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INEQUALITY AVERSE COLLECTIVE CHOICE

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Abstract: The Lorenz-Pareto optimal frontier of a collective choice problem identifies a (usually quite large) subset of all Pareto optimal outcomes which are not *inegalitarian* according to the Lorenz criterion. An inequality averse choice function should thus be Lorenz-Pareto optimal in the sense that it should never "choose" an outcome outside the Lorenz-Pareto frontier of any choice problem. We study the basic properties of Lorenz-Pareto optimal choice functions and, in particular, obtain necessary and sufficient conditions for rationalizability and representability of such functions.

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1. *INTRODUCTION*

Among the numerous normative criteria developed within the body of welfare economics, the notion of Pareto optimality is undoubtedly the most well-known and compelling one. Given an abstract collective choice problem that is represented by the set of all possible distributions of (utility or monetary) payoffs of individuals (as in Nash's bargaining problem), one thus exclusively concentrates on the Pareto frontier of the choice set. Yet it is, of course, not reasonable to view all outcomes on the Pareto frontier as equally desirable for the society; a social planner with some egalitarian concerns would surely not be satisfied with certain Pareto optimal allocations of individual payoffs. Identification of the *egalitarian* Pareto optimal solutions of a collective choice problem is, therefore, of interest. Put differently, what we need is a refinement of the Pareto frontier of a choice problem by means of eliminating those Pareto optimal outcomes that would never be selected by any inequality averse social choice procedure. In this paper, we shall propose one such refinement.

The notion of "egalitarianism" we employ in the present paper is defined in terms of the celebrated Lorenz criterion.[2] Given a collective choice problem S (i.e., a compact, comprehensive and convex subset of $\mathbf{R}^n_+$, $n \geq 2$, with a nonempty interior), a distribution of payoffs $z \in S$ is viewed "more equal than" another outcome y in S whenever z strongly Lorenz dominates y (see below for formal definitions). If both y and z are Pareto optimal, egalitarianism clearly demands that y cannot be viewed as a social best, for z is a better outcome than y relative to the Lorenz criterion (while y and z are not distinguishable by the Pareto criterion). Now suppose y is Pareto optimal in S while z is not. But one can then show that there must exist a Pareto optimal $x \in S$ such that x Pareto dominates z which is, in turn, *more egalitarian* than y. Consequently, from an ethical angle that combines the notions of Pareto and Lorenz optimality in a particular way, y is again dominated by another feasible outcome, namely x. If the choice situation under consideration is a cooperative bargaining problem, for instance, it seems reasonable that an egalitarian arbitrator would not view y as a socially preferred outcome over x, let alone proposing y as the solution to the problem. We are thus led to refine the Pareto frontier of a choice problem S to obtain what may be called the *Lorenz-Pareto frontier* which contains all Pareto optimal outcomes that are not Lorenz dominated by any feasible outcome in S.

Given the popularity of the Lorenz domination concept in the theory of income inequality measurement, there is reason to believe that the Lorenz-Pareto frontier might prove to be an interesting refinement of the Pareto frontier of a collective choice problem. As is evident from the fact that the Lorenz-Pareto frontier of a problem S is simply the collection of all second order stochastically (i.e. generalized Lorenz) undominated outcomes in S, this refinement, after all, contains all the outcomes which do not perform unacceptably in either equality or efficiency fronts. Moreover, it is always nonempty and

[2]Analogous formulations of egalitarianism can be found in Moulin (1988) in a context which is very similar to the present one, and in Dutta and Ray (1989) in the framework of TU games in coalitional form. Blackorby, Bossert and Donaldson (1994), on the other hand, point to the importance of inequality aversion in cooperative bargaining problems, and develop a solution concept which is "egalitarian" in precisely the same sense that is advanced in this paper; see Example 4.

compact, and as all egalitarian refinements of the Pareto frontier should do, it reduces to a singleton (which is composed of Kalai's egalitarian solution) when the problem at hand is symmetric.

It may thus be argued that an "inequality averse" solution concept (choice function) should never pick an outcome that is not in the Lorenz-Pareto frontier of a collective choice problem. We call such solution concepts *Lorenz-Pareto optimal*, and elaborate on their properties in the sequel. First we demonstrate that Lorenz-Pareto optimality is not a degenerate postulate; while the egalitarian, Nash, and the generalized Gini solutions are all Lorenz-Pareto optimal (whenever they are well-defined), the Kalai-Smorodinsky solution concept fails to satisfy Lorenz-Pareto optimality. Second, we show that if a Lorenz-Pareto optimal solution concept satisfies independence of irrelevant alternatives (strong axiom of revealed preference), then it is rationalizable by a strictly monotonic and strictly Schur-concave (complete and transitive) reflexive binary relation. Unfortunately, such a solution concept need not be representable in the sense that it may not be obtained as a result of maximizing a particular social welfare function. However, by using an important representability result due Peters and Wakker (1991), one can show that all Lorenz-Pareto optimal solution concepts which satisfy the strong axiom of revealed preference and a weak continuity axiom are, in fact, representable by an upper semicontinuous, strictly monotonic and strictly Schur-concave social welfare function. This observation generalizes the main representability theorem of Bossert (1994) to n-person bargaining problems.

All in all, the notion of Lorenz-Pareto optimality leads to an interesting refinement of that of Pareto optimality. It has a natural interpretation which integrates two key ethical criteria of egalitarianism and efficiency. Moreover, it is observed here that this notion has rather nice properties and implications, and hence, one may gain considerable analytic ground by replacing Pareto optimality with Lorenz-Pareto optimality in certain axiomatic inquiries. We conclude that while Lorenz-Pareto optimality is admittedly not the only way one can introduce a "concern for equality" into the collective choice process, it paves way towards a potentially fruitful "ethical" approach to the problem of collective choice.

The organization of the paper is as follows. In Section 2, we review some preliminary concepts like collective choice problems, solution concepts, Lorenz and generalized Lorenz ordering and Schur-concavity. We introduce the key notion of Lorenz-Pareto frontier as a correspondence on the class of all choice problems in Section 3. After establishing some basic properties of this correspondence, we move to study the Lorenz-Pareto optimal solution concepts in Section 4. Several examples of such solution concepts are provided in this section along with some comments concerning characterizations of these choice functions with the aid of the Lorenz-Pareto optimality axiom. Finally, we report our main findings on the rationalizability and representability of Lorenz-Pareto optimal solution concepts in Section 5. The paper concludes with a number of remarks on the possible extensions of our main representation theorem.

2. *PRELIMINARIES*

2.1 *Collective Choice Problems*

Let $n \geq 2$. The class of all compact, comprehensive (with respect to the origin) and convex sets in $\mathbf{R}^n_+$ which have a nonempty intersection with $\mathbf{R}^n_{++}$ is denoted by $\mathbf{\Sigma}^n$.[3] We refer to each $S \in \mathbf{\Sigma}^n$ as a **collective choice problem** (or a choice situation), and call any member of S an **outcome**.

There are several specific models of group-decision making which fall well within the boundaries of this abstract setting. For example, under some plausible assumptions, the standard *utility possibility sets* of classical welfare economics are well-defined collective choice problems (see Bossert, 1996).[4] Alternatively, each member of $\mathbf{\Sigma}^n$ can be interpreted as a cooperative bargaining problem in the sense of Nash (1950) where the disagreement point is taken to be the origin. (See Thomson, 1994 for an excellent survey of the related literature.)

One may also think of problems in $\mathbf{\Sigma}^n$ which model certain instances of division of monetary incomes. Consider, for example, the problem of distributing welfare payments among a certain subgroup (of presumably poor individuals). The *social cost* of providing the payment distribution $x \in \mathbf{R}^n_+$ need not be simply $\sum_{i=1}^n x_i$, for this would ignore that the characteristics of the poor individuals (like employment histories, etc.) and how the money is collected from the rest of the society. More generally, the social cost of the transfer payment distribution x can be considered as $C(x)$ for some $C : \mathbf{R}^n_+ \to \mathbf{R}_+$ such that $C(0) = 0$, and in this case, the space of *feasible benefit distributions* would be $S_c \equiv \{x \in \mathbf{R}^n_+ : C(x) \leq c\}$ for some $c > 0$. Notice that $S_c \in \mathbf{\Sigma}^n$ provided that C is continuous, increasing and convex.

A **solution concept** (or a choice function) on $\mathbf{\Sigma}^n$ is defined as any function $F : \mathbf{\Sigma}^n \to \mathbf{R}^n_+$ such that $F(S) \in S$ for all $S \in \mathbf{\Sigma}^n$. For any $S \in \mathbf{\Sigma}^n$, we define

$$PO(S) \equiv \{x \in S : \neg\exists y \in S : [y > x]\}$$

and say that the solution concept F is **strongly Pareto optimal** if, for any S, $F(S) \in PO(S)$.[5] On the other hand, F is said to be **Pareto optimal** if, for any S, there does not exist an x such that $x \gg F(S)$. We say that F is **symmetric** if, for all $S \in \mathbf{\Sigma}^n$ such that $S = \{x\Pi : x \in S\}$ for any $n \times n$ permutation matrix Π, we have $F_i(S) = F_j(S)$, $i, j = 1, ..., n$.

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[3] $S \subset \mathbf{R}^n_+$ is called *comprehensive* if, for any $x \in S$, $x \geq y \geq 0$ implies that $y \in S$; it is called *strictly comprehensive* if it is comprehensive and for any $x, y \in S$, $x \geq y$ implies that there exists a $z \in S$ such that $z_i > y_i$ for all i.

[4] Where $X \subset \mathbf{R}^n_+$ is a closed convex set of feasible alternatives, for all continuous, concave and strictly increasing $u_i : X \to \mathbf{R}_+$ such that $\inf_{\omega \in X} u_i(\omega) = 0$, $i = 1, ..., n$, we have $\{(u_1(\omega), ..., u_n(\omega)) : \omega \in X\} \in \mathbf{\Sigma}^n$. If u_is are affine, the associated utility possibility set coincides with the classical *budget set* of a consumer. $\mathbf{\Sigma}^n$, therefore, contains the linear budget problems (along with the nonlinear ones) which are extensively studied within the domain of revealed preference theory.

[5] Vector inequalities: For all $x, y \in \mathbf{R}^n$, $x \geq y$ iff $x_i \geq y_i$ for all i; $x > y$ iff $x \geq y$ and $x \neq y$; $x \gg y$ iff $x_i > y_i$ for all i.

2.2 *The Lorenz Ordering and Schur-Concavity*

Let $\mathbf{R}_n^+ \equiv \{x \in \mathbf{R}_+^n : x_1 \leq \cdots \leq x_n\}$, and for any $x \in \mathbf{R}_+^n$, let $x_{(.)}$ stand for $x\Pi$ where Π is an $n \times n$ permutation matrix such that $x_{(.)} \in \mathbf{R}_n^+$. Thus, $x_{(1)} \leq \cdots \leq x_{(n)}$ for any $x \in \mathbf{R}_+^n$. The **Lorenz ordering** $\succcurlyeq_L$ is defined on $\mathbf{R}_+^n$ as: $x \succcurlyeq_L y$ if, and only if,

$$\sum_{i=1}^{s} x_{(i)} \geq \sum_{i=1}^{s} y_{(i)} \text{ for all } s = 1, ..., n-1 \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

As usual, we define $\sim_L \equiv \succcurlyeq_L \cap \{(y, x) : x \succcurlyeq_L y\}$ and $\succ_L \equiv \succcurlyeq_L \setminus \sim_L$.

When $x \succ_L y$, we contend that the elements of x are more equally distributed than those of y, for by a result due Hardy, Littlewood and Polya (1934), $x \succ_L y$ holds if, and only if, x can be obtained from y by a finite sequence of transfers from y's larger entries to its smaller ones such that the rank orders of the elements of y remain unaltered. (In the literature on income inequality, such transfers are usually referred to as the *Pigou-Dalton* transfers; see Dasgupta, Sen and Starrett, 1973, Fields and Fei, 1978 and Marshall and Olkin, 1979). Especially when the elements of $\mathbf{R}_+^n$ are interpreted as income distributions, therefore, $\succcurlyeq_L$ ensues to be a compelling *equality ordering.*

A binary relation $\succcurlyeq \subset \mathbf{R}_+^n \times \mathbf{R}_+^n$ is called **(strictly) Schur-concave** if $\succcurlyeq_L \subseteq \succcurlyeq$ (and $\succ_L \subseteq \succ$), where $\succ$ stand for the asymmetric factor of $\succcurlyeq$. On the other hand, a real-valued function W on $\mathbf{R}_+^n$ is said to be **strictly Schur-concave** if, for all $x, y \in \mathbf{R}_+^n$, $x \succ_L (\sim_L) y$ implies $W(x) > (=) W(y)$. In the context of income distributions, a strictly Schur-concave function on $\mathbf{R}_+^n$ is usually interpreted either as an inequality averse social evaluation function or as an equality index (cf. Foster, 1985). We note that a strictly Schur-concave function is necessarily symmetric in the sense that $W(x) = W(x_{(.)})$ for all $x \in \mathbf{R}_+^n$.

Notice that x and y in $\mathbf{R}_+^n$ can be connected with $\succcurlyeq_L$ only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ so that $\succcurlyeq_L$ completely disregards the Pareto optimality characteristics of the distributions. For instance, while $(5,5)$ is clearly a better income (or utility) distribution than $(1,1)$ for a 2-person society, this intuition is totally missed by $\succcurlyeq_L$, for the Lorenz ordering focuses on only the equality of distributions of a fixed size. One way of remedying for this is to extend $\succcurlyeq_L$ to what is usually called the **generalized Lorenz** (or *second order stochastic dominance*) ordering, $\succcurlyeq_{GL} \subset \mathbf{R}_+^n \times \mathbf{R}_+^n$, which is defined as

$$x \succcurlyeq_{GL} y \quad \text{if and only if} \quad \sum_{i=1}^{s} x_{(i)} \geq \sum_{i=1}^{s} y_{(i)} \text{ for all } s = 1, ..., n.$$

(The asymmetric part of $\succcurlyeq_{GL}$ is denoted by $\succ_{GL}$.) It is well-known that the generalized Lorenz ordering, in effect, integrates the inequality aversion embodied in $\succcurlyeq_L$ with the basic premise of vector dominance (see Shorrocks, 1983). This argument can be formalized as follows:

Lemma 1: *For any $S \in \Sigma^n$ and any $x, y \in \mathbf{R}^n_+$, the following statements are equivalent:*

(*i*) $x \succ_{GL} y$

(*ii*) $W(x) \geq W(y)$*for all strictly increasing and strictly Schur-concave* $W : \mathbf{R}^n_+ \rightarrow \mathbf{R}$

(*iii*) *There exists a* $z \in S$*such that* $x \geqslant z \succcurlyeq_L y$*with either* $\geqslant$*or* $\succcurlyeq_L$ *holding strictly.*

Proof. Since the equivalence of (*i*) and (*ii*) is established by Shorrocks (1983, Theorem 2), and that (*iii*) implies (*i*) is obvious, it is enough to show that (*i*) $\Rightarrow$ (*iii*).[6] Take any $x, y \in S \in \Sigma^n$ such that $x \succ_{GL} y$, and assume that $x, y \in \mathbf{R}^+_n$ without loss of generality. Let

$$\{\ell_1\} \equiv \arg\min_{k \in \{1,...,n\}} \sum_{i=1}^{k} (x_i - y_i)$$

and define $w^1 \in \mathbf{R}^{\ell_1}_+$ such that $w^1_i \equiv x_i,\ i = 1, ..., \ell_1 - 1$ and $w^1_{\ell_1} \equiv x_{\ell_1} - \sum_{i=1}^{\ell_1} (x_i - y_i)$. (That $w^1_{\ell_1} \geq 0$ easily follows from the defnition of ℓ_1.) Notice that if $\ell_1 = n$, the result is trivially established upon choosing $z = w^1$, so we assume that $\ell_1 \leq n - 1$.

We wish to show now that $w^1 \succcurlyeq_L (y_1, ..., y_{\ell_1})$. Consider first the case where $w^1_{\ell_1} \geq w^1_{\ell_1 - 1}$ (so that $w^1_1 \leq \cdots \leq w^1_{\ell_1}$). In this case, $x \succcurlyeq_{GL} y$ implies that $\sum_{i=1}^k w^1_i = \sum_{i=1}^k x_i \geq \sum_{i=1}^k y_i$ for all $k = 1, ..., \ell_1 - 1$, and by definition of w, $\sum_{i=1}^{\ell_1} w^1_i = \sum_{i=1}^{\ell_1 - 1} x_i + x_{\ell_1} - \sum_{i=1}^{\ell_1} (x_i - y_i) = \sum_{i=1}^{\ell_1} y_i$. Consequently, we may conclude in this case that $w^1 \succcurlyeq_L (y_1, ..., y_{\ell_1})$.

Assume now that $w^1_{\ell_1} < w^1_{\ell_1 - 1}$, and let

$$s \equiv \# \left\{ i \in \{1, ..., \ell_1 - 1\} : w^1_i \leq w^1_{\ell_1} \right\} + 1.$$

(Consequently, s is the rank order of $w^1_{\ell_1}$ in w^1.) By the hypothesis that $x \succcurlyeq_{GL} y$, we have $\sum_{i=1}^k w^1_i \geq \sum_{i=1}^k y_i$ for all $k = 0, ..., s - 1$. (Throughout this proof, we adopt the convention that $\sum_{i=a}^b f(i) = 0$ when $b < a$.) On the other hand, for any $k \in \{s, ..., \ell_1 - 1\}$, $\sum_{i=1}^k w^1_i \geq \sum_{i=1}^k y_i$ holds if and only if

$$\sum_{i=1}^{s-1} (x_i - y_i) + (x_{\ell_1} - y_s) + \sum_{i=s+1}^{k} (x_i - y_i) - \sum_{i=1}^{\ell_1} (x_i - y_i) \geq 0.$$

But since $x_{\ell_1} \geq x_s$, we have

$$\sum_{i=1}^{s-1} (x_i - y_i) + (x_{\ell_1} - y_s) + \sum_{i=s+1}^{k} (x_i - y_i) - \sum_{i=1}^{\ell_1} (x_i - y_i) \geq \sum_{i=1}^{k} (x_i - y_i) - \sum_{i=1}^{\ell_1} (x_i - y_i) \geq 0,$$

the last inequality following from the definition of ℓ_1. Consequently, we may conclude that $\sum_{i=1}^k w^1_i \geq \sum_{i=1}^k y_i$ for all $k = 1, ..., \ell_1 - 1$. Since we have $\sum_{i=1}^{\ell_1} w^1_i = \sum_{i=1}^{\ell_1} y_i$ by construction, it follows that $w^1 \succcurlyeq_L (y_1, ..., y_{\ell_1})$.

[6] Although (*i*) $\Rightarrow$ (*iii*) can hardly be considered as a new result, we were unable to find a proof of this particular form of the assertion in the literatures on inequality measurement and majorization. For completeness, therefore, we present here a proof of the claim which is similar to Fan's decomposition of the submajorization ordering (cf. Fan, 1951, and Marshall and Olkin, 1979, p. 123.)

Now, by definition of ℓ_1, we must have $\sum_{i=\ell_1+1}^{k} x_i \geq \sum_{i=\ell_1+1}^{k} y_i$ for all $k = \ell_1+1, ..., n$ so that $(x_{\ell_1+1}, ..., x_k) \succcurlyeq_{GL} (y_{\ell_1+1}, ..., y_k)$. But then by defining

$$\{\ell_2\} \equiv \underset{k\in\{\ell_1+1,...,n\}}{\arg\min} \sum_{i=\ell_1+1}^{k} (x_i - y_i)$$

we may find a $w^2 \in \mathbf{R}_+^{\ell_2-\ell_1}$ such that $(x_{\ell_1+1}, ..., x_{\ell_2}) \geqslant w^2 \succcurlyeq_L (y_{\ell_1+1}, ..., y_{\ell_2})$. Continuing this way, we obtain a finite sequence $\ell_0, \ell_1, ..., \ell_r$ (where $\ell_0 = 0$ and $r \leq n$) such that $\ell_r = n$ and there exist $w^i \in \mathbf{R}_+^{\ell_{i+1}-\ell_i}$, $i = 1, ..., r$, with the property that

$$(x_{\ell_i+1}, ..., x_{\ell_{i+1}}) \geqslant w^{i+1} \succcurlyeq_L (y_{\ell_i+1}, ..., y_{\ell_{i+1}}) \quad \text{for all} \quad i = 0, ..., r-1.$$

But then defining $z = (w^1, ..., w^r)$, it is easily observed that $x \geqslant z \succcurlyeq_L y$. Since S is comprehensive and $x \in S$, we must have $z \in S$. Moreover, $x = z \sim_L y$ contradicts (i), and hence (iii) is established. ■

3. *THE LORENZ-PARETO FRONTIER*

Consider a collective choice problem $S \in \Sigma^n$, and let $y \in PO(S)$. From an efficiency perspective, therefore, it is not feasible to improve upon y in S. Yet y can be an unacceptably unequal distribution of payoffs, it can even be a dictatorial solution to the problem at hand. It may thus be possible to "improve upon" y from an egalitarian point of view. For instance, it may well be the case that there exists a $z \in S$ such that $z \succ_L y$; that is, z represents a 'more equal' distribution than y in S. But of course, z may itself be vastly Pareto inoptimal, and hence discarding y just because there exists a more egalitarian payoff distribution in S seems hardly reasonable. But it is easy to see that there must then exist another $x \in PO(S)$ such that $x > z$.[7] There is then a clear sense in which x "dominates" y, for the Pareto optimal x Pareto dominates z which, in turn, Lorenz dominates y; one may thus say that a *Lorenz-Pareto improvement* over y is feasible. It appears that one who wishes to eliminate the unacceptably unequal outcomes in $PO(S)$ would then discard y. In other words, no inequality averse collective choice procedure would propose y as a solution to the problem S as opposed to an alternative like x.

This discussion leads us to refine the Pareto optimal frontier of a collective choice problem $S \in \Sigma^n$ to obtain what we shall henceforth refer to as the **Lorenz-Pareto**

[7] If there exists a $w \in S$ such that $w \gg z$, then by compactness of S,

$$\max\{\lambda > 0 : \lambda z \in S\}z \in \partial S \cap \mathbf{R}_{++}^n,$$

so by convexity and comprehensiveness of S, we may assume without loss of generality that z is weakly Pareto optimal. Then there must exist a $w \in S$ such that $w_i > z_i$ for all $i \in K$ and $w_i = z_i$ for all $i \in \{1, ..., n\} \backslash K$ for some $K \equiv \{a_1, ..., a_k\} \subset \{1, ..., n\}$, $1 < k < n$. Let $x^1 \equiv (\lambda_1 w_{a_1}, w_{-a_1})$ where $\lambda_1 \equiv \max\{\lambda > 0 : (\lambda w_{a_1}, w_{-a_1}) \in S\}$ and define

$$x^j \equiv (\lambda_j w_{a_j}, x_{-a_j}^{j-1}) \text{ where } \lambda_j \equiv \max\{\lambda > 0 : (\lambda_j w_{a_j}, x_{-a_j}^{j-1}) \in S\}, j = 2, ..., k.$$

By compactness of S, $x^k \in PO(S)$ and $x^k > z$.

frontier of S:

$$LPO(S) \equiv \{y \in PO(S) : \neg\exists x \in S : [x \succ_L y]\}$$
$$= \{y \in S : \neg\exists(z^1, z^2) \in PO(S) \times S : [z^1 \geqslant z^2 \succ_L y]\}.$$

The members of $LPO(S)$ are said to be **Lorenz-Pareto optimal** in S. Clearly, an outcome x is Lorenz-Pareto optimal in S if, and only if, a Lorenz-Pareto improvement over x is not feasible in S.[8]

Remarks: (1) Not all outcomes in $LPO(S)$ can be thought of as egalitarian. Indeed, one can easily see that there exists an $S \in \Sigma^n$ such that $\arg\max_{x\in S} x_i \subset LPO(S)$ for some i; that is, there may exist a dictatorial outcome in the Lorenz-Pareto frontier of a particular choice problem; a telltale sign of the potential existence of inegalitarian outcomes in the Lorenz-Pareto optimal frontier. In fact,

$$\left(\arg\max_{x\in S} \sum_{i=1}^{n} x_i\right) \bigcap LPO(S) \neq \emptyset \quad \text{for all } S \in \Sigma^n,$$

that is, there is always a utilitarian choice in any $LPO(S)$. Nevertheless, it must be clear that all outcomes in $PO(S)\backslash LPO(S)$ are surely inegalitarian according to the Lorenz criterion. Put differently, the notion of Lorenz-Pareto frontier is useful in determining a (usually quite large) subclass of inegalitarian Pareto optimal outcomes rather than identifying only the egalitarian ones. Implicit in $LPO(\cdot)$ are, therefore, some efficiency considerations stemming from presumably the shape of the problem at hand.

(2) When a collective choice problem is interpreted as comprised of utilities of the individuals (as in Nash's bargaining problem), information invariance required by the $LPO(\cdot)$ correspondence becomes a relevant issue. Evidently, the definition of a Lorenz-Pareto frontier in the case of such problems necessitates *cardinal full comparability*, i.e., invariance with respect to identical affine transformations. Nevertheless, it is important to note that $LPO(\cdot)$ admits translation and scale invariant selections; see Examples 1 and 2 below.

(3) $LPO(S)$ is nonempty for all $S \in \Sigma^n$. To see this, take any $S \in \Sigma^n$ and let $\phi : \mathbf{R}^n_+ \to \mathbf{R}$ be any continuous, symmetric, strictly increasing and strictly concave function. It is well-known that any such function is strictly Schur-concave (cf. Marshall and Olkin, (1979), p. 68, Proposition C.2.d), and hence, for any $x, y \in \mathbf{R}^n_+$ such that $x \succ_{GL} y$, we have $\phi(x) > \phi(y)$. But by continuity of ϕ and compactness of S, $T \equiv \arg\max_{x\in S} \phi(x) \neq \emptyset$, and in fact, by strict concavity of ϕ and convexity of S, T is a

[8] The notion of the Lorenz-Pareto optimal frontier is also discussed in Moulin (1988, p.48) and Dutta and Ray (1989). We refer the reader to the former reference for interesting examples of choice problems the Lorenz-Pareto frontiers of which are explicitly computed (see, in particular, Example 2.3 and Exercises 2.5-6). We note that the way we wish to utilize the Lorenz-Pareto optimality concept is very similar to Dutta and Ray (1989). While Dutta and Ray employ this notion to identify egalitarian allocations in TU cooperative games in characteristic function form, we wish to implement Lorenz-Pareto optimality to find out which Pareto optimal payoff distributions can be deemed *inegalitarian* in collective choice problems.

singleton; say $T = \{y\}$. Therefore, by the previous observation, there does not exist an $x \in S$ such that $x \succ_{GL} y$, and thus, $y \in LPO(S)$.[9] ■ .

In view of Lemma 1, we readily observe that the Lorenz-Pareto frontier of a collective choice problem has, in fact, quite a simple algebraic structure:

Proposition 1: *For all $S \in \Sigma^n$,*

$$LPO(S) = \{y \in S : \neg\exists x \in S : [x \succ_{GL} y]\}.$$

Proof. By using Lemma 1 and compactness of S, we obtain

$$\begin{aligned} \exists z \in S : [z \succ_L y] \;&\Leftrightarrow\; \exists (x, z) \in PO(S) \times S : [x \geqslant z \succ_L y] \\ &\Leftrightarrow\; \exists x \in PO(S) : [x \succ_{GL} y] \\ &\Leftrightarrow\; \exists x \in S : [x \succ_{GL} y] \end{aligned}$$

for all $y \in S$, and the proposition follows. ■

Proposition 1 sheds further light into the basic structure of $LPO(.)$. Since $\succcurlyeq_{GL}$ is a dominance relation that brings together the notions of equality and efficiency with respect to Lorenz and Pareto criteria, respectively, $LPO(S)$ can be thought of as the set of all outcomes in S which survive the equality-efficiency dilemma. To see this from a different angle, notice that by Lemma 1, $y \notin LPO(S)$ if and only if $W(x) > W(y)$ for *any* inequality averse (i.e. strictly Schur-concave) and Paretian (i.e. strictly increasing) social welfare function $W : \mathbf{R}_+^n \to \mathbf{R}$. In other words, when $y \notin LPO(S)$, both Rawlsians and utilitarians would agree that there is a better outcome than y in S.[10] It is in this sense we argue that the notion of Lorenz-Pareto optimality integrates the key notions of equality and efficiency.

4. *LORENZ-PARETO OPTIMALITY*

The basic premise behind Lorenz-Pareto optimality is that, for any collective choice problem S, $PO(S) \backslash LPO(S)$ is comprised of (Lorenz-wise) inegalitarian Pareto optimal outcomes. It is then natural to require an inequality averse solution concept not to choose an outcome outside the boundaries of the Lorenz-Pareto frontier. This motivates the following axiom:

Lorenz-Pareto Optimality (LPO): For all $S \in \Sigma^n$, $F(S) \in LPO(S)$.

[9] One can also show that $LPO(S)$ is compact for all $S \in \Sigma^n$. Since we shall not use this property in the sequel, however, we omit the proof here.

[10] It is immediate from Proposition 1 that the Lorenz-Pareto optimal frontier of a 2-person choice problem $S \in \Sigma^2$ is simply that section of $PO(S)$ which falls between the egalitarian and the utilitarian outcomes in S (see Moulin, 1988, Figure 2.2).

We refer to a solution concept which satisfies this axiom as **Lorenz-Pareto optimal.**

In this section we shall study several examples of Lorenz-Pareto optimal solution concepts. But before doing this, let us highlight two important properties of such solution concepts.

Proposition 2: If F *is a Lorenz-Pareto optimal solution concept on* Σ^n*, then it is strongly Pareto optimal and symmetric, but not conversely.*

Proof. If F satisfies LPO, then it is trivially strongly Pareto optimal. To see symmetry, take any $S \in \Sigma^n$ such that $S = \{x\Pi : x \in S\}$ for any $n \times n$ permutation matrix Π. Pick any $x \in S$ and notice that, by convexity of S,

$$y \equiv \frac{1}{n!} \sum_{i=1}^{n!} x\Pi_i \in S$$

where $\{\Pi_i : i = 1, ..., n!\}$ is the set of all $n \times n$ permutation matrices. But since $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$, it is immediate that $y \succ_L x$ unless $x_1 = \cdots = x_n$, and therefore, $x \notin LPO(S)$ unless $x_1 = \cdots = x_n$. But as shown above $LPO(S) \neq \emptyset$, and hence, $x \in LPO(S)$ if and only if $x \in PO(S)$ and $x_1 = \cdots = x_n$.

To show that the converse of the assertion fails to hold, choose any symmetric and strongly Pareto optimal solution concept which coincides with the Kalai-Smorodinsky solution concept on Σ^2 so that, for any $S \in \Sigma^2$, $F(S)$ is in $PO(S)$ and is on the segment connecting $(0,0)$ to $(a_1(S), a_2(S))$, where $a_i(S) = \arg\max_{x \in S} x_i$.[11] Choose $S = \{x \in \mathbf{R}_+^2 : x_2 \leq 1 - x_1, x_1 \leq 1/2\}$, and notice that $LPO(S) = \{(1/2, 1/2)\}$ while $F(S) = (1/3, 2/3)$. ■

This proposition shows that LPO is an axiom with a considerable 'bite'; it combines the widely used properties of Pareto optimality and symmetry, among others. Moreover, it has a certain selection power; for instance, the celebrated solution concept of Kalai and Smorodinsky (1975) does not satisfy LPO, as demonstrated in the proof of Proposition 2. In what follows, we shall present four examples of Lorenz-Pareto optimal solution concepts along with some observations about their characterizations.

Example 1. (*The Egalitarian Solution*) If F satisfies LPO and Kalai's *monotonicity* (i.e. if, for all $S, T \in \Sigma^n$, $T \subseteq S$ implies $F(S) \geq F(T)$), then for any strictly comprehensive problem S, $F(S) \in PO(S)$ and $F_1(S) = \cdots = F_n(S)$; that is, all such solution concepts agree with Kalai's *egalitarian* solution concept. The proof is immediate from Proposition 2 and the well-known characterization of the egalitarian solution by Kalai (1977). ■

Example 2. (*The Nash Solution*) If F satisfies LPO, independence of irrelevant alternatives (defined below), and *scale invariance* (i.e. if $F(\{xA : x \in S\}) = F(S)A$ for

[11] The Kalai-Smorodinsky solution concept does not provide the counter-example we seek here in general, for it is not strongly Pareto optimal on Σ^n, $n \geq 3$.

any diagonal $n \times n$ matrix A with strictly positive diagonal entries), then, and only then, F is the *Nash* solution concept: $F(S) \in \arg\max_{x \in S} \prod_{i=1}^{n} x_i$ for all $S \in \Sigma^n$. The proof trivially follows from the classic characterization theorem of Nash (1950) and Proposition 2. This example is interesting in that it shows that while $LPO(\cdot)$ correspondence is clearly not scale invariant, it admits a scale invariant selection. ■

Example 3. (*The Collectively Rational Separable Solution*) Let $\Sigma \equiv \bigcup_{n=2}^{\infty} \Sigma^n$ and define a *generalized solution concept* as any function $F : \Sigma \to \bigcup_{n=2}^{\infty} \mathbf{R}_+^n$ such that $F(S) \in S$ for all $S \in \Sigma$ (cf. Thomson, 1983). A generalized solution concept F satisfies LPO, continuity (defined below) and *multilateral stability*[12], if and only if, there exists a continuous, strictly increasing and strictly concave $f : \mathbf{R}_+ \to \bar{\mathbf{R}}$ such that $F(S) \in \arg\max_{x \in S} \sum_{i=1}^{n} f(x_i)$ for all $S \in \Sigma$. The proof is again straightforward: If F satisfies these properties, then by Theorem 1 of Lensberg (1988), $F(S) \in \arg\max_{x \in S} \sum_{i=1}^{n} f_i(x_i)$ for all $S \in \Sigma$, for some continuous and strictly increasing $f_i : \mathbf{R}_+ \to \bar{\mathbf{R}}$, $i = 1, ..., n$. But by Proposition 2, F must be symmetric, and therefore, $f_1 = \cdots = f_n$. Moreover, by LPO, the mapping $x \mapsto \sum_{i=1}^{n} f(x_i)$ is strictly Schur-concave on $\mathbf{R}_+^n$ (see Lemma 2 below) so that f must be strictly concave by Proposition C.1.a of Marshall and Olkin (1979).[13] ■

Example 4. (*The Generalized Gini Solution*) Let Σ_C^n denote the class of strictly convex choice problems in Σ^n, and let Σ_*^n be the set of all compact, and convex sets in $\mathbf{R}^n$ such that $S_* \cap \mathbf{R}_+^n \in \Sigma^n$ whenever $S_* \in \Sigma_*^n$. Consider a solution concept F defined on Σ_*^n, and let it be generated by maximizing a continuous function $g^n : \mathbf{R}_+^n \to \mathbf{R}$ in the sense that $F(S) \in \arg\max_{x \in S} g^n(x)$ for all $S \in \Sigma^n$. Assume further that F satisfies LPO and *linear invariance* (i.e., for all $S \in \Sigma^n$ and $i = 1, ..., n$, $F(S + \{\delta_i e^i\}) = F(S) + \delta_i e^i$ where e^i is the ith unit vector, and $\delta_i \in \mathbf{R}$ satisfies $F(S) + \delta_i e^i \in \mathbf{R}_+^n$ and $F_i(S) + \delta_i e^i \geq F_j(S)$ iff $F_i(S) \geq F_j(S)$ for all $j \neq i$), and in addition, suppose that there exists an $S \in \Sigma^n$ such that $x, y \in S$ and $F(S) \in \{x, y\}$, for all $x, y \in \mathbf{R}_n^+$. Then, one can easily show that $F(S)$ must be a *generalized Gini* solution on Σ_C^n (cf. Blackorby, Bossert and Donaldson, 1994, 1996); that is,

$$F(S) \in \arg\max_{x \in S} \sum_{i=1}^{n} \alpha_i^n x_{(i)} \quad \text{for all } S \in \Sigma_C^n$$

[12]Let $n \geq 2$ and $\sigma \subset \{1, ..., n\}$, and denote the projection of any $x \in \mathbf{R}^n$ on $\mathbf{R}^{\#\sigma}$ by x_σ. Define

$$H_\sigma^x \equiv \{y \in \mathbf{R}^n : y_{\{1,...,n\} \backslash \sigma} = x_{\{1,...,n\} \backslash \sigma}\} \quad \text{for all } x \in \mathbf{R}^n,$$

and, for any $x \in T \subset \mathbf{R}^n$, let $t_\sigma^x(T)$ stand for the projection of $H_\sigma^x \cap T$ on $\mathbf{R}^{\#\sigma}$. *Multilateral stability* is a consistency condition on F defined as follows: For all $n \geq 2$, $\sigma \subset \{1, ..., n\}$, $S \in \Sigma^n$ and $T \in \Sigma^{\#\sigma}$, $T = t_\sigma^{F(S)}(S)$ implies $F(T) = F(S)_\sigma$ (cf. Lensberg, 1988, and Thomson and Lensberg, 1989).

[13]This example is, in fact, fully captured by Theorem 1 of Lensberg (1988), for this theorem yields that $F(S) \in \arg\max_{x \in S} \sum_{i=1}^{n} f_i(x_i)$ for all $S \in \Sigma$ where, in addition to the properties stated above, f_is have the property that $x \mapsto \sum_{i=1}^{n} f_i(x_i)$ is a strictly quasiconcave mapping on $\mathbf{R}_+^n$ which, in turn, implies that at most one f_i can fail to be strictly concave (cf. Yaari, 1977). Adding the symmetry axiom, therefore, readily establishes the observation noted in Example 3.

for some $\alpha_1^n > \cdots > \alpha_n^n > 0$.[14] To see this, consider the ordering $\succcurlyeq \in \mathbf{R}^n \times \mathbf{R}^n$ defined by $x \succcurlyeq y$ iff $g^n(x) \geq g^n(y)$. $\succcurlyeq$ is, of course, a continuous and complete preorder which is, by LPO, anonymous (invariant under permutations) and strictly Schur-concave (see Lemma 2 below). Now take $x, y \in \mathbf{R}_n^+$ and define $x', y' \in \mathbf{R}_+^n$ such that $x'_j = x_j$ and $y'_j = y_j$ for all $j \neq i$ and $x'_i = x_i + t$ and $y'_i = y_i + t$ for some $t \in \mathbf{R}$ such that $x', y' \in \mathbf{R}_n^+$. Notice that if $x \succ y$, then by hypothesis, there exists an $S \in \Sigma^n$ such that $x, y \in S$ and $F(S) = x$. But then by linear invariance, $F(S + \{te^i\}) = F(S) + te^i$, and since $y + \delta_i e^i \in S + \{te^i\}$, we must have $x' \succ y'$. That $x' \succ y'$ implies $x \succ y$ is similarly observed. Therefore, $x \succcurlyeq y$ if and only if $x' \succcurlyeq y'$, and hence we can apply Theorem C of Ben Porath and Gilboa (1994) to conclude that, for all $x \in \mathbf{R}_+^n$, $g^n(x) = \sum_i^n \alpha_i^n x_{(i)}$ for some $\alpha_1^n > \cdots > \alpha_n^n > 0$. Since $\# \arg\max_{x \in S} g^n(x) = 1$ whenever S is strictly convex, F must be the generalized Gini solution concept on Σ_C^n. ■

5. *RATIONALIZABILITY OF LORENZ-PARETO OPTIMAL SOLUTION CONCEPTS*

In this section, we shall combine the LPO axiom introduced above with some other axioms for solution concepts which are studied extensively in the literatures on axiomatic bargaining and revealed preference. Our aim is to obtain a tight set of axioms that will characterize the Lorenz-Pareto optimal solution concepts which are obtained by maximizing a real-valued function. The rest of the present study is, therefore, closely connected to the analyses of Peters and Wakker (1991) and Bossert (1994).

We say that a solution concept is **rationalizable** if there exists a $\succcurlyeq \subset \mathbf{R}_+^n \times \mathbf{R}_+^n$ such that

$$\{F(S)\} = \mathcal{G}(S; \succcurlyeq) \quad \text{for all } S \in \Sigma^n,$$

where

$$\mathcal{G}(S; \succcurlyeq) \equiv \{x \in S : \forall y \in S : [x \succcurlyeq y]\} \quad \text{for all } S \in \Sigma^n.$$

The relation $\succcurlyeq$ is then said to **rationalize** F. The following observation identifies the influence of LPO on rationalizable solution concepts.

Lemma 2: *Let F be a choice function defined on Σ^n which is rationalizable by a reflexive binary relation $\succcurlyeq$ on $\mathbf{R}_{++}^n$. F satisfies LPO if, and only if, $\succcurlyeq$ is strictly monotonic and strictly Schur-concave.*

Proof. Sufficiency is readily established. To see the necessity, let F be rationalizable by $\succcurlyeq$, and satisfy *LPO*. Since *LPO* implies strong Pareto optimality, that $\succcurlyeq$ must be strictly monotonic is obvious. We now wish to show that $\succ_{GL} \subseteq \succ$. Assume that $x \succ_{GL} y$ and that $\neg(x \succ y)$. Letting $S \equiv cch\{x, y\}$, we must have $F(S) = \lambda x + (1 - \lambda)y$ for some $\lambda \in [0, 1]$. Moreover, since $F(S) \succcurlyeq z$ for all $z \in S$, and F is single-valued, we must have $F(S) \succ z$ for all $z \in S$. Therefore, since $\neg(x \succcurlyeq y)$ while $y \in S$, we must have $F(S) \neq x$. By LPO, $y = F(S)$ cannot hold either, and hence we conclude

[14] Such functions are usually referred to as the *generalized Gini (social) evaluation functions* since for $\alpha_i^n = 2(n-i)+1$, $i = 1, ..., n$, g^n reduces to the (absolute) Gini equality index on $\mathbf{R}_+^n$ (Weymark, 1981).

that $F(S) = \lambda x + (1-\lambda)y$ for some $0 < \lambda < 1$. But since for any $0 \leq \lambda \leq 1$ and $k \in \{1, ..., n\}$, $\sum_{i=1}^{k} x_{(i)} \geq (>) \sum_{i=1}^{k} y_{(i)}$ implies

$$\sum_{i=1}^{k} x_{(i)} \geq (>)\ \lambda \sum_{i=1}^{k} x_{(i)} + (1-\lambda) \sum_{i=1}^{k} y_{(i)} = \sum_{i=1}^{k} \left(\lambda x_{(i)} + (1-\lambda) y_{(i)}\right),$$

we must have $x \succ_{GL} \lambda x + (1-\lambda)y$ for all $\lambda \in [0,1)$. Consequently, we obtain $x \succ_{GL} F(S)$ which contradicts LPO by Proposition 1. ■

A widely used necessary and sufficient condition for rationalizability is:

Independence of Irrelevant Alternatives (IIA): For all $S, T \in \Sigma^n$, *if* $T \subset S$ *and* $F(S) \in T$, *then* $F(T) = F(S)$.

In view of Lemma 2, the combined effect of LPO and IIA on solution concepts is straightforward:

Proposition 3: *Let F be a solution concept defined on Σ^n. F satisfies LPO and IIA if, and only if, it is rationalizable by a strictly monotonic and strictly Schur-concave binary relation $\succcurlyeq$ on $\mathbf{R}_+^n$.*

Proof. Sufficiency is again obvious. To see necessity, assume that F satisfies LPO and IIA, and notice that Σ^n is closed under finite intersections. But IIA is equivalent to the *weak axiom of revealed preference* (WARP) on a domain which is closed under finite intersections (cf. Hansson, 1968, Theorem 4), and hence, F satisfies WARP. But it is well known that the binary relation

$$\succcurlyeq \equiv \{(x,y) \in \mathbf{R}_{++}^{2n} : \exists S \in \Sigma^n : [x = F(S) \text{ and } y \in S]\} \tag{1}$$

rationalizes F so long as F satisfies WARP. Moreover, if F is strongly Pareto optimal, $\succcurlyeq$ must be reflexive, for then, $F(cch\{x\}) = x$ for any $x \in \mathbf{R}_{++}^n$. The proof is thus complete in view of Lemma 2. ■

Remark: From the analyses of Peters and Wakker (1991, 1994), we know that a strongly Pareto optimal solution concept that satisfies LPO need not be rationalizable by a complete preorder. Unfortunately, the examples constructed in these papers do not establish the insufficiency of LPO and IIA in guaranteeing rationalizability by a complete preorder. We are presently unable to settle this issue; the question of whether a solution concept on Σ^n which satisfies LPO and IIA must be rationalizable by a complete preorder is open at the moment. ■

To guarantee the rationalizability of a solution concept by a complete preorder, we shall employ another well-known postulate which is stronger than IIA. To introduce this particular strengthening of IIA, let us define the following two binary relations on $\mathbf{R}_+^n$ given a solution concept F on Σ^n:

$$xR_F y \quad \text{if and only if} \quad x = F(S) \text{ and } y \in S \text{ for some } S \in \Sigma^n$$

and

$$xR_F^\bullet y \quad \text{if and only if} \quad xR_Fx^1R_F\cdots R_Fx^mR_Fy \ \text{ for some } m \geq 1 \text{ and } x^1,...,x^m \in \mathbf{R}^n_+.$$

Notice that $R_F^\bullet$ is simply the transitive closure of R_F. As usual, we denote the asymmetric part of R_F by P_F.

We are now ready to state our next axiom:

Strong Axiom of Revealed Preference (SARP): For all $x, y \in \mathbf{R}^n_+$, if $xR_F^\bullet y$, then $\neg(yP_Fx)$.

This rationality postulate is sometimes called the *congruence* axiom (mostly when the solution concepts under consideration are multi-valued). We note that the present formulation of SARP is equivalent to its standard version which is, in turn, equivalent to rationalizability by a complete preorder. (See Wakker (1989), Theorem I.2.5.) The following observation is thus straightforward.

Proposition 4: *Let F be a choice function defined on Σ^n. F satisfies LPO and SARP if, and only if, it is rationalizable by a strictly monotonic and strictly Schur-concave complete preorder on $\mathbf{R}^n_+$.*

We say that a solution concept F is **representable** if there exists a $W : \mathbf{R}^n_+ \to \mathbf{R}$ such that

$$\{F(S)\} = \arg\max_{x \in S} W(x) \quad \text{for all } S \in \Sigma^n.$$

W is said to **represent** F, and is usually thought of as a social welfare (or evaluation) function (SWF). We shall adopt this interpretation in what follows.

Clearly, a solution concept which is represented by a strictly increasing and strictly Schur-concave SWF is rationalizable by a strictly monotonic and Schur-concave complete preorder. We demonstrate next that the converse of this observation fails to hold.

Proposition 5: *Let $\succcurlyeq^l$ be the leximin preorder*[15] *and define the solution concept*

$$\{l(S)\} \equiv \mathcal{G}(S; \succcurlyeq^l) \quad \textit{for all } S \in \Sigma^n.$$

While l is rationalizable by a strictly monotonic and strictly Schur-concave complete preorder, it is not representable by any monotonic SWF.

Proof. Let $x \succ_{GL} y$ for some $x, y \in \mathbf{R}^n_+$. By definition, we must then have $\sum_{i=1}^k x_{(i)} \geq \sum_{i=1}^k y_{(i)}$ for all $k = 1, ..., n$, with at least one of the inequalities holding strictly. But

[15] Define $\sim^l \equiv \{(x,y) \in \mathbf{R}^n_+ : x_{(.)} = y_{(.)}\}$ and

$$\succ^l \equiv \{(x,y) \in \mathbf{R}^n_+ : \exists k \in \{1,...n\} : [x_{(i)} = y_{(i)} \text{ for } i = 1,...,k-1 \text{ and } x_{(k)} > y_{(k)}]\}.$$

The *leximin preorder* is defined as $\succcurlyeq^l \equiv \sim^l \cup \succ^l$.

then $x_{(i)} = y_{(i)}$ for all $i = 1, ..., s-1$ and $x_{(s)} > y_{(s)}$ where

$$s \equiv \min \left\{ k \in \{1, ..., n\} : \sum_{i=1}^{k} x_{(i)} > \sum_{i=1}^{k} y_{(i)} \right\},$$

and hence, $x \succ^l y$. Therefore, $\succ_{GL} \subseteq \succ^l$, and l is rationalizable by a strictly monotonic and strictly Schur-concave complete preorder. To complete the proof, we wish to show that l is not representable by any increasing SWF. If we assume the contrary, we would have

$$\mathcal{G}(S; \succcurlyeq^l) = \arg\max_{x \in S} W(x) \quad \text{for all } S \in \Sigma^n \tag{2}$$

for some increasing $W : \mathbf{R}^n_+ \to \mathbf{R}$. Now let $x \succcurlyeq^l y$ for arbitrary $x, y \in \mathbf{R}^n_+$. If $x_i \geq y_i$ for all i, then $W(x) \geq W(y)$ by monotonicity of W. If, on the other hand, $x_{(i)} > y_{(i)}$, $i = 1, ..., k$ and $x_{(k+1)} \leq y_{(k+1)}$, observe that $\mathcal{G}(cch\{x, y\}; \succcurlyeq^l) = \{x\}$ so that, by (4), $W(x) \geq W(y)$ obtains again. Conversely, assume that $W(x) \geq W(y)$ for arbitrary $x, y \in \mathbf{R}^n_+$. Since $y \succ^l x$ would yield $W(y) > W(x)$ by a similar argument to the previous one, we must have $x \succcurlyeq^l y$. Thus, if l is representable by an increasing $W : \mathbf{R}^n_+ \to \mathbf{R}$, then W must represent $\succcurlyeq^l$. But this is well-known to be impossible. ■

In view of Proposition 5, it is clear that F must satisfy a certain 'continuity' property in order to be representable. The following continuity axiom is introduced in Peters (1986):

Pareto Continuity (PC): For all $\{S_m\}_{m=1}^{\infty}$ in Σ^n,

$$\lim_{m \to \infty} F(S_m) = F\left(\lim_{m \to \infty} S_m \right),$$

provided that $\lim_{m \to \infty} S_m \in \Sigma^n$ *and* $\lim_{m \to \infty} PO(S_m) = PO(S)$.[16]

PC says that infinitesimal alterations of the choice problem must result in infinitesimal changes in the solution, provided that the changes in the Pareto frontier of the problem are also small. It should be noted that this axiom is weaker than the standard continuity axioms widely used in axiomatic bargaining theory.

Combining PC with LPO and SARP, one may considerably improve upon Proposition 5; the main result of the present paper follows:

Theorem 1: *If F is a solution concept defined on Σ^n which satisfies LPO, SARP and PC, then it must be representable by a strictly monotonic, strictly Schur-concave and upper semicontinuous SWF.*

[16]The limits are taken with respect to the topology induced by the Hausdorff metric. (The *Hausdorff metric*, ρ, on the set of all nonempty and compact subsets of $\mathbf{R}^n_+$ is defined as

$$\rho(S, T) \equiv \max \left\{ \sup_{x \in S} d(x, T), \ \sup_{y \in T} d(y, S) \right\}$$

where $d(x, A) \equiv \inf_{y \in A} \|x - y\|$ for any compact $A \subset \mathbf{R}^n_+$; Berge, 1963, pp. 126-7.)

Proof.[17] By Theorem 5.3 of Peters and Wakker (1991), there must exist an $f : \mathbf{R}^n_+ \to \mathbf{R}$ which represents F by representing the indirect preference relation $R^\bullet_F$, i.e., $x \, R^\bullet_F \, y$ if and only if $f(x) \geq f(y)$. By Lemma 5.4 of Peters and Wakker (1991), on the other hand, f must be strictly monotonic and strictly quasiconcave.[18] Define

$$W(x) \equiv \lim_{\epsilon \downarrow 0} \sup_{z \in N_\epsilon(x)} g(z) \quad \text{for all } x \in \mathbf{R}^n_+,$$

where $N_\epsilon(x)$ is the ϵ-neighborhood of x relative to $\mathbf{R}^n_+$, and where

$$g(z) \equiv \frac{f(z)}{1 + |f(z)|} \quad \text{for all } z \in \mathbf{R}^n_+.$$

We wish to show that W is a strictly monotonic, strictly Schur-concave and upper semicontinuous function which represents F. First note that $f(x) \geq f(y)$ if and only if $g(x) \geq g(y)$ so that g is a strictly monotonic and strictly quasiconcave function which represents $R^\bullet_F$ (and hence represents F). Moreover, since the range of g is bounded, W is well-defined, and is hence upper semicontinuous.[19]

To show that W represents F, we shall show that

$$\arg\max_{z \in S} W(z) = \arg\max_{z \in S} g(z) \qquad \text{for all } S \in \Sigma^n.$$

Pick any S in Σ^n, and note that $\arg\max_{z \in S} W(z) \neq \emptyset$ since W is upper semicontinuous and S is compact. Now let $x \in \arg\max_{x \in S} W(x)$ and $x \neq F(S)$. Since g represents F and F is single-valued, we must have $g(F(S)) > g(x)$. Let $y = (x + F(S))/2$ and notice that $y \in S$ since S is convex so that $g(F(S)) > g(y)$.

Let $U(y) \equiv \{w \in \mathbf{R}^n_+ : g(w) \geq g(y)\}$, and notice that this set is convex by quasiconcavity of g. Since y is on the boundary of $U(y)$, therefore, by the supporting hyperplane theorem, there exist a $p \in \mathbf{R}^n$ such that $pw \geq py$ for all $w \in U(y)$. In fact, we can show that $p \in \mathbf{R}_{++}$, for if $p_i < 0$ for some i, say $i = 1$, then by choosing w_1 large enough, we obtain $p_1 w_1 + \sum_{i=2}^n p_i y_i < py$ while $(w_1, y_2, ..., y_n) \in U(y)$ by monotonicity of g, a contradiction. On the other hand, if $p_i = 0$ for some i, say $i = 1$, then $(y_1 + 1, y_2, ..., y_n)$ must be on the hyperplane supporting $U(y)$ which means that $(y_1 + 1, y_2, ..., y_n) \notin \operatorname{int} U(y)$, contradicting strict monotonicity of g. Therefore, we have $p \in \mathbf{R}^n_{++}$.

Now let $T \equiv \{w \in \mathbf{R}^n_+ : pw \leq py\}$, and notice that $y = F(T)$ since g represents F. But since g represents $R^\bullet$ and $g(F(S)) > g(y)$, we must have $F(S) \notin T$, i.e. $py < pF(S) = p(2y - x)$ which yields $py > px$. Thus, there must exist an $\epsilon > 0$ such that $N_\epsilon(x) \subset \operatorname{int} T$ which implies that

$$g(y) \geq \sup_{z \in N_\epsilon(x)} g(z) \geq W(x).$$

[17] The method of proof is basically due Hurwicz and Richter (1971); Proof B, p. 69.

[18] Strictly speaking, the results of Peters and Wakker (1991) do not apply readily to our setting for their domain of choice situations include non-comprehensive sets as well. But given strong Pareto optimality (entailed by LPO), we may use their results without loss of generality.

[19] See Haaser and Sullivan (1971), Proposition 2.5, p. 220.

Consequently, we have

$$W(F(S)) \geq g(F(S)) > g(y) \geq W(x)$$

which contradicts that $x \in \arg\max_{z\in S} W(z)$. It follows that

$$\arg\max_{z\in S} W(z) = \{F(S)\} = \arg\max_{z\in S} g(z),$$

that is, W represents F.

To complete the proof we need to demonstrate that W is strictly monotonic and strictly Schur-concave, i.e., we shall be done if we can show that $x \succ_{GL} y$ implies $W(x) > W(y)$ for all $x, y \in \mathbf{R}^n_+$. So let $x \succ_{GL} y$. Since by Lemma 2, f, and hence g, is strictly monotonic and strictly Schur-concave, $g(x) > g(y)$ holds, and since g represents $R^{\bullet}_F$, we have $x \, R_F \, x^1 R_F \cdots R_F \, x^m R_F \, y$ for some $x^1, ..., x^m \in \mathbf{R}^n_+$. But W represents F, and hence, we must have $W(x) > W(x^1) > \cdots > W(x^m) > W(y)$ by definition of R_F. ■

We conclude with several remarks concerning the tightness and extensions of Theorem 1.

Remarks: **(1)** Let F be a solution concept on Σ^n such that $F(S) \gg 0$ for all $S \in \Sigma^n$. Lemmata 5.4 and 5.5 of Peters and Wakker (1991) and Theorem 1 culminate in the following characterization: F *satisfies* LPO, SARP and PC, *if and only if, it is represented by a strictly monotonic, strictly quasiconcave, symmetric and upper semicontinuous SWF on* $\mathbf{R}^n_{++}$.[20]

(2) Lemma 5.4 of Peters and Wakker (1991) and the proof of Theorem 1 show that Theorem 2 of Bossert (1994) generalizes to n-person bargaining problems.

(3) Proposition 8 is tight in the sense that LPO, SARP and PC constitute an independent set of axioms. The solution concept defined in Proposition 7 satisfies LPO and SARP, but not PC. The solution concept defined as

$$\{F_1(S)\} \equiv \arg\max_{x\in S} \left(\sum_{i=1}^{n-1} \sqrt{x_i} + 2\sqrt{x_n} \right) \quad \text{for all } S \in \Sigma^n$$

satisfies SARP and PC, but fails to satisfy LPO since $x \mapsto \sum_{i=1}^{n-1} \sqrt{x_i} + 2\sqrt{x_n}$ is not a Schur-concave mapping on $\mathbf{R}^n_+$. Finally, let $a_i(S) \equiv \max_{x\in S} x_i$, $i = 1, ..., n$, and define

$$\{F_2(S)\} \equiv \arg\min_{x\in LPO(S)} \sum_{i=1}^{n} (a_i(S) - x_i)^2 \quad \text{for all } S \in \Sigma^n$$

which is a variant of a Yu solution (cf. Yu, 1973). F_2 is well-defined since $LPO(S)$ is compact for all $S \in \Sigma^n$, and it clearly satisfies LPO and PC. F_2, however, fails to satisfy IIA, and hence SARP.[21]

[20] By Lemma 5.3 of Peters and Wakker (1991), and Theorem 1, it is immediate that strict positivity hypothesis is necessary only for the sufficiency part of this characterization.

[21] Let $S \equiv \{x \in \mathbf{R}^2_+ : x_2 \leq 2 - 2x_1, \, x_1 \leq 1\}$ and $T \equiv \{x \in \mathbf{R}^2_+ : x_2 \leq 2 - 2x_1, \, x_1 \leq 1/2\}$. While $T \subset S$, we have $LPO(S) = LPO(T) = PO(T)$, and $(1/3, 4/3) = F_2(S) \neq F_2(T) = (2/9, 14/9)$.

(4) The representing SWF found in Theorem 1 need not be continuous due to the potential presence of "poles" (see Hurwicz and Richter, 1971, Remark 4). Moreover, it seems difficult to find "natural" axioms on F which would guarantee the existence of a continuous SWF that represents F. One may, of course, use the earlier results proved within the body of the revealed preference theory, but the additional axioms used there seem much less appealing in our general setting. For the sake of completeness, however, we shall state two immediate observations.

Define $B(p,c) \equiv \{x \in \mathbf{R}^n_+ : px \leq c\}$ for any $(p,c) \in \mathbf{R}^n_{++} \times \mathbf{R}_{++}$. Of course, $B(p,c)$ is nothing but the budget set of a consumer with income c when the price vector is p. In our context, $\{B(p,c) : (p,c) \in \mathbf{R}^n_{++} \times \mathbf{R}_{++}\}$ is thought of as the class of all strictly comprehensive collective choice problems the Pareto optimal frontier of which is a hyperplane. A frequently used assumption in the theory of revealed preference posits that any given strictly positive n-vector is the best choice in one and only one budget set (see, for instance, Uzawa, 1971, D.II′). A straightforward adaptation of this postulate in our setting would read as:

Smoothness (S): For any $x \in \mathbf{R}^n_+$. there exists a unique $(p,c) \in \mathbf{R}^{n+1}_{++}$ such that $\sum_{i=1}^n p_i = 1$ and $F(B(p,c)) = x$.[22]

(i) *Let F be a solution concept defined on Σ^n which satisfies* LPO, SARP, PC and S. *If the mapping $c \mapsto F(B(p,c))$ is Lipschitz continuous on $\mathbf{R}_{++}$, then F is supported by a strictly monotonic, strictly Schur-concave and continuous SWF.*

Proof is immediate from Theorem 2 of Uzawa (1971) and Theorem 1.[23]

(ii) *Let F be a solution concept defined on Σ^n which satisfies* LPO, PC and S, *and assume that F is represented by $W : \mathbf{R}^n_+ \to \mathbf{R}$. If, there exists a $\delta > 0$ such that $W(F(B(p,c-\delta))) > W(y)$ for any $x,y \in \mathbf{R}^n_+$ such that $x = F(B(p,c))$, $y \notin B(p,c)$ and $W(y) > W(x)$, then F is represented by a strictly monotonic, strictly Schur-concave and continuous SWF.*

To prove this claim, we may assume that W is upper semicontinuous, for otherwise, we can obtain an upper semicontinuous function that represents F and that is isotonic to W as in the proof of Theorem 1. Moreover, by Lemma 5.4 of Peters and Wakker (1991), W must be strictly quasiconcave. Therefore, if we can show that for any $x \in \mathbf{R}^n_+$ such that $x = F(B(p,c))$ and any $y \in \mathbf{R}^n_+$ such that $W(x) > W(y)$, there exists a $\delta > 0$ such that $W(F(B(p,c-\delta))) > W(y)$, then the claim will be established by the main Lemma of Sonnenschein (1971) and Theorem 1. To this end, take any $x,y \in \mathbf{R}^n_+$ such that $x = F(B(p,c))$ and $W(x) > W(y)$. If $y \notin B(p,c)$, then we are done by hypothesis, so let $y \in B(p,c)$, assume without loss of generality that $\sum_{i=1}^n p_i = 1$, and notice that

$$z \equiv \frac{x}{2} + \frac{y}{2} \in \operatorname{int}\{w \in \mathbf{R}^n_+ : W(w) \geq W(y)\}$$

[22]This postulate can intuitively be thought of as a *smoothness* condition. For instance, if W is a SWF such that the boundaries of all upper contour sets of W are C^1 manifolds, then any F which is supported by W would satisfy S, for then $\partial B(p,c) \cap \mathbf{R}^n_{++}$ coincides on $\mathbf{R}^n_{++}$ with the tangent space to $\{y \in \mathbf{R}^n_+ : W(y) \geq W(x)\}$ at $x = F(B(p,c))$.

[23]In this proposition, one may replace S with the following boundary condition: For all $c > 0$ and $p^m \in \mathbf{R}^n_{++}$ such that $\lim_{m\to\infty} p^m = p \in \partial\mathbf{R}^n_+$ and $p \neq \mathbf{0}_n$, $\lim_{m\to\infty} \|F(B(p^m,c))\| = \infty$; see Mas-Colell (1978), Theorem 2.

by strict quasiconcavity of W. Thus there exists a $\delta > 0$ such that $W(z - \delta\mathbf{1}_n) > W(y)$ where $\mathbf{1}_n \equiv (1, ..., 1) \in \mathbf{R}^n$. But recalling that $pz \leq c$ and $\sum_{i=1}^n p_i = 1$,

$$z - \delta\mathbf{1}_n \in B(p, p(z - \delta\mathbf{1}_n)) \subseteq B(p, c - \delta)$$

so that $W(F(B(p, c - \delta))) \geq W(z - \delta\mathbf{1}_n) > W(y)$ holds, and we are done.

(5) Let Ω^n denote the class of all compact and comprehensive sets in $\mathbf{R}^n_+$ with a nonempty interior. Since in many instances of bargaining problems, randomization over all alternatives (and hence, convexification of the choice situation) is not possible, it is of interest to see if Theorem 1 would generalize to solution concepts defined on Ω^n (which includes non-convex problems). Unfortunately, such a generalization is not viable, for as shown in Ok and Zhou (1996, Theorem 1), there does not exist a solution concept on Ω^n which satisfies LPO (or strong Pareto optimality), SARP and PC. However, if F is multi-valued (so that $F(S) \subset S$ for all $S \in \Omega^n$) and $F(S)$ is connected for all $S \in \Sigma^n$, then one can show that F satisfies suitably generalized versions of LPO, IIA and PC if, and only if, there exists a continuous, strictly monotonic and Schur-concave SWF such that $F(S) = \arg\max_{x \in S} W(x)$ for all $S \in \Omega^n$; see Ok and Zhou(1996). ■

References

Ben Porath, E. and I. Gilboa, 1994, Linear measures, the Gini index, and the income-equality trade-off, *Journal of Economic Theory* 64, 443-467.

Berge, C., 1963, *Topological Spaces*, (MacMillan, New York).

Blackorby C., W. Bossert and D. Donaldson, 1994, Generalized Ginis and cooperative bargaining solutions, *Econometrica* 62, 1161-1178.

Blackorby C., W. Bossert and D. Donaldson, 1996, Consistency, replication invariance and generalized Gini bargaining solutions, *Journal of Economic Theory* 69, 367-386.

Bossert, W., 1994, Rational choice and two-person bargaining solutions, *Journal of Mathematical Economics* 23, 549-563.

Bossert, W., 1996, The Kaldor compensation test and rational choice, *Journal of Public Economics* 59, 265-276.

Chipman, J.S., L. Hurwicz, M.K. Richter and H.F. Sonnenschein, 1971, *Preferences, Utility and Demand* (Harcourt Brace Jovanovich, New York).

Dasgupta, P., Sen, A.K. and D. Starrett, 1973, Notes on the measurement of inequality, *Journal of Economic Theory* 6, 180-187.

Dutta, B. and D. Ray, 1989, A concept of egalitarianism under participation constraints, *Econometrica* 57, 615-635.

Fan, K., 1951, Maximum properties and inequalities for the eigenvalues of completely continuous operators, *Proceedings of the National Academy of Sciences, USA* 37, 760-766.

Fields, G. and J. Fei, 1978, On inequality comparisons, *Econometrica* 46, 303-316.

Foster J.E., 1985, Inequality measurement, in: P. Young, ed., *Fair Allocations, American Mathematical Society Proceedings in Applied Mathematics*, Vol. 33. (American Mathematical Society, Providence).

Haaser, N. and J. Sullivan, 1971, *Real Analysis* (Dover, New York).

Hansson, B., 1968, Choice structures and preference relations, *Synthese* 18, 443-458.

Hardy, G., J.E. Littlewood and G. Polya, 1934, *Inequalities* (Cambridge University Press, Cambridge).

Hurwicz, L. and M.K. Richter, 1971, Revealed preference without demand continuity assumptions, in: J.S. Chipman, et al., eds., *Preferences, Utility and Demand*, 59-76.

Kalai, E., 1977, Proportional solutions to bargaining situations: interpersonal utility comparisons, *Econometrica* 45, 1623-1630.

Kalai, E. and M. Smorodinsky, 1975, Other solutions to Nash's bargaining problem, *Econometrica* 43, 513-518.

Lensberg, T., 1987, Stability and collective rationality, *Econometrica* 55, 935-961.

Marshall, A.W. and I. Olkin, 1979, *Inequalities: Theory of Majorization and Applications* (Academic Press, New York).

Mas-Colell, A., 1978, On revealed preference analysis, *Review of Economic Studies* 45, 121-131.

Moulin, H., 1988, *Axioms of Cooperative Decision Making* (Cambridge University Press, Cambridge).

Nash, J.F., 1950, The bargaining problem, *Econometrica* 18, 15-62.

Ok, E.A. and L. Zhou, 1996, Revealed group preferences on non-convex choice problems, mimeo, New York University.

Peters, H., 1986, Simultaneity of issues and additivity in bargaining, *Econometrica* 54, 153-169.

Peters, H. and P. Wakker, 1991, Independence of irrelevant alternatives and revealed group preferences, *Econometrica* 59, 1787-1801.

Peters, H. and P. Wakker, 1994, WARP does not imply SARP for more than two commodities, *Journal of Economic Theory* 62, 152-160.

Shorrocks, A.F., 1983, Ranking income distributions, *Economica* 50, 3-17.

Sonnenschein, H., 1971, On the lower semicontinuity of utility functions derived from demand data, in: J.S. Chipman, et al., eds., *Preferences, Utility and Demand*, 77-81.

Thomson,W., 1983, The fair division of a fixed supply among a growing population, *Mathematics of Operations Research* 8, 319-326.

Thomson, W., 1994, Cooperative models of bargaining, in: R.J. Aumann and S. Hart, eds., *Handbook of Game Theory*, Vol. 2 (Elsevier, Amsterdam), 1237-1284.

Thomson, W. and T. Lensberg, 1989, *Axiomatic Theory of Bargaining with a Variable Number of Agents* (Cambridge University Press, Cambridge).

Uzawa, H., 1971, Preference and rational choice in the theory of consumption, in: J.S. Chipman, et al., eds., *Preferences, Utility and Demand*, pp. 7-28.

Wakker, P., 1989, *Additive Representation of Preferences: A New Foundation of Decision Analysis* (Kluwer Academic, Dordrecht).

Weymark, J.A., 1981, Generalized Gini Inequality Indices, *Mathematical Social Sciences* 1, 409-430.

Yaari, M.E., 1977, A note on separability and quasiconcavity, *Econometrica* 45, 1183-1186.

Yu, P.L., 1973, A class of solutions for group decision problems, *Management Science* 19, 936-946.

Zhou, L., 1996, Bayesian utilitarianism, Core discussion paper 9611.