest utility.* The equilibrium or equilibria resulting from this S must in fact *be* the boundary equilibrium unless there are still other equilibria that are higher than any of those constructed by our process from any S. We rule out this possibility in Appendix B. Therefore, we have proved:

<u>Theorem 3.3 - Boundary Structure Theorem</u>: If $x^S(Z_1)$ maximizes utility over the subsets S of S^C that satisfy (3.1), its equilibria E(S) are the boundary equilibria at Z_1

This theorem helps us in three ways. (1) It leads immediately to a first general result about the shape of the region, (2) it enables us to study in detail the boundaries of small models, and (3) it leads directly to methods of calculation that enable us to compute the boundaries of large problems. We will give examples of these three effects now.

The Monotone Increase Theorem: The first general result about the regional shape is

Theorem 3.4 - Monotone Increase Theorem: The Country 1 boundary curve $C_1(Z_1)$ is monotone increasing with Z_1 for $Z_1<Z_C$. The Country 2 boundary is monotone increasing with Z_2 for $Z_1>Z_C$.

Both behaviors can be seen in Figures 3.1b, 3.1c, 3.2 and 3.3. Proof: The Country 1 utility of any x^S is given from (2.1),(2.2), and (2.3) by:

$$Lu_1(x^S,Z_1,\varepsilon^{\max}) = u_1(x^S,Z_1,\varepsilon^{\max}) = \sum_i x^S_{i,1}\ (d_{i,1}\ln\ d_{i,1}e_{i,1}L_1)\ +\ x^S_{i,2}\ (d_{i,1}\ln\ d_{i,1}e_{i,2}L_2\frac{Z_1}{Z_2}).$$

We can see by inspection that for any fixed S (and therefore fixed $x^S_{i,j}$) all the terms are either constant or increase with Z_1. Therefore the utility of any fixed x^S increases with Z_1 . Also we can easily see that if x^S satisfies (3.1) for Z_1 it continues to satisfy it for all larger Z_1. Therefore the collection of sets S we maximize over only gets larger as Z_1 increases. This means that utility increases steadily with Z_1 with an occasional upward jump if a new set joins in the maximization and provides the new maximum. This gives us monotone increase and proves the theorem.

Next we discuss a specific small model.

The Ricardo Example and the Outcome Best for Both Countries: Figures 3.1a, 3.1b, and 3.1c show the regions for a particular Ricardian textile-wine model. There are only two industries and two equal sized countries. Country 1 (England) is assumed to excel in the first of them (textiles) with $e^{max}_{1,1}=1$ while its wine production is characterized by $e^{max}_{1,2}=0.55$. The other country (Portugal) excels in the other industry (wine), with $e^{max}_{2,2}= 1$ and $e^{max}_{2,1}=0.45$. The demand parameters for textiles and for wine are the same in both countries, $d_{i,1}=.55$ for textiles, $d_{i,2}=.45$ for wine.

The boundaries and the region below them include all equilibria with $e_{i,j}\leq e^{max}_{i,j}$. In Figure 3.1a we have plotted world output as measured by Country 1's utility function. In Figure 3.1b, we show Country 1's utility, and in Figure 3.1c we show Country 2's utility measured in Country 2 utility units. We will discuss Figure 3.1 and its boundary points for different values of Z_1.

For very small Z_1 the classical assignment assigns both industries to the very low wage England, so S^C contains both industries. But since both textiles and wine have a relative income share of .45 or more, either one alone provides too much relative income for the given Z_1 and violates the constraint (3.1). The only subset S with revenue not exceeding income share is the empty set, so the only and therefore maximizing S is the empty set. That is, neither industry can be assigned exclusively to England. Both industries therefore belong in S^C-S and equilibrium requires $e_{i,1}=(w_1/w_2)e^{max}_{i,1}$ and $e_{i,2}=e^{max}_{i,2}$ for both industries. England, with both low wage and low productivity competes with Portugal in both textiles and wine, getting a small share of the market for each. Then, as Z_1 increases, England's wage increases and its productivity rises to keep it competitive in both industries.

The first assignment change occurs at Z_1=.3548. To the right of Z_1=.3548 the classical assignment now gives textiles to England and wine to Portugal. The increases in England's wages and productivity have brought England to its maximal wine productivity, but despite that, its higher wages make it no longer competitive with Portugal in wine. S^C assigns only textiles to England and assigns wine to Portugal. However, the only subset of S^C that does not violate (3.1) is still the empty set. So Portugal takes the entire wine market, England shifts to taking a larger slice of the textile industry which is the one industry left in S^C-S. The next possible change is at Z_1=.45 where for the first time there is a possible subset S other than the empty set. Assigning the wine industry to England is now possible without violating (3.1). Direct evaluation of the two possibilities would show us that the greater utility is still associated with the empty set, so we go on as before, with England increasing both wage and productivity and taking an ever larger slice of the textile market. Finally, at Z_1=.55, which is the classical level Z_C, it is possible for the first time to assign the textile industry to England alone. This assignment turns out to be the maximizing S. Therefore, textiles move from S^C-S into S itself, and the boundary point now has the productivity parameter $e^{max}{}_{1,1}=1$ in textiles rather than $(w_1/w_2)\, e^{max}{}_{1,2}= (.55/.45).55=.672$. Consequently, the boundary curve jumps up discontinuously. This reflects the fact that England is capable of improving its textile productivity even after capturing the entire world market from Portugal with less than its maximal productivity. Since both S^C and S only contain the textile industry, S^C-S is empty and there are no industries to share. England has all of textiles and Portugal all of wine. Both are fully developed. It is the classical point.

The classical point is so high that the best outcome for each country is attained there, as Figures 3.1b and 3.1c show. So in this two-product case the classical specialized outcome is the best possible result for *both* countries. But this result is far from typical as we will see when our methods of computation allow us to deal with larger problems

A Method of Computation: The Boundary Structure Theorem next gives us a method of computation.

Theorem 3.5 - Exact Boundary Theorem: The utility of each point of the boundary curve $C_1(Z_1)$ for $Z_1 \le Z_C$ is obtained by solving the maximization problem in *integer* x:

$$C_1(Z_1) = Max_x\ u(x,Z_1,\epsilon^{max}) = Max_x\ Lu(x,Z_1,\epsilon^{max}) \tag{3.2}$$
$$\textstyle\sum_i x_{i,1}(d_{i,1}Z_1+d_{i,2}Z_2)\le Z_1, \qquad x_{i,1}+x_{i,2}=1$$

There are similar statements for $Z_1>Z_C$ and for $C_2(Z_1)$ and $C_w(Z_1)$.

Proof: This is very close to a restatement of what we already know. The *integer* $x_{i,1}$ select the industries that make up the set S. The use in (3.2) of the inequality (3.1) makes sure that revenue from this set of industries does not exceed Z_1. The maximization criterion causes the selection of the maximizing S.[9] We can use either utility or linearized utility as the objective function as we

[9]To complete the proof we would have to verify one point. Our method of constructing an equilibrium from S, required S to be a subset of the industries that the classical assignment would assign to Country 1. We need to show that the S given by (3.2) meets that criterion. However, this is straightforward. If the S selected by (3.2) contained any industry in which Country 1 was

are dealing with specialized assignments and the Linearized Utility Theorem applies.

Solving (3.2) is a straightforward dynamic programming calculation which yields the utility of the boundary equilibrium for each Z_1. The resulting boundary curves are those of Figures 3.1a,b,c , 3.2, and 3.3.

Larger Examples and an Emerging Shape: The Exact Boundary Theorem enables us to look at larger models. Figure 3.2 is the region for a 4-industry model. In it Country 1 is more productive in two industries and Country 2 is more productive in two industries. In Figure 3.2 we have combined all the boundaries in a single figure. Country 1 utility is read from the right vertical axis and Country 2 utility from the left. In contrast with the two industry model, in this 4 industry model *the equilibria best for the individual countries are not the same* - they are point P_1 for Country 1 and point P_2 for Country 2. Figure 3.3 depicts a 6-industry model. Again, different widely separated equilibria are best for the two countries.

These larger models have show that an increase in the number of traded commodities can change entirely change the implications of the model. We have seen that two industries is not enough. It takes more than a pair of goods to bring out the trade conflicts that, as we will see, always play a significant role in larger models. This can be contrasted with, for example, the early indifference map analysis of consumer purchase behavior, where it was shown that two or three good models reveal a great deal that single good analysis cannot show, but that larger numbers of goods offer no additional major insights.

In addition something else is happening. The boundary curves are becoming smoother and a definite shape is emerging. We will capture that underlying shape by a simple approximation.

Consider a modification of (3.2) in which we replace the inequality by equality and, importantly, allow continuous rather than only integer x. We again use the linearized utility. This gives us instead of (3.2):

(3.3)
$$\ln B_1(Z_1) = Max_x \; Lu_1(x, Z_1, \epsilon^{max})$$
$$\textit{subject to} \;\; \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\} \, x_{i,1} = Z_1 \;\; \textit{and} \;\; x_{i,1} + x_{i,2} = 1.$$

We will show below that the curve $B_1(Z_1)$ that emerges from (3.3) lies above and very close to the exact regional boundary $C_1(Z_1)$, and that it gets closer and closer as n, the number of industries, increases. First we will show that $B_1(Z_1)$ is very easily obtained.

Calculating $B_1(Z_1)$: (3.3) represents the problem of assigning industries to Country 1 in such a way that Country 1 maximizes its utility for a given national income Z_1. With continuous variables x and linearized utility this is done very easily. First assign all industries to Country 2. Then order the industries by calculating for each industry the amount that shifting it to Country 1 would increase Country 1's utility (this is a coefficient in the objective function) divided by the

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the higher unit cost producer, then we could shift that industry to Country 2. This decreases Country 1's revenue so (3.1) still holds, but it increases utility by the Production Shift Lemma. This contradicts the optimality of the set S.

increase that shift would make in Country 1's national income (this is a coefficient in the equation). Then move industries to Country 1 in this order, the largest first, until the national income Z_1 is exactly obtained.[10] Entire industries are shifted by this process until the last one. This last industry, which we will refer to as the special industry or the shared industry, is usually only partly moved in order to produce a national income that is exactly Z_1. The identity of the special industry varies with Z_1.

The result depends strongly on the ordering. When both countries have the same demand structure the ordering turns out to depend on comparative advantage alone. If the demand structures are not the same the ordering still takes into account comparative advantage, but also gives weight to the relative importance of the industry to the two countries. *Country 1 may do better to acquire an industry in which it has a small comparative advantage and a strong domestic demand, and therefore a strong interest in a lower price for the good, than to acquire an industry in which it has a large comparative advantage but, due to low domestic demand, is relatively indifferent to the price of that good.*

The first result about the $B_1(Z_1)$ calculated in this way is:

Theorem 3.6 - Approximate Boundary Theorem: $B_1(Z_1)$ lies above all the equilibria of the linear BP family.

Proof: Every equilibrium (x, Z_1, ϵ) must satisfy (1.1) which is the same as the equality in (3.3). Therefore, every equilibrium x is considered in the maximization with ϵ^{max} instead of ϵ, a change that only increases its utility.

However, $B_1(Z_1)$ is not only above the equilibria, it is close to them. We show this by producing from the optimizing x an equilibrium that has the sane Z_1 and almost the same utility.

Converting the x to an Equilibrium: We convert the optimizing x to an equilibrium using the same basic approach as before. If in the ith industry the assignment x matches the classical assignment, we leave the $e_{i,j}$ at their maximal productivity values. If the assignment is the opposite of the classical assignment so that the cheaper potential producer does not produce, we lower the productivity of this non-producer to stabilize production in that industry. This change in a non-producer has no effect on utility. Finally in the one shared industry we lower the productivity of the cheaper producer to match the cost of the other producing country. This step does affect utility, it reduces it, but it also stabilizes this last industry and gives us an equilibrium.

The only productivity change that affected linearized utility in going from the maximizing x to the equilibrium was the last step involving industry k. If each industry has only a small share of the total demand, then this difference in utility will be small and the resulting equilibrium will be near the original boundary point. We can expect this to happen as the number of industries increases and demand for any one industry's products becomes a small part of national income. In Appendix C we give a precise statement of the very wide range of conditions under which this convergence of the approximate boundary to the region of equilibria occurs.

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[10](3.3) is a special linear programming problem, a knapsack problem, for which the standard simplex method takes on this simple form.

The Boundary Shapes: We have used (3.3) to calculate over 100 examples ranging in size from 5 industries to over 40. The calculation is simple and rapid. We always see the same characteristic shapes, the ones displayed in Figures 3.5 and 4.2 for different 22 industry models.

The Country 1 boundary $B_1(Z_1)$ always starts from a zero utility level at $Z_1=0$, rises steadily to a point that is always well to the right of the classical level Z_C, and then declines to the unit level which represents Country 1's utility in autarky. $B_2(Z_1)$ has a similar (reversed) shape.

Later we will give more intuitive economic reasons for these characteristic regional shapes. Here we will state the fundamental theorem, based on (3.3), that gives the regional shape for $B_1(Z_1)$ and $B_2(Z_1)$.

Theorem 3.7 - Country Peak Theorem: $B_1(Z_1)$ increases monotonically from zero at $Z_1=0$ to a peak value at $Z_{p1} \geq Z_C$ and is monotone decreasing thereafter to the Country 1 autarky value at $Z_1 =1$. There is a similar statement for Country 2.

Proof Outline: That $B_1(Z_1)$ is monotone increasing for $Z_1<Z_C$ is certainly what we would expect since the exact boundary $C_1(Z_1)$ is. Similarly the zero value at $Z_1=0$ and the autarky value at $Z_1=1$ are clear from the fact that at $Z_1=0$ Country 1 has, by definition, zero percent of the total income of the two countries and at $Z_1=1$ it is the sole producer. The difficult part is showing the monotone increase and to the peak to the right of Z_C and the decrease thereafter. We do this by proving the quasi-concavity of $B_1(Z_1)$ in Appendix D. That then completes the proof of Theorem 3.7.

The quasi-concavity of $B_1(Z_1)$ is not easy to prove, there seems to be no useful way to work with linear combinations of equilibria or of solutions to (3.3) as one would naturally try to do. In fact, any proof must depend on the difference between (3.3) and (3.2), since, as the figures show, $C_1(Z_1)$ the exact regional boundary is clearly not quasi-concave, but $B_1(Z_1)$ the approximate boundary is. Also, as we show by an example below, even the approximate world boundary $B_W(Z_1)$ is *not* quasi-concave.

The world boundary always shows up in our computations as very roughly dome shaped. The shapes of the world and country boundaries are somewhat linked. At any Z_1, Country 1 has a share Z_1 of world income and it is plausible that this will give it roughly a Z_1 share of world utility. With symmetric Cobb-Douglas demand ($d_{i,1}=d_{i,2}$) this connection is exact. We see this in Figures 3.4 and 3.5 which are models with symmetric demand. The linkage is much less clear in Figures 4.2 and 4.3 which are models with non-symmetric demand. In Figure 4.3 , an 11-industry model, the world boundary has a local *minimum* almost exactly at the classical level which shows that it is not quasi-concave. Nevertheless, the world boundary gives a useful intuitive view. If we accept its roughly dome shape, then we can interpret the increase and then decrease in the Country 1 boundary as Country 1 getting an ever larger share of a world output that first increases but eventually, for large Z_1, becomes small.

IV. Economic Consequences of the Regional Shape

Inherent Conflict: Clearly, the best outcomes for Country 1 are always to the right of Z_C, those at or near the peak of the Country 1 boundary in Figure 3.5. Figure 3.5 also shows the corresponding region for Country 2, with the best equilibria for Country 2 near its peak to the left of the classical level. Because of the position of the peaks, the outcomes best for Country 1 are always poor for Country 2 and vice versa. A country that is successful in maximizing its utility does so at the expense of its trading partner. Thus the shape of the region, and the location of the peaks of the two countries shows that there is inherent conflict in the interests of the trading partners.

Trading with a Fully Developed Partner: Next we assert that if Country 1 trades with a fully developed Country 2, (i.e., $e_{i,2}=e^{max}_{i,2}$), it is confined to the outcomes to the left of Z_C.

Theorem 4.1- Fully Developed Partner Theorem: If Country 1 (2) trades with a fully developed Country 2, the resulting equilibrium will always have $Z_1 \leq Z_C$. ($Z_1 \geq Z_C$) .

Proof: Suppose there is an equilibrium with $Z_1 > Z_C$. If County 1 were fully developed (i.e., $e_{i,1}=e^{max}_{i,1}$) as Country 2 is, the classical assignment would already give Country 1 less revenue than its share Z_1 (Classical Assignment Theorem). If Country 1 is not fully developed, it will at this equilibrium have even fewer industries and less revenue than in the classical assignment of production.. In either case its revenue does not match its share, equilibrium condition (1.1) does not hold, and there is no equilibrium.

As the figures clearly show all Country 1's best outcomes are to the right of Z_C, so trading with a developed partner confines it to relatively poor outcomes. So it is *not* good to be a partly developed country trading with a fully developed partner.

Trading with a Partly Developed Partner: Next we will show that it *is* good to be fully developed, especially if the trading partner is only partly developed.

Partly Developed Partner Theorem - Theorem 4.2: If Country 1 is fully developed, then the possible equilibria all lie to the right of Z_C and above the Country 1 autarky level. The area of possible equilibria is area A in Figure 4.1

Proof : If Country 1 is fully developed, then from Theorem 4.1 the possible equilibria all have $Z_1 \geq Z_C$. Since there is trade, and hence gains from trade, the outcomes will be better than autarky.

Even within the region A there is a wide range of outcomes depending on the state of development of Country 2, i.e. its productivity parameters. In Figure 4.1, which is a typical example, the gains from trade to Country 1 at its peak P_1 exceed the gains from trade at the classical equilibrium C by more than the amount that the gains from trade at the classical equilibrium exceed autarky.

We know that a fully developed County 2 trading with Country 1 gives us the classical equilibrium. But what kind of a Country 2 will give us the equilibrium at P_1? We will call this Country 2 "The Ideal Trading Partner" and we will investigate its characteristics.

Characteristics of the Ideal Trading Partner: With what we already know we can find the ideal trading partner for any given Country 1. As an example we choose a 22 industry BP model with non-symmetric Cobb-Douglas utility, and with countries of almost equal in size We assume that Country 1 is fully developed and look for its ideal trading partner. Using (3.3) we first calculate the regional frontier $B_1(Z_1)$and locate the Z_1 value of its peak. As Figure 4.2 shows, the peak is at Z_1=.74, so at the peak Country 1's wage is roughly 3 times that of Country 2. Because of its high wage, Country 1 would be the cheaper producer in only 4 of the 22 industries if Country 2 were fully developed. Next we follow our usual procedure, described in Section 3, and convert the assignment x of (3.3) into an equilibrium, reducing some of Country 2's productivities from their maximum values in the process. At the resulting equilibrium, the fully developed Country 1 is the sole producer in its 4 industries but, in addition, has almost the entire market in 13 other industries in which Country 2 is *not* fully developed. Country 2 is the sole producer in 5 industries and has a very small sliver of the market represented by the 13 other industries in which it would be the cheaper producer if fully developed. The incomplete development of Country 2 permits Country 1 to hold the market in those industries and to have its large relative national income of .74.

This example is typical, the partly developed productivities of the ideal trading partner usually allow Country 1 to make most of the world's goods while Country 2 produces at its maximum productivity only in the smaller share of industries that it has to itself. A high-technology country making most things for itself but trading for a few goods with an agricultural country can be an example.. At the peak outcome *any* change in Country 2's productivities has a detrimental effect on Country 1. If Country 2's production parameters *increase*, this hurts Country 1, if Country 2's parameters *decrease*, that too hurts Country 1.

Peak Gains and History: Economic historians have long debated such questions as whether the U.K. lost out or benefitted from the relative rise in productivity since the 19th century in countries like Germany. Our analysis suggests that the effect cannot be determined simply from the change in German productivity, but requires knowing whether Germany, for example, moved closer to or further from being an ideal partner for the U.K.

Decreasing Productivities: In discussing departure from peak gains we mentioned the possibility of production parameters *decreasing* as well as *increasing*. It is natural to imagine a country's productivities *increasing* as it learns the latest methods of production or distribution, however *decreases* in productivity also have a realistic economic interpretation. We need only reinterpret the productivities of the BP family to allow for industry wide learning. We do this by allowing the *unit* of output in each industry to increases over time, while the $e^{max}_{i,j}$ remain constant. A country with fixed real output per labor hour in industry i, i.e. a country that does not keep up with the new methods, would move to new equilibria with smaller $e_{i,j}$.[11]

Quantifying Peak Gains: Our ability to compute using (3.3) and our ability to work out a rather complete theory of the symmetric demand case (see Gomory and Baumol [1998]) enable us to make quantitative statements about the importance of peak gains from trade. On the basis of both experiment and theory we assert that *over a wide range of country sizes and maximal productivity parameters the peak gains are a very considerable addition to the classical gains from trade, often exceeding the classical gains by more than the amount that the classical gains exceed autarky.*

The location of the peak also matters. As we will see in the next section, the location of

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[11]More formally: Let the new unit in the ith industry be the old unit multiplied by $p_i(t)\geq 1$, $p_i(0)=1$. At any equilibrium (x,Z_1,ϵ), at time t, the productivities in new units are $e_{i,j}$, and in t=0 units, $p_i(t)e_{i,j}$. Utility has been increased by $\Pi_i p_i(t)$. Since we plot vertical position in autarky units, which are also multiplied by $\Pi_i p_i(t)$, the (Z_1,U_1) diagram does not change at all. But a country with fixed productivity in old units has productivity in new units of $e_{i,j}/p_i\{t\}$, a decrease.

any equilibrium relative to the peak indicates whether the further development of its trading partner is likely to be helpful or harmful to Country 1.

V. Maximal Productivity Equilibria and the Subregion of Maximal Productivity

The effect of limits on productivity is felt well into the region, not only near the boundary. These limits have some effect on all equilibria where producers are producing at maximal productivity. They have the most effect at the equilibria at which the *producing* industries in each country are all at their maximal productivities, i.e., $x_{i,j}>0$ implies $e_{i,j}=e^{max}_{i,j}$. We call these the *maximal productivity equilibria.*

If a country is an actual producer in an industry, it will tend to learn how to produce better. Over time its productivity will tend to approach the limit imposed by its fundamental capabilities in that industry. That is to say, hat countries will tend to be near maximal productivity equilibria. For that reason we will focus our attention on the part of the region of equilibria which contains maximal productivity equilibria.

We might well expect the maximal productivity equilibria to lie in some sort of a band directly under the upper boundary, and indeed they do. We show this band for Country 1 as the subregion M_1 in Figure 5.1. It is not hard to show that this band is the subregion that lies between the upper boundary $B_1(Z_1)$ and the lower boundary $BL_1(Z_1)$ obtained by minimizing instead of maximizing in (3.3).

(5.1)
$$\ln BL_1(Z) = Min_x\ Lu_1(x, Z_1, \epsilon^{max})$$
$$subject\ to\ \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}\ x_{i,1} = Z_1 \quad and \quad x_{i,1} + x_{i,2} = 1$$

(5.1) is (3.3) with the Max replaced by Min.

Conflict and Cooperation: In equilibria below the region of maximal productivity there are always some producers who can improve their productivity. In most cases they can do this without changing either x, the pattern of production, or relative national income Z_1. Such changes are quite benign, providing more output with fixed x and Z_1, benefitting both countries.

Within the region of maximal productivity, where we generally expect to find the trading countries, such benign changes are scarcer. At maximal productivity equilibria they are unavailable. There, increases in productivity are only possible for non-producers. These increases generally have no effect at first, since they are increasing the productivities of those who do not produce, but if the increase is sufficiently large a new equilibrium can emerge with the former non-producer becoming a producer. Then x and Z_1 must change, entailing migration of industries and change in relative national income.

The subregion of maximal productivity, then, separates the region of equilibria into two parts: (1) the part below the subregion, where increases in productivity usually benefits both countries, and where an analysis that does not consider technological limits can often be used, and (2) the subregion of maximal productivity itself where increases in productivity are usually constrained in one country or the other by maximal productivity limits and where industry productivity increases in one country will tend to cause shifts in industries, utility and relative national income.

The effect of these shifts depends strongly on relative income as we see in Figure 5.2. Figure 5.2 shows that if Country 1 can *increase* its share anywhere within the region of maximal productivity to the left of its peak, it will generally increase its utility and *decrease* that of its trading partner. However if its relative national income is sufficiently large, or equivalently if its trading partner has a relatively small share, so Country 1 is in the region to the right of its peak, *decreases* in its share generally *increase utility for both countries* since the regions of both countries slope down to the right.

So within the subregion of maximal productivity itself we have a further division (Figure 5.2) into the *region of conflict* and the *regions of cooperation*. The region of conflict lies between the two peaks, where gain in one country usually comes at the expense of the other. The the regions of cooperation lie outside the two peaks, there shifts in relative national income tend to either benefit both countries or harm them both.

The effect of loss or gain of industries does not usually depend on the distinction between import and export oriented industries. In the region to the right of Country 1's peak, improvements in Country 2's productivities, that enable it to take over industries in which it formerly imported from Country 1, benefit both countries. The improvement in the cost of those goods, because they are now made by the low wage Country 2, overwhelms the effect on Country 1 of reduced share or, equivalently, shifting exchange rates. In contrast, between the two peaks the relationship is reversed. The increases in Country 2's productivity that enable it to export what it formerly imported still benefit it and still make those newly exported goods cheaper, but the effects on exchange rate, by making Country 2's other exports more expensive, are such as to produce a net loss in Country 1's welfare.

In the regions of cooperation, where one country tends to be highly developed while the other is relatively undeveloped, the development of the relatively undeveloped country helps both. Between the peaks however, where countries are more similar to each other in their stage of development, we see the inherent conflict in international trade.

VI. The Effect of Different Maximal Productivities and Country Sizes on the Shape of the Region.

So far we have confined ourselves to quite general results, results that depend only on the general regional shape. In this section we take a more detailed approach and consider the effect on the regional shape and its economic consequences of different sets of maximal productivities ϵ^{max} and different country sizes. We have seen that analyzing regional boundaries gets simpler as we move from examining the exact boundaries of small models to looking at the approximate boundary for large models. It gets simpler still if we follow Dornbush, Fischer, and Samuelson [1977] in assuming symmetric demand, $d_i=d_{i,1}=d_{i,2}$, and going to a continuous model with an infinity of goods each infinitesimally small.

Under the assumptions of symmetric demand and a continuous model we will give a remarkably complete characterization of the possible regional shapes. We will obtain explicit formulas for the boundary curves. We will show simple graphical methods for locating the regional peaks that allow us to see the effect of different maximal productivities on peak location and hence on the regions of conflict and cooperation. We will describe the dependence of the regional shapes on country size. We will see that the magnitude of peak gains often exceed the classical gains by more than the amount that the classical gains exceed autarky, and that, to a good approximation, peak location can be determined from knowledge of the exchange rate alone.

All this requires some preparation.

Preliminaries - Symmetric Demand and Normalized Productivities: When we described the method of solution of (3.3) we observed that, for symmetric demand only, the solution x is obtained by assigning industries to Country 1 in the order of their comparative advantage until the national income Z_1 is reached. To exploit this property we (1) normalize maximal productivities and (2) introduce a new diagram, the e-Z_1 diagram.

We normalize the maximal productivities in each industry by dividing each $e^{max}_{i,j}$ by $(e^{max}_{i,1}+e^{max}_{i,2})$. The new normalized productivities cannot exceed 1 and for normalized productivities $e^{max}_{i,1}+e^{max}_{i,2}=1$. A normalized productivity of .95 for County 1 in an industry implies that Country 2 has a normalized productivity of .05, so Country 1 has a 20:1 comparative advantage in that industry. The advantage of the normalization is that arranging the normalized $e^{max}_{i,1}$ in order of decreasing size now arranges them in order of decreasing comparative advantage for Country 1. We no longer have to consider the $e^{max}_{i,2}$. Normalization changes the scale of the utility by a fixed amount and does not affect the shape of the region. In fact, since we plot utility in autarky units, the regional diagrams are completely unchanged.

Next we introduce the e-Z_1 diagram in which we plot productivity vertically against Z_1 horizontally. In Figure 6.1a we plot the normalized $e^{max}_{i,1}$ values from left to right in descending order, which is the order of comparative advantage, and give each the horizontal length d_i that represents the demand. Since we have symmetric demand, the $d_i=(d_{i,1}Z_1+d_{i,2}Z_2)$ also represent the total revenue for each industry. The maximizing x can be read from the diagram for any Z_1 by drawing a vertical line through Z_1 and giving Country 1 all the industries to the left of that line. The shared or special industry is the one whose horizontal segment is pierced by the vertical line at Z_1 and its revenue is divided between the two countries.

Preliminaries - The Continuous Model and Three Maximal Productivity Curves: Now we move to a continuous model in which the industries become very fine grained. Instead of the descending succession of horizontal line segments $e^{max}_{i,1}$ of Figure 6.1a, we have a monotone decreasing curve[12] $e=e^{max}_1(Z_1)$. The three dashed plots, E_1, E_2 and E_3 of Figure 6.1b, represent three such maximal productivity curves. Each curve $e=e^{max}_1(Z_1)$, like the set of horizontal segments $e^{max}_{i,1}$ in Figure 6.1a, contains not only the productivity information, but also, in its slope, the demand information as well. Instead of a special or shared industry $e^{max}_{k,1}$ we have the e-value $e=e^{max}(Z_1)$ which divides the industries assigned to Country 1 from those assigned to Country 2 at the boundary point $B_1(Z_1)$.

Just as in the n-industries case we can convert this assignment of industries into an equilibrium by changing some of the productivities to produce stability. The procedure is

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[12] We will use the general word "curve" since any descending curve is allowed. Our examples, for simplicity, are made up of connected straight line segments.

virtually the same as in the n-industries case so we will postpone a more detailed description of it until we can describe the structure of peak equilibria which we will do below.

While we can deal perfectly well with arbitrary functions $e^{max}{}_1(Z_1)$, which produce arbitrary descending curves in our e-Z_1 diagram, for concreteness in our analysis we will discuss the three maximal productivity functions of Figure 6.1b. The first, E_1, is a straight line of uniformly decreasing ultimate comparative advantage, Country 1 has an enormous, almost infinite, comparative advantage near the left edge which decreases to the same overwhelming advantage for Country 2 near the right edge. E_2 has a first section where Country 1 has significant greater productivity potential than Country 2, perhaps attributable to its natural resources, then there is a middle section where the countries productivity potentials are not very different from each other. Perhaps this section represents industries like textiles, autos, or semiconductors which depend on skills and knowledge that can be imported or otherwise acquired by either country. Then there is a final section where Country 2 has significantly more potential than Country 1. The third curve, E_3 is the curve for countries with identical potentials $e^{max}{}_1(Z_1).=e^{max}{}_2(Z_1)$.

The Derivative of the Boundary Curve for the Continuous Model: In Appendix D we derived a general formula for the derivative of the country boundary curves with non-symmetric demand. In Appendix E.1 we simplify this result to the present case and extend it to the world boundary. The resulting derivatives are

(6.1)

$$\frac{d\ lnB_1(Z_1)}{dZ_1} = \frac{1}{Z_1} + \ln\frac{L_1 e^{max}{}_1(Z_1)/(Z_1)}{L_2(1-e^{max}{}_1(Z_1))/(1-Z_1)}$$

$$\frac{l\ lnB_2(Z_1)}{dZ_1} = \frac{-1}{1-Z_1} + \ln\frac{L_1 e^{max}{}_1(Z_1)/(Z_1)}{L_2(1-e^{max}{}_1(Z_1))/(1-Z_1)}; \qquad \frac{d\ lnB_W(Z_1)}{dZ_1} = \ln\frac{L_1 e^{max}{}_1(Z_1)/(Z_1)}{L_2(1-e^{max}{}_1(Z_1))/(1-.}$$

(6.1) will help us in two ways. (1) For any given $e^{max}{}_1(Z_1)$, the derivatives in (6.1) become functions of Z_1 only. Therefore, we can integrate them and obtain explicit formulas for the boundary curves $B_1(Z_1)$, $B_2(Z_1)$, $B_W(Z_1)$. We will see these formulas below. (2) For equal size countries we will construct from the three equalities in (6.1) three special curves which we will plot in the e-Z_1 diagram. The intersection of *any* maximal productivity curve $e^{max}{}_1(Z_1)$ with these three fixed curves will locate the Country 1, Country 2, and World peaks associated with that $e^{max}{}_1(Z_1)$.

Equal Size Countries, Locating Zc and the Country Peaks: To separate the effect of different maximal productivity functions $e^{max}{}_1(Z_1)$ from the very strong effects of country size, we first assume both countries have the same size ($L_1=L_2$) and then analyze the effect of country size separately below.

We are now ready to find the classical level and the regional peaks. We can do this in an interesting way. Consider the expression for the derivative of $lnB_1(Z_1)$ from (6.1) simplified by $L_1=L_2$. This is:

(6.2) $$\frac{d\ lnB_1(Z_1)}{dZ_1} = \frac{1}{Z_1} + \ln\frac{e^{max}{}_1(Z_1)/(Z_1)}{(1-e^{max}{}_1(Z_1))/(1-Z_1)} = \Psi_1(e^{max}{}_1(Z_1),\ Z_1).$$

Let us add to the e-Z_1 diagram the curve P_1 determined by $\Psi_1(e, Z_1)=0$, ,which we will call the Country 1 Peak Determining Curve. P_1 appears in Figure 6.2b where we can see its rather special shape. We assert that:

Theorem 6.1 - Intersection Theorem: If in the e-Z_1 diagram the intersection of $\Psi_1(e, Z_1)$ with the maximum productivity curve $e^{max}{}_1(Z_1)$ is the point (e,Z_1), then that Z_1 is the peak Z_1 value of the boundary curve $B_1(Z_1)$.

Proof: At the intersection (e,Z_1) we have both $e=e^{max}{}_1(Z_1)$ since the intersection lies on the maximum productivity curve, and $\Psi_1(e, Z_1)=0$ because the intersection lies on P_1. Therefore, Z_1 satisfies $\Psi_1(e^{max}{}_1(Z_1),Z_1)= 0$. But from (6.2) $\Psi_1(e^{max}{}_1(Z_1),Z_1)$ is the derivative of $\ln B_1(Z_1)$ the Country 1 boundary curve. Therefore, at Z_1 the boundary curve $B_1(Z_1)$ has a zero derivative and, by the Country Peak Theorem, Z_1 must be the peak Z_1 value for Country 1.

We can produce a similar curve P_2 for the Country 2 peak, and another curve (actually a straight line) P_W for the world boundary peak. All three peak determining curves are shown in Figure 6.2b together with E_1, E_2, and E_3. The successive intersections of (for example) E_1 with P_1, P_2, and P_3 determine the Country 2 peak, the World peak, and the Country 1 peak for the boundary determined by E_1

In Appendix E.1 we also showed that the peak of the world boundary curve occurs at Z_C the classical level. Therefore, the intersection of any maximal productivity curve with the curve P_W gives us its Z_c. We illustrate this for E_2 in Figure 6.2a.

Unit Size Countries,The Effect of Different Maximal Productivities: The regions of cooperation and of conflict are determined by the location of the Country 1 and Country 2 peaks which we can now find easily.

As we can see from Figure 6.2b, a rapid drop in Country 1's maximal productivity curve (and therefore its comparative advantage) between its intersection with P_2 and its intersection with P_1 tends to pull the peaks together. This contracts the range of conflict and increases the range of cooperation. On the other hand, ranges of large comparative advantage for one country or the other, separated by a range of industries in which both countries have little natural advantage, as in curve E_2, gives more widely separated peaks. To the extent that modern goods depend more on skill and knowledge, and less on natural resources, the $e^{max}{}_1(Z_1)$ curves become flatter and more like E_2 or E_3 and produce wider peak separation.

The economic consequences of moving from E_1 to E_2 or E_3 are (1) an enlargement of the region of conflict between the peaks and a decrease in the region of cooperation outside the peaks, and (2) a decrease in the gains from trade because of the decreases in possible comparative advantage. In Figure 6.2c we show R_1, Country 1's region and R_2, Country 2's region for E_1, E_2, and E_3. The black dots indicate the R_1 and R_2 peaks and, therefore, the boundary between the region of conflict and the region of cooperation. The widening of the region of conflict and the decrease in gains from trade are both apparent.

Although this describes the direction of change with different maximal productivities, another point that emerges from looking at the intersections in Figure 6.2b is the relative insensitivity of the location of the peaks to significant changes in maximal productivities. The e-Z_1 diagram suggests that for a wide range of capabilities the peak for Country 1, for example, is always in the range of Z_1=.65 to .75. With $L_1=L_2=1$, the real exchange rate is $w_1/w_2=Z_1/Z_2 =Z_1/(1-Z_1)$, that is, the exchange rate equals the real relative wage. Then we can say that, for a wide range of capabilities, a real exchange rate or relative wage of between 2:1 and 3:1 is likely to give Country 1 its maximal gains from trade. Similarly an exchange rate of w_2/w_1 between 2:1 and 3:1 is likely to give Country 2 its maximal gains from trade. However, both countries cannot simultaneously have this exchange rate (or per capita income) advantage. This is another way of looking at the element of conflict that exists in international trade.

Fully Normalized Productivity Curves: All three of our standard productivity curves describe BP families whose maximal productivities $e^{max}{}_1(Z_1)$ and $e^{max}{}_2(Z_1)$ are in some sense balanced. In all three cases, when both countries are fully developed, the resulting equilibrium, the classical point, gives each country half of world production. Neither country, when fully developed, dominates the other. This shows up in the e-Z_1 diagram where all three productivity curves cross the P_w curve at Z_1=.5, which implies that the world peak, and, therefore, the classical level, are at Z_1=.5. We will call these balanced productivity curves, curves with $e^{max}{}_1(.5)$=.5, *fully* normalized productivity curves.

We can expect fully normalized production curves to play a major role in the modern world. In any trade situation involving both goods in which one country has a natural advantage, and also goods whose methods of production can be learned and perfected in either country, we should see successively (1) a first part of the productivity curve where Country 1 has an inherent productivity advantage (normalized productivity >.5), then (2) a range of industries in which neither country has an in inherent advantage, (normalized productivity at or near .5), and then (3) a range of industries where Country 2 has an inherent advantage (normalized productivity <.5). If the industries that can be learned and perfected play a significant role, the flat middle part of this curve will be large and will be the part that intersects the world line P_w., producing a fully normalized productivity curve.

Fully normalized productivity curves also play a special role in this theory. We will show below that all the regions resulting from productivity curves that are *not* fully normalized are very simply related to the regions obtained from BP families with fully normalized production curves.

Before we analyze these more general regions we will observe that our productivity plot gives us not only the location of regional peaks but also a rapid way to read out the structure of boundary equilibria.

Reading Out the Structure of Boundary Equilibria: We can quickly read out from the e-Z_1 diagram the industry assignments x, and the productivities ϵ, of any boundary equilibrium. We will illustrate this process in Figure 6.3 by finding the structure of the Country 1 peak produced by the maximal productivity curve E_2

The intersection of E_2 with P_1, marked x in Figure 6.3, determines the Z_1 value, $Z^p{}_1$ of the Country 1 peak. We put a vertical line V_1 through that intersection and also a horizontal line H_1 through the intersection of V_1 with P_W. The two lines V_1 and H_1 divide the plot into four quadrants. These four quadrants determine the assignments x and the productivities ϵ of the equilibrium.

The assignments and productivities are determined as follows: (1) The industries in the upper left quadrant are assigned to Country 1 and for these $e_1(Z_1)=e^{max}_1(Z_1)$, and $e_2(Z_1)=e^{max}_2(Z_1)$. (2) The industries in the lower right quadrant are assigned to Country 2 with $e_1(Z_1)=e^{max}_1(Z_1)$, and $e_2(Z_1)=e^{max}_2(Z_1)$. (3) The industries in the lower left quadrant are also assigned to Country 1, but for these industries Country 2 needs to be incompletely developed so that Country 1 can be the producer. Giving Country 2 a productivity of zero in these industries will certainly accomplish that, but any values $e_2(Z_1)$ with $e_2(Z_1)/w_2 \le e^{max}_1(Z_1)/w_1$ will do just as well. This completely describes the equilibrium structure for the E_2 peak.[13]

In this case we see that Country 1's ideal trading partner, the country that provides peak gains from trade to Country 1, is one that is fully developed in the industries in which it has strong natural advantages, and underdeveloped in those that in which Country 2 is more closely competitive.

Utility at the Boundary Points: For unit sized countries we have been able to locate the peaks and determine their structure and those of any boundary points. Both for unit sized countries and for countries of arbitrary size we will also need to determine the utility of boundary points. Remarkably enough, the assumption of symmetry, by giving us simple expressions in (6.1) enables us to obtain exact formulas giving us the utility of boundary points. We obtain these formulas by integrating the derivatives in (6.1).

The first formula we obtain is for the world boundary. In stating the formulas we will use $B_w(Z_1, e^{max}_1, e^{max}_2, L_1, L_2)$ to indicate the world boundary for the BP family with productivities e^{max}, and e^{max}_2, and country sizes L_1 and L_2. We will use $A_1(e^{max}_1, L_1)$ to denote for that family the utility of Country 1 in autarky. We will use S^1_w for world utility measured in Country 1 autarky units -- this is B_w/A_1. The formula, which we derive by integrating the world boundary derivative from (6.1), is:

[13] Here is the reasoning that leads to this process. Industries above the horizontal line H_1, which is the line $e=Z^P_1=w^P_1$, have $e^{max}_1(Z_1)>w^P_1$, hence with our normalizations, $e^{max}_2=1-e^{max}_1<1-w^P_1=w^P_2$.
Combining these gives $e^{max}_1/w^P_1>e^{max}_2/w^P_2$. This means that the classical assignment assigns these to Country 1. Similarly those below H_1 are assigned by the classical assignment to Country 2. But to the left of V_1 all industries must be assigned to Country 1, therefore, those in the lower left quadrant must be switched over to Country 1 by lowering Country 2's productivity in these industries. This explains the process for a Country 1 peak. If the productivity curve had passed through the upper right quadrant instead of the lower left, as would have happened if we had chosen to examine the peak of Country 2 instead of the peak of Country 1, the industries in that quadrant would have to be switched from Country 1 to Country 2, and to make this happen we would decrease the Country 1 productivity in those industries.

$$B_w(Z_1,e^{max}{}_1,e^{max}{}_2,L_1,L_2) = A_1(e^{max}{}_1,L_1)\ S^1{}_w(Z_1,e^{max}{}_1,e^{max}{}_2,L_1,L_2)$$

(6.3)
$$\text{with}\quad S^1{}_w = \left(\frac{L_2}{L_1}\right)^{Z_2} (1/Z_1)^{Z_1}\ (1/Z_2)^{Z_2}\ \exp\left(\int_{Z_1}^{1} \ln \frac{e^{max}{}_2(S)}{e^{max}{}_1(S)}\ dS \right).$$

The Country 1 utility boundary, as we know from is simply the expression for the world boundary in (6.3) multiplied by Z_1. So $B_1=Z_1B_w=A_1(Z_1S^1{}_w)=A_1S^1{}_1$, where $S^1{}_1$ denotes the Country 1 boundary curve in Country 1 autarky units. When we plot the boundary curves in Country 1 autarky units, as we usually do, we are simply plotting the world curves $S^1{}_w$ or the Country 1 curves $S^1{}_1$.

We will also need world utility expressed in terms of Country 2's utility in autarky, A_2, so we need the corresponding formulas:

$$B_w(Z_1)= A_2(e^{max}{}_2,L_2)\ S^2{}_w(Z_1,e^{max}{}_1,e^{max}{}_2,L_1,L_2)$$

(6.4)
$$\text{with}\quad S^2{}_w = \left(\frac{L_1}{L_2}\right)^{Z_1} (1/Z_1)^{Z_1}\ (1/Z_2)^{Z_2}\exp\left(\int_{0}^{Z_1} \ln\frac{e^{max}{}_{i,1}(S)}{e^{max}{}_{i,2}(S)}\ dS \right).$$

Again Country 2's utility boundary B_2 and Country 2 curve $S^2{}_2$ are simply the expressions in (6.4) multiplied by Z_2.

Now that we have our utility formulas we will begin to deal with general BP families. We start by analyzing the effect on the regional shape of a uniform across the board increase in productivity.

Uniform Increases in Productivity: Both Hicks [1953] and Dornbush, Fischer and Samuelson [1977] conclude that a uniform across the board increase in the productivity of a trading partner is beneficial for both countries. Since their result applied to each individual equilibrium we can hope to find some fairly uniform change in the region of equilibria if we replace Country 1 by a new Country 1 whose maximal productivities have all been increased by a uniform multiple $\lambda>1$.

So we now introduce a new BP family whose Country 1 maximal productivities have been multiplied by λ while those of Country 2 remain the same. Direct substitution in (6.3) and (6.4)[14] shows that the new region has boundary curves $B^*{}_w=A^*{}_1S^*{}_w$ with

(6.5)
$$\frac{B^*{}_w}{A_1}=\lambda^{Z1}\ S^1{}_w\ ,\qquad \frac{B^*{}_w}{A^*{}_1}=\left(\frac{1}{\lambda}\right)^{Z2}\ S^1{}_w\ ,\ \text{and}\qquad \frac{B^*{}_2}{A_2}=\lambda^{Z1}\ S^2{}_w.$$

In Figure 6.4, B_1 and B_2 are the regional boundaries based on the productivity curve E_1, and $B^*{}_1$

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[14] This substitution shows that the new autarky level $A^*{}_1$ and the new curve shape $S^{*1}{}_w$ are given in terms of the old by

$$A^*{}_1 = \lambda\ A_1(e^{max}{}_1,L_1)\quad \text{and}\ S^{*1}{}_w = (1/\lambda)^{Z2}\ S^1{}_1(e^{max}{}_1,e^{max}{}_2,L_1,L_2).$$

and B^*_2 are the boundaries when the new Country 1 is 4 times more productive.

The first equation in (6.5) shows that, because of the multiplication by λ^{Z1}, the new Country 1 now has a region allowing more total utility than the original Country 1 when measured in original Country 1 autarky units. Nevertheless, its boundary curve B^*_1 is lower than B_1 in Figure 6.4. This is because, as usual, we have plotted Country 1 utility in Country 1 autarky units, and the second equation in (6.5) shows that its gains from trade have decreased relative to its new and higher autarky level. The decrease ranges from a factor of $1/\lambda$ at the left edge to no change at the right edge. This decrease means that trade has become less important to Country 1.

However, both Figure 6.4, and the third equation in (6.5), also show that a productive Country 1 opens up the possibility of extremely large gains from trade for Country 2 relative to its unchanged autarky level $A_2=A^*_2$. Multiplying the Country 2 boundary by λ^{Z2} increases the height of the boundary. This increase evolves steadily from no change at the extreme left to a factor of λ at the right edge. The gains for Country 2 will be especially large if Country 2 can obtain its peak gains. If Country 2 can trade with a partially developed Country 1 that is very productive in the few industries in which it is actually the producer, it can obtain enormous benefits.

In the modern world it is more difficult than before to talk about a "potentially rich country" and to find examples of a new Country 1 with four times the maximal productivities of the original. However, what we will see next is that different country sizes (measured by the size of the labor force) produce exactly the same effects as different maximal productivities, and even the modern world has no shortage of countries that are of different sizes.

Effect of Country Size: As we would expect, especially from Dornbush, Fischer, and Samuelson [1977], country size has a strong effect. The effect of country size is in fact almost indistinguishable from the case of a uniform increase in productivity. This is plausible since it is a well known observation that we can change units of labor and of productivity together. If we are measuring labor input in person hours, we can change to measuring the input in terms of an hour's work of a group of 10 provided we increase the productivity by 10 also. So a uniform increase in productivity is equivalent at the total country level to an increase in country size. The two changes are different if instead of total income we consider exchange rate or per capita income since the total national income in a small very productive country is spread over fewer workers than the same national income obtained from a large unproductive country. So our figures, which represent total national income level, will respond the same way to an increase in size and an increase in productivity, but when we translate Z_1 into wage $w_1=Z_1/L_1$ we will get very different results.

In fact, as we can easily verify by direct substitution in (6.3) and (6.4), an increase in Country 1 size by a factor λ yields new regions whose boundaries are given by the equations in (6.5). Therefore, we can reinterpret Figure 6.4 as giving the effect of an increase in the size of Country 2 rather than as giving the effect of an across-the-board increase in productivity.

We can summarize the reinterpretation as follows: A small Country 1 trading with a large County 2 has the possibility of very large gains from trade, especially if the large Country 2 is relatively undeveloped. However, in accord with common sense, Country 2 matters little to the large Country 1 if Country 1 can ever become fully developed.

General Regional Shape: We have discussed the effect on the regions of simple across-the-board changes in productivity curves, and also of changes in country size. If we use this together with our knowledge of fully normalized BP families (families having completely normalized productivity curves) we can elucidate the general structure of all possible regions.

Our first theorem states that the world boundary for any BP family is closely related to the boundary obtained from a fully normalized BP family with unit size countries. In fact, it is obtained from the completely normalized BP family's boundary by multiplying that boundary by β^{Z2} where β is an easily determined constant. Furthermore, the completely normalized production function that is used in the BP family is also easily obtained from the original maximal production curve by dividing by a constant and renormalizing. More precisely:

Theorem 6.2 - General Regional Shape Theorem: For any BP family with parameters e^{max}_1, e^{max}_2, L_1, and L_2, the world boundary curve in Country 1 autarky units is $S^1_w = \beta^{Z2} S_w(en^{max}_1, en^{max}_2, 1, 1)$. Here $S_w(en^{max}_1, en^{max}_2, 1, 1)$ is the world curve for unit sized countries using the fully normalized en^{max}_1. This en^{max}_1 derived from e^{max}_1 by dividing by $e^{max}_1(.5)$ and renormalizing is:

$$en^{max}_1(Z_1) = \frac{e^{max}_1(Z_1)/e^{max}_1(.5)}{e^{max}_1(Z_1)/e^{max}_1(.5) + e^{max}_2(Z_1)/e^{max}_2(.5)},$$

and β is the constant $\dfrac{L_2\ e^{max}_2(.5)}{L_1\ e^{max}_1(.5)}$.

It is β that makes the adjustment both for different country sizes and for different overall productivities. Clearly there is a similar statement for the world boundary in Country 2 autarky units. And, as we would expect, the country boundaries for Countries 1 and 2 are obtained from the world boundaries by multiplying by Z_1 and Z_2 respectively. We give a derivation of Theorem 6.2 in Appendix E-2

An equivalent form of the theorem, which we will use in the application below, is one that expresses any boundary curve, whatever the actual size of the countries, in terms of unit sized countries and a changed, but not fully normalized, maximal productivity curve. In this case, there is no term β^{Z2} to affect the shape of the boundary curve.

Theorem 6.2a.-Second Regional Shape Theorem: For any BP family with parameters e^{max}_1, e^{max}_2, L_1, and L_2, the world boundary curve in Country 1 autarky units is $S^1_w = S_w(eb^{max}_1, eb^{max}_2, 1, 1)$, with

$$eb^{max}_1(Z_1) = \frac{(1/\beta)en^{max}_1(Z_1)}{(1/\beta)en^{max}_1(Z_1) + en^{max}_2(Z_1)}.$$

In this form of the theorem the countries are still unit sized, but the factor β has been absorbed into the production function, which, as a result, is no longer fully normalized. The advantage of this form is that, since we have S^1_w exactly equal[15] to a boundary curve involving only unit size countries, we can use our peak diagram (Figure 6.2b). We can find all the peaks by intersecting the changed (but easily obtained) production functions eb^{max}_1 with P_w, P_1 and P_2. We derive this form too in Appendix E-2.

As an application of this general theorem we will find approximate peak exchange rates for a Country 2 trading with a large Country 1.

The Peak Exchange Rate when Trading with a Large Trading Partner: Knowing the peak exchange rate, even approximately, enables each country to know where it is in its regional diagram. If the current exchange rate is more favorable to Country 2 than its peak exchange rate, then the current trade regime is in the region of cooperation and Country 2 will benefit from Country 1's increased productivity. If the exchange rate is less favorable than Country 2's peak rate, the countries are in the region of conflict and Country 2 is likely to lose from Country 1's increases in productivity.

We start with a productivity curve $e^{max}_1(Z_1)$. Let us assume that $L_2=1$ and that L_1 is the larger trading partner, so $L_1 \geq 1$. In Figure 6.5 we show three productivity curves E^*_1, E^*_2, and E^*_3. Each E^*_i is derived from one of our standard fully normalized productivity curves E_1, E_2, and E_3, using Theorem 6.2a with a $\beta=1/4$. This is the β we would have, for example if Country 1 were four times the size of Country 2 and the productivity curve $e^{max}_1(Z_1)$ was fully normalized so $k=e^{max}_2(.5)/e^{max}_1(.5)= 1$. The E^*_i are the eb^{max}_i of Theorem 6.2a, and the E_i are the en^{max}_1. Whatever our original e^{max}_1 may have been, fully normalized or not, its fully normalized version en^{max}_1 is likely to lie somewhere between E_1 and E_3, and therefore the corresponding eb^{max}_1 would lie between E^*_1 and E^*_3. Then, as we can see from Figure 6.5, eb^{max}_1 would intersect P_2 somewhere between .4 and .5, locating the Country 2 peak between those Z_1 values.

Since $w_2/w_1= Z_2L_1/Z_1L_2$, this locates the peak exchange rate w_2/w_1 in the interval $(.5/.5)L_1 \leq w_2/w_1 \leq (.6/.4)L_1$. Since $\beta=1/4$, we have $1/4= k/L_1$ or $L_1=4k$, so that interval is $4k \leq w_2/w_1 \leq 6k$. As we have argued before in discussing fully normalized productivity curves, in a modern world we are not likely to see a k very different from 1, so in fact the peak exchange rate for Country 2 is likely to be somewhere between 4:1, and 6:1.

We can repeat this argument with ever larger L_1's. If we do we will find that Country 2's peak exchange rate continues to increase steadily with the size of Country 1. When Country 1 is 10 times the size of Country 2, and k is near 1, Country 2's peak is obtained at an exchange rate in the neighborhood of 9. A good approximation over a fairly wide range of Country 1 sizes is that w_2/w_1 at the Country 2 peak is $L_1(2/3 + 5/3L_1)k$. We, therefore, have the following observation: *when Country 1 is much larger (or inherently more productive) than Country 2, Country 2's peak is attained at wages that are a large multiple of the wage in Country 1.*

This contrasts strongly with the exchange rate obtained when both countries are fully

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[15] We are referring to the fact that there is no longer the factor β^{Z2} to reshape the curve as in Theorem 6.2.

developed. This can be shown by the same methods always to remain not far from 1 as Country 1 increases in size. In this case we have the following observation : *as Country 1 approaches full development the real wages in the two countries will approach closely to one another, as real wage in country 2 declines*[16] *and that in Country 1 simultaneously increases.*

This more general analysis is consistent with what we have shown earlier with fully normalized production curves and unit sized countries. With a disparity of size the effects seen there become even more pronounced. A small country trading with a large underdeveloped country can obtain far greater gains from trade at its peak than it can obtain when its partner is fully developed, and to obtain these gains it requires an exchange rate that is very strongly in its favor.

<u>The Importance of Peak Gains from Trade</u>: Our methods allow us to find the peak and classical point locations for a wide range of country sizes. We can then obtain from (6.3) the gains from trade, both at the classical point and at the peaks and see directly how large both gains are.

In Figure 6.6 we have plotted utility for Country 1 at the classical point, and also at the Country 1 peak, for country sizes L_1 ranging from $L_1=.2$ and $L_2=.8$, to $L_1=.8$ and $L_2=.2$. This covers country size ratios ranging from 1:4 to 4:1 with equal sized countries at $L_1=.5$. As usual, the utilities are in Country 1 autarky units, so for each curve it is the vertical height above 1 that measures gains from trade. The solid curves C_1, C_2, and C_3 show the utility at the classical point using productivity curves E_1, E_2 and E_3. The dashed curves P_1, P_2 and P_3 show the corresponding utilities at the Country 1 peak. We observe that: *Over a wide range of country sizes the peak gains are a very considerable addition to the classical gains from trade, often exceeding the classical gains by more than the amount that the classical gains exceed autarky.* The relative importance of peak gains increases strongly as we move from E_1 to E_2 to E_3.

The picture that emerges from this analysis, from the point of view of a developed country, is that in the early stages of the development of its trading partner, the developed country realizes very large gains from trade, which then diminish appreciably as the trading partner becomes more developed. Knowledge of the exchange rate enables a country to know roughly where it is in this process. These peak gains are considerable over a wide range of productivities and country sizes, and they become even stronger as trade becomes less natural resource based and more based on skills and knowledge.

In this section we have provided the tools that permit a detailed exploration of possible regional shapes and boundary equilibrium structures. Next we will discuss, in a less detailed way, the underlying economics that lies behind the basic regional shape.

<u>VII. Some Underlying Economic Factors and a Connection With Economies Models</u>

Most of the work we have done so far is quite quantitative. We calculate actual regional

[16]The curves of Figure 6.6 described below give us some measure of that decline.

boundaries, we prove rigorously that the rise to the peak is monotone, we can be quantitative about peak gains from trade. In this section we will take a different tack. We will attempt to give some intuitive economic reasoning that bring our results in line with ordinary economic intuition.

The simplest intuitive explanation of the characteristic regional shape involves the generally dome-like shape of the world utility frontier. If we have a roughly dome-like world utility we can see why, as Country 1's share grows it should at first gain utility, it is getting a larger share of an increasing world output. We can see why it would later lose utility as it gets an ever larger share of a world output that is now decreasing toward the autarky level. But why, in economic terms, would we expect the world region to be dome shaped? One reason comes immediately to mind. The allocation of fair shares of industries to each country allows the exploitation of comparative advantage far better than when one country does most things for itself. We would expect this for any utility, not only Cobb-Douglas.

Certainly we would expect comparative advantage to play a role but there is also something else at work. Both our formulas and our figures show that even when the two countries in the BP model are identical i.e. $e^{max}{}_1(Z_1)=e^{max}{}_2(Z_1)$ we still have the same characteristic regional shape.

The second economic influence that contributes to the dome shape of the world utility frontier is the diminishing marginal rate of substitution usually assumed for consumer utilities. This indicates that the world's consumers will have a relatively low valuation of a boundary equilibrium near the extreme right or extreme left of the graph. In such an equilibrium the country with small share specializes in a very few goods, producing large quantities of this small number of items because it uses its entire labor force on them. The other country divides up its labor force giving a small amount of its labor to each of its many industries, so only small quantities of these goods will be produced. Diminishing marginal rate of substitution implies that consumers will prefer a more balanced bundle of outputs.

Neither of these arguments depend on special properties of Cobb-Douglas demand. So that it seems likely that these two economic factors, comparative advantage and diminishing marginal utility, will produce the same effect in models with other utilities.

However, there are limits to how far these general economic arguments can go. The arguments we have just given make the dome shape, and, therefore, the element of conflict in the regional boundaries of the two countries, seem plausible. But these arguments apply to small models as well as to large, and while it is true that the world frontier of Figure 3.1a is monotone ascending to a peak, the economic consequences of that is not conflict but rather a central peak that is good for both countries. Also, with the detailed knowledge that we have about the effect of different maximal productivity choices on large models, we can also construct, even for large models, extremely artificial choices of productivity parameters that eliminate the characteristic two peak shape by merging the two peaks at Z_C [17]. There is room f or much more exploration of

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[17]For example if $e^{max}{}_1(Z_1)$ intersects P_2 and then plummets vertically down to P_1 the two peaks will come together.

the demand structures and productivities that produce a dome shaped world utility with separated peaks for the two countries.

A third general economic factor that would push the world frontier toward a dome shape is economies of scale. Since we have linear production functions, economies are not present in our models, but if they were they would add further to the dome shape, since economies tend to be reduced when one country makes almost all the goods. There is in fact a very close tie between the region of equilibria that we have discussed and the multiple equilibria and the regional shape that emerges in the presence of scale economies Gomory [1994]. We now explain that connection.

VIII. The Correspondence Principle.

The Correspondence Principle, indicating that the same equilibrium can arise in both a linear and a scale-economies model, suggests itself in the following way. A given *specialized* equilibrium can be stable for two very different reasons. In a model with linear production functions, it can be stable because the $e_{i,j}$ satisfy the stability conditions (1.5), with the producing industries the low-cost producers. Alternatively, an equilibrium can be stable because its production functions have scale-economies. These stabilize the specialized equilibrium by preventing new producers from entering industries on a small scale, where there is an established large-scale producer.

This suggests that the same specialized equilibrium with the same assignments of industries and perhaps even the same output of goods can be obtained from a linear model and from a model with scale economies. To show this we must define our scale-economies model and its equilibria.

The Scale-Economies Model and its Equilibrium Conditions. We say that a *scale-economies model $M(f_{i,j})$ corresponds to a linear BP family model* if it has the same labor-force sizes L_1 and L_2 and the same country demand values $d_{i,j}$. However, instead of linear production functions $e_{i,j}l_{i,j}$, the model $M(f_{i,j})$ has production functions $f_{i,j}(l)$ with economies of scale, defined as non-decreasing average productivity, $f_{i,j}(l)/l$. We assume that there is a well defined derivative $df_{i,j}(l)/dl$ at l=0, and that $f_{i,j}(L_j)/L_j$, which is the largest productivity value that $f_{i,j}(l)/l$ can attain in the model, is $e^{max}_{i,j}$.

We adapt the equilibrium requirements (1.1)-(1.5) for this model. The conditions (1.1)-(1.4) can be retained unchanged; we need only remember that $q_{i,j}$, the quantity produced, now equals $f_{i,j}(l_{i,j})$ not $e_{i,j}l_{i,j}$. The conditions (1.5) that stabilize the equilibria also translate easily. If there are two producers of good I, we require them to have the same average cost, and we do not allow a non-producer to have an average cost on entering lower than the current producer. The average productivity for small scale entry is $df_{i,j}(0)/dl$, so (1.5) becomes:

$$\text{(8.1)}\qquad \begin{aligned} &\textit{if } x_{i,1}{>}0 \quad \textit{and} \quad x_{i,2}{=}0 \quad \textit{then} \quad \frac{f_{i,1}(l_{i,1})}{l_{i,1}w_1} \geq \frac{df_{i,2}(0)/dl_{i,2}}{w_2} \\ &\textit{if } x_{i,2}{>}0 \quad \textit{and} \quad x_{i,1}{=}0 \quad \textit{then} \quad \frac{df_{i,1}(0)/dl_{i,1}}{w_1} \leq \frac{f_{i,2}(l_{i,2})}{l_{i,2}w_2} \\ &\textit{if } x_{i,2}{>}0 \quad \textit{and} \quad x_{i,1}{>}0 \quad \textit{then} \quad \frac{f_{i,1}(l_{i,1})}{l_{i,1}w_1} = \frac{f_{i,2}(l_{i,2})}{l_{i,2}w_2}. \end{aligned}$$

Conditions (1.1)-(1.4) and (8.1) are equilibrium conditions for a stable zero-profit equilibrium. The equilibria (x,Z_1) of such an economies of scale model $M(f_{i,j})$ can be very numerous.

An important special case occurs when all the production functions $f_{i,j}(l)$ entail startup costs. With output zero for small l values, the production functions have $df_{i,j}(0)/dl=0$. Using this in (8.1) we see that any *specialized* x automatically satisfies the conditions (8.1). In economic terms, startup costs stabilize *any* specialized production pattern.

Now we relate the many equilibria that arise in $M(f_{i,j})$ to the linear equilibria.

Corresponding Equilibria. We say that a *specialized* equilibrium point from the linear BP family and a *specialized* equilibrium of a corresponding scale-economies model are *corresponding equilibria* if the Z_1, the market share variables $x_{i,j,}$ the wages, the quantities of labor $l_{i,j}$ employed in each industry and the prices and the quantities produced are the same in both equilibria. Clearly, any two corresponding equilibria are represented by the same point in our graph. We assert that for each equilibrium of the economies model there is a corresponding equilibrium of the linear BP family.

Theorem 8.1 - Correspondence Theorem: From any specialized equilibrium (x,Z_1) of the scale-economies model we can construct a corresponding equilibrium (x,Z_1,ϵ) of the linear BP family having the same x and Z_1 and an ϵ given by: (1) the $e_{i,j}$ for producers is average productivity at the economies equilibrium, so $e_{i,j}=f_{i,j}(l_{i,j})/l_{i,j}$, and (2) the $e_{i,j}$ for non-producers is average productivity at output zero, so $e_{i,j}=df_{i,j}(0)/dl_{i,j}$.

Proof: We can verify directly that the x and Z_1 with this ϵ satisfy the equilibrium conditions (1.1)-(1.4) and (8.1) so (x,Z_1,ϵ) is a linear equilibrium. Since $e_{i,j}=f_{i,j}(l_{i,j})/l_{i,j}\leq f(L_j)/L_j = e^{max}_{i,j}$ this is one of the equilibria of the linear BP family. Since the x and Z_1 are the same in both equilibria they yield the same labor quantities through (2.2) and then, because of the choice of the $e_{i,j}$, the same quantities are produced at both equilibria. Since the demands are the same, so are the prices. Therefore, (x,Z_1) and (x,Z_1,ϵ) are corresponding equilibria.

Many Corresponding Equilibria: If the economies model has many equilibria, each will clearly correspond to a different equilibrium (x,Z_1,ϵ) of the linear model. One economies model is, therefore, a way of looking at a large sample of the equilibria of a BP family of linear models. Figure 8.1 shows the equilibria corresponding to one rather small economies model.

The location of the equilibria corresponding to $M(f_{i,j})$ in the region of equilibria of the linear BP family depends on the nature of the scale economies. If the production functions $f_{i,j}(l)$ have productivities $f_{i,j}(l)/l$ that go on increasing until $l=L_j$, the corresponding equilibria tend to be low in the region of equilibria of the linear model. This is because equilibrium labor quantities $l_{i,j}$ are generally small compared to the entire work force L_j. Therefore, the $e_{i,j}=f_{i,j}(l_{i,j})/l_{i,j}$ they produce in the corresponding equilibria will tend to be small compared with $e^{max}_{i,j}= f_{i,j}(L_{i,j})/L_{i,j}$. This results in equilibria with relatively low productivity and low utility. On the other hand, if the production functions have already reached full economies of scale when each country is supplying its own needs in autarky, the corresponding equilibria are high up in the region. In fact they are all maximal productivity equilibria, because $e_{i,j}=f_{i,j}(l_{i,j})/l_{i,j} = f_{i,j}(L_{i,j})/L_{i,j}=e^{max}_{i,j}$. Figure 8.1 is a case with mild scale economies.

We can also look at the correspondence in the other direction. Given a set of equilibria of the linear model, when do these *all* correspond to equilibria from *one* economies model? We give a general discussion and a general theorem in Appendix F. Here, however, we will state a theorem, proved in Appendix F, to indicate the connection between families of linear models and economies of scale models.

Theorem 8.3 - Maximal Productivity Correspondence Theorem: The 2^n-2 specialized equilibria of the region of maximal productivity always correspond to the equilibria of a single economies model.

This theorem shows the tight connection between families of linear models and economies models with startup costs. The region of maximal productivity and its equilibria are virtually identical with the equilibria of such an economies model. This tends to explain why we obtain such similar economic results, such as conflict in the interests of trading partners, in both settings.

IX. Summary and Conclusions

By taking explicitly into account the limits of productivity we have shown that the equilibria of a BP family of linear models form a well defined region with a robust shape. The shape of the region is such that the best outcomes for one country are always poor ones for its trading partner, so that policy or circumstance that succeeds in attaining the best outcomes for one country inherently involves conflict with the interests of the other. In fact, all the best outcomes for a developed country require its trading partner to be in a only a partly developed state.

We have introduced the concept of the ideal trading partner for Country 1, and have shown how the productivity parameter values of a country's ideal trading partner can be determined. Any departure from these values, whether through increases or decreases in productivity of Country 2, the partner, will harm Country 1. Thus the welfare of a country is sometimes enhanced and sometimes reduced by a rise in productivity of its trading partner, but these outcomes follow a systematic pattern that is easily understood.

Within the region of equilibria there lies a maximal-productivity subregion. Within the region the interests of the two countries are in conflict over a wide range, but there is also a range where improvements in productivity in either country tend to benefit both. This beneficial range occurs when one trading partner is in the early stages of development.

We then analyzed the effect of the parameters of our linear models on the more detailed regional shape and on special subregions. Our analyses indicate that the tendency of modern industry to be based on acquirable skills rather than fixed natural resources tends to intensify the inherent conflicts we have described. However, the relative insensitivity of regional shape to different plausible maximal productivity curves also enables us to estimate the location of regional peaks based mainly on relative country size. Thus a country can tell whether it is in the range of cooperation or in the range of conflict from knowing the current exchange rate.

Finally we have introduced the correspondence principle which connects the outcomes obtained from economies of scale models with those we have obtained here for families of linear models. This shows that the patterns of multiple equilibria and the regional shape that emerges in the presence of scale economies are not peculiar to that state of affairs. On the contrary, we see that even in the linear models that characterize the classical theory of international trade the large body of equilibria and the region that contains them has its direct counterpart, with all of its direct implications for theory and policy.

Appendix A - General Regional Structure

The first step in showing the regional structure is:

Lemma A.1 There are equilibria for every Z_1, $0<Z_1<1$. Proof: To construct the desired equilibrium choose *any* x satisfying (1.1) for the given Z_1. That is, choose any production pattern x whatsoever that produces the given national income Z_1. That there are (many) such x for any $0 \leq Z_1 \leq 1$ follows from the fact that $\sum (d_{i,1}Z_1+d_{i,2}Z_2)x_{i,1}=1$ when $x_{i,1}=1$ all I, and $\sum(d_{i,1}Z_1+d_{i,2}Z_2)x_{i,1}=0$ when $x_{i,1}=0$ all I. Now to construct an equilibrium from x we simply choose the $e_{i,j}$ to produce stability. For example, if $x_{i,1}=1$ and $x_{i,2}=0$, choose any $e_{i,1}>0$ and $e_{i,2}=0$. If $x_{i,2}=1$ and $x_{i,1}=0$, choose $e_{i,2}>0$ and $e_{i,1}=0$. If both variables are positive for some industry I, choose $e_{i,1} = w_1 = Z_1/L_1$ and $e_{i,2} = w_2 = Z_2/L_2$. These choices clearly satisfy the stability condition (1.5). By Remark (2) of Section 1, the wages, prices, etc. that this x, Z1 and $w_j=Z_j/L_j$ generate satisfy the remaining equilibrium conditions, so this is an equilibrium.

Lemma A.2 If (x,Z,ϵ) is an equilibrium on the Z_1 vertical line with utility U_j, all the points below (Z_1,U_j) on that vertical line are also equilibria.

Proof: If (x,Z_1,ϵ) is an equilibrium of the BP family, so is $(x,Z_1,\lambda\epsilon)$ for all positive $\lambda \leq 1$ because the equilibrium conditions (1.1)-(1.5) remain satisfied. So given any equilibrium (x,Z_1,ϵ) if we steadily decrease the $e_{i,j}$ by multiplying them by a $\lambda<1$, the new equilibria have the same x and Z_1, and consequently the same wages w_j and the same $l_{i,j}$. However, the quantities produced, the $q_{i,j}$, will steadily decrease because the $q_{i,j}=\lambda e_{i,j} l_{i,j}$ and λ is decreasing. Therefore, the utilities U_1, U_2 and U_w all decrease and the point representing the equilibrium in any of the regions R_1, R_2, or R_w moves steadily down, tracing out a vertical line of equilibrium points with decreasing utility.

Now we can establish our theorem:

Theorem A.1- General Regional Structure Theorem: There is a curve $C_j(Z_1)$ such that every point of the (Z_1,U_1) diagram on or under $C_j(Z_1)$, and no point above that curve, is an equilibrium point. Proof: There are equilibrium points on any vertical line by Lemma A.1. These points are bounded above because the $e_{i,j}$ and the $l_{i,j}$ are bounded, and, therefore, the $q_{i,j}$ and U_j are also. Define $C_j(Z_1)$ to be the point in the diagram that is the supremum of the equilibrium points on Z_1. Suppose there is a point p on Z_1 somewhere below the supremum curve that is *not* an equilibrium point. Since $C_j(Z_1)$ is the supremum, there are equilibrium points on Z_1 between it and p. But then lemma A.2 asserts that p must also be an equilibrium. This contradiction proves that every point strictly below $Cj(Z_1)$ must be an equilibrium point. The points on $C_j(Z_1)$ itself are limit points of these equilibria. Now if we have a bounded sequence of equilibria satisfying (1.1) - (1.5), then their limit point will also satisfy (1.1) -(1.5), so that the points on $C_j(Z_1)$ are also equilibria.

Appendix B - Completing the Proof of Optimality

We now rule out the possibility of there being an equilibrium with greater utility than those we have constructed. We start from any equilibrium $(x,Z_1.\epsilon)$. We select from this equilibrium only those industries in which Country 1 is both the sole producer and the cheaper producer. Country 1's revenue based on these industries will be less than or equal to what it was at the equilibrium, and less than or equal to share Z_1. Therefore, we can take this as the set S in the assignment $x(Z_1,S)$ or x^S in which the industries in S are assigned to Country 1, the remaining industries are assigned to Country 2, and all productivities are maximal.

We assert that the utility of x^S, using maximal productivity in every industry, will have a larger utility than the original equilibrium (x,Z_1,ϵ). This follows from (1) the utility has not changed in the industries in S. (2) In the remaining industries either (a) Country 1 was not the cheaper producer, or (b) it was not the sole producer. In case (a) the new assignment gives all production in that industry to County 2 which is the cheaper producer, so this increases utility. In case (b) Country 2 had the same unit cost in equilibrium as Country 1, but this cost has decreased as Country 2 is now the sole producer and its productivity has gone from $e_{i,j}$ to $e^{max}{}_{i,j}$. So the utility of x^S is equal to or greater than the utility of (x,Z_1,ϵ).

Since the equilibria derived from x^S have the same utility as the utility of the assignment x^S, (x,Z_1,ϵ) can not have a greater utility than these equilibria. This completes the proof.

Appendix C - Convergence of the Approximate Boundary

The utility change from $B_1(Z_1)$ to $(x,Z_1\epsilon)$ must come entirely from the one shared industry. Either Country 1 reduces its productivity to match Country 2's cost or Country 2 reduces its productivity to match Country 1's cost. In either case the effect on utility is easily calculated. The ratio of approximate boundary utility to utility of the constructed equilibrium

$(x,Z_1\epsilon)$ is, using k for the shared industry,

(C.1)
$$\frac{B_1(Z_1)}{U_1(x,Z_1,\epsilon)} \leq \left(\max\left\{\frac{e^{max}{}_{k,1}/w_1}{e^{max}{}_{k,2}/w_2},\frac{e^{max}{}_{k,2}/w_2}{e^{max}{}_{k,1}/w_1}\right\}\right)^{u_{k,1}}.$$

Since $(x,Z_1\epsilon)$ must lie in or under the exact boundary curve $C_1(Z_1)$, $C_1(Z_1)$ is even nearer to $B_1(Z_1)$ than $(x,Z_1\epsilon)$ is. This leads to the:

Approximation Corollary: The ratio $B_1(Z_1)/ C_1(Z_1)$ is always $\leq (\max(R_1/(w_1/w_2),R_2/(w_2/w_1))^D$. Here $D=\max_i d_{i,1}$, $R_1=\max_i (e^{max}{}_{i,1}/e^{max}{}_{i,2})$, and $R_2 = \max_i(e^{max}{}_{i,2}/e^{max}{}_{i,1})$.

Proof: This is merely (C.1) with the worst possible values inserted.

Now we are ready to address the convergence of $B_1(Z_1)$ to $C_1(Z_1)$ in large problems.

Convergence in Large Problems: We will say that an n-industry BP family has *extremeness* bounded by K if its parameters satisfy:

(C.2)
$$L_1/L_2 \leq K, \quad L_2/L_1 \leq K \quad d_{i,j} \leq K\left(\frac{1}{n}\right), \quad \frac{e^{max}{}_{i,1}}{e^{max}{}_{i,2}} \leq K, \quad \frac{e^{max}{}_{i,2}}{e^{max}{}_{i,1}} \leq K.$$

K restricts the amount of variation in country size, productivity advantage, or in the case of the restriction on demand, the amount by which the demand for one good can exceed the average demand 1/n. If we restrict the extremeness of our models, their boundaries B_1 and C_1 converge as n, the number of industries, becomes large. More precisely:

Theorem E1 - Convergence Theorem: For any sequence of n-industry models with increasing n

and with extremeness bounded by K , the ratio $B^n{}_1(Z_1)/C^n{}_1(Z_1)$ of the nth model approaches 1 as $n \to \infty$.
Proof: The formula in the Approximation Corollary shows that any n-industry model in the sequence has its $B^n{}_1(Z_1)/ C_1{}^n(Z_1)$ bounded by:

$$\textbf{(C.3)} \quad 1 \le \frac{B^n{}_1(Z_1)}{C^n{}_1(Z_1)} \le (\max\ (R^n{}_1/(Z_1L_2/Z_2L_1),\ R^n{}_2/(Z_2L_1/Z_1L_2)\ )^D \le (K^2 \max\ (Z_1/Z_2, Z_2/Z_1)^{K/n}$$

The exponent in the last term in (C.3) approaches 0 so the term approaches 1. So for models with large numbers of industries the region of equilibria becomes almost identical with the region under the approximating curve of utility $B_1(Z_1)$.

Appendix D - Proving Quasi-Concavity of the Approximate Country Boundaries

Outline of the proof: The approximate boundary $B_1(Z_1)$ is defined to be $\ln B_1(Z_1)=u_1(x,Z_1,\epsilon^{max})$ where $x=x(Z_1)$ is the solution of the linear programming problem (3.3) for each Z_1. We prove quasi-concavity by showing that no local minima of u_1 are possible.

To do this we first obtain an expression for the derivative du_1/dZ_1. From this expression we show that u_1 has a continuous derivative except at a finite number of exceptional points where it has a well defined, left derivative (LD) and right derivative (RD). Using properties of the linear programming solution we will show that at these exceptional points we cannot have both LD<0 and RD>0, so that a local minimum cannot occur at an exceptional point. Next we obtain the second derivative at the non-exceptional (regular) points. We show that when $du_1/dZ_1 = 0$, the second derivative is always negative. This rules out the possibility of a local minimum at regular points and completes the proof.

Obtaining the First Derivative: We will repeatedly use the fact that the linear programming solution has at most only one variable $x_{k,1}$ that is neither 0 or 1. At a finite number of points (exceptional points) the non-integer variable can change from index k to some other index. For points in the intervals between exceptional points (regular points) the solution x changes only in the $x_{k,1}$ (and $x_{k,2}=1-x_{k,1}$) variables, the other $x_{i,1}$ and $x_{i,2}$ are fixed at their integer values. To obtain the derivative at regular points we start with

$$\textbf{(D.1)} \qquad u_1 = \ln U_1 = \sum_i d_{i,1}\{x_{i,1}\ln\ d_{i,1}e_{i,1}L_{i,1} + x_{i,1}\ln\ d_{i,1}e_{i,2}L_{i,2}\frac{Z_1}{Z_2}\}.$$

Differentiating with respect to Z_1 and using the fact that $dx_{i,1}/dZ_1 = 0$ except for I=k gives:

$$\textbf{(D.2)} \qquad \frac{du_1}{dZ_1} = -\sum_i d_{i,1}x_{i,2}\frac{1}{Z_1Z_2} + d_{k,1}\frac{dx_{k,1}}{dZ_1}\ln\frac{e_{i,1}L_1Z_2}{e_{i,2}L_2Z_1}$$

We will make two substitutions in (D.2). First, we need an expression for $dx_{k,1}/dZ_1$: If we differentiate

$$\sum_i x_{i,1}(d_{i,1}Z_1+d_{i,2}Z_2)=Z_1 \quad \textit{with respect to } Z_1 \textit{ we obtain}$$

$$\frac{dx_{k,1}}{dZ_1} = \frac{1-\sum_i x_{i,1}(d_{i,1}-d_{i,2})}{(d_{k,1}Z_1+d_{k,2}Z_2)} = \frac{1-\gamma(x)}{TD_k}$$

Here we use TD_k for the denominator which is the total demand for the kth good and $\gamma(x)$ for the sum in the numerator. We know a fair amount about $\gamma(x)$ from Gomory [1994]. For symmetric demands $\gamma=0$. The opposite of symmetric demand is orthogonal demand, i.e. $d_{i,1}d_{i,2}=0$ for all I. It is clear that $-1 \le \gamma(x) \le 1$ and that equality is only possible with orthogonal demands. If we exclude orthogonal demands, which we will, we have $(1-\gamma(x))$ always positive. We also have, since $\Sigma_i(d_{i,1}-d_{i,2})(x_{i,1}+x_{i,2})=0$, that $\Sigma_i\, x_{i,2}(d_{i,1}-d_{i,2})= -\gamma(x)$ an identity we will use below.

We will need one more identity: This time it is for $\Sigma_i d_{i,1}x_{i,2}$.

$$Z_2=\sum_i (d_{i,1}Z_1+d_{i,2}Z_2)x_{i,2}=\sum_i (d_{i,1}x_{i,2})Z_1+\sum_i (d_{i,2}x_{i,2})Z_2 \quad so$$

$$\sum_i (d_{i,1}x_{i,2})Z_1+\sum_i (d_{i,1}x_{i,2})Z_2-\sum_i ((d_{i,1}-d_{i,2})x_{i,2})Z_2=Z_2 \quad so$$

$$\sum_i (d_{i,1}x_{i,2})=Z_2(1+\sum_i (d_{i,1}-d_{i,2})x_{i,2})=Z_2(1-\gamma(x))$$

Substituting the two identities in (D.2) gives us a remarkably simple expression for the derivative at regular points:

(D.3)
$$\frac{du_1}{dZ_1}=(1-\gamma(x))\{\frac{1}{Z_1}+\frac{d_{k,1}}{TD_k}\ln\frac{e_{i,1}L_1Z_2}{e_{i,2}L_2Z_1}\}.$$

An important property of (D.3) is that the sign of the derivative is the same as the sign of the term in the brackets because $(1-\gamma(x))$ is always positive.

Exceptional Points: (3) clearly shows that we have a continuous derivative du_1/dZ_1 except possibly at Z_1 values where the index k changes. At these exceptional Z_1 values we have both left (LD)and right derivatives (RD) given by (3). We need to show that we cannot have both LD<0 and RD>0. Let us first suppose that the index changes at the exceptional point Z'_1 because, as Z_1 increases, the increasing $x_{k,1}$ has reached a value of 1. For $Z_1 > Z'_1$ we must, therefore, choose a new k. The index change will produce a smaller value for the expression

$$R=\frac{d_{k,1}}{TD_k}\ln\frac{e_{i,1}L_1Z_2}{e_{i,2}L_2Z_1}$$

because R is the expression used to pick the next shared or special variable in the linear programming algorithm, and the R value with the new index cannot be larger than the current R value or it would have been picked before it for Z_1 just less than Z'_1. So in this case there can be a discontinuity in the R-value with the change of index, but it can only lead to a new index and a new R-value that is not larger. Since the term in brackets in (D.3) determines the sign of the derivative, this decrease in R cannot change the derivative from negative to positive as Z_1 passes through Z'_1.

The other case is that the shared variable changes index before reaching its maximum value. This means that some other industry comes to have a larger R as Z_1 (and hence the wage) changes. However, it does this by first developing an R equal to that of the current shared variable and then surpassing it. The index change occurs at equality and, therefore, does not produce a discontinuity in the bracket in (D.3) although it may produce a discontinuity in γ. However, since the bracket determines the sign of the derivative, the derivative may change value discontinuously but it cannot change sign.

We have now ruled out a local minimum at exceptional points. Next we discuss regular points.

Analyzing the Regular Points: With the first derivative in a this form, we look next at the second derivative, d^2u_1/dZ_1^2.

(D.4)
$$\frac{d^2u_1}{dZ_1^2}=\{(\frac{d}{dZ_1}(1-\gamma(x)))\ (\frac{1}{Z_1}+\frac{d_{k,,1}}{TD_k}\ln\frac{e_{i,1}L_1Z_2}{e_{i,2}L_2Z_1})\} + \{(1-\gamma(x))\ \frac{d}{dZ_1}(\frac{1}{Z_1}+\frac{d_{k,1}}{TD_k}\ln\frac{e_{i,1}L_1Z_2}{e_{i,2}L_2Z_1})\}.$$

(D.4) consists of two terms. Since we are only interested in the second derivative when the first derivative is zero, we can entirely disregard the first bracket. From (D.3) it will be 0 whenever du_1/dZ_1 is. This leaves us for the second derivative when the first derivative is 0,

(D.5)
$$\frac{d^2u_1}{dZ_1^2}=(1-\gamma(x))\ \frac{d}{dZ_1}(\frac{1}{Z_1}+\frac{d_{1,k}}{TD_k}\ln\frac{e_{i,1}L_1Z_2}{e_{i,2}L_2Z_1})\quad so$$
$$\frac{d^2u_1}{dZ_1^2}=(1-\gamma(x))\{-\frac{1}{Z_1^2}-\frac{d_{k,1}(d_{k,1}-d_{k,2})}{TD_k^2}\ln\frac{e_{i,1}L_{i,1}Z_2}{e_{i,2}L_{i,2}Z_1}-\frac{d_{k,1}}{TD_k}(\frac{1}{Z_1Z_2})\}.$$

When the first derivative is 0, we have from (D.3), $\ln\frac{e_{1,1}L_1Z_2}{e_{i.2}L_2Z_1}=(-\frac{TD_k}{Z_1d_{k.1}})$. Using this in (D.5) gives

(D.6)
$$\frac{d^2u_1}{dZ_1^2}=(1-\gamma(x))\{-\frac{1}{Z_1^2}+\frac{(d_{k,1}-d_{k,2})}{Z_1TD_k}-\frac{d_{k,1}}{TD_k}(\frac{1}{Z_1Z_2})\}\quad so$$
$$\frac{d^2u_1}{dZ_1^2}\le(1-\gamma(x))\{-\frac{1}{Z_1^2}+\frac{d_{k,1}}{Z_1TD_k}-\frac{d_{k,1}}{TD_k}(\frac{1}{Z_1Z_2})\}=(1-\gamma(x))\{.-\frac{1}{Z_1^2}+\frac{d_{k,1}}{Z_1TD_k}(1-\frac{1}{Z_2})\}.$$

Since $1/Z_2 > 1$ this shows that d^2u_1/dZ_1^2 is negative. This rules out a local minimum at a regular point and proves the theorem.

Appendix E.1 - The Derivative Formulas with Symmetric Demand

With symmetric demand we can modify the formula (D.3) for du_1/dZ_1 by setting $\gamma(x)=0$ and also use $TD_k=d_{k,1}Z_1+d_{k,2}Z_2=d_{k,1}$. Therefore, we have for du_1/dZ_1 or equivalently for dB_1/dZ_1

(E1.1) $$\frac{dB_1}{dZ_1}=\frac{1}{Z_1}+\ln\frac{e_{i,1}L_1Z_2}{e_{i,2}L_2Z_1}.$$

This gives us one of the needed formulas. However, one is enough because with symmetric demand world utility, Country 1 utility, and Country 2 utility are closely tied together. Country 1's utility is

$U_j=\Pi_i\, y_{i,1}{}^{di,j}$ with $y_{i,1}$ Country 1's consumption of the ith good. With symmetric demand Country 1 consumes a fraction $F_{i,j}(Z_1) = d_{i,1}Z_1/(d_{i,1}Z_1+d_{i,2}Z_2)=Z_1$ of total world production. So

(E1.2) $$U_1=\Pi_i\, y_{i,1}{}^{di,1}=\Pi_i\, Z_1(q_{i,1}+q_{i,2})^{di,1}=Z_1\Pi_i\, 1(q_{i,1}+q_{i,2})^{di,1}=Z_1U_W$$

To the relation $U_1=Z_1U_W$ we can add $U_2=Z_2U_W$ which is similarly derived. This means that when we maximize (3.3) for any fixed Z_1 for Country 1 utility, we maximize U_2, and U_w simultaneously. It follows immediately that the boundary curves have the same relation so $B_1(Z_1)=Z_1B_W(Z_1)$ and $B_2(Z_1)=Z_2B_W(Z_1)$. So:

(E1.3) $$\ln B_1(Z_1)=\ln Z_1+\ln B_W(Z_1)\quad \textit{and therefore}\quad \frac{d\ln B_1}{dZ_1}=\frac{1}{Z_1}+\frac{d\ln B_W}{dZ_1}.$$

Substituting the formula for dB_1/dZ_1 from (E1.1) into (E1.3) gives us the desired formula for dB_W/dZ_1. The formula for dB_2/dZ_1 is obtained similarly.

Appendix E.2 - Proof of General Regional Shape Theorems

If we define

$$en^{max}{}_1(Z_1)=\frac{e^{max}{}_1(Z_1)/e^{max}{}_1(.5)}{e^{max}{}_1(Z_1)/e^{max}{}_1(.5)+e^{max}{}_2(Z_1)/e^{max}{}_2(.5)}$$

Then in (6.3) we can replace in the integral the production functions $e^{max}{}_1$ and $e^{max}{}_2$ using:

$$\frac{e^{max}{}_2(Z_1)}{e^{max}{}_1(Z_1)}=\frac{e^{max}{}_2(.5)}{e^{max}{}_1(.5)}\,\frac{en^{max}{}_2(Z_1)}{en^{max}{}_1(Z_1)}.$$

Then we can move the constant term $e^{max}{}_2(.5)/e^{max}21(.5)$ out of the integral and into the term

L_2/L_1 of (E.5) to form the β of the General Regional Shape Theorem.

To prove the Second Form of the General Regional Shape Theorem, we simply take the β term, which is a constant, and move it up into the integral and incorporate it into the production function which becomes $eb^{max}{}_2/eb^{max}{}_1$ with

$$eb^{max}{}_1 = \frac{(1/\beta)en^{max}{}_1}{(1/\beta)en^{max}{}_1 + en^{max}{}_2}.$$

This establishes the second form of the General Regional Shape Theorem.

Appendix F - Multi-Equilibrium Correspondence Theorem

Any one linear equilibrium (x, Z_1, ϵ) has well determined labor inputs $l_{i,j}$ and the resulting outputs $q_{i,j}=e_{i,j}l_{i,j}$. There is, therefore, a well determined input-output pair $(l_{i,j}, q_{i,j})$ for each industry in each country. Any m linear equilibria provide m such pairs. We refer to each collection of m pairs as the set $S_{i,j}$.

Given an $S_{i,j}$ with kth element (l^k, q^k) we say that $S_{i,j}$ is an *economies set* if there is a scale-economies production function $f(l)$, such that $f(l^k)=q^k$ for all m of the (l^k, q^k) pairs. From Figure F.1 it is clear that the points (l^k, q^k) can have a single economies curve passing through all of them, (and therefore have such a production function), if and only if the slopes from the origin (average productivities) of successive points, when they are arranged in order of increasing l, are non-decreasing.

Theorem 8.2 - Multi-Equilibrium Correspondence Theorem: m specialized equilibria of a BP family of linear models will correspond to m equilibria of a single economies of scale model $M(f_{i,j})$ if and only if each $S_{i,j}$ is an economies set.

Proof: If there are m corresponding equilibria in some economies model $M(f_{i,j})$, each one has the same input and output as its corresponding linear equilibrium. Therefore, together these equilibria generate the same set $S_{i,j}$. However, they produce each $S_{i,j}$ by assigning the various input quantities l^k of the m equilibria of $M(f_{i,j})$ to a *single* production function $f_{i,j}(l^k)$ and obtaining the corresponding outputs q^k. This is only possible if $S_{i,j}$ is an economies set. So the condition is clearly necessary.

To show it is also sufficient, we will construct an $M(f_{i,j})$ that satisfies Theorem 8.2. To do this we add (see Figure F.2) to each of the given $S_{i,j}$ the pair (0,0), (if it is not already included), and also a pair $(l^*_{i,j}, 0)$ which lies on the l axis halfway between the origin and the first pair that is not (0,0). Then we will add the pair $(L_j, e^{max}{}_{i,j}L_j)$ which lies further to the right than any existing pair. The pairs we have added on the left have zero slopes and are to the left of any successive pairs with positive slopes. The new pair on the right has a larger l than any of the other pairs and also a larger slope. This augmented set of points has increasing slopes with increasing l and is therefore an economies set. Any production function $f_{i,j}(l)$ that passes through this augmented set not only has economies of scale but also zero derivative at l=0, and $f_{i,j}(L_j)/L_j=e^{max}{}_{i,j}$. We will use these $f_{i,j}$ in our economies model $M(f_{i,j})$.

Now take (x, Z_1, ϵ), one of the set of m linear equilibria, and use its x and Z_1 as a candidate equilibrium for the economies model $M(f_{i,j})$. With (x, Z_1) in the economies model we will get the same demand and hence the same labor inputs as at (x,Z_1,ϵ). Because of the construction of the $f_{i,j}$ we will have the same outputs from those inputs, and hence the same prices. Furthermore, the candidate equilibrium is stable. This is because it is specialized, and the $f_{i,j}$ have been constructed with setup costs. So we have $df_{i,j}(0)/dl= 0$ and (8.1) is satisfied. This shows that (x,Z_1) is a stable equilibrium and that it corresponds to (x,Z_1,ϵ). This ends the proof.

If we apply this theorem to the maximal productivity equilibria we get:

Theorem 8.3 - Maximal Productivity Correspondence Theorem: The 2^n-2 specialized equilibria of the region of maximal productivity always correspond to the equilibria of a single economies model.

Proof: Each $S_{i,j}$ contain points of the form $(l,e^{max}_{i,j}l)$ when Country j produces in industry I, and also the point (0,0) when Country j is a non-producer. These points give us a constant slope $e^{max}i,j$ for every l, so $S_{i,j}$ is an economies set. Theorem 8.2 then gives the result.

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Figure 1.1a - 14 Equilibria Showing Country 1 Utility

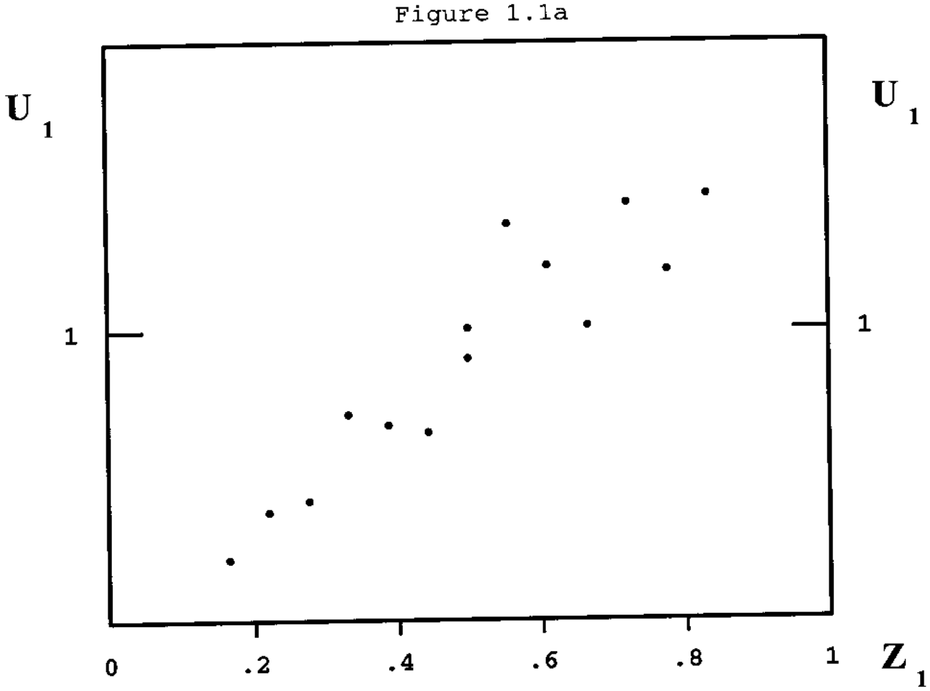

Figure 1.1b - 14 Equilibria Showing Country 2 Utility

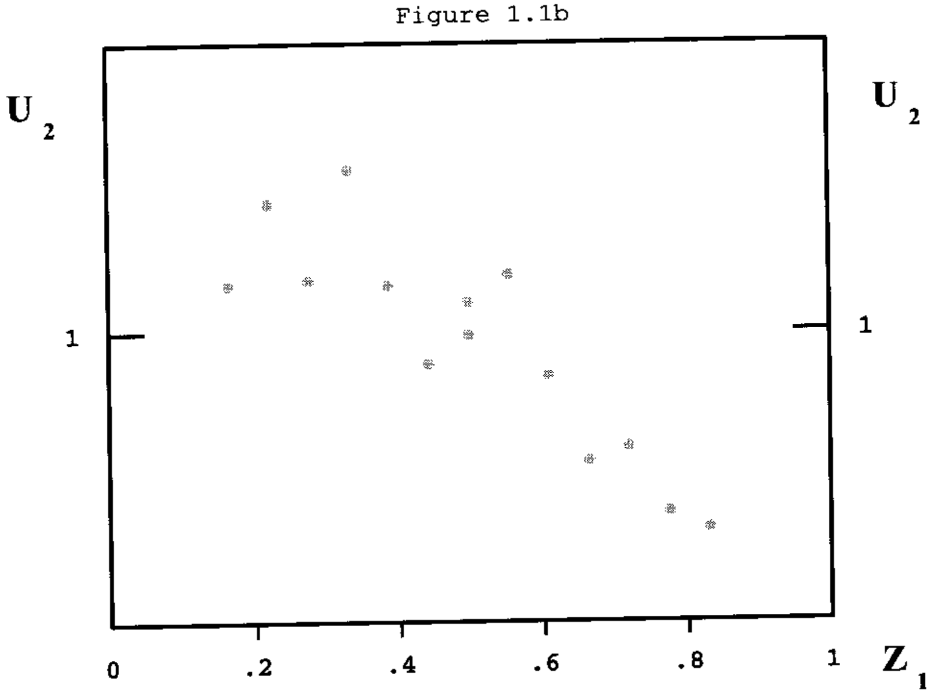

Figure 1.1c - 14 Point Pairs Showing Country 1 and Country 2 Utility

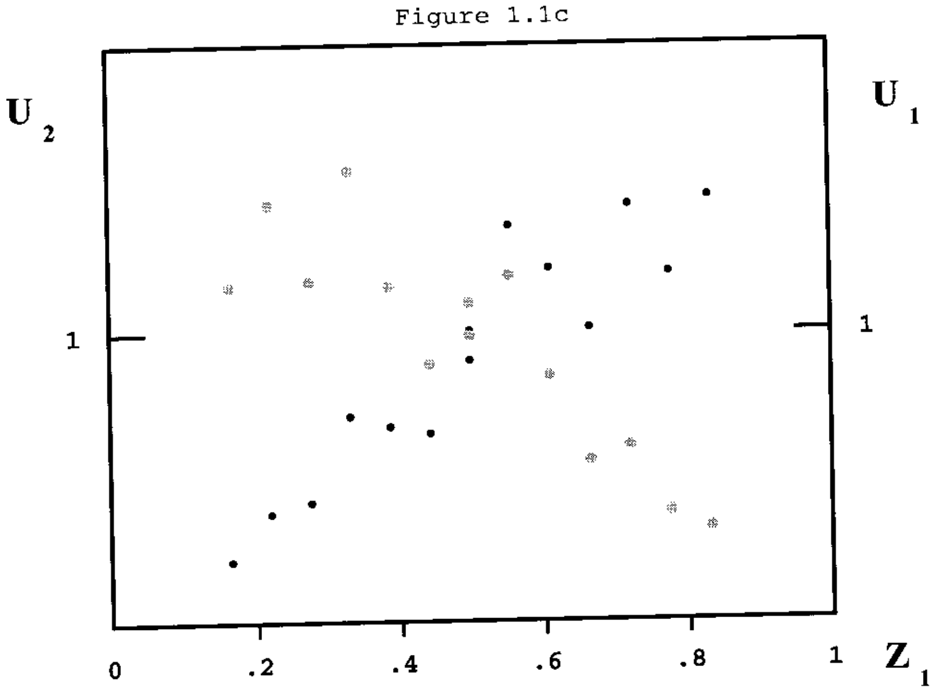

Figure 3.1a - World Utility Boundary for a Two Industry Model

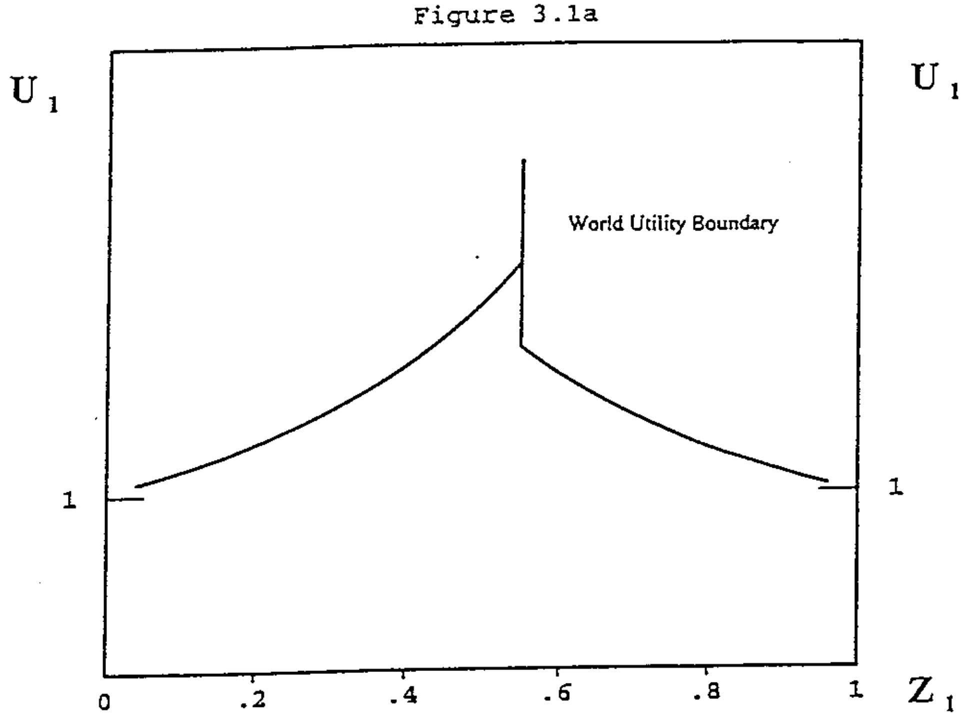

Figure 3.1b - Country 1 Utility Boundary for a Two-Industry Model

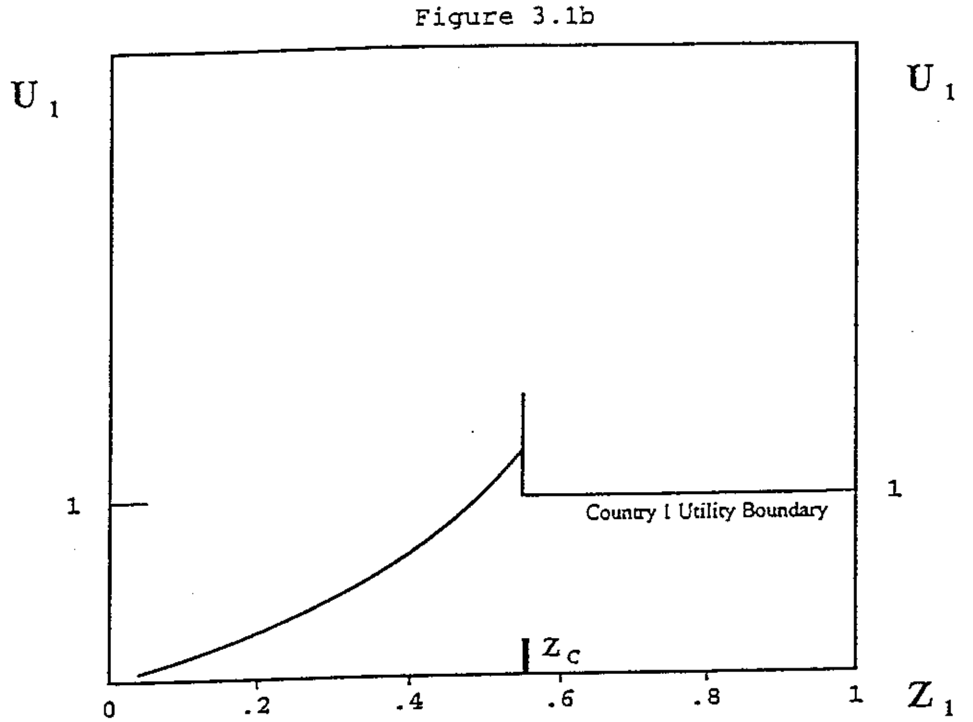

Figure 3.1c - Country 2 Utility Boundary for a Two-Industry Model

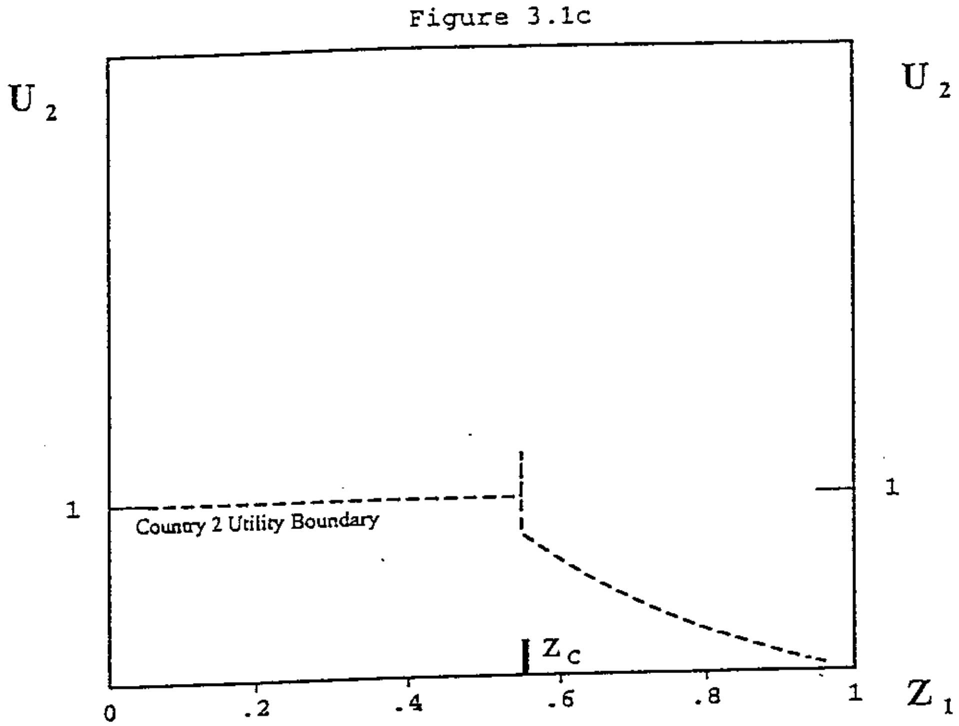

Figure 3.2 - Utility Boundaries for a Four Industry Model

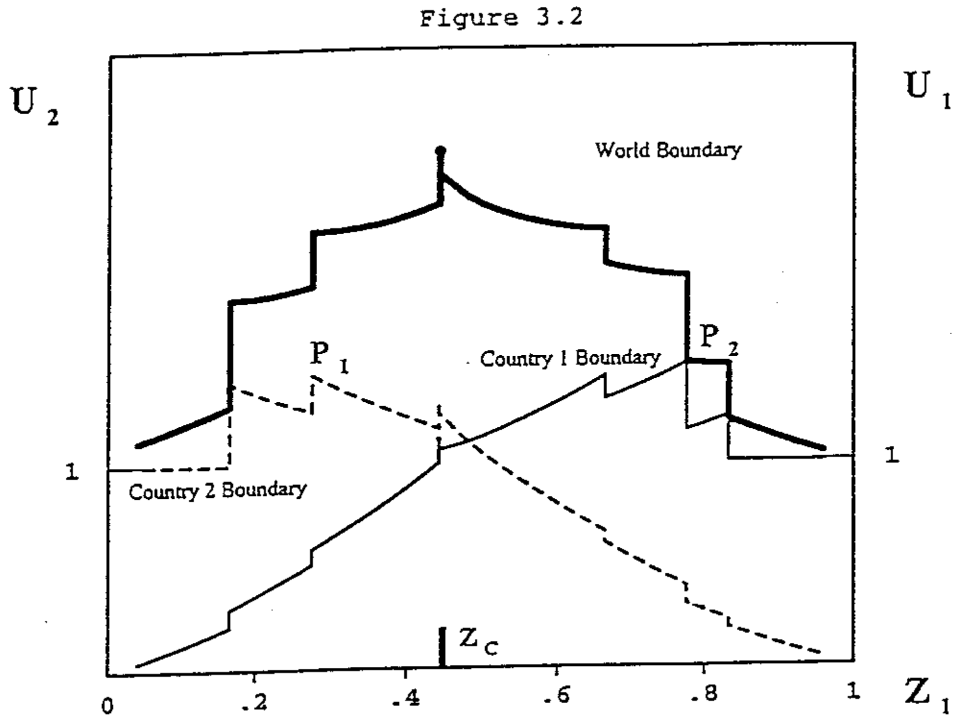

Figure 3.3 - Utility Boundaries for a Six Industry Model

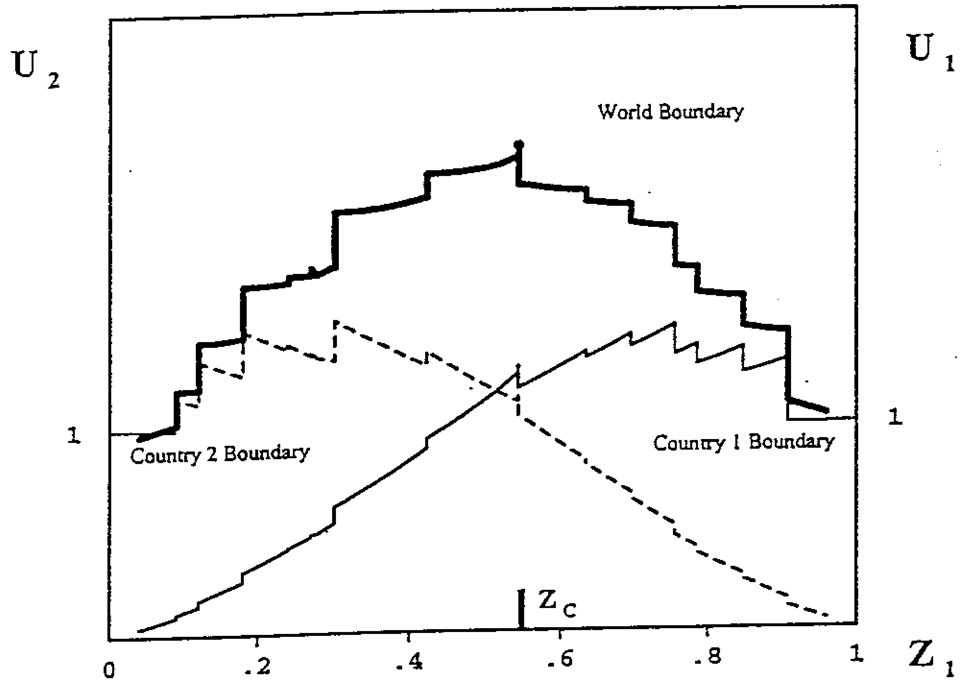

Figure 3.4 - The Approximating Curves

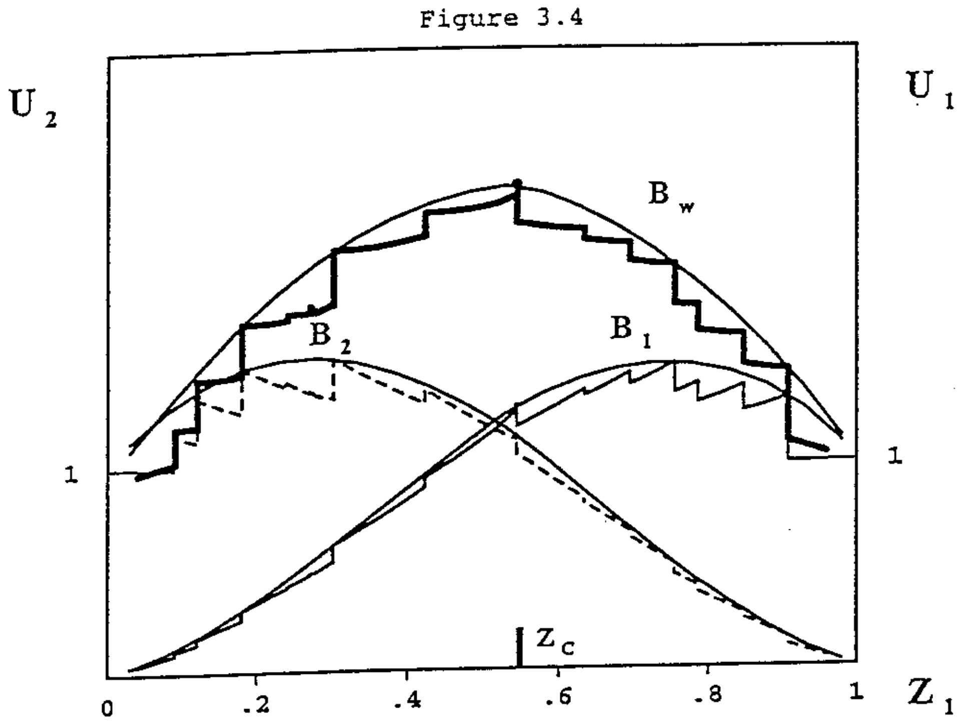

Figure 3.5 - A 22 Industry Model

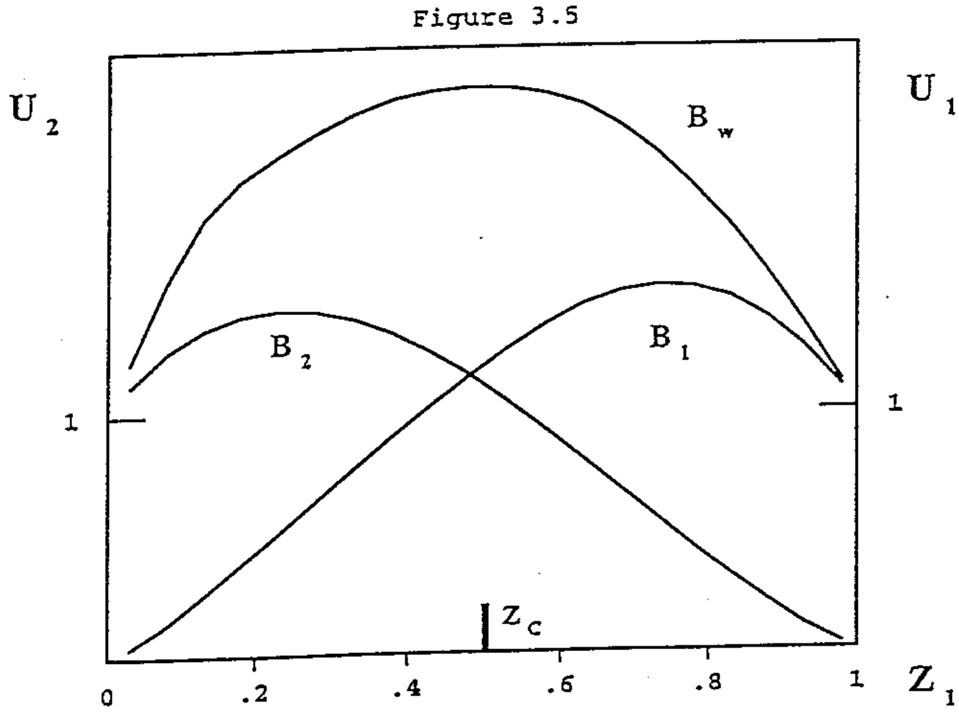

Figure 4.1 - Outcomes for a Fully Developed Country

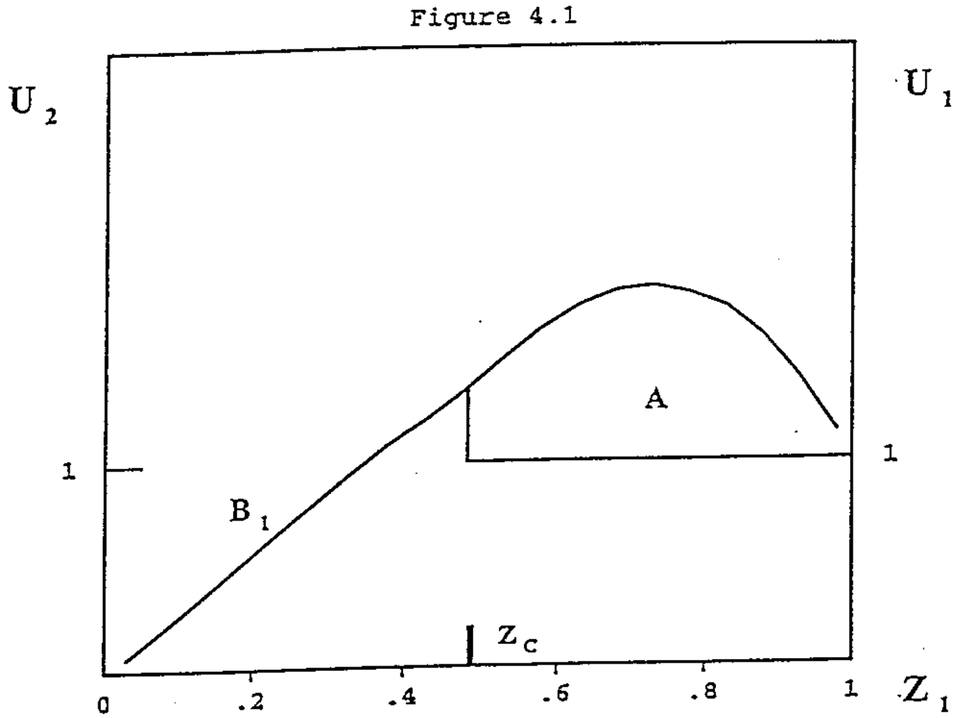

Figure 4.2 - The Ideal Trading Partner

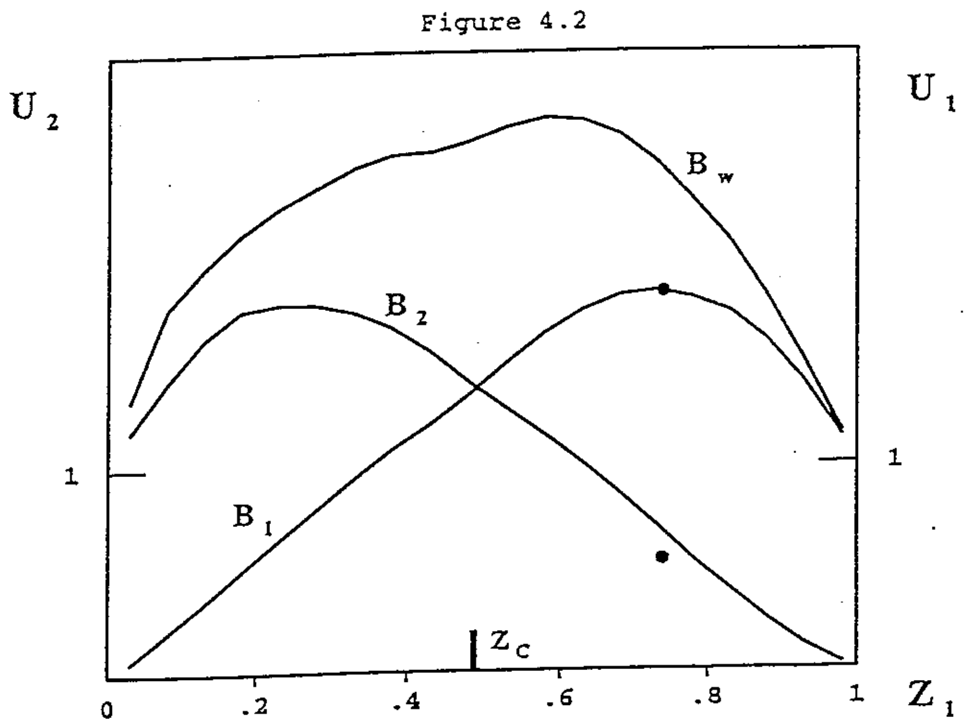

Figure 4.3 - An 11 Industry Model

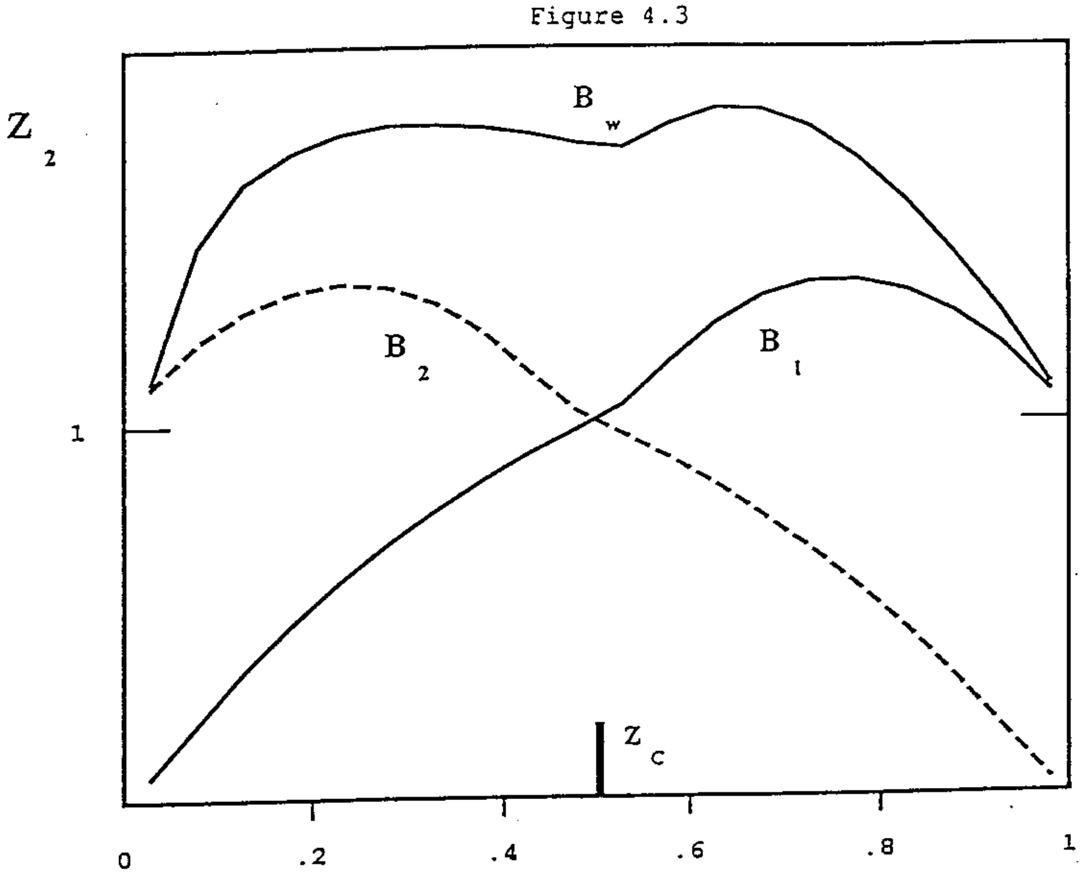

Figure 5.1 - The Region of Maximal Productivity of Country 1

Figure 5.1

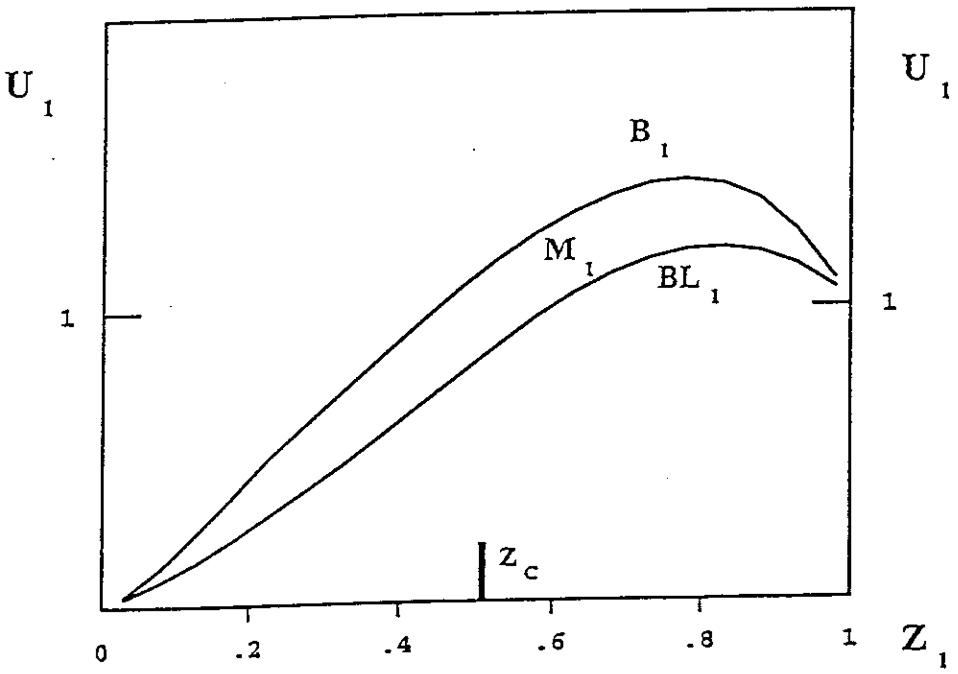

Figure 5.2 - Ranges of Conflict and of Cooperation

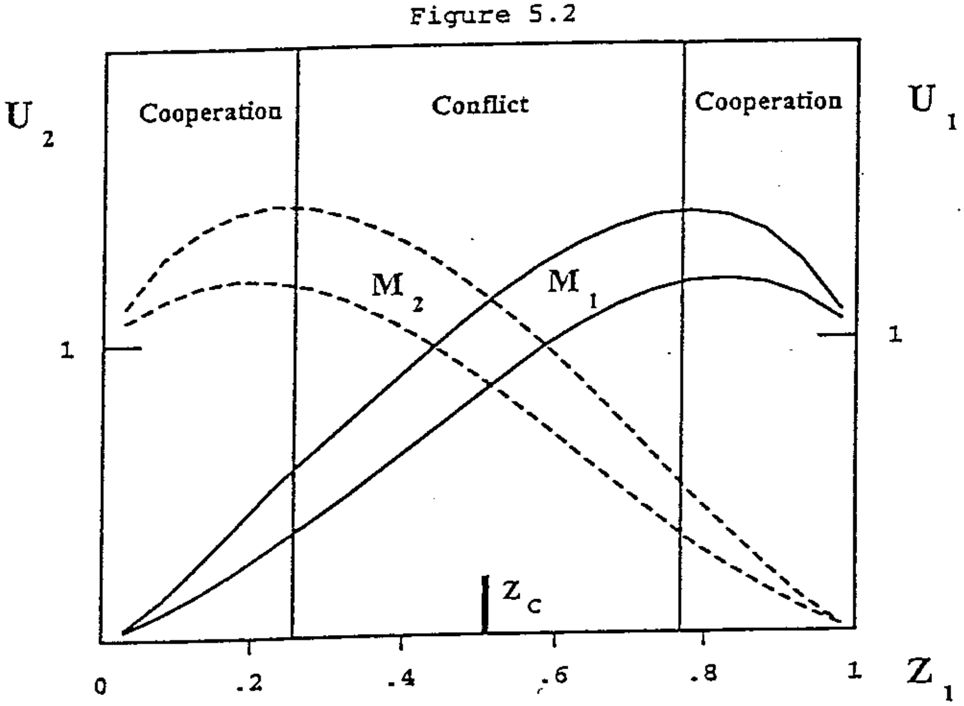

Figure 6.1a - The e-Z1 Diagram

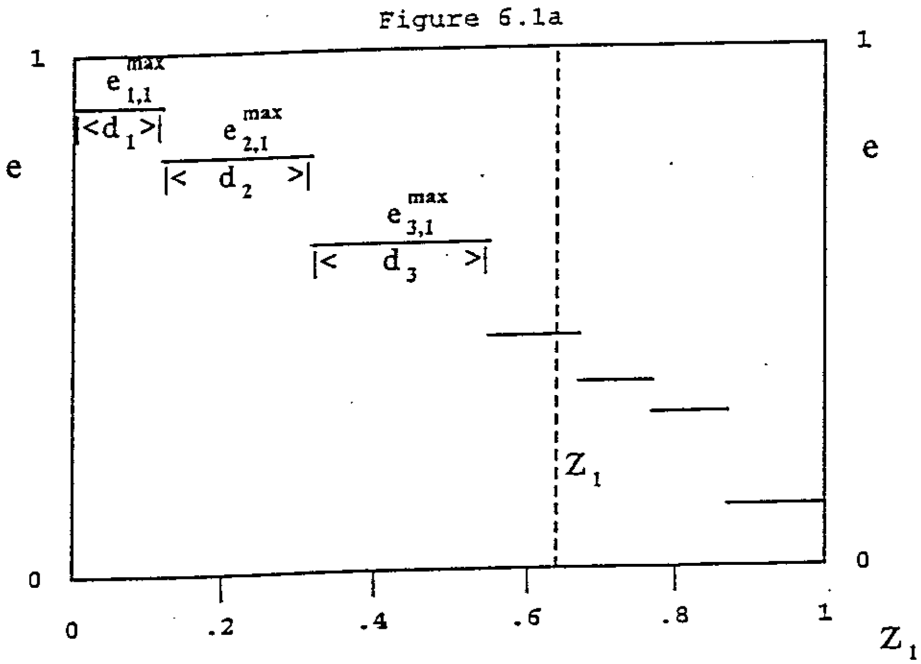

Figure 6.1b - Three Productivity Curves

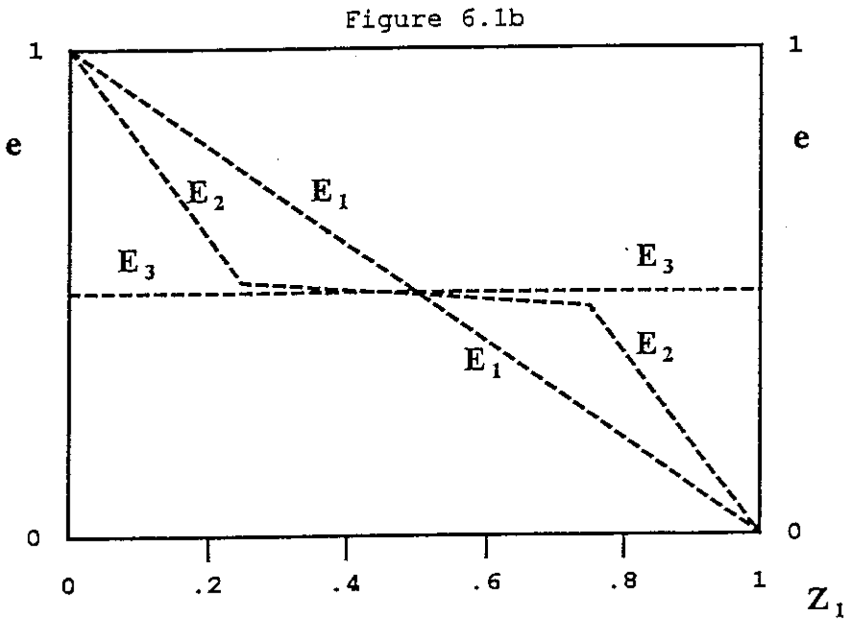

Figure 6.2a - Intersecting a Productivity Curve with the World Peak Curve to Determine the Classical Level.

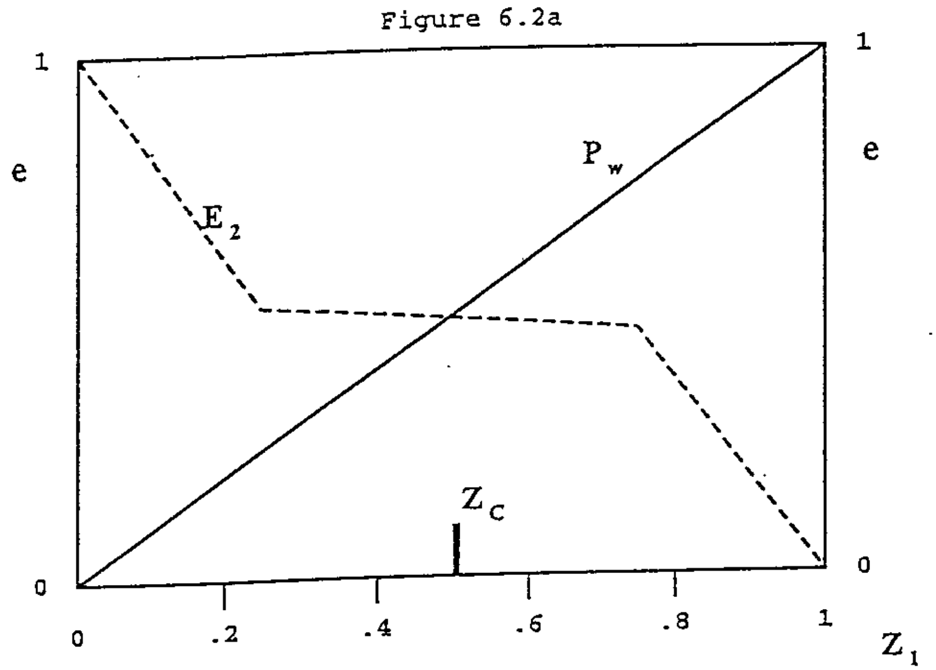

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Figure 6.2b - The Productivity Curves and their Intersection with the Peak Curves

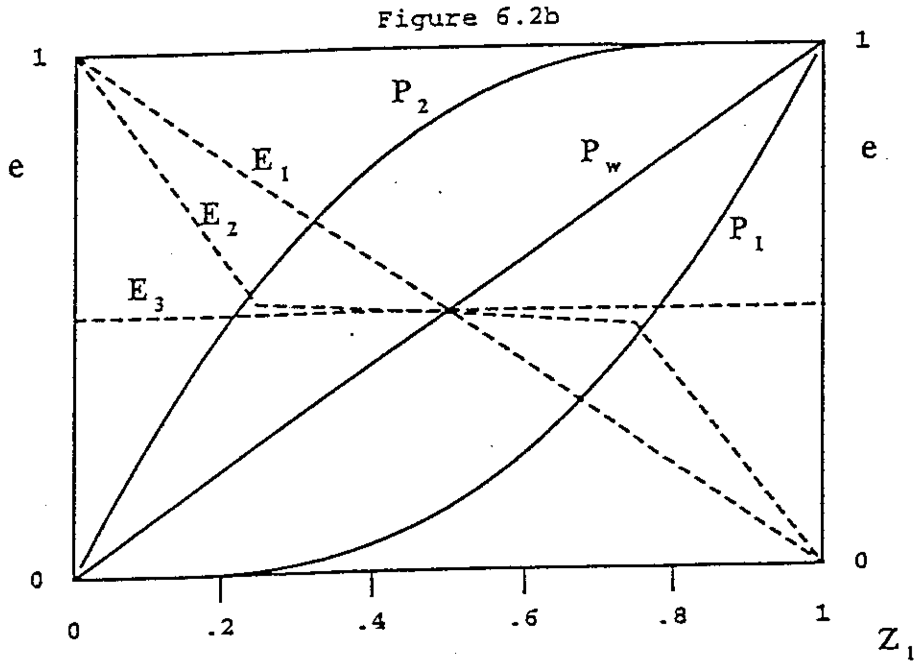

Figure 6.2c -Country 1 and Country 2 Regions for the 3 Productivity Curves

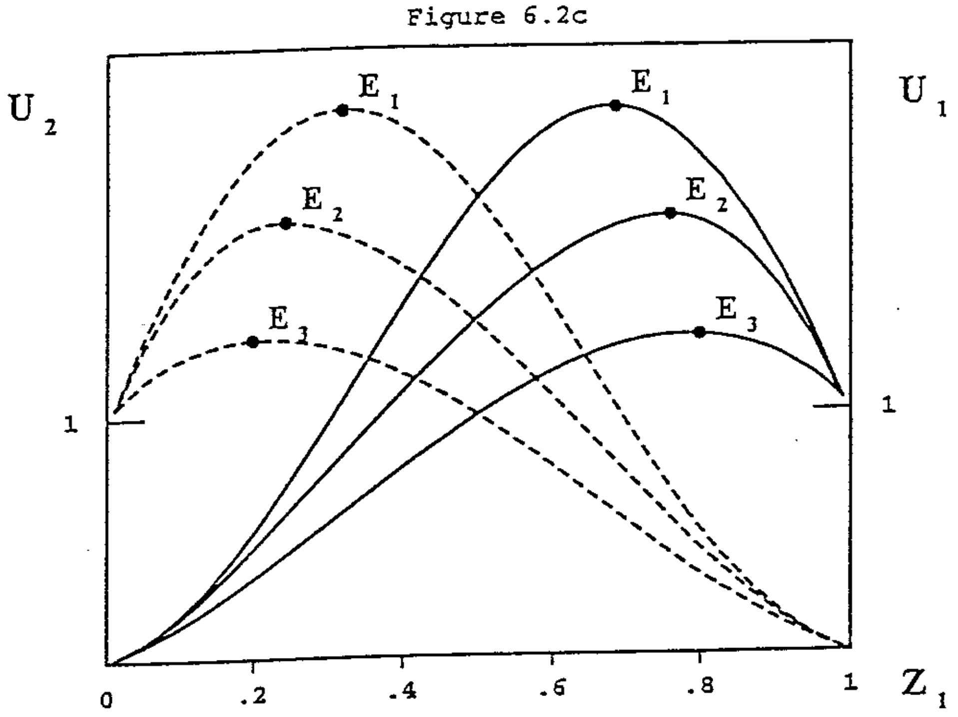

Figure 6.3 - Structure of the Peak of Region 1 when the Productivity Curve is E1

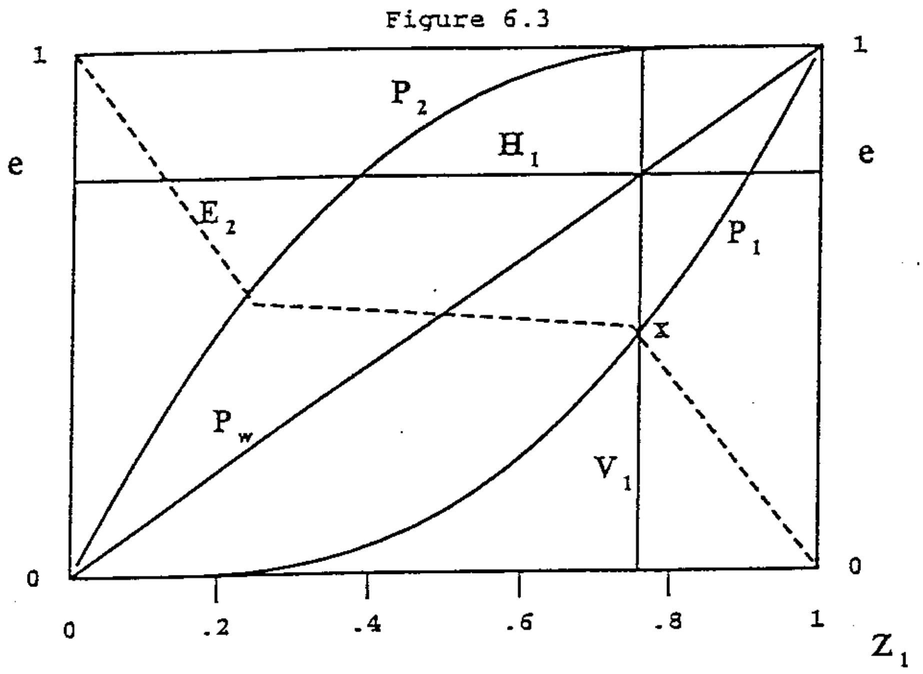

Figure 6.3 - Structure of the Peak of Region 1 when the Productivity Curve is E1

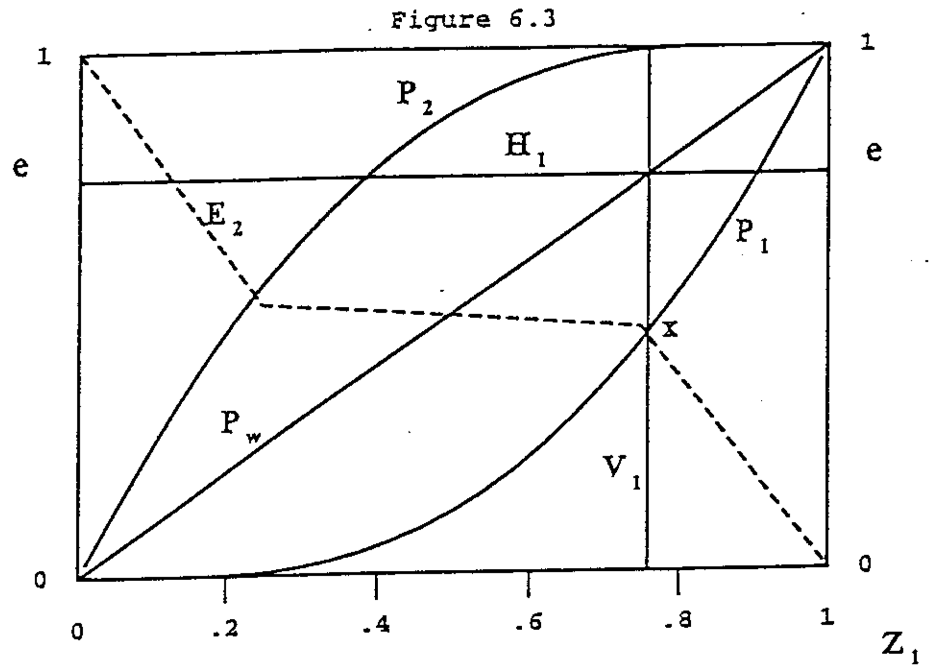

Figure 6.4 - Effect of Changing Country Size

Figure 6.4

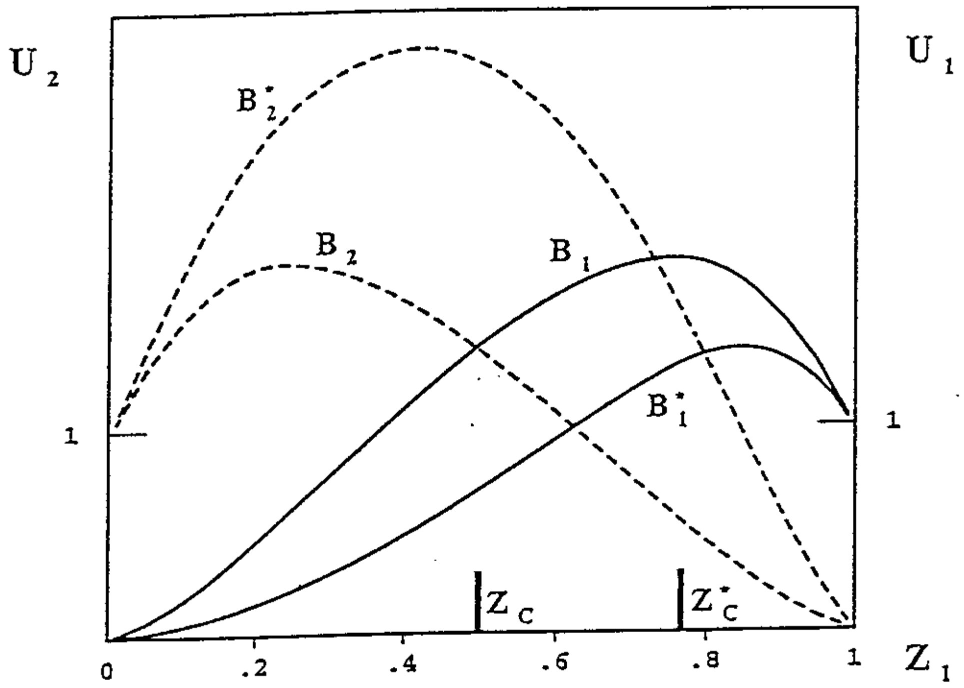

Figure 6.5 - The Changed Productivity Curves

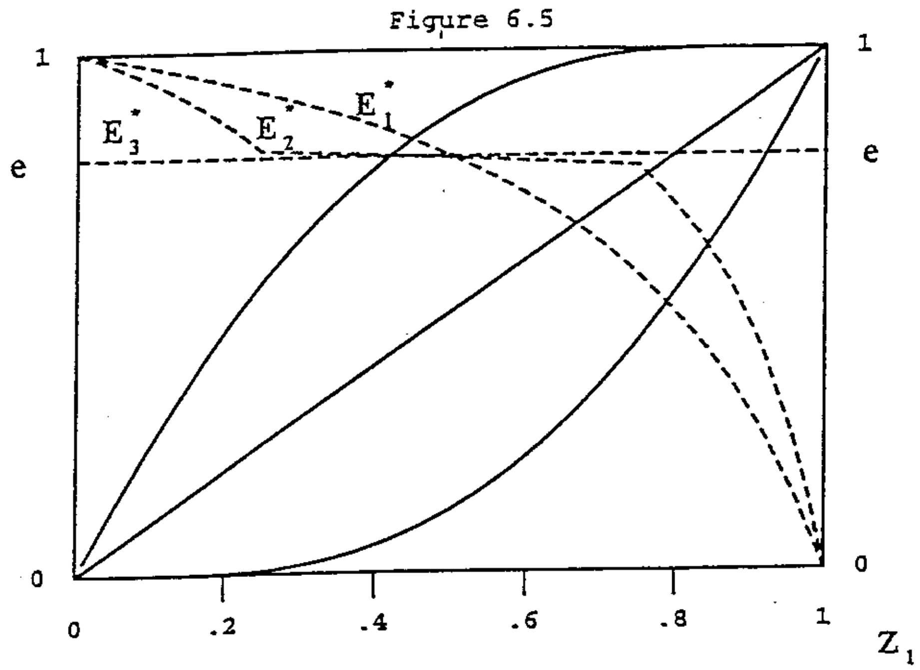

Figure 6.6 - Utility at the Peak and at the Classical Point

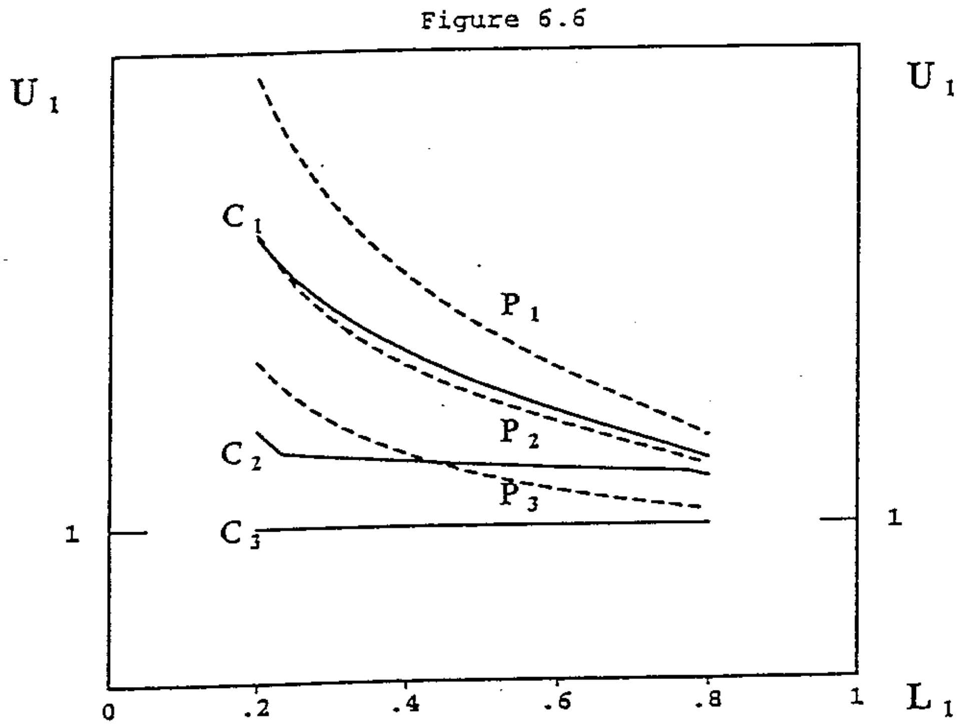

Figure 8.1 - Equilibria from an Economies Model Corresponding to Equilibria of a Linear Family

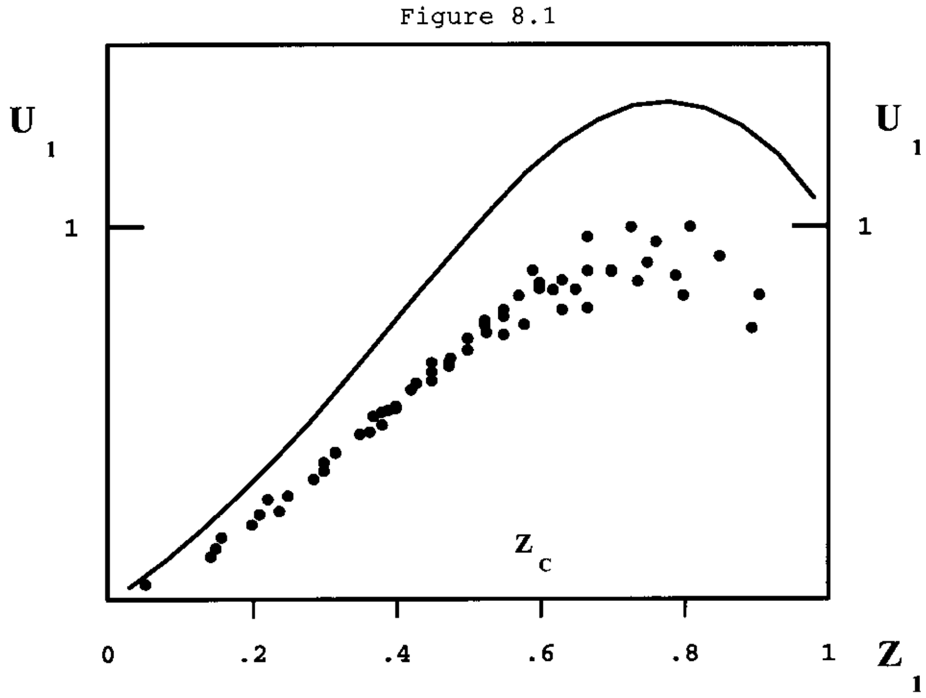

Figure F.1 - The (l,q) Diagram

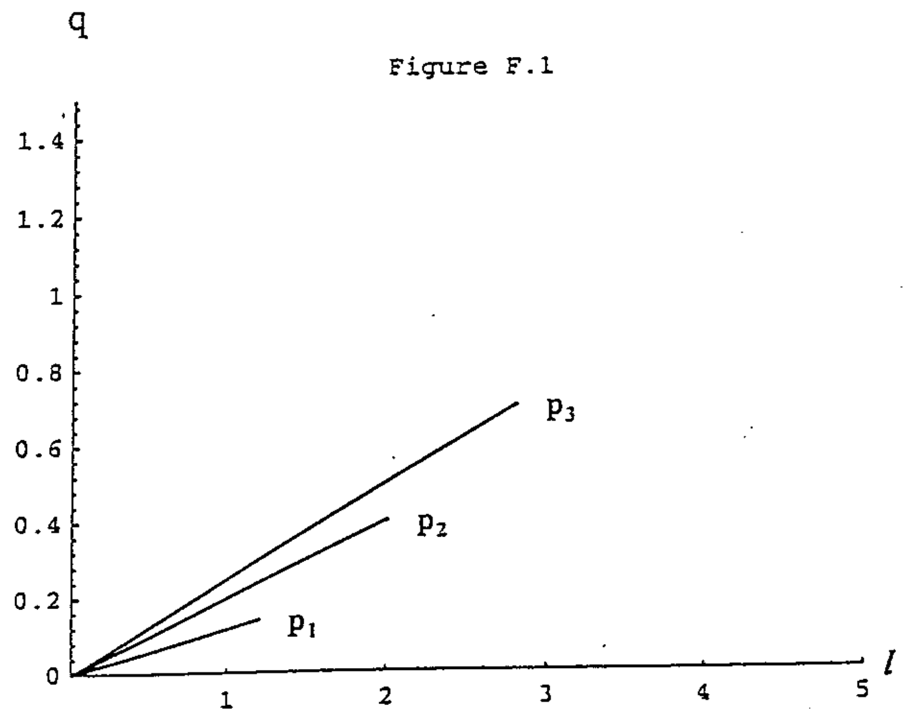

Figure F.2 - The Extended (l,q) Diagram