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OLD AND NEW MOVING-KNIFE SCHEMES

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INTRODUCTION

Over the past fifty years, the mathematical theory of fair division has often been formulated in terms of cutting a cake. More specifically, one seeks ways to divide a cake among n people so that each person is satisfied, in some sense, with the piece he or she receives, even though different people may value certain parts of the cake differently. (See [BT], [G], [K], [O] and [S3].)

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We focus in this paper on "moving-knife" schemes for fair division, but the earliest fair-division methods were quite different. When there are only two people ($n = 2$), the parental solution for appeasing two quarreling children of "one cuts, the other chooses" is well known, going back at least to Hesiod's Theogeny some 2,800 years ago [L,pp. 126-131]. A half century ago, Hugo Steinhaus [S1] first asked about generalizing cut-and-choose to more than two people. Since that time, three kinds of results have been obtained:

1. Existence results. One of the earliest results of this kind was Neyman's theorem [N], which established the existence, for n countably additive probability measures defined on the same space C, of a partition of C into n sets that is even (meaning that each of the sets is of measure $1/n$ with respect to all the measures). The proof, however, in the words of Rebman [R; p. 33], gives "no clue as to how to accomplish such a wonderful partition."

Note that any even division is envy-free (meaning that each person thinks he or she received a piece at least as large as those received by the other people), and that any envy-free division is proportional (meaning that each person thinks he or she received at least $1/n$ of the cake). The reader is invited to construct, for $n = 3$, examples showing that neither implication is reversible. For $n = 2$, of course, proportionality and envy-freeness are equivalent.

2. Discrete algorithms. The so-called last-diminisher procedure of Stefan Banach and Bronislaw Knaster (see [S1]) provides an early example of this kind of result, which is called a "protocol" in [BT], [G] and a "game-theoretic algorithm" in [B]. Under the Banach-Knaster procedure, the first person cuts a piece from the cake [that he or she considers to be of size $1/n$].[1] This piece is then passed, in turn, to each of the other people. Upon receiving such a piece, a person has the option either to pass it along unaltered to the next person [which is done if the person holding it considers it to be of size at most $1/n$], or to trim it [to size $1/n$ in his or her measure] and then pass it

[1]Our use of brackets will be explained later.

along. The last one to trim it - or the first person, if no one trimmed it - gets that piece as his or her share. The trimmings are returned to the cake, and the procedure is then repeated for the remaining cake. Note that a participant may receive a "piece" consisting of many subpieces that were widely separated in the original cake, which is a feature common to many discrete algorithms.

We leave it to the reader to check that if a player follows his or her strategy - and it is precisely these strategic aspects that we have placed in square brackets - then that player will receive a piece he or she thinks is of size at least 1/n, regardless of what strategy the other players employ. We shall say more later about the difference between strategies and rules.

3. Moving-knife procedures. This kind of continuous procedure seems first to have been proposed in 1961, when Lester Dubins and Edwin Spanier [DS] presented the following elegant version of the Banach-Knaster protocol: A knife is slowly moved across the cake, say from left to right. (Figure 1, borrowed from [A], illustrates this.) At any time, any player can call "cut" and then receive the piece to the left of the knife, with ties broken by some kind of random device. It is easy to see that if a player employs the obvious strategy of calling "cut" any time the piece so determined is of size exactly 1/n in his or her measure, then this will certainly yield him or her a piece of size at least 1/n. In contrast to discrete algorithms, moving-knife schemes tend, by exploiting continuity and intermediate values, to minimize the number of chunks each person receives.

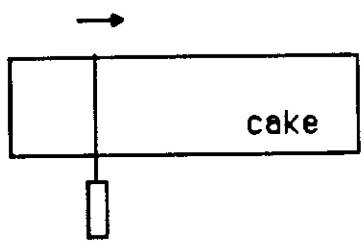

Figure 1

Any discrete algorithm or moving-knife procedure consists of both rules and strategies. Rules incorporate those parts of the procedure that can be enforced by a referee (because the referee can tell whether or not a rule has been followed without knowing the measures of the various players), whereas strategies are good advice to the players (which they can follow by using their knowledge of only their measure). To keep the distinctions clear, strategies are put in square brackets whereas rules are not. For example, a rule and strategy we shall use later is, "Player 1 divides the cake into three pieces [each of which is 1/3 of the cake in his or her opinion]."

For the sake of brevity, in what follows we ignore the possibility that all of the players involved in some part of a procedure abandon their strategies by choosing never to call "cut." The problem is easily addressed via an added rule that allocates the cake in some manner (e.g., according to some random device) should this situation arise.

We are not concerned, in this paper, with how an individual might exploit knowledge of another's measure to do even better than he or she would by following the suggested strategy. (Even cut-and-choose is sensitive to this kind of information [BT2].) In effect, then, we assume that none of the players has knowledge of others' measures, and we seek procedures that guarantee a certain payoff under this assumption. In fact, in the schemes we present, each player's strategy will guarantee him or her the appropriate payoff (either at least 1/n of the cake, or freedom from envy), even in the face of a conspiracy by the other players.

Our goal in the present paper is to present eight moving-knife schemes, in addition to that of Dubins and Spanier [DS] described earlier, several of which are new. Our main focus is on obtaining an envy-free division among three people - that is, one in which each player not only considers the piece he or she gets to be of size at least 1/3 but also to be a piece at least tied for largest among the three pieces allocated. Neither the

Banach-Knaster [S1] scheme, nor the Dubins-Spanier [DS] moving-knife version of it, guarantees an envy-free allocation. Five of the schemes to be presented yield such an allocation for three people.

Two of the envy-free schemes that we will present employ Austin's two-person scheme, to be described next.

AUSTIN'S TWO-PERSON EQUALIZING SCHEME

Recall that Neyman's [N] result guarantees the existence of a partition of the cake into n pieces such that every person thinks every piece is of size 1/n. For n = 2, this yields a single piece of cake that both people agree is of size exactly 1/2. In 1982, A. K. Austin [A] produced the following elegant scheme that achieves this using a pair of moving knives, wherein each player's strategy will guarantee that he or she receives a piece of size exactly 1/2:

Assume there is a single knife that moves slowly across the cake from the left edge toward the right edge, as in the Dubins-Spanier [DS] procedure, until one of the players - assume it is player 1 - calls "stop" [which he or she does at the point when the piece so determined is of size exactly 1/2]. At this time, a second knife is placed at the left edge of the cake. Player 1 then moves both knives across the cake in parallel fashion [in such a way that the piece between the two knives remains of size exactly 1/2 in player 1's measure], subject to the requirement (superfluous, if the strategies are followed) that when the knife on the right arrives at the right-hand edge of the cake, the left-hand knife lines up with the position that the first knife was in at the moment when player 1 first called "stop" (see Figure 2). While the two knives are moving, player 2 can call "stop" at any time [which he or she does precisely when the measure of the piece between the two knives is of size exactly 1/2 in his or her measure].

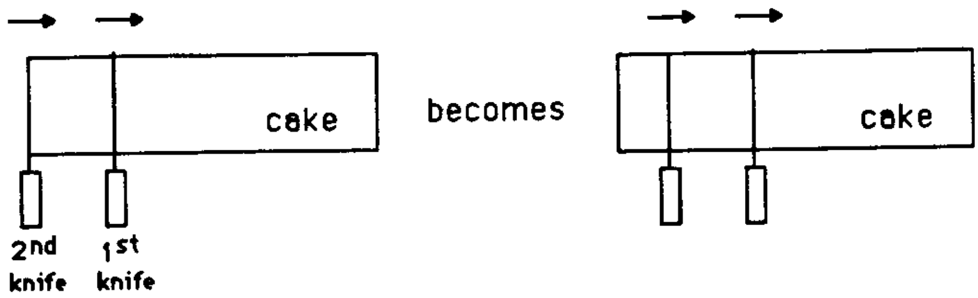

Figure 2

Now, what guarantees that there will be a point where player 2 thinks the piece between the knives is of size exactly 1/2? Notice that at the instant when the two knives start moving, player 2 thinks the piece between the knives is of size strictly less than 1/2 (assuming he or she has followed the strategy given). At the point when the two knives stop moving, the piece between the knives is the complement of what it was when the knives started moving. Hence, player 2 thinks the piece between the knives is now of measure strictly greater than 1/2. Thus, with an appropriate continuity assumption, there must be a point where the measure of the piece between the knives is <u>exactly</u> 1/2.

No generalization of Austin's scheme to $n > 2$ people is known. (We give later an additional reason why a generalization would be of interest.) However, Austin himself noted - and we will need this observation later - that a simple extension of his scheme produces a single piece of cake that each of two players thinks is of size exactly 1/k for any k. This extension proceeds as follows:

Player 1 first makes a sequence of k-1 parallel marks on the cake [in such a way that he or she thinks the k pieces so determined are all of size 1/k]. Now, player 2 cannot possibly think all k pieces are of size less than 1/k, and player 2 cannot possibly think all k pieces are of size greater than 1/k. Thus, either player 2 thinks one of the pieces is of size exactly 1/k - in which case we are done - or we can assume, without loss of generality, that he or she thinks the first piece is of size less than 1/k and the second piece is of size greater than 1/k. But now we can can have player 1 place knives on the left and right edges of the first piece, and move them as before [so as to keep the measure of the piece between the two knives at exactly 1/k], subject to the same sort of requirement as before. This argument shows that, at some point, player 2 will think the piece between the two knives has measure exactly 1/k.

An iteration of Austin's two-person scheme allows two players to partition the cake into j pieces, each of which is of size 1/j according to both players. For example, if j = 3, we begin by using Austin's two-person scheme to obtain a single piece of cake that both players think is of size 1/3. Now we apply the k = 2 version of Austin's two-person scheme to the rest of the cake. These latter two pieces have size 1/2 x 2/3 = 1/3 according to both players, as desired. In what follows, we refer to both the original and iterated version as Austin's two-person scheme.

AUSTIN'S VERSION OF FINK'S ALGORITHM

A. M. Fink [F] devised a clever alternative to the Banach-Knaster [S1] procedure. Fink's scheme is, like the Banach-Knaster procedure, a discrete algorithm that yields an allocation for n players wherein each player receives a piece of cake that he or she thinks is of size at least 1/n.

Austin [A] introduced his two-person scheme into Fink's algorithm, obtaining a moving-knife scheme that yields an allocation among n players in which each player thinks he or she receives a piece of size exactly 1/n. The scheme proceeds as follows:

Players 1 and 2 use Austin's two-person scheme to divide the cake into two pieces, A and B [so that both think A and B are of size 1/2]. Players 1 and 3 then cut a piece A' from A [that they both think is exactly 1/3 of A]. Players 2 and 3 now do exactly the same thing to B to obtain B' (see Figure 3).

Player 1 now receives A - A', player 2 receives B - B', and player 3 receives A' ∪ B'. It is easy to see that each player receives a piece of cake that he or she thinks is of size exactly 1/3.

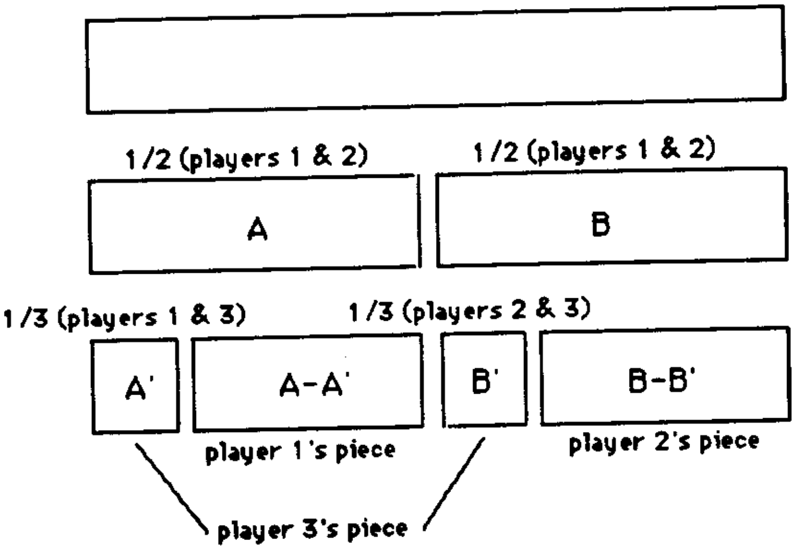

Figure 3

If a fourth person now comes along, each of the three earlier players simply gets together with this fourth person and cuts a small piece that he or she and the fourth person agree is 1/4 of the piece held by that player. The fourth person then gets the union of the three small pieces, etc.

STROMQUIST'S ENVY-FREE SCHEME

Probably the best known but most complicated moving-knife scheme is the envy-free procedure for three players due to Walter Stromquist [St]. This procedure begins with a referee holding a knife at the left edge of the cake. Each of the three players holds a knife parallel to the referee's [at a point that that player thinks exactly halves the remainder of the cake to the right of the referee's knife]. The referee moves his or her knife slowly across the cake, as was the case with all the previous procedures. The three players move their knives in the same way as the referee, with each player's keeping his or her knife to the right of the referee's [so that it exactly halves the piece to the right of the referee's knife, with respect to that player's measure].

At any time, a player can call "cut" and receive the piece to the left of the referee's knife (X in Figure 4). A cut is then also made by whichever of the three players' knives is in the middle (yielding Y and Z in Figure 4). Of the other two players, the one whose knife was closer to the referee's knife gets Y, and the other gets Z.

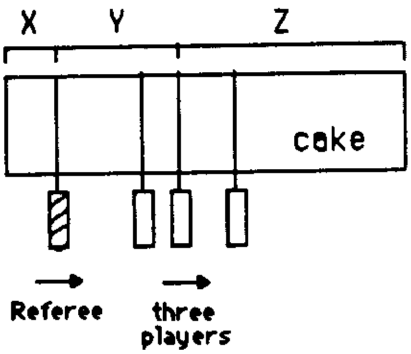

Figure 4

The strategy is for a player to call "cut" only if he or she thinks the left-hand piece X is at least as large as both the middle piece Y and the right-hand piece Z. (Misinterpretations of this strategy have caused some confusion in the literature; see [O] and [St].) Hence, the player calling "cut" will never envy the other two. Since neither of the other two players called "cut," they must each think the largest piece is either Y or Z. To see that neither of them experiences any envy, notice that each thinks he or she is getting the larger of Y and Z, or there is a tie. This is easy to check and left to the reader.

THE LEVMORE-COOK ENVY-FREE SCHEME

There is another procedure for producing an envy-free division among three people. It is due to Saul X. Levmore and Elizabeth Early Cook [LC] and seems to have been largely overlooked. It is essentially a moving-knife algorithm, although they describe it as a process with "infinitely small shavings." It can be described as follows:

Player 1 divides the cake into three pieces P, Q, R [that he or she considers equal]. Each of the other two players selects a piece [that he or she considers largest]. If they choose different pieces, we are done. Otherwise, we can assume they both choose P. Now player 1 starts a vertical moving knife, as in the Dubins-Spanier [DS] scheme, but at the same time he or she places a second knife perpendicular to the first and over the portion of the cake over which the vertical knife has already swept (see Figure 5).

Notice that if cuts were to be made from such a positioning of the knives, the piece of cake labeled P would be cut into three pieces, exactly two of which would involve both knives. Let S denote one of these two pieces and let T denote the other. The second knife is moved up and down [in such a manner that player 1 thinks Q ∪ S is the same size as R ∪ T].

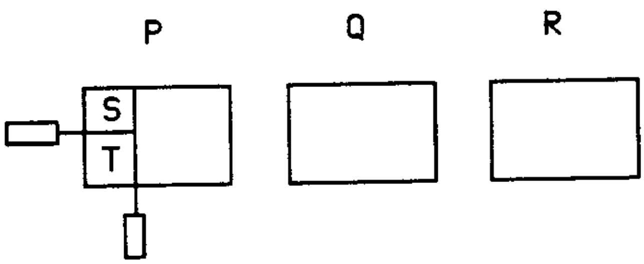

Figure 5

When the process begins, both S and T are the empty set, so player 2 and player 3 both think P-(S∪T) is larger than both Q∪S (giving two inequalities) and R∪T (giving two inequalities). Now let either player 2 or player 3 call "stop" [when he or she knows that any of these four inequalities first reverses], and take Q∪S or R∪T [whichever he or she thinks is bigger]. Player 1 gets the other composite piece, and the player who did not call "stop" gets P-(S∪T).

WEBB'S ENVY-FREE SCHEME

It turns out that by combining the basic idea in the Dubins-Spanier [DS] scheme with Austin's [A] two-person scheme, one can obtain a fairly simple moving-knife procedure that guarantees an envy-free allocation among three people. This scheme was first discovered by William Webb [W], although he was unaware of Austin's work and thus recreated the part of it he needed. The version we give next uses Austin's original scheme:

A knife is slowly moved across the cake, as in the Dubins-Spanier procedure, until some person - assume it is player 1 - calls "cut" [because he or she thinks the piece so determined is of size 1/3]. Call the piece resulting from this cut A1, and notice that players 2 and 3 both think A1 is of size at most 1/3.

We now have player 1 and either one of the other two players - assume for definiteness it is player 2 - apply Austin's two-person scheme to the rest of the cake, resulting in a partition of it into two sets, A2 and A3 [that players 1 and 2 think is a 50-50 division of the rest of the cake]. Notice that if the bracketed strategies are followed, then

1. Player 1 thinks that all three pieces are of size exactly 1/3.

2. Player 2 thinks A2 and A3 are tied for largest (since each is exactly 1/2 of a piece that is at least 2/3 of the whole cake).

An envy-free division is now easily obtained by having the players choose among the three pieces in the following order: player 3, player 2, player 1. Notice that player 3 envies no one, because he or she is choosing first; player 2 envies no one, because he or she had two pieces tied for largest; and player 1 envies no one, because he or she thinks all three pieces are the same size.

A PIE SCHEME FOR ENVY-FREE DIVISIONS WITH n = 3

Another conceptually simple envy-free moving-knife scheme for 3 players can be achieved by picturing a round cake (or pie, as in [G]) instead of a rectangular one. The idea of using a pie and radial knives seems to be a part of the cake-division folklore, but the following scheme is, as far as we know, new.

Start by having player 1 hold three knives over the round cake as if they were hands of a clock [in such a way that he or she considers the three wedged-shaped pieces to be all of size exactly 1/3]. Now have player 1 start moving all three knives in a clockwise fashion [so that each piece remains of size exactly 1/3 in his or her measure], subject to the requirement (superfluous if strategies are followed) that the moment any knife reaches the initial position of some other knife, all knives line up with such initial positions.

The claim is that, at some point, player 2 must think at least two of the wedges are tied for largest. That is, if player 2 thinks a single wedge (call it A) is largest at the instant when the knives start moving, then A is eventually transformed, as in Austin's scheme, to the wedge immediately clockwise. Thus, at some point prior to this, the piece determined by the two knives that originally determined A loses its position as largest to another piece. At the instant when this happens, we have the desired two-way tie for largest in the eyes of player 2.

The envy-free allocation is now obtained by having the players choose in the following order: player 3, player 2, player 1. This scheme can be recast as one in which three knives move in parallel across a rectangular cake (with the understanding that as a knife slides off the right edge, it immediately jumps back onto the left edge).

AN EASY THREE-PERSON ENVY-FREE SCHEME

Our final envy-free moving-knife scheme for three people is an immediate consequence of Austin's [A] two-person scheme for dividing a cake into three pieces such that each of two players thinks that the division is even. One simply has players 2 and 3 use that scheme to obtain a partition of the cake into three pieces [that they both think are all of size 1/3]. The players then choose the piece they want in the following order: player 1, player 2, player 3. Player 1 experiences no envy, because he or she is choosing first; and neither player 2 nor player 3 will experience envy, because each thinks all three pieces are the same size.

Perhaps the most important aspect of this three-person scheme is that it can be extended, at the cost of some complexity, to a moving-knife scheme that produces an envy-free allocation among <u>four</u> players. This scheme is described in [BTZ].

AN ALMOST ENVY-FREE SCHEME FOR $n > 3$

As we pointed out earlier, the Dubins-Spanier moving-knife scheme does not guarantee an envy-free allocation. The reason is that as soon as a player calls "cut," he or she is relegated to spectator status for the remainder of the procedure. Thus, if a larger piece should arise later, he or she has no recourse but to sit quietly by and watch one of his or her competitors get it.

Might we not alter the Dubins-Spanier [DS] procedure by allowing a player to re-enter the process in some way? One possibility that suggests itself is to allow a player to call "cut" again, even though he or she already has done so at least once and thus received a piece of cake. That player would then be required to take the new piece determined by this most recent cut, returning his or her previous piece to the cake.

What this yields is the following: Given n people and some $\varepsilon > 0$, there is a moving-knife scheme that will guarantee each player a piece of cake that he or she thinks is at most ε smaller than the largest. The scheme is simply the one we just described, and the concomitant strategy: each player calls "cut" initially whenever he or she thinks the piece this will yield is of size $1/n$, and thereafter calls "cut" whenever he or she thinks the new piece is ε larger than the one he or she presently holds. Ties are broken at random.

Unfortunately, the rules of this scheme - as opposed to the strategies just described - would allow a player to call "cut" infinitely many times. However, the strategies described are not affected by an additional rule which asserts that each player can call "cut" at most $1/\varepsilon$ times. Thus, if $\varepsilon = 1/100$, a player need never call "cut" more than 100 times to ensure that his or her piece is "almost" the largest - that is, smaller than the largest piece by at most 1/100 of the entire cake.

TOWARD AN ENVY-FREE SCHEME FOR ARBITRARY n

If Austin's [A] two-person scheme could be extended to n players, then one could immediately obtain an envy-free moving-knife scheme for n + 1 players by simply mimicking what we did earlier. More generally:

> If there exists a moving-knife scheme $\mathcal{A}$ that will divide a cake into n pieces so that each of n players think all the pieces are of size $1/n$, then there exists a moving-knife procedure for producing an envy-free division of the cake among n + 1 players.

To see how the envy-free procedure would work for n + 1 players, we begin - as in the Dubins-Spanier [DS] procedure - by obtaining a piece of cake that, say, player 1 thinks is of size exactly $1/(n+1)$ and everyone else thinks is of size at most $1/(n+1)$. We now have player 1, together with any n - 1 of the other players, divide up the rest of the cake into n pieces, using

$\mathscr{A}$, so that each thinks all n pieces are the same size. The player not involved in the application of $\mathscr{A}$ then gets to choose first, while player 1 is forced to choose last. The order in which the others choose is immaterial. Envy-freeness follows as before.

The above result shows that if we had a moving-knife scheme to divide a cake into four pieces so that each of four players would think it is an even division, then we could produce a moving-knife scheme that would yield an envy-free allocation among any five people. In fact, by a considerably more complicated argument, we show in [BTZ] that such an envy-free moving-knife scheme for five people would follow from a minimal extension of Austin's scheme: a partition of the cake into two pieces so that each of three players (instead of two) thinks it is a 50-50 division.

CONCLUSIONS

In general, moving-knife schemes seem to be easier to come by than pure existence results (like Neyman's [N] theorem), but harder to come by than discrete algorithms (like the Dubin's-Spanier [DS] last-diminisher method). In the case of envy-free allocations for four or more people, however, the order of difficulty might actually be reversed. Neyman's existence proof (for any n) goes back to 1946, the discovery of a discrete algorithm for all $n \geq 4$ is quite recent [BT 1,3], and a moving-knife solution for $n = 4$ was found only as this paper was being prepared (see [BTZ]). We are unaware of a moving-knife scheme giving an envy-free division for more than four players, however, so we conclude with a question:

Is there a moving-knife scheme that yields an envy-free division for five (or more) players?

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