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BY

Yaw Nyarko

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C. V. STARR CENTER FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

Bayesian Learning Without Common Priors and Convergence To Nash Equilibria

Yaw Nyarko[1]
New York University
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ABSTRACT.

Consider an infinitely repeated game where each player is characterized by a "type" which may be unknown to the other players in the game. Suppose that players obey the axioms of Savage (1954): I.e., players have prior probability beliefs over the set of types and actions that will be chosen by all players and maximize their expected utility given these beliefs. Assume also that any event which has probability zero under any one player's beliefs has probability zero under the beliefs of all other players. Suppose further that each player's beliefs about others are independent of that player's type. Then any limit point of beliefs of players about the future of the game conditional on the past lies in the set of Nash equilibria.

Mailing Address:
New York University
269 Mercer St. Rm 723
New York, NY 10003.

1. Introduction. Consider a finite collection of players in an infinitely repeated game. Suppose that each player is characterized by a "type" which is not necessarily known to the other players of the game. Impose only two conditions on players. First, suppose each player obeys the Savage (1954) axioms: In particular, each player has a prior probability belief over the set of types of all the players as well as the actions over time each player-type will choose. Each player then maximizes its expected utility for the infinite horizon game given those beliefs. Second, suppose that the beliefs of players are such that if one player assigns probability zero to an event, then all other players assign probability zero to that event. We show that under these two conditions the beliefs of players converge to the set of Nash equilibria.

The motivation of this paper is the same as that of Blume and Easley (1984) much earlier: We seek to determine conditions under which players initially not in equilibrium can "learn" their way to an equilibrium. The results reported in this paper are a generalization of the results of Jordan (1991a and b). Jordan studied the same model as presented here, but assumed that players' priors are the same. We generalize the results of Jordan by showing that convergence to the set of Nash equilibria still occurs when we weaken the common prior assumption by requiring merely that players' "uncommon" priors a mutually absolutely continuous with respect to each other; (i.e., they assign probability zero to the same events). The **common** prior assumption is due to Harsanyi (1968). We therefore refer to our condition as the Generalized Harsanyi Condition or Condition (GH).

Allowing players to have different priors is important. Imposing the common prior assumption begs the question of where these priors come from. Players are solving complicated optimization problems. The less is the amount of initial coordination of beliefs we model players as having, the better is any result we obtain on the convergence to equilibrium over time.

Like Jordan (1991a and b) our results shall state that **beliefs** of players converge to the set of Nash equilibria. The **actual** play over time does not necessarily converge to the set of Nash equilibria. In section 2 we provide an example to illustrate all this. Kalai and Lehrer (1990), (K-L) also obtain results on the convergence to the set of Nash equilibria. However the assumptions imposed in (K-L) are stronger than the ones we impose in this paper. Indeed, our example in section 2 violates the (K-L) assumptions but satisfies those of this paper. Without our very weak condition (GH) we do not believe much can be said the players' limiting behavior. Indeed, for an example of what could go wrong without condition (GH) see Nyarko (1991a).

Now suppose that there are no "types" or, alternatively, suppose that the types are common knowledge. Under the common priors framework of Jordan (1991a and b), since players have no "types" to condition their actions on, each player will know the actions that will be chosen by the other players. Hence under common priors and common knowledge of types, players will be in a Nash equilibrium from date one. When priors are not common, each player may be unsure of the actions that will be chosen by others even when types are common knowledge. Indeed, consider a game where at each date no player has a strictly dominated action. Then in any fixed **finite** time period any play of players could be optimal given **some** beliefs (which can be made to obey our condition (GH)). Our results show that in the limit, the play of the game must be Nash. Any disagreement over the play of the game will disappear over time! Our main result states that **beliefs** converge to a Nash equilibrium. When types are common knowledge more is true: **Actual** play also converges to a Nash equilibrium. (See section 8.8 for details.)

Like Jordan (1991a and b) we shall suppose that each player's beliefs about the other players is independent of that player's own type. In Nyarko (1992) this assumption is relaxed. However, without the type-independence assumption, the limit points of beliefs are not necessarily Nash equilibria. Instead they are correlated equilibria. Nyarko (1992) extends the work of this paper and of Jordan (1991a and b) in another dimension. Nyarko (1992) also obtains the convergence of sample path averages (i.e., the empirical distributions) to the set of Nash (or, without type-independence, correlated) equilibria.

This paper is organized as follows: In the next section an example is provided to illustrate the results of this paper. The rest of the paper contains the details. In section 9 we provide some comments on the existence of the behavior described here under our assumptions.

2. An Example.

Suppose there are three players, A, B and C, each with two actions, L and R. Player i could be any type τ_i in the interval $T_i=[\underline{\tau}_i, \overline{\tau}_i]$ where $0<\underline{\tau}_i<\overline{\tau}_i<\infty$. The utility or payoff function is given by the following matrix box, where in each box the payoffs are those for players A, B and C respectively.

If C goes L and

Player A	Player B: L	Player B: R
L	$1,1,-\tau_C$	0,0,0
R	0,1,0	0,0,1

If C goes R and

A	Player B: L	Player B: R
L	-1,0,0	$-\tau_A,0,0$
R	$0,-\tau_B,0$	0,0,0

Note that the complete information game with a given fixed and known vector of types (τ_A, τ_B, τ_C) does not have a Nash equilibrium in **pure** strategies. Also notice that if any player chooses the action R then that player receives a payoff of zero regardless of the actions the other players choose.

For each $i\epsilon I$, let π_i be any probability distribution over the type space, T. π_i will be Player i's prior probability over the type space. At the beginning of the initial period, Player i realizes its type, τ_i, and is given no other information (and in particular is not told of the types of the other players, τ_{-i}). Player i's beliefs about the other players' types will be given by the conditional probability, $\pi_i(.|\tau_i)$. We shall suppose that for each i, π_i is a product measure. This implies that $\pi_i(.|\tau_i)$ is independent of τ_i. We shall suppose further that π_i admits a strictly positive and continuous Lebesgue density function over T. Each player has a zero discount factor so at each date seeks to maximize its expected payoffs within that period.

Consider player A. Suppose that Player A assigns probability p_{AB} (resp. p_{AC}) to player B (resp. C) choosing action L. The expected return to A choosing action L is

$$p_{AB}p_{AC}-p_{AB}(1-p_{AC})-\tau_A(1-p_{AC})(1-p_{AB}). \qquad (2.1)$$

Since A receives a payoff of zero if A plays R, Player A will choose action L if $\tau_A<\tau_A^*$ where

$$\tau_A^*=[p_{AB}p_{AC}-p_{AB}(1-p_{AC})]/(1-p_{AC})(1-p_{AB}) \qquad (2.2)$$

(and where, whenever the denominator in this expression is zero, we define $\tau_A^*=+\infty$ or $-\infty$ depending upon whether the numerator in the expression is positive or negative). Similarly, if p_{ij} is the probability assigned by player i to the event that player j chooses the action L then player i chooses action L (resp. R) if $\tau_i<\tau_i^*$ (resp. $\tau_i>\tau_i^*$), and player i is indifferent between actions L and R if $\tau_i = \tau_i^*$, where τ_A^* is as in (2.2) above, and where

$$\tau_B^* = [p_{BA} p_{BC} + (1-p_{BA}) p_{BC}] / [(1-p_{BA})(1-p_{BC})] \quad \text{and} \tag{2.3}$$

$$\tau_C^* = (1-p_{CA})(1-p_{CB}) / [p_{CA} p_{CB}] . \tag{2.4}$$

The vector of numbers $\tau^* = (\tau_A^*, \tau_B^*, \tau_C^*)$ determines the behavior of each player. We shall construct our example so that there is agreement as to how each player will behave as a function of that player's type. However there will be imperfect information on what the types of each player actually are, and there will in general be disagreement about the relative likelihoods of each type (e.g., B and C may have different beliefs about A's type). Hence, in our example, the vector of critical types $\tau^* = (\tau_A^*, \tau_B^*, \tau_C^*)$ will be known to each player. Players i's beliefs about the type space is given by π_i. If player i knows the critical type vector, τ^*, this will determine i's beliefs about the **actions** of other players, $\{P_{ij}\}_{j \neq i}$. In particular, we shall have

$$p_{AB} = \pi_A([\underline{\tau}_B, \tau_B^*]) \quad \text{and} \quad p_{AC} = \pi_A([\underline{\tau}_C, \tau_C^*]) \tag{2.5}$$

$$p_{BA} = \pi_B([\underline{\tau}_A, \tau_A^*]) \quad \text{and} \quad p_{BC} = \pi_B([\underline{\tau}_C, \tau_C^*]) \tag{2.6}$$

$$p_{CA} = \pi_C([\underline{\tau}_A, \tau_A^*]) \quad \text{and} \quad p_{CB} = \pi_C([\underline{\tau}_B, \tau_B^*]) \tag{2.7}$$

Given any tuple of beliefs about the type space $\{\pi_i\}_{i \in I}$, the critical vector of types, $(\tau_A^*, \tau_B^*, \tau_C^*)$, will be determined by the simultaneous solution of the equations (2.2)-(2.7). From the results of section 8 we may conclude that such critical types exist. Each player i will have beliefs over the vector of initial types **and** date one actions, TxS, induced by π_i and the critical types, τ^*. Denote this measure by μ_i^1. Then player i's behavior as described by i's critical type, τ_i^*, is a **best-response** to i's beliefs, μ_i^1.

At the beginning of date 2 the players will observe some vector of actions s^1. This will indicate to each player that the vector of types is in some rectangle $T_2 \equiv \Pi_{i \in I}[\underline{\tau}_{2i}, \overline{\tau}_{2i}]$. (For example, if the vector $s^1=(L,R,L)$ is observed then the vector of types must lie in the set $T_2=[\underline{\tau}_A, \tau_A^*] \times [\tau_B^*, \overline{\tau}_B] \times [\underline{\tau}_C, \tau_C^*]$.) Each player's posterior distribution over the type space will then be the prior conditional on this information.

For date 2 in history s^1, we may mimic the construction for date 1 to obtain some critical types $(\tau_{2A}^*, \tau_{2B}^*, \tau_{2C}^*)$ such that under the following behavior at date 2 each player is best responding to its beliefs: Player i chooses action L at date 2 if its type, τ_i, is less than or equal to τ_{2i}^* and chooses the action R otherwise.

This process may be continued in each and every period to construct critical types at each date n in every history, $(\tau_{nA}^*, \tau_{nB}^*, \tau_{nC}^*)$ in a manner similar to that obtained for date 1. It should of course be noted that the values of the critical types at each date depend upon the past history. Each player i will have beliefs over the vector of initial types, T, **and** play of the game over the infinite time horizon, induced by π_i and the critical types, $\{(\tau_{nA}^*, \tau_{nB}^*, \tau_{nC}^*)\}^{\infty}_{n=1}$. Denote this measure by μ_i. Then player i's behavior as described by i's critical types, $\{\tau_{in}^*\}^{\infty}_{n=1}$, is a

best-response to i's beliefs, μ_i. In particular, if we suppose that each player begins the game with beliefs given by μ_i, then under μ_i each player's action at each date is a best response to the beliefs determined by μ_i. However, since we allow for $\pi_i \neq \pi_j$, players' beliefs about the game may differ.

Whatever is the true vector of types notice that players are choosing a **pure** strategy (L or R) at each date. (The set of types where players are indifferent at any date is a countable set and has probability zero under π_i for each i.) Any limit point of the players' actions will therefore be some vector of pure strategies. However, for the true game with given vector of types $(\tau_A, \tau_B, \tau_C) \epsilon T$, there does not exist a Nash equilibrium in **pure** strategies. Hence along each sample path the limit points of actions chosen by the players do **NOT** constitute a Nash equilibrium for the true game with the given vector of types $\tau \epsilon T$.

Note that we are allowing the priors π_A, π_B and π_C to differ; hence this is a model without common priors. In particular, two players, e.g., A and B, may disagree about the probabilities that the type of the third player, player C, lies in the set $[\underline{\tau}_C, \tau_C^*]$; players A and B may therefore disagree over the probability that player C will choose the action L in equilibrium. Let p^n_{ij} denote the probability assigned by player i conditional on the observed history, s^{n-1}, to the event that at date n player j chooses the action L. Since the priors of players are different, in general $p^n_{ij} \neq p^n_{kj}$ for $i \neq k$ for all $n < \infty$. In section 7 we show that the probabilities assigned by any two players to the event that the third player chooses L **merge** in the following sense: If along some sub-sequence the beliefs of one player (i.e., the probability assigned to the third player choosing action L) converges to some limit point then along the same sub-sequence the beliefs of the other player will converge to the same limit point. In particular,

$$|p^n_{AC}-p^n_{BC}| \to 0, \ |p^n_{AB}-p^n_{CB}|\to 0, \text{ and } |p^n_{BA}-p^n_{CA}|\to 0 \qquad \text{as } n\to\infty. \qquad (2.8)$$

Fix a true vector of types $(\tau_A, \tau_B, \tau_C) \in T$ and a sample path s^∞. Suppose along some sub-sequence of dates $p^n_{AC} \to p_C^\infty$, $p^n_{AB} \to p_B^\infty$ and $p^n_{BA} \to p_A^\infty$. Since beliefs of players' merge, we conclude that along the given sample path and sub-sequence of dates, $p^n_{BC} \to p_C^\infty$, $p^n_{CB} \to p_B^\infty$ and $p^n_{CA} \to p_A^\infty$. The results of section 7 show that $(p_A^\infty, p_B^\infty, p_C^\infty)$ is a (mixed strategy) Nash equilibrium for the normal form game with true vector of types (τ_A, τ_B, τ_C). So, loosely speaking, "beliefs of the players converge to a Nash equilibrium".

3. Some Terminology.

I is the **finite** set of players. Given any collection of sets $\{X_i\}_{i\in I}$, we define $X \equiv \Pi_{i\in I} X_i$ and $X_{-i} \equiv \Pi_{j\neq i} X_j$. Given any collection of functions $f_i: X_i \to Y_i$ for $i \in I$, $f_{-i}: X_{-i} \to Y_{-i}$ is defined by $f_{-i}(x_{-i}) \equiv \Pi_{j\neq i} f_j(x_j)$. The Cartesian product of metric spaces will always be endowed with the product topology. Let X be any metric space. We let $\mathcal{P}(X)$ denote the set of (Borel) probability measures on X. Unless otherwise stated the set $\mathcal{P}(X)$ will be endowed with the weak topology. Given any $\sigma\epsilon\mathcal{P}(X)$ we let $\sigma(dx)$ denote integration: $\int h(x)\sigma(dx)$ is the integral of the real-valued function h on X with respect to σ. If X is a cartesian product X=YZ we let $\sigma(dy)$ denote integration over Y with respect to the marginal of σ on Y. The latter will often be denoted by $Marg_Y\ \sigma$. $\Re$ denotes the real line.

4. The Basic Structure.

4.1. Following Jordan (1991b) we have the following basic structure of the game. I is the **finite** set of players. S_i represents the **finite** set of actions available to player i at each date n=1,2,...; $S=\Pi_{i\epsilon I}S_i$. Even though the action space S_i is independent of the date we shall sometimes write S_i as S_{in} when we seek to emphasize the set of action choices at **date n**. We define $S^N\equiv\Pi^N{}_{n=1}S_n$ and $S^\infty\equiv\Pi^\infty{}_{n=1}S_n$, the set of date n and infinite histories, respectively. s^0 will denote the null history, (at date 0, when there is no history)! In summary, s or S with a "superscript" (e.g., s^n) denotes the history, while with a "subscript" (e.g., s_n) denotes the current period.

Next, we define $F_{iN}\equiv\{f_{iN}:S^{N-1}\rightarrow\mathcal{P}(S_i)\}$; $F_N\equiv\Pi_{i\epsilon I}F_{iN}$; $F\equiv\Pi^\infty{}_{N=1}F_N$; $F_i\equiv\Pi^\infty{}_{N=1}F_{iN}$. F_i is the set of all behavior strategies for player i. F_{iN} is endowed with the topology of pointwise convergence; F_N, F_i and F are endowed with their respective product topology. The mapping $m:F\rightarrow\mathcal{P}(S^\infty)$ defines the probability distribution m(f) on S^∞

resulting from the behavior strategy profile f; i.e., induced by the following transition equation: For each subset D of S_{N+1},

$$m(f)(D|s^N) \equiv f_{N+1}(s^N)(D) \tag{4.2}$$

Perfect recall is assumed; in particular, at date n when choosing the date n action s_{iN}, player i will have information on $s^{N-1}=\{s_1,\ldots,s_{N-1}\}$. For each $i\epsilon I$, we define equivalence class relation, ~, on F_i as follows: For each f_i and $f_i' \epsilon F_i$, $f_i \sim f_i'$ if for all $f_{-i}\epsilon F_{-i}$, $m(f_i,f_{-i}) = m(f_i',f_{-i})$. Let $F_i\sim$ denote the set of equivalence classes of ~. From Kuhn (1953) and (Aumann (1964) for the infinite horizon case) we may conclude that there is a function $\kappa_i:\mathcal{P}(F_i)\to F_i\sim$ such that for any $\phi_i\epsilon\mathcal{P}(F_i)$ and any $f_i\epsilon\kappa(\phi_i)$ and any $f_{-i}\epsilon F_{-i}$, the probability distribution on S^∞ induced by ϕ_i and f_{-i} is equal to $m(f_i,f_{-i})$.

The shift operator $\sigma_{iN}:S^N x F_i \to F_i$ is defined by setting for each $s^N\epsilon S^N$ and $f_i\epsilon F_i$, $\sigma_{iN}(s^N,f_i)\equiv f_i'$ where the date n coordinate of f'_i is defined by $f'_{in}(.) \equiv f_{iN+n}(s^N,.)$. We also use the shift operator over measures. We define the shift operator $\sigma_N:S^N x\mathcal{P}(S^\infty)\to\mathcal{P}(S^\infty)$ for any date N as follows: Fix date N history s^N. Let q be any probability measure over S^∞. Denote the probability distribution over the "future", $s^{N++}\equiv\{s_n\}^\infty_{n=N+1}$ conditional on the past, s^N, by $q(ds^{N++}|s^N)$. We define $\sigma_N(s^N,q)$ to be the probability distribution obtained by viewing the game as beginning at date one where the play of the game has the same distribution as the date N "future" under $q(ds^{N++}|s^N)$. In particular we define $\sigma_N:S^N x\mathcal{P}(S^\infty)\to\mathcal{P}(S^\infty)$ by setting for all subsets D of S^∞, $\sigma_N(s^N,q)(D)=q(D(s^N)|s^N)$ where $D(s^N)\equiv\{s^{\infty\prime}\epsilon S^\infty | s^{N\prime}=s^N$ *and there exists some* $s^{\infty\prime\prime}\epsilon D$ *such that* $s_n{}''=s_{N+n}{}'$ for all n=1,2,....}. We denote $\sigma_N(s^N,q)$ by $q_N(.|s^N)$. I.e., we use a **subscript N** to signify that $q_N(.|s^N)$ is equal to the conditional probability $q(.|s^N)$ but **"shifted"** by N-coordinates.

4.3. The Type Space. Each player has an attribute vector which is some element θ_i of the set Θ_i. The attribute vector will represent the parameter of its utility function unknown to other players in the game. $u_i:\Theta_i x S\rightarrow\Re$ is player i's (within period or instantaneous) utility function which depends upon its attribute vector, θ_i, as well as the vector of actions, $s\epsilon S$, chosen by the players. We assume that u_i is continuous and uniformly bounded on it's domain. The player has a discount factor which is a continuous function, $\delta_i:\Theta_i\rightarrow[0,1)$, of the player's attribute vector. We shall suppose that Θ_i is a compact subset of finite dimensional Euclidean space. This is without loss of generality since the set of joint actions, S, is assumed finite.

We suppose that players know the functional forms of each player's utility function, $\{u_i\}_{i\epsilon I}$. Each player i knows its own attribute vector θ_i but does not know those of other players, θ_{-i}. Player i's type, τ_i, specifies that player's attribute vector, θ_i; it also specifies that player's beliefs about other players' attribute vectors; it specifies that player's beliefs about other players' beliefs about the attribute vectors; and beliefs about beliefs about beliefs ...; etc. In particular an player's type specifies a hierarchy of beliefs about θ. We let T_i denote the set of possible types of player i, and set $T \equiv \Pi_{i \epsilon I} T_i$. We let $\theta_i(\tau_i)$ denote the attribute vector of player i of type τ_i; $\theta_i: T_i \to \Theta_i$ is therefore the projection mapping from the type space T_i representing the i-th players attribute vector into Θ_i. (See Mertens and Zamir, 1985, or Nyarko, 1991b, for details).

4.4. Payoffs. We define $U_i: \Theta_i x S^\infty \to \Re$ by

$$U_i(\theta_i, s^\infty) = \Sigma_{n=1}^{\infty} [\delta_i(\theta_i)]^{n-1} u_i(\theta_i, s_n)$$

where $s^\infty \equiv \{s_n\}_{n=1}^{\infty}$. We define $V_i: \theta_i x F \to \Re$ by

$$V_i(\theta_i, f) = \int U_i(\theta_i, s^\infty) m(f)(ds^\infty)$$

where $m(f)(ds^\infty)$ denotes integration with respect to the measure $m(f)$ over S^∞ (as defined in (4.2)).

4.5. Nash Equilibria. Define for each $i \epsilon I$ and $\theta = \{\theta_i\}_{i \epsilon I}$,

$$N_i(\theta_i) \equiv \{f = \Pi_{j \epsilon I} f_j \epsilon F: \; f_i \; \epsilon \; \text{argmax} \;\; V_i(\theta_i, f_{-i}, .)\}$$

$$N(\theta) \equiv \cap_{i \epsilon I} N_i(\theta_i).$$

$N(\theta)$ is the set of Nash equilibrium behavior strategies for the game with attribute vector $\theta = \{\theta_i\}_{i \epsilon I}$.

5. Bayesian Strategy Processes Without Common Priors.

5.1. Let $\{\mu_i\}_{i\epsilon I}$ be a collection of probability distributions over TxF. Each such probability distribution, μ_i, of course induces a probability distribution over $TxFxS^\infty$ via the measure m(f) over S^∞ conditional $f\epsilon F$, as defined in (4.2). Hence we sometimes consider μ_i as a measure over $TxFxS^\infty$. We shall consider μ_i to be the ex ante beliefs player i has about the evolution of the game **before** i has realized its own type. Define $\pi_i = Marg_T\ \mu_i$, the marginal of μ_i on the type space T. We may consider π_i to be player i's ex ante beliefs about the distribution of types, (before i has realized its own type). $\mu_i(.|\tau_i)$ is player i's ex post beliefs.

5.2. We consider as fixed the vector of ex ante beliefs of players over the type space, $\{\pi_i\}_{i\epsilon I}$, where for each $i\epsilon I$, $\pi_i \epsilon \mathcal{P}(T)$. The collection of measures, $\{\mu_i\}_{i\epsilon I} \epsilon \Pi_{i\epsilon I}\mathcal{P}(TxFxS^\infty)$, is a Bayesian Strategy Process (BSP) for the Repeated Game with (not necessarily common) priors $\{\pi_i\}_{i\epsilon I}$, if the following conditions hold for each $i\epsilon I$:

(5.3) $Marg_T\ \mu_i = \pi_i.$

(5.4) $\mu_i(df|\tau) = \Pi_{j\epsilon I}\mu_i(df_j|\tau_j).$

(5.5) $\mu_i(\{(\tau,f)\epsilon TxF|f_i \text{ maximizes } \int V_i(\theta_i(\tau_i),.,f_{-i})\mu_i(df_{-i}|\tau_i)\})=1.$

Condition (5.3) insists that π_i indeed be the prior beliefs of player i over the type space. Condition (5.4) requires under player i's beliefs, each player chooses actions, conditional on their realized type, which is independent of the choices of other players. This of course does not imply independence under i's beliefs conditional on only i's type; i.e., $\mu_i(df|\tau_i)$ need not be a product measure over F. For example consider a three player model where Player A believes that Players B and C went to the same school so either their types are (β',γ') in which case they choose respectively the strategies $(f_{\beta'},g_{\gamma'})$ or else their types are $(\beta",\gamma")$ in which case they choose respectively the strategies $(f_{\beta"},g_{\gamma"})$. Under Player A's beliefs the strategies of the others are not independent. However, conditional on other Players types, their strategies are (trivially) independent (since they are uniquely and non-randomly determined by their types).

Condition (5.5) insists that given player i's beliefs about the evolution of the game, $\mu_i(.|\tau_i)$, player i maximizes its expected utility. (5.5) by itself does not imply that under i's beliefs about the game other players j≠i are maximizing their expected utility. (However, this will be true under a "condition (GH)" which will be introduced in a subsequent section.)

If player i has a zero discount factor we will need to strengthen condition (5.5) in the definition of the BSP to the following, which requires optimization period-by-period:

$$(5.5') \quad \mu_i(\{(\tau,f,s^\infty)\in TxFxS^\infty \,|\, s_{in+1} \text{ maximizes } \int u_i(\theta_i(\tau_i),..,s_{\text{-}in+1})\mu_i(ds_{\text{-}in+1}|s^n,\tau_i)\}) = 1.$$

5.6. Remark: The collection of measures $\{\mu_i\}_{i\epsilon I}$ shall be said to be a Bayesian Nash Equilibrium (BNE) if it is a BSP and in addition $\mu_i(.|\tau)=\mu_j(.|\tau)$ for all i and j in I and all initial types of players $\tau\epsilon T$. In a BNE players agree on the evolution of play of the game as a function of the initial vector of types of players in the game. The only uncertainty in a BNE is the initial vector of types of players in the economy.

Define for each μ_i, $\pi_i \equiv Marg_T\ \mu_i$. Let P_τ denote the common conditional probability $\mu_i(.|\tau)=\mu_j(.|\tau)$ in a BNE. Then for each $i\epsilon I$, $\mu_i=\pi_i.P_\tau$. $\{\mu_i\}_{i\epsilon I}$ is said to be a BNE with common priors if it is a BNE and in addition $\pi_i=\pi_j$. Observe that a Bayesian Strategy Process with common priors (i.e., where $\mu_i=\mu_j$ for all $i,j\epsilon I$) is also a Bayesian Nash equilibrium with common priors and vice versa. The example of section 2 was a BNE without common priors. Nyarko (1991b) has studied a BNE without common priors.

Jordan (1991a and b) studied a Bayesian Strategy Process with common priors (which, as remarked above is also a BNE with common priors). Indeed what is defined in Jordan to be a Bayesian Strategy Process is what here we refer to as a BSP with common priors. Our definition of a Bayesian Strategy Process above is more general than a Bayesian Nash Equilibrium (with or without common priors).

6. The Generalized Harsanyi Consistency Condition.

We will impose the following condition on the beliefs of players, $\{\mu_i\}_{i \in I}$, which requires that, ex ante, the players agree on probability zero events. The Harsanyi (1968) common prior assumption requires $\mu_i = \mu_j$ for all i and j. Our condition (GH) below is therefore a generalization of the Harsanyi assumption. The common prior assumption is used by Jordan (1991).

Given any two probability measures μ' and μ'' on some (measure) space Ω, we say that μ' is absolutely continuous with respect to μ'' if for all (measurable) subsets D of Ω, $\mu'(D)>0$ implies that $\mu''(D)>0$. We then write $\mu' << \mu''$. We say that μ' and μ'' are mutually absolutely continuous with respect to each other if $\mu' << \mu''$ and $\mu'' << \mu'$.

6.1. Condition (GH): *The measures $\{\mu_i\}_{i \in I}$ in $\mathcal{P}(T \times F)$ are mutually absolutely continuous with respect to each other.*

Under condition (GH) any event which has positive (ex ante) probability with respect to any μ_i will have positive (ex ante) probability with respect to each μ_j for all $j \in I$. Note that condition (GH) does not require the ex post probabilities, $\mu_i(.|\tau_i)$ and $\mu_j(.|\tau_j)$, to be mutually absolutely continuous.

We shall use the following much weaker version of condition (GH):

6.2. Condition (GGH): *There exists a measure μ^* over TxF such that for all $i \in I$, μ^* is absolutely continuous with respect to μ_i.*

One may wish to interpret μ^* as the "true" distribution of the types and play while μ_i is player i's beliefs. Any event which has positive probability under μ^* in condition (GGH) will have strictly positive probability under μ_i for each $i \in I$. The converse however need not be true under the weaker condition (GGH). When condition (GGH) holds we shall state our results in terms of the measure μ^*; that condition should therefore be thought of as providing such a measure. If condition (GH) holds, to obtain condition (GGH) we may take the measure μ^* to be equal to any of the μ_i's or indeed any measure over TxF which is mutually absolutely continuous with respect to any (and therefore all) of the μ_i's; e.g., μ^* may be taken to be the average measure $\Sigma_{i \in I}\mu_i/(\#I)$.

6.3. Remark. As will soon become apparent, the principal use of condition (GH) or (GGH) is to ensure agreement in the limit about **play** of the game; (in particular its main use will be to prove Theorem 7.2 below). Hence, for all the main results of this paper, we may replace conditions (GH) and (GGH) above with assumptions which require only absolute continuity of the **marginals of μ_i on S^∞** and not necessarily over all of TxS^∞. In particular, beliefs of players about **types** are by themselves unimportant. We use assumptions (GH) and (GGH) as stated above because it is expositionally more convenient to do so.

6.4. Remark. Let μ^* denote the true distribution of play for the various player-types. The absolute continuity assumption of Kalai and Lehrer (1990) requires that for each i in I and for each $\tau=\{\tau_i\}_{i\in I}\in T$, $\mu^*(.|\tau) << \mu_i(.|\tau_i)$. In particular, their assumption requires **ex post** absolute continuity. Our condition (GGH) is weaker and requires only ex ante absolute continuity. In particular the example of section 2 obeys condition (GH) but violates the Kalai and Lehrer assumptions. For their assumption to hold the set of ("behavior equivalent classes of") types must be finite or countably infinite. (See Nyarko 1991b for details.)

6.5. When Does Condition GGH hold? Suppose that the players' prior beliefs over the **type space** are mutually absolutely continuous with respect to each other. Suppose further that each player's behavior strategy is uniquely determined by that player's type; (i.e., players are in a Bayesian Nash equilibrium with not necessarily common priors). Then from Proposition 6.6. below, condition (GH) is then necessarily satisfied. Analogous remarks are true for condition (GGH). This enables us to obtain conditions (GH) or (GGH) merely in terms of beliefs over **types** in any Bayesian Nash equilibrium (BNE). (The existence of BNE's follows from our discussion in section 9 below.)

6.6. Proposition. Let $\{\pi_i\}_{i\epsilon I}$ be a finite collection of (Borel) probability measures on a metric space T, which are mutually absolutely continuous with respect to each other. Let P_τ be a regular conditional probability measure over a metric space F conditional on $\tau\epsilon T$. Define for each $i\epsilon I$, the (Borel) probability measure μ_i over TxF by $\mu_i(.)\equiv\int P_\tau(.)\pi_i(d\tau)$. Then $\{\mu_i\}_{i\epsilon I}$ are mutually absolutely continuous with respect to each other.

Proof: Fix any set D in TxF and suppose that for some $i\epsilon I$, $\mu_i(D)>0$. Define $D'\equiv\{\tau\epsilon T|P_\tau(D)>0\}$. Then from the definition of μ_i we have $\pi_i(D')>0$. Fix any $j\epsilon I$. Under the mutually absolute continuity assumption on π_i and π_j, we conclude that $\pi_j(D')>0$. From the definition of μ_j this in turn implies that $\mu_j(D)>0$. Hence $\mu_i(D)>0$ implies $\mu_j(D)>0$. Hence $\{\mu_i\}_{i\epsilon I}$ are mutually absolutely continuous with respect to each other. //

The following is also easy to show:

6.7. Proposition. Let $\{\pi_i\}_{i\epsilon I}$, T, P_τ and $\{\mu_i\}_{i\epsilon I}$ be as in Proposition 6.6. However in place of the mutual absolute continuity assumption on $\{\pi_i\}_{i\epsilon I}$ assume merely that there exists a measure $\pi^* \epsilon \mathcal{P}(T)$ such that $\pi^* << \pi_i$ for all $i\epsilon I$. Define $\mu^*\epsilon\mathcal{P}(TxF)$ by $\mu^*(.)\equiv\int P_\tau(.)\pi^*(d\tau)$. Then $\mu^* << \mu_i$ for all $i\epsilon I$.

7. (GGH) Implies that Beliefs about the Future " Merge."

The following result follows immediately from Blackwell and Dubins (1963) theorem on "Merging of Opinions": Let $\mu_{iN}(ds^{N++}|s^N)$ denote the probability distribution over the "future", $s^{N++} \equiv \{s_{N+1}, s_{N+1}, \ldots\} \epsilon S^\infty$ conditional on the "past," s^N, with respect to the measure $\mu_i(.|s^N)$. The norm $||.||$ denotes the total variation norm on S^∞; i.e., given $p, q \epsilon \mathcal{P}(S^\infty)$,

$$||p|| \equiv Sup_E|p(E)-q(E)| \qquad (7.1)$$

where the supremum is over (measurable) subsets E of S^∞.

The theorem below implies that for each i and j in I, the beliefs of the players about the future of the game conditional on the past, $\{\mu_{iN}(ds^{N++}|s^N)\}^\infty_{N=1}$ and $\{\mu_{jN}(ds^{N++}|s^N)\}^\infty_{N=1}$ have the same limiting behavior and share the same limit points along any sub-sequence of dates in each sample path. Observe that these conditional probabilities are **NOT** conditioned on players' types.

7.2. Theorem. (Blackwell and Dubins). *Suppose that the measures $\{\mu_i\}_{i\epsilon I}$ on $TxFxS^\infty$ obey condition (GGH) and let μ^* be as in that condition. Define*

$$W \equiv \{(\tau, f, s^\infty) \epsilon TxFxS^\infty : \ lim_{n\to\infty} ||\mu_{iN}(ds^{N++}|s^N) - \mu_{jN}(ds^{N++}|s^N)|| = 0\}. \qquad (7.2')$$

Then $\mu^*(W) = 1$.

8. Convergence To Nash Equilibrium.

We shall now suppose that under each player's prior beliefs the set of types of players are independent.

8.1. A Type-Independence Assumption. $\pi_i \equiv \text{Marg}_T\ \mu_i$ is a product measure on the type space T; i.e., $\pi_i = \Pi_{j\epsilon I}[\text{Marg}_{T_j}\ \pi_i]$

8.2. Remark. The independence assumption (8.1) and the property (5.4) of a Bayesian Strategy Process implies that for each i, μ_i is a product measure over $\Pi_{j\epsilon I} T_j x F_j$; i.e., $\mu_i = \Pi_{j\epsilon I}[\text{Marg}_{T_j x F_j}\ \mu_i]$.
This latter condition is part of the definition of a Bayesian Strategy Process in Jordan (1991b).

8.3. Let $\{f^k\}^{\infty}_{k=1}$ be a sequence of elements in F. Let D be any subset of F. We write $f^k \overset{c}{\to} D$ if every cluster point of f^k lies in the set D, where the convergence in F is with respect to the weak topology on F. Fix a Bayesian Strategy Process $\{\mu_i\}_{i\epsilon I}$. For each $i\epsilon I$ let f_j^{*i} be any **Kuhn Strategic representation** of the marginal of μ_i on F_j; i.e., we define

$$f_j^{*i} \quad \epsilon \quad \kappa_i(\text{Marg}_{F_j}\ \mu_i) \qquad \text{and} \qquad f^{*i} \equiv \Pi_{j\epsilon I} f_j^{*i}. \tag{8.4}$$

where κ_i is as defined in (4.1). Under the independence assumption

8.1, $m(f^{*i}) = \text{Marg}_F\, \mu_i$; i.e., the distribution over play, S^∞, generated by f^{*i} is the same as the marginal distribution of μ_i on S^∞. Hence f^{*i} is the strategic representation of player i's beliefs (not conditioning on i's own realized type)! Suppose condition (GGH) holds and let μ^* be as in that condition. From Theorem 7.2. there is a set W with $\mu^*(W)=1$ such that on W,

$$\lim_{N\to\infty} \| \sigma_N(s^N, f^{*i}) - \sigma_N(s^N, f^{*j}) \| \to 0 \text{ for all } i \text{ and } j \in I. \qquad (8.5)$$

Hence on W the sequences $\{\sigma_N(s^N, f^{*i})\}^\infty_{N=1}$ and $\{\sigma_N(s^N, f^{*j})\}^\infty_{N=1}$ have the same limiting behavior. If along a sub-sequence of dates one of the sequences converges, then the other will also converge along that sub-sequence and the convergence will be to the same limit. We now have our main theorem:

8.6. Theorem. (Convergence to Nash Equilibria.) Let $\{\mu_i\}_{i\in I}$ be a BSP. Suppose condition (GGH) holds and let μ^* be as in that condition. Suppose also that the independence assumption (8.1) holds. Then $\mu^*(\{(\tau, s^\infty) \in T \times S^\infty \mid (8.5) \text{ holds and } \sigma_N(s^N, f^{*i}) \to^c N(\theta(\tau)) \text{ for all } i \in I\}) = 1$.

Proof: Define

$$G_i \equiv \{(\tau, s^\infty) : \sigma_N(s^N, f^{*i}) \to^c N_i(\theta_i)\} \qquad (8.6.1)$$

On G_i any convergent sub-sequence of $\{\sigma_N(s^N, f^{*i})\}^\infty_{N=1}$ converges to the set $N_i(\theta_i)$. From Theorem 7.2 there is a set W with $\mu^*(W)=1$, on which the measures $\sigma_N(s^N, f^{*i})$ and $\sigma_N(s^N, f^{*j})$ become closer and closer to each other as $N \to \infty$ for each i and j in I. Hence it is easy to show that on $G_i \cap W$,

$$\sigma_N(s^N, f^{*j}) \to^c N_i(\theta_i) \qquad \textit{for all } j \in I. \qquad (8.6.2)$$

Hence on $\cap_{i\in I} G_i \cap W$, **(8.6.2)** holds for each i and j in I. Since $N(\theta) = \cap_{i\in I} N_i(\theta_i)$, we conclude that on $\cap_{i\in I} G_i \cap W$, $\sigma_N(s^N, f^{*j}) \to^c N(\theta)$ for all $j \in I$, and in particular there is convergence to the set of Nash equilibria.

Hence it suffices to show that $\mu^*(\cap_{i\in I} G_i \cap W) = 1$. We already have $\mu^*(W) = 1$. Hence it remains only to show that $\mu^*(G_i) = 1$ for each i in I. For this, from condition (GGH) it is easy to see that it suffices to show that $\mu_i(G_i) = 1$. The convergence result of Jordan (1991b, Theorem 3.2) is for a model with common priors. However a careful reading of the proof of that Theorem shows that what is indeed proved is the result below, Theorem 8.7. This therefore concludes the proof of Theorem 8.6.//

8.7. Theorem (Jordan, 1991b). Fix any $i \in I$ and suppose that $\mu_i \in \mathcal{P}(T \times F \times S^\infty)$ obeys conditions (5.4)-(5.5) and the independence assumption 8.1. Let G_i be as in (8.6.1). Then $\mu_i(G_i) = 1$.

8.8. Model with Types Common Knowledge. Suppose now that there is only one vector of possible types (or, alternatively, that the vector of types is common knowledge). Then, trivially, condition (8.1) holds. From Theorem 8.6, limit points of beliefs not conditioning on types lie in the set of Nash equilibria. Since there is only one vector of types, each player's beliefs about it's own play not conditioning own types is equal to beliefs conditioning on types which in turn is equal to actual play of that player. Hence **actual play**, and not merely beliefs about play, converges to a Nash equilibrium. I.e.,

8.9. Corollary. *Let $\{\mu_i\}_{i\in I}$ be a BSP and suppose condition (GGH) holds. Also suppose that the type space T is a singleton (or equivalently the true vector of types is common knowledge.) Define μ^{**} to be the true distribution of play, i.e., that induced by $\{\mu_i\}_{i\in I}$. Define*

$$\overline{G}^{**} \equiv \{(\tau, s^{\infty}) : \mu_N^{**}(ds^{N++} \mid s^N) \twoheadrightarrow N(\theta(\tau)) \text{ for all } i \in I\}$$

Then $\mu^{**}(\{(\tau, s^{\infty}) \in \overline{G}^{**}$ *and (8.5) holds*$\}) = 1$.

9. On The Existence of Bayesian Nash Equilibria for Games Without Common Priors.

We now study the question of the existence of a Bayesian Nash equilibrium (BNE) for a game with given (not necessarily common) ex ante prior beliefs over the set of types. As remarked all BNE's are BSP's, but not vice versa. Hence the existence of BNE's implies the existence of BSP's. The existence of BNE's follows immediately from the results of Milgrom and Weber (1985), (henceforth (M-W)). The M-W result requires each player i's beliefs over types, π_i, to be absolutely continuous with respect to the product measure induced by π_i:

9.1. An Absolute Continuity Assumption (M-W). Let π_{ij} denote the marginal distribution of π_i over T_j. Let $\Pi_{j\epsilon I}\pi_{ij}$ be the product distribution on T of these marginal distributions. Then π_i is absolutely continuous with respect to $\Pi_{j\epsilon I}\pi_{ij}$; i.e., $\pi_i << \Pi_{j\epsilon I}\pi_{ij}$.

Under our type independence assumption (8.1) $\pi_i = \Pi_{j\epsilon I}\pi_{ij}$. So (8.1) implies (9.1). Under condition (9.1) (M-W) are able to show the existence of Bayesian Nash equilibria. The results in (M-W) were stated in terms of common priors over types. However, as is mentioned in the paper itself (see p. 631), their result generalizes immediately to the model without common priors, so long as (9.1) holds, and the proof follows with obvious changes to the proof of their result. In particular,

9.2. Proposition (Milgrom and Weber). Fix any collection of (not necessarily common) priors $\{\pi_i\}_{i\epsilon I}$ over the type space T, and assume (9.1) for all $i \epsilon I$. Assume the type space, T, is compact. Then there exists a Bayesian Nash equilibrium for the game with priors over types equal to the given $\{\pi_i\}_{i\epsilon I}$.

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