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Abstract: This paper studies the conditions under which the basic results of the revealed preference theory can be established on the domain of choice problems which include non-convex feasible sets; the exercise is closely related to the works of Peters and Wakker (1991) and Bossert (1994). We show that while no continuous choice function can satisfy strong Pareto optimality and independence of irrelevant alternatives over the class of all compact and comprehensive choice problems, strong Pareto optimality, Arrow's choice axiom, upper hemicontinuity and a weak compromisation postulate turn out to be necessary and sufficient to represent choice correspondences by *continuous*, *strictly increasing* and *quasiconcave* real-valued functions. Some applications of our main findings to axiomatic bargaining theory are also studied.

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1. *Introduction*

The traditional revealed preference theory starts from the premise that one may be able to uncover the preferences of a consumer by observing the choices she makes when choosing a commodity bundle from her budget set. The basic insight, however, need not apply only to standard consumer choice problems; it is equally forceful with respect to more complex choice situations where the feasible sets may possess different characteristics than the usual budget sets. Peters and Wakker (1991) and Bossert (1994), for instance, advanced the idea that revealed preference theory can be generalized to incorporate cooperative bargaining problems á la Nash where the bargaining solutions are thought of as the recommendations of an impartial arbitrator, or as predictions about the outcome of the underlying non-cooperative bargaining game. In either case, the solutions of a Nash bargaining problem can be viewed as choices from the set of all feasible utility profiles, and therefore, it appears natural to study the conditions under which such solutions can be rationalized by a well-behaved preference ordering. Indeed, it is possible to pose the basic question of revealed preference theory in terms of arbitrary utility possibility sets familiar from theoretical welfare economics (cf. Bossert, 1996). One is thus led to extend the basic results of classical revealed preference theory to the case where the feasible set is not anymore restricted to a be a budget set in $\mathbf{R}^n_+$, but rather, it may take the form of any convex, compact and comprehensive subset of $\mathbf{R}^n_+$. This is indeed roughly the framework of the penetrating analyses of Peters and Wakker (1991) and Bossert (1994).[2]

While the choice situations covered by arbitrary convex, compact and comprehensive sets are far more general than the so-called budget problems, there are other interesting choice problems that do not fit in this framework. Indeed, a number of authors have argued that convexity of a bargaining problem is a stringent hypothesis, for it is solely motivated by appealing to a randomization argument which is valid only when the constituent parties are expected utility maximizers. When the expected utility hypothesis (which does not have a commendable experimental support) is dropped, the convexity postulate becomes suspect. Moreover, one may think of numerous instances where the second-best utility possibility set of a given economy is non-convex, and where convexification of this set by randomizing over the set of all alternatives is not possible. In consequence, the problem of extending the findings of Peters and Wakker (1991) and Bossert (1994) to a setting which includes non-convex choice situations arises. It is, in fact, precisely this problem that we aim to provide a solution in the present paper.

After introducing some preliminary concepts and nomenclature in Section 2, we present a collective choice problem (choice situation) as any compact and comprehensive set in $\mathbf{R}^n_+$ with a nonempty interior in Section 3. A choice function is any function that associates a choice problem with a feasible allocation while a choice correspondence is any multi-valued function that maps each choice situation to a nonempty set of feasible outcomes. We say

[2] A well-known branch of revealed preference theory takes as primitives a *finite* number of observations of individual choices from certain budget sets, and studies the conditions under which these *particular* choices can be rationalized by means of a well-behaved utility function (cf. Afriat, 1967). Indeed, there are even some studies that incorporate generalized (nonlinear) budget sets within this framework (cf. Matzkin, 1991). The present approach, however, differs from these studies; as it will become apparent shortly, it rather accords with the general treatment of the theory of "rational choice" as outlined in Richter (1971).

that a choice correspondence is (regularly) representable if it is obtained by maximizing a (continuous, strictly increasing and quasiconcave) real-valued function; see Section 4 for formal definitions. In Section 5, to characterize such choice correspondences, we consider the widely used properties of strong Pareto optimality (SPO) and Arrow's choice axiom (ACA) along with two weak regularity conditions, namely, upper hemicontinuity (UHC) and compromisation (C).

In Section 6 we show that no choice correspondence that satisfy these axioms can be single-valued over the class of choice problems under consideration. Since Arrow's choice axiom is equivalent to Nash's independence of irrelevant alternatives (IIA) axiom in the case of choice *functions*, this observation demonstrates that one may not hope to develop a useful revealed preference theory in our non-convex framework in terms of continuous choice functions which satisfy the compelling properties of SPO and IIA. Given the main findings of Peters and Wakker (1991) and Bossert (1994), it is thus evident that expanding the domain of choice problems to include non-convex situations entails considerable differences with regard to conclusions concerning representability of choice functions.

Interestingly, switching our focus from choice functions to correspondences, we can draw a markedly different picture. Indeed, the main result of this paper establishes that a choice correspondence satisfies SPO, ACA, UHC and C if, and only if, it is regularly representable; see Section 7. This result appears to indicate that the primitives of a revealed preference analysis for non-convex choice problems should be choice correspondences rather than choice functions.

Given our characterization of regularly representable choice correspondences, it is possible to learn more about the representing function by imposing further properties on the induced correspondence. To illustrate the potential of this observation, in Section 8, we use our characterization theorem in conjunction with a result due Weymark (1981) to obtain a complete characterization of the generalized Gini bargaining solutions of Blackorby, Bossert and Donaldson (1994). This concludes the present paper.

2. *Preliminaries*

Throughout this paper, we shall treat $n \in \{2, 3, ...\}$ as a fixed parameter and adopt the following notation for vector inequalities: For all $x, y \in \mathbf{R}^n$, $x \geq y$ iff $x_i \geq y_i$ for all i; $x > y$ iff $x \geq y$ and $x \neq y$; $x \gg y$ iff $x_i > y_i$ for all i. We also define $\mathbf{0}_n \equiv (0, ..., 0) \in \mathbf{R}^n$ and $\mathbf{1}_n \equiv (1, ..., 1) \in \mathbf{R}^n$. The ith unit vector is denoted by e^i, $i = 1, ..., n$. Finally, given a set X_m, by the notation $(a_m) \in X_m$, we mean that (a_m) is a sequence the mth term of which is $a_m \in X_m$, $m = 1, ...$.

Let R be a binary relation on $\mathbf{R}^n_+$: $R \subset \mathbf{R}^n_+ \times \mathbf{R}^n_+$. R is said to be a *complete preorder* if it is complete and transitive; it is called *asymmetric* if $\{(x, y), (y, x)\} \not\subseteq R$ for any $x, y \in \mathbf{R}^n_+$. We say that R is *upper* (resp. *lower*) *semicontinuous* if $\{y : (y, x) \in R\}$ (resp. $\{y : (x, y) \in R\}$) is closed. R is continuous if it is both upper and lower semicontinuous. Finally, the *transitive closure* of R is defined as $\bigcup_{m=0}^{\infty} R_m$ where $R_0 = R$, and $x\, R_m\, y$ if and only if there exist an $m \geq 1$ and $z^1, ..., z^m \in \mathbf{R}^n_+$ such that $x\, R\, z^1\, R \cdots z^m\, R\, y$.

A set $S \subset \mathbf{R}^n_+$ is called *comprehensive* if, for any $x \in S$, $x \geq y \geq 0$ implies that $y \in S$; it is called *strictly comprehensive* if it is comprehensive and for any $x, y \in S$, $x \geq y$ implies that there exists a $z \in S$ such that $z \gg y$. Given any $S \subset \mathbf{R}^n_+$, the *comprehensive hull* of S, *ch* S, is defined as the smallest comprehensive set containing S. The *convex comprehensive hull* of S, *cch* S, is defined as the smallest comprehensive and convex set containing S.

Let $\mathcal{C}^n$ denote the set of all nonempty and compact subsets of $\mathbf{R}^n$. $\mathcal{C}^n$ is made a metric space by the *Hausdorff distance*, ρ, defined as

$$\rho(S,T) \equiv \max\left\{\sup_{x\in S} d(x,T),\ \sup_{y\in T} d(y,S)\right\} \quad \text{for all } S,T \in \mathcal{C}^n$$

where $d(x,A) \equiv \inf_{y\in A} \|x-y\|$ for any $A \in \mathcal{C}^n$. It is easy to see that $\rho(S,T) \equiv \inf\{\delta > 0 : S \subseteq N_\delta(T) \text{ and } T \subseteq N_\delta(S)\}$ where $N_\delta(A) \equiv \{y \in \mathbf{R}^n : \exists x \in A : [\|x-y\| < \delta]\}$ for any $A \in \mathcal{C}^n$ (cf. Kuratowski, 1966).

3. *Collective Choice Problems*

The class of all compact and comprehensive sets in $\mathbf{R}^n_+$ which have a nonempty intersection with $\mathbf{R}^n_{++}$ is denoted by $\mathbf{\Omega}^n$. In what follows, we treat $\mathbf{\Omega}^n$ as a metric subspace of $\mathcal{C}^n$. The set of all convex members of $\mathbf{\Omega}^n$ is denoted by $\mathbf{\Omega}^n_{\text{con}}$.

A set $S \in \mathbf{\Omega}^n$ is referred to as a *collective choice problem* (or as a *choice situation*). A choice problem may be interpreted as a set of feasible utility levels of the constituents of the society, although other interpretations are possible. The comprehensiveness postulate reflects the free-disposability of utilities (alternatives). While compactness is assumed for mathematical convenience, the existence of at least one $x \gg \mathbf{0}_n$ in $S \in \mathbf{\Omega}^n$ eliminates degenerate problems.

There are several specific models of group-decision making which fall within the boundaries of this abstract setting. For example, under some plausible assumptions, the standard *utility possibility sets* of classical welfare economics are well-defined collective choice problems (see Bossert, 1996). Alternatively, each member of $\mathbf{\Omega}^n$ can be interpreted as a cooperative bargaining problem in the sense of Nash (1950) where the disagreement point is taken to be the origin. (See Thomson, 1994, for an excellent survey of the related literature.) Classical choice sets like linear and/or nonlinear budget sets are also included in $\mathbf{\Omega}^n$.

The main difference of our approach from the standard collective choice models lies in the fact that $\mathbf{\Omega}^n$ includes non-convex problems along with the convex ones. Consequently, the cooperative bargaining problems where randomization over the whole set of alternatives is not possible, or where some individuals are not expected utility maximizers, are included in our domain of problems.[3] Similarly, the utility possibility sets induced by classical economies with externalities may be non-convex even when utility functions are well-behaved. In such situations, it is conceivable that stochastic policy-making may not be admissible, and if so, one would be forced to deal with non-convex choice problems once again.

[3]The relevance of non-convex bargaining sets are well-recognized in the literature on axiomatic bargaining theory; see, for instance, Kaneko (1980), Herrero (1989), and Conley and Wilkie (1991, 1996).

4. *Choice Functions and Correspondences*

A *choice function* on $\mathbf{\Omega}^n$ is defined as any function $F : \mathbf{\Omega}^n \to \mathbf{R}^n_+$ such that $F(S) \in S$ for all $S \in \mathbf{\Omega}^n$. The interpretation of F depends on the choice problem under consideration. If $S \in \mathbf{\Omega}^n$ is interpreted as a bargaining problem, for instance, $F(S)$ can be thought of as the resolution suggested by an impartial arbitrator, or a prediction concerning the outcome of the underlying strategic bargaining game. If S is, on the other hand, a budget problem, $F(S)$ can be interpreted as the demand of the individual.

A natural generalization of the notion of a choice function leads us to that of a *choice correspondence* which is defined as any correspondence on $\mathbf{\Omega}^n$ such that $\emptyset \neq F(S) \subseteq S$ for all $S \in \mathbf{\Omega}^n$. Where $S \in \mathbf{\Omega}^n$ is interpreted as a bargaining problem, for instance, $F(S)$ could be viewed as the set of *all* recommendations of an impartial arbitrator.

As in the classical revealed preference theory, we are particularly interested in this paper with choice functions that are generated by an optimization exercise. A choice function F is said to be *rationalizable* if there exists a complete preorder $\succcurlyeq$ on $\mathbf{R}^n_+$ such that $F(S) \succcurlyeq y$ for all $y \in S$ and all $S \in \mathbf{\Omega}^n$. A well-known necessary and sufficient condition for rationalizability of F is the following: Where $R_F \subset \mathbf{R}^n_+ \times \mathbf{R}^n_+$ is defined as $x \, R_F \, y$ iff $y \in S$ and $x = F(S)$ for some $S \in \mathbf{\Omega}^n$, the transitive closure of R_F is asymmetric. (See, for instance, Richter, 1971, Corollary 1.) This condition is usually referred to as Houthakker's *strong axiom of revealed preference* (SARP). If R_F is itself asymmetric, then F is said to satisfy the *weak axiom of revealed preference* (WARP) on $\mathbf{\Omega}^n$.

A choice correspondence F is said to be *rationalizable* if there exists a complete preorder $\succcurlyeq$ on $\mathbf{R}^n_+$ such that $F(S)$ is the set of $\succcurlyeq$-greatest elements in S for all $S \in \mathbf{\Omega}^n$. A necessary and sufficient condition for rationalizability of a choice correspondence is Richter's *congruence axiom* which states that, for all $S \in \mathbf{\Omega}^n$, if $x \in S$ and $x \, R_F \, y$ for all $y \in S$, then $x \in F(S)$. (See, Richter, 1971, Theorem 8).

A choice function F is said to be *representable* if there exists a $W : \mathbf{R}^n_+ \to \mathbf{R}$ such that

$$F(S) \in \arg\max_{a \in S} W(a) \quad \text{for all } S \in \mathbf{\Omega}^n,$$

whereas a choice correspondence F is called *representable* if there exists a $W : \mathbf{R}^n_+ \to \mathbf{R}$ such that

$$F(S) = \arg\max_{a \in S} W(a) \quad \text{for all } S \in \mathbf{\Omega}^n.$$

In either case, W is said to *represent* F. (Of course, a representable choice correspondence (or function) is necessarily rationalizable.) We shall say that F is *regularly representable* if it is representable by a strictly monotonic, quasiconcave and continuous real-valued function. It is important to note that while a regularly representable choice correspondence is single-valued over strictly convex choice situations, it cannot be single-valued on $\mathbf{\Omega}^n$.

5. *A Basic Set of Axioms*

In this section, we introduce a basic set of axioms for choice functions and correspondences. To provide a coherent presentation, we formulate these axioms in terms of choice correspondences with the understanding that a single-valued choice correspondence is equivalent to a choice function. Since most of the following axioms are studied extensively in the literature, we shall keep our related discussion brief.

Our first axiom is certainly very widely used, and is hardly exceptionable.

Strong Pareto Optimality (SPO): For all $S \in \Omega^n$ and all $x \in F(S)$, there does not exist a $y \in S$ such that $y > x$.

Another widely used axiom (introduced by Arrow, 1959) requires a choice correspondence to be consistent in its choices with respect to contractions of the choice set:

Arrow's Choice Axiom (ACA): For all $S, T \in \Omega^n$, if $T \subseteq S$ and $F(S) \cap T \neq \emptyset$, then $F(T) = F(S) \cap T$.

Notice that ACA reduces to Nash's *independence of irrelevant alternatives* (IIA) axiom in the case of choice functions: For all $S, T \in \Omega^n$, if $T \subseteq S$ and $F(S) \in T$, then $F(T) = F(S)$.

We note that ACA, and hence IIA, are much stronger conditions on Ω^n than they are on Ω^n_{con}. Indeed, by a result due Hansson (1968), IIA is equivalent to WARP on Ω^n since Ω^n is closed under intersections, while WARP is equivalent to SARP on Ω^n since Ω^n is closed under finite unions. (It is well-known that the latter equivalence fail on Ω^n_{con}.) Therefore, a choice function F on Ω^n satisfies IIA if, and only if, it is rationalizable. Zhou (1996) shows that if F further satisfies SPO, then it is in fact representable by a real-valued function. As we shall see, similar results hold for ACA as well.[4]

Our next axiom is a commonly used continuity requirement:

Upper Hemicontinuity (UHC): For any $S \in \Omega^n$ and open $O \subset \mathbf{R}^n_+$ such that $F(S) \subseteq O$, there exists a $\delta > 0$ such that $F(T) \subseteq O$ for all $T \in N_\delta(S)$.

In words, UHC says that infinitesimal changes in the choice problems should not cause dramatic changes in the choices that F dictates. It may thus be viewed as a rather weak regularity condition for choice correspondences. We should also note that UHC reduces to the usual continuity axiom familiar from cooperative bargaining theory in the case of choice functions. (We thus refer to single-valued choice correspondences that satisfy UHC simply as continuous choice functions.)

Our final axiom concerns the possibility of ceratin compromises between two choices in convex choice problems.

Compromisation (C): For any $S \in \Omega^n_{\text{con}}$, $\#F(S) \neq 2$.

[4]The following weakening of ACA is sometimes used in the related literature: (*Dual Chernoff Axiom*) For all $S, T \in \Omega^n$, if $T \subseteq S$ and $F(S) \cap T \neq \emptyset$, then $F(T) \subseteq F(S) \cap T$. We note that all of the results reported in this paper remain valid if we replace ACA with the dual Chernoff axiom. This is because an upper hemicontinuous choice correpondence that satisfies SPO and the dual Chernoff axiom necessarily satisfies ACA; the proof of this claim is identical to that of Lemma 1 of Blackorby et al. (1994).

The intuition behind this axiom is fairly straightforward. If $\#F(S) = 2$, then only two "extreme" outcomes are chosen, and no compromise between these outcomes are allowed. Axiom C, therefore, requires that at least one compromise between two choices are always in the choice set in convex choice situations.[5]

We should note that Blackorby et al. (1994, 1996) use the property of *connected-valuedness* of F on $\mathbf{\Omega}^n_{\text{con}}$ to study choice correspondences which allow for compromisation. This property is, however, technically demanding, and C (which is obviously a far weaker postulate) seems to be more readily acceptable in comparison. In fact, all of the results in this paper would remain true, if we have replaced C with the requirement that $F(S)$ be *convex* for all $S \in \mathbf{\Omega}^n_{\text{con}}$. (See Lemma 2.) What is more, C can be dropped from our basic list of postulates so long as one is not interested in *quasiconcave* representations of choice correspondences. (Compare Theorems 1 and 2).

We conclude this section by noting a number of elementary observations which clarify the basic structure that axioms SPO, UHC and ACA bring in.

Lemma 1: *Let F be a choice correspondence defined on $\mathbf{\Omega}^n$ which satisfies SPO and UHC. For any $x \in \mathbf{R}^n_+$ and any sequence $(y^m) \in \mathbf{R}^n_+$ such that $y^m \to y$, we have*

$$\exists M > 0 : [(\{y\} = F(ch\{x,y\}) \text{ and } x \neq y) \Rightarrow (\forall m \geq M : [\{y^m\} = F(ch\{x,y^m\})])] \quad (1)$$

and

$$\exists M > 0 : [(\{x\} = F(ch\{x,y\}) \text{ and } x \neq y) \Rightarrow (\forall m \geq M : [\{x\} = F(ch\{x,y^m\})])]. \quad (2)$$

If, in addition, F satisfies ACA, then, for any $x \in \mathbf{R}^n_+$,

$$\forall \alpha > 0 : [(\{x, \alpha \mathbf{1}_n\} = F(ch\{x, \alpha \mathbf{1}_n\}) \Rightarrow (\forall \gamma \in (0, \alpha) : [\{x\} = F(ch\{x, \gamma \mathbf{1}_n\})])] \quad (3)$$

and

$$\forall \alpha > 0 : [(\{x, \alpha \mathbf{1}_n\} = F(ch\{x, \alpha \mathbf{1}_n\}) \Rightarrow (\forall \beta > \alpha : [\{\beta \mathbf{1}_n\} = F(ch\{x, \beta \mathbf{1}_n\})])]. \quad (4)$$

Proof. To see (1), take an $\epsilon > 0$ such that $x \notin N_\epsilon(y)$. (Here $N_\epsilon(y)$ stands for the ϵ-neighborhood of y in the standard topology.) By UHC, there exists a $\delta > 0$ such that $F(T) \subseteq N_\epsilon(y)$ for all $T \in N_\delta(ch\{x,y\})$. But since $y^m \to y$ implies $ch\{x,y^m\} \to ch\{x,y\}$ as $m \to \infty$, there must exist an $M > 0$ such that $ch\{x,y^m\} \in N_\delta(ch\{x,y\})$ for all $m \geq M$. Thus, $F(ch\{x,y^m\}) \subset N_\epsilon(y)$ for all $m \geq M$, and since $x \notin N_\epsilon(y)$, $x \notin F(ch\{x,y^m\})$ for all $m \geq M$. By SPO, therefore, we have $\{y^m\} = F(ch\{x,y^m\})$ for all $m \geq M$. (2) is shown to hold similarly.

(3) readily follows from the fact that $x \in F(ch\{x, \gamma \mathbf{1}_n\})$ holds by ACA for any $\alpha > \gamma > 0$ and $\{x, \alpha \mathbf{1}_n\} = F(ch\{x, \alpha \mathbf{1}_n\})$. So if (3) was false, by SPO and ACA, we would have

$$\{x, \gamma \mathbf{1}_n\} = F(ch\{x, \gamma \mathbf{1}_n\}) = F(ch\{x, \alpha \mathbf{1}_n\}) \cap ch\{x, \gamma \mathbf{1}_n\} = \{x\}$$

[5]Notice that C conditions choice correspondences only with respect to convex problems, for it is evident that compromises between alternative solutions may sometimes be inefficient in non-convex choice situations. (Consider a choice problem like $ch\{x,y\}$ where x and y are equally desirable, for instance.)

yielding $x = \gamma\mathbf{1}_n$. But then by SPO, $x \notin F(ch\{x, \alpha\mathbf{1}_n\})$ since $\alpha > \gamma$, contradiction.

Clearly, if $x = \alpha\mathbf{1}_n$, then (4) trivially holds, so assume that $x \neq \alpha\mathbf{1}_n$, and take any $\beta > \alpha$. If $\beta\mathbf{1}_n \notin F(ch\{x, \beta\mathbf{1}_n\})$, by SPO, $\{x\} = F(ch\{x, \beta\mathbf{1}_n\})$ must hold, and thus by ACA,

$$\{x, \alpha\mathbf{1}_n\} = F(ch\{x, \alpha\mathbf{1}_n\}) = F(ch\{x, \beta\mathbf{1}_n\}) \cap ch\{x, \alpha\mathbf{1}_n\} = \{x\}$$

contradicting $x \neq \alpha\mathbf{1}_n$. If, on the other hand, $\{x, \beta\mathbf{1}_n\} = F(ch\{x, \beta\mathbf{1}_n\})$, we must have $\{x\} = F(ch\{x, \alpha\mathbf{1}_n\})$ by (3), and this contradicts $\{x, \alpha\mathbf{1}_n\} = F(ch\{x, \alpha\mathbf{1}_n\})$. □

6. *A Difficulty with Continuous Choice Functions*

There are, of course, many choice functions on $\mathbf{\Omega}^n$ which satisfy SPO and IIA; for instance, any choice function which is representable by a strictly increasing real-valued function trivially satisfies these axioms. However, it is easily observed that a *continuous* choice function cannot satisfy both SPO and IIA. This is demonstrated next.

Proposition 1: *There does not exist a continuous choice function on $\mathbf{\Omega}^n$ which satisfies SPO and IIA.*

Proof. Let F be a choice function on $\mathbf{\Omega}^n$ which satisfies SPO, IIA and C. As we have noted in Section 5, IIA is equivalent to rationalizability on $\mathbf{\Omega}^n$; there must then exist a complete preorder $\succcurlyeq$ on $\mathbf{R}^n_+$ which rationalizes F. Let $\sim \equiv \succcurlyeq \cap \{(x,y) : (y,x) \in \succcurlyeq\}$. It is readily observed that $[x]_\sim \equiv \{y \in \mathbf{R}^n_+ : x \sim y\}$ is a singleton for any $x \in \mathbf{R}^n_+$. Indeed, let $x \sim y$ for some $x \neq y$, and suppose $F(ch\{x,y\}) = x$ without loss of generality. But then by continuity, SPO and the fact that $\succcurlyeq$ rationalizes F,

$$x = F(ch\{x,y\}) = F\left(\lim_{m\to\infty} ch\left\{\left(1 - \frac{1}{m}\right)x, y\right\}\right) = \lim_{m\to\infty} F\left(ch\left\{\left(1 - \frac{1}{m}\right)x, y\right\}\right) = y,$$

contradiction. So, $\#[x]_\sim = 1$.

We now proceed to show that $\succcurlyeq$ must be continuous. Let us first establish that $L(x) \equiv \{y \in \mathbf{R}^n_+ : x \succcurlyeq y\}$ is closed for any $x \in \mathbf{R}^n_+$. Take any $(y^m) \in L(x)$ and let $y^m \to y$. If $y \succ x$, then clearly $y = F(ch\{x,y\})$, and by (1), there exists an $M > 0$ such that $y^m = F(ch\{x, y^m\})$ for all $m \geq M$. Since $\succcurlyeq$ rationalizes F, we must have $y^m \succcurlyeq x$ for all $m \geq M$, and hence, we obtain $y^m \sim x$ for all $m \geq M$. But since $\#[x]_\sim = 1$, $y^m = x$ for all $m \geq M$ so that $y = x$ contradicting $y \succ x$. $L(x)$ is thus closed, and we may conclude that $\succcurlyeq$ is lower semicontinuous. The upper semicontinuity of $\succcurlyeq$ is observed similarly. $\succcurlyeq$ is thus continuous. Since a monotonic and continuous complete preorder on $\mathbf{R}^n_+$ cannot have singleton indifference classes, Proposition 1 follows.[6] □

This result, of course, does not hold for choice functions defined only for convex choice problems in $\mathbf{\Omega}^n$. For example, both the Nash and the utilitarian choice functions are continuous, and satisfy both SPO and IIA on $\mathbf{\Omega}^n_{\text{con}}$.

[6] For any $x \in \mathbf{R}^n_+$ such that $x_i \neq x_j$ for some $i \neq j$, by SPO, there must exist $\beta > \alpha > 0$ such that $\beta\mathbf{1}_n \succ x \succ \alpha\mathbf{1}_n$, and by continuity of $\succcurlyeq$, $x \sim \gamma\mathbf{1}_n$ for some $\gamma \in (\alpha, \beta)$ while $x \neq \gamma\mathbf{1}_n$.

7. *Regularly Representable Choice Correspondences*

The main result of this section is an axiomatic characterization of the regularly representable choice correspondences. Interestingly, while there does not exist a choice function on $\mathbf{\Omega}^n$ which satisfies SPO, ACA and UHC, one can easily see that any regularly representable choice correspondence on $\mathbf{\Omega}^n$ satisfies SPO, ACA and UHC. Moreover, we shall show in this section that the converse of this statement would also hold if we confine our attention to correpondences that satisfy C. Put precisely, the class of all regularly representable choice correspondences on $\mathbf{\Omega}^n$ is equal to the class of all choice correspondences on $\mathbf{\Omega}^n$ which satisfy SPO, ACA, UHC and C.

We begin by proving two lemmas that we shall use in proving the main theorem.

Lemma 2: *Let* $W : \mathbf{R}^n_+ \to \mathbf{R}$ *be a continuous, quasiconcave and strictly increasing function. If* F *is a choice correspondence on* $\mathbf{\Omega}^n$ *such that* $F(S) = \arg\max_{a \in S} W(a)$ *for all* $S \in \mathbf{\Omega}^n$*, then it satisfies SPO, ACA, UHC and is convex-valued on* $\mathbf{\Omega}^n_{\text{con}}$.[7]

Proof. That F must satisfy SPO and ACA is obvious. On the other hand, the upper hemicontinuity of F is an immediate consequence of the maximum theorem (Berge, 1963, p. 116). Finally, since $F(S)$ is the intersection of S and an upper contour set of W, $F(S)$ must be convex for all $S \in \mathbf{\Omega}^n_{\text{con}}$. □

The next lemma reports a more general observation than what is needed for our main result. In particular, it is the counterpart of a result obtained for choice functions on $\mathbf{\Omega}^n_{\text{con}}$ by Peters and Wakker (1991, Lemma 5.4) in the case of choice correspondences.

Lemma 3: *Let* $W : \mathbf{R}^n_+ \to \mathbf{R}$ be a *continuous and strictly increasing function, and let* F *be a choice correspondence on* $\mathbf{\Omega}^n$ *that satisfies C. If* $F(S) = \arg\max_{a \in S} W(a)$ *for all* $S \in \mathbf{\Omega}^n_{\text{con}}$*, then* W *must be quasiconcave.*

Proof. Take any $z \in \mathbf{R}^n_+$ and consider the upper contour set of z, $U(z) \equiv \{y \in \mathbf{R}^n_+ : W(y) \geq W(z)\}$. Suppose that $U(z)$ is not convex, that is, there exist $x, y \in U(z)$ and a $\lambda \in [0,1]$ such that $\lambda x + (1-\lambda)y \equiv w \notin U(z)$. We define

$$\alpha \equiv \max\{\theta \in [0,1] : \theta w + (1-\theta)x \in U(z)\}$$

and

$$\beta \equiv \max\{\theta \in [0,1] : \theta w + (1-\theta)y \in U(z)\}.$$

By continuity of W, both α and β are well-defined, and $0 < \alpha, \beta < 1$. Define

$$x' \equiv \alpha w + (1-\alpha)x \quad \text{and} \quad y' \equiv \beta w + (1-\beta)y.$$

[7]This proposition might at first seem at odds with Theorem 1. But notice that a choice correspondence that satisfies the hypotheses of Lemma 2 *cannot* be single-valued.

Since w is a convex combination of x and y, it is obvious that $x' \neq y'$. Moreover, $W(x') = W(y')$ since $x', y' \in \partial U(z)$ by the choice of α and β. Therefore, since W is strictly increasing and since int $(co\{x', y'\}) \cap U(z) = \emptyset$, we have

$$F(cch\{x', y'\}) = \underset{a \in cch\{x', y'\}}{\arg\max}\ W(a) = \{x', y'\}$$

contradicting C. □

The following two results constitute our main findings.

Theorem 1: *Let F be a choice correspondence defined on $\mathbf{\Omega}^n$. F satisfies SPO, ACA and UHC if, and only if, there exists a continuous and strictly increasing $W : \mathbf{R}^n_+ \to \mathbf{R}$ that represents F.*

Proof. Sufficiency is proved in Lemma 2. To establish necessity, we define

$$W(x) \equiv \sup\{\alpha > 0 : x \in F(ch\{x, \alpha\mathbf{1}_n\})\} \quad \text{for all } x \in \mathbf{R}^n_+,$$

and note that $0 < W(x) \leq \max_i x_i$ for all $x \in \mathbf{R}^n_+$ by SPO. It is easy to see that if

$$\forall x, y \in \mathbf{R}^n_+ : [F(ch\{x, y\}) = \underset{a \in ch\{x,y\}}{\arg\max}\ W(a)], \tag{5}$$

then we must have $F(S) = \arg\max_{a \in S} W(a)$ for all $S \in \mathbf{\Omega}^n$. To see this, assume that (5) holds, and let $F(S) \not\subseteq \arg\max_{a \in S} W(a)$ for some $S \in \mathbf{\Omega}^n$, i.e., let there be an $(x, y) \in F(S) \times S$ such that $W(y) > W(x)$. But then by ACA, SPO and (5),

$$x \in F(S) \cap ch\{x, y\} = F(ch\{x, y\}) = \{y\},$$

contradiction. If, on the other hand, $\arg\max_{a \in S} W(a) \not\subseteq F(S)$ for some $S \in \mathbf{\Omega}^n$, there must exist an $x \in S \backslash F(S)$ such that $W(x) \geq W(y)$ for all $y \in S$. But then by ACA and (5),

$$x \notin F(S) \cap ch\{x, y\} = F(ch\{x, y\}) = \{x\}$$

holds for any $y \in S$, contradiction.

Consequently, that W represents F will be established if we can prove (5). To this end, we shall demonstrate a further preliminary observation:

$$\forall x \in \mathbf{R}^n_+ : [F(ch\{x, W(x)\mathbf{1}_n\}) = \{x, W(x)\mathbf{1}_n\}]. \tag{6}$$

Take any $x \in \mathbf{R}^n_+$ and let $\alpha_m \equiv W(x) - 1/m$, $m = 1, ...,$ and notice that by definition of W and ACA, we have

$$x \in F(ch\{x, \alpha_m\mathbf{1}_n\}) \quad \text{for all } m = 1, \dots . \tag{7}$$

But then if $\alpha_m\mathbf{1}_n \in F(ch\{x, \alpha_m\mathbf{1}_n\})$ for some $m \geq 1$, (4) implies that $\{\alpha_{m+1}\mathbf{1}_n\} = F(ch\{x, \alpha_{m+1}\mathbf{1}_n\})$ contradicting (7). We must thus have $\{x\} = F(ch\{x, \alpha_m\mathbf{1}_n\})$ for all

m, and since $\lim_{m\to\infty} ch\{x, \alpha_m \mathbf{1}_n\} = ch\{x, W(x)\mathbf{1}_n\}$, UHC yields $x \in F(ch\{x, W(x)\mathbf{1}_n\})$. But $W(x)\mathbf{1}_n \in F(ch\{x, W(x)\mathbf{1}_n\})$, because otherwise $\{x\} = F(ch\{x, W(x)\mathbf{1}_n\})$, and by (2), $x \in F(ch\{x, (W(x) + 1/m)\mathbf{1}_n\})$ for large enough m, which is impossible by definition of W. (6) is thus established.

We are now ready to prove (5). Take any $x, y \in \mathbf{R}^n_+$ such that $\alpha \equiv W(x) = W(y)$. If $\{x, \alpha\mathbf{1}_n\} \not\subseteq F(ch\{x, y, \alpha\mathbf{1}_n\})$, then by ACA and (6),

$$\{x, \alpha\mathbf{1}_n\} \not\subseteq F(ch\{x, y, \alpha\mathbf{1}_n\}) \cap ch\{x, \alpha\mathbf{1}_n\} = F(ch\{x, \alpha\mathbf{1}_n\}) = \{x, \alpha\mathbf{1}_n\},$$

contradiction. Thus, $\{x, \alpha\mathbf{1}_n\} \subseteq F(ch\{x, y, \alpha\mathbf{1}_n\})$, and that $y \in F(ch\{x, y, \alpha\mathbf{1}_n\})$ as well is similarly observed. But then by ACA

$$F(ch\{x, y\}) = F(ch\{x, y, \alpha\mathbf{1}_n\}) \cap ch\{x, y\} = \{x, y, \alpha\mathbf{1}_n\} \cap ch\{x, y\} = \{x, y\},$$

and (5) is verified. Now take any $x, y \in \mathbf{R}^n_+$ such that $W(x) > W(y)$ so that there exists a $\beta > 0$ such that $W(x) > \beta = W(\beta\mathbf{1}_n) > W(y)$. But then, by SPO and the definition of W, $F(ch\{y, \beta\mathbf{1}_n\}) = \{\beta\mathbf{1}_n\}$. Moreover, since $F(ch\{x, W(x)\mathbf{1}_n\}) = \{x, W(x)\mathbf{1}_n\}$ by (6), we must have $F(ch\{x, \beta\mathbf{1}_n\}) = \{x\}$ by (3). Consequently, applying ACA twice we learn that $y, \beta\mathbf{1}_n \notin F(ch\{x, y, \beta\mathbf{1}_n\})$ so that

$$F(ch\{x, y\}) = F(ch\{x, y, \alpha\mathbf{1}_n\}) \cap ch\{x, y\} = \{x\} \cap ch\{x, y\} = \{x\},$$

and (5) is established. We therefore conclude that W represents F.

Since strict monotonicity of W is immediate from SPO, all we need to prove is its continuity. We thus claim that $\{y \in \mathbf{R}^n_+ : W(y) \leq W(x)\}$ is closed for any x. To see this, let $y^m \to y$ (as $m \to \infty$) where $W(y^m) \leq W(x)$ for all m. If $W(y) > W(x)$, there must exist an $\alpha > 0$ such that $W(y) > \alpha = W(\alpha\mathbf{1}_n) > W(x)$ so that, by (6) and (3), $F(ch\{y, \alpha\mathbf{1}_n\}) = \{y\}$. But then by (1), $y^m \in F(ch\{y^m, \alpha\mathbf{1}_n\})$ for m large enough, and since W rationalizes F, $W(y^m) \geq W(\alpha\mathbf{1}_n) > W(x)$ for such m, contradiction. Therefore, W is lower semicontinuous. The upper semicontinuity of W is observed similarly. □

Theorem 2: *Let F be a choice correspondence defined on $\mathbf{\Omega}^n$. F satisfies SPO, ACA, UHC and C if, and only if, it is regularly representable.*

Proof. Immediate from Lemma 3 and Theorem 1. □

8. *Application: Characterization of Generalized Gini Bargaining Solutions*

Given the representation theorem obtained above, it is evident that postulating further conditions on a choice correspondence F that satisfies SPO, ACA and C, translates into imposing additional properties on the real function representing F. It may thus be possible to exploit Theorem 1 and certain characterizations of social welfare functions (already established in the literature on social choice) to obtain direct characterizations of some interesting regularly representable choice correspondences. To demonstrate the potential of this approach, we shall provide in this section a complete characterization of the generalized Gini bargaining solutions (introduced by Blackorby et al., 1994) on our extended domain which includes non-convex bargaining problems.

For any $x \in \mathbf{R}^n$, let us write $x_{(.)} = (x_{(1)}, ..., x_{(n)})$ for $x\Pi$ where Π is an $n \times n$ permutation matrix such that $x_{(1)} \leq \cdots \leq x_{(n)}$. The function $W_\alpha : \mathbf{R}^n \to \mathbf{R}$ defined as

$$W_\alpha(x) \equiv \sum_{i=1}^{n} \alpha_i x_{(i)} \quad \text{for some } \alpha_1 \geq \cdots \geq \alpha_n > 0$$

is called a *generalized Gini (social) evaluation function* (Weymark, 1981). We note that a generalized Gini evaluation function is linear in rank-ordered subspaces of $\mathbf{R}^n_+$, and is identical to the (absolute) Gini equality index on $\mathbf{R}^n_+$ when $\alpha_i = 2(n-i)+1$ for all i.

Let $\mathbf{\Omega}^n_* \equiv \{S \in \mathcal{C}^n : S \cap \mathbf{R}^n_+ \in \mathbf{\Omega}^n\}$, i.e., $S \in \mathbf{\Omega}^n_*$ if and only if S is compact, $S \cap \mathbf{R}^n_{++} \neq \emptyset$ and $S \cap \mathbf{R}^n_+$ is comprehensive. A *generalized Gini bargaining solution* on $\mathbf{\Omega}^n_*$, G_α, is a choice correspondence on $\mathbf{\Omega}^n_*$ that is represented by a generalized Gini evaluation function W_α, i.e.,

$$G_\alpha(S) \equiv \arg\max_{x \in S} \sum_{i=1}^{n} \alpha_i x_{(i)} \quad \text{for all } S \in \mathbf{\Omega}^n_*$$

for some $\alpha_1 \geq \cdots \geq \alpha_n > 0$ (Blackorby et al., 1994, 1996). The generalized Gini bargaining solutions are of interest, for they provide solution concepts which yield a certain compromise between the lack of inequality aversion of the utilitarian solution and the egalitarian solution's insensitivity to the shape of the choice problems. We refer the reader to Blackorby et al. (1994) for an extensive discussion of these cooperative bargaining solutions.

As we shall see, our earlier development lets us readily obtain an axiomatic characterization of the class of generalized Gini bargaining solutions by adding the following two postulates to our basic set of axioms.

Anonymity (A): For all $S \in \mathbf{\Omega}^n$, if $x \in F(S)$, then $x\Pi \in F(\{y\Pi : y \in S\})$ for any $n \times n$ permutation matrix Π.

Linear Invariance (L.INV): For all $(S, x) \in \mathbf{\Omega}^n \times F(S)$,

$$x + t_i e^i \in F(\{y + t_i e^i : y \in S\}), \quad i = 1, ..., n$$

for all real t_i such that $x + t_i e^i \in \mathbf{R}^n_+$ and $x_i + t_i e^i \geq x_j$ if and only if $x_i \geq x_j$ for all $j \neq i$.

Anonymity is a natural requirement guaranteeing the impartial treatment of the individuals. Linear invariance, on the other hand, requires that expansion (or contraction) of the choice situation in any coordinate direction by a certain amount entails the expansion (or contraction) of the choices in exactly the same way, provided that the rank orders of the agents' payoffs are preserved. L.INV is thus an additivity postulate which applies only on rank ordered subspaces of $\mathbf{R}^n_+$. It is demanding and harder to justify relative to the previous axioms; we refer the reader to Blackorby et al. (1994) for a related discussion.

Theorem 3: *A choice correspondence F defined on $\mathbf{\Omega}^n_*$ satisfies SPO, ACA, UHC, C, A and L.INV if, and only if, it is a generalized Gini bargaining solution.*

Proof. Sufficiency is readily checked. To see the necessity, notice that by Theorem 2, there exists a continuous $W : \mathbf{R}^n \to \mathbf{R}$ such that $F(S) = \arg\max_{a \in S} W(a)$ for all $S \in \mathbf{\Omega}^n_*$. By A, W must be a symmetric function in the sense that it is invariant under permutations of its arguments. Moreover, by Lemma 3, W must be quasi-concave, and since it is symmetric, it must, in fact, be Schur-concave on $\mathbf{R}^n_+$.[8]

We claim that for all $x^i \in \mathbf{R}^n_+$ such that $x^i = x^i_{(.)}$, $i = 1, 2, 3$,

$$W(x^1) \geq W(x^2) \quad \text{if and only if} \quad W(x^1 + x^3) \geq W(x^2 + x^3).$$

To see this, recall that $W(x^1) \geq W(x^2)$ implies that $x^1 \in F(ch\{x^1, x^2\})$. Therefore, letting $t_i = x^3_i$ and applying L.INV successively, we have

$$x^1 + x^3 = x^1 + \sum_{i=1}^n t_i e^i \in F\left(\left\{y + \sum_{i=1}^n t_i e^i : y \in ch\{x^1, x^2\}\right\}\right)$$

while $x^2 + x^3 \in \{y + \sum_{i=1}^n t_i e^i : y \in ch\{x^1, x^2\}\}$. But then, $W(x^1 + x^3) \geq W(x^2 + x^3)$ must indeed be the case. The converse of the claim can proved by reversing the steps in this argument.

We may therefore apply Theorem 3 of Weymark (1981) to conclude that $W(x) = \sum_{i=1}^n \alpha_i x_{(i)}$ for all $x \in \mathbf{R}^n_+$ for some real $\alpha_1, ..., \alpha_n$. The Schur-concavity and strict monotonicity of W, on the other hand, imply that $\alpha_1 \geq \cdots \geq \alpha_n > 0$. Therefore, $F(S) = \arg\max_{a \in S} W_\alpha(a)$ for all $S \in \mathbf{\Omega}^n$. By SPO, it follows that $F(S) = \arg\max_{x \in S} W_\alpha(x)$ for all $S \in \mathbf{\Omega}^n_*$ as well. □

It should be noted that the coefficients of a generalized Gini social evaluation function on $\mathbf{R}^n$ is parametric over n. In other words, Theorem 3 provides a characterization of a sequence of choice correspondences (F_n) such that $F_n(S) = \arg\max_{x \in S} \sum_{i=1}^n \alpha^n_i x_{(i)}$ for all $S \in \mathbf{\Omega}^n_*$, $n = 2, ...,$ where $\alpha^n_1 \geq \cdots \geq \alpha^n_n > 0$. If $\alpha^n_i = \alpha_i$ for all n, then (F_n) is called a *single-series Gini bargaining solution.* We conclude by noting that single-series Gini bargaining solutions can also be characterized on $\mathbf{\Omega}^n_*$ along the lines of Blackorby et al. (1996, Theorem 2) with the aid of Theorem 2 and an independent characterization of the single-series Gini evaluation functions as in Bossert (1990).

[8] A real-valued function f is *Schur-concave* on $\mathbf{R}^n_+$ if and only if $f(x) \geq f(y)$ for all $x, y \in \mathbf{R}^n_+$ such that $\sum_{i=1}^s x_{(i)} \geq \sum_{i=1}^s y_{(i)}$ for all $s = 1, ..., n-1$, and $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$.

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