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A RICARDO MODEL WITH ECONOMIES OF SCALE

BY

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A Ricardo Model With Economies of Scale[1]

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Introduction

There are significant and unavoidable technical difficulties in working with large models having scale economies, and this paper represents an attack on those difficulties.

Work that has appeared in the last decade has contributed greatly to the understanding of the theoretical consequences of scale economies for trade theory. This important recent literature has generally worked with small models, typically two countries and two goods. In this paper we deal directly with two-country models having large numbers of traded goods and, consequently, very large numbers of equilibria. Our model is directly analogous to the classical Ricardo model but here we will assume economies of scale in place of the classical linear or diseconomies assumptions.

We will provide algorithms that, even for large problems, select from the resulting great array of equilibria, those that tend to maximize utility for each country. Our algorithmic approach will then show that, contrary to what one might expect, equilibria do not occur just anywhere. Rather they lie in a well defined

[1] A summary of some of the results of this paper appeared as Gomory [1991]. This paper is a revision of an earlier (09/30/91) version, which was also circulated as C. V. Starr Economic Research Report RR #92-04. This revision has been greatly improved by suggestions from William J. Baumol.

[2] Alfred P. Sloan Foundation, 630 Fifth Avenue, New York, N.Y. 10111.
The author mentions with pleasure the many contributions of Herbert E. Scarf without which this paper would not have been written.

region of a graph of utility versus relative national income which we will describe. This region of equilibria has a well defined shape that persists across many different models and has an economic explanation and significant economic consequences.

Summary of Results

Though for simplicity in what follows we will often express results in terms of Country 1, the statements apply, with obvious changes, to Country 2.

Our main results are:

1. We give simple and rapid algorithms for obtaining from among the very large number of possible equilibria, those equilibria that are very good for one country or the other. These chosen equilibria will be shown to approach the best possible utility for a particular country in large problems. We will see that the best possible equilibrium for one country is usually poor for the other. The algorithms will also show that the two countries' interests are less opposed when their demand structures are similar, and more opposed when they are dissimilar.

2. In the presence of economies of scale of the type we will specify below, the set of equilibrium solutions can be described by a *region* in which these equilibrium points lie, and which they tend to *fill up*. The equilibrium points tend to fill up the solution region in the sense that, given any arbitrarily selected point, P, in the region, then, with a sufficient number of commodities traded, an equilibrium point will appear within any preselected distance (however small) from P.

3. A simple algorithm will be described that allows the calculation of the boundaries of the region. This calculation gives the specific shape and location of the region rapidly even for large models.

4. The characteristic shape of the region of equilibria has the following consequences:

(a) The region always contains a large subregion of equilibria that are advantageous and often strongly advantageous to Country 1 relative to autarky.

(b) There is always a central subregion, often a very large one, within which *the interests of the trading partners are generally opposed.* At the center of this subregion is an area of points that are *above autarky for both countries* as in the classical model. However moving to the right in this subregion, which means that Country 1 captures more and more industries, generally results in further increases in utility for Country 1 and losses in utility for Country 2. The subregion contains at its extreme right the utility maximizing point for Country 1, which, as mentioned above, is usually a very poor point for Country 2.

(c) As Country 1 captures an ever larger share of export industries from Country 2 there comes a point beyond which any further acquisition of industries by Country 1 is disadvantageous to *both* Country 1 and Country 2. It follows from this that the region of equilibria always contains at the extreme right a subregion in which there are equilibria that are relatively disadvantageous for both countries.

(d) The region *always* contains a subregion at whose equilibria Country 1 receives *less* utility than it would in autarky. This region can sometimes be substantial in size. This effect is particularly pronounced for the country that is the larger trading partner.

The results pertaining to the various subregions and the behavior of the equilibria in them simply can not be obtained for very small problems. This is because in small problems, for example two-industry models, the subregions referred to will often be completely empty of equilibria.[3]

The *general picture* that will emerge from the analysis of regions and subregions *is one of a considerable range of conflict in the interests of the two trading partners.*

Assumptions of the Model and of the Related Literature

We will now attempt to relate the foundations of this model to the existing literature. That literature has employed at least three different models of the nature of scale economies.

[3]Experience indicates that for models with six or more industries the subregions are reasonably populated.

The first of these, assumes that firms are perfectly competitive, that they operate as individual entities under constant returns to scale, and that the scale economies are produced by externalities that benefit the firms within a single industry in a given country. Under such circumstances prices will, be set at levels that yield zero profits. This model implies that there will be strong forces making for specialization and non uniqueness of equilibrium. Examples of work using this approach include Kemp [1969] and Ethier [1982].

The second widely-used scale economies model assumes them to be internal to the firm. As is well known, this leads us to expect that the market will be monopolistic or subject to monopolistic competition, and unless the markets are perfectly contestable, it is likely to entail non-zero profits. Helpman and Krugman [1985] have been the leading users of this approach. (see also, e.g., Krugman [1979] and Helpman [1984]).

The third scale economies model, less widely used, also assumes perfect competition and externalities. However, in contrast to the first of the models, it is assumed here that the externalities are not a function of the size of the output of the industry within a single country. Rather, those externalities are generated by the industry's output world wide. In such a case the tendency for extreme specialization within individual countries disappears and some of the policy problems that characterize the case of externalities within particular countries also vanish. This approach was used by Viner [1937], and more recently by Ethier [1979].

In addition to the work just cited, the 1980's produced much more. Notably, the work of Helpman and Krugman and that of Grossman and Helpman have markedly expanded the field. From all this work we know that scale economies tend to preclude uniqueness in trade equilibria, and that the several equilibria obtained are apt to include some that are locally stable. These stable equilibria may well contain some that are mutually disadvantageous, sometimes substantially so, relative to some other equilibria, some being even worse than autarky.

The results of this paper are entirely consistent with this literature. We will however be able to deal directly with large problems and will be able to discuss their properties, many of which are new. Our emphasis will be on the region that we introduce here, and that is needed to encompass the equilibria as the number of traded commodities grows.

The model we will use here is a one input two country model with production functions $f_{i,j}(l)$, for the ith industry in the jth country, that have economies of scale. We will also use a zero-profit pricing assumption. As is well known these assumptions are the ones that can be derived from the first model described above. They also match quite well the author's direct observations of industries containing small numbers of large firms which have do have significant internal economies at low levels of input but are very competitive with each other at the higher levels at which they actually operate. At these levels the internal economies of scale have been realized and the firm's cost structure tends to be linear.

Contents of the Sections

The paper is organized as follows:

Section 1 makes some preliminary remarks about the model and introduces the basic graph that is used throughout the paper.

Section 2 specifies exactly what assumptions are made on our one-input production functions, gives a precise definition of equilibrium, and shows that the number of equilibria grows exponentially with problem size.

Section 3 introduces the normalized variables that facilitate the analysis of the equilibrium region, discusses perfectly specialized equilibria, and describes the calculations that produce the boundaries of the equilibrium region, and how they also select good equilibria.

Section 4 shows the tight relation of the boundary to the equilibrium points in it by proving that for large problems there are perfectly specialized equilibria arbitrarily close to every boundary point. One consequence is that the algorithms previously described approach optimality as the problems become large.

Section 5 describes briefly the actual computational effort required for the various algorithms. The result is that problems of almost any reasonable size are in fact conveniently solvable.

Section 6 represents a shift in emphasis away from algorithms and toward the properties of regions. In this section we show that the whole region fills in with perfectly specialized equilibria as the number of goods grows. This opens the way for a discussion of the economic meaning of the various subregions as we will know that they are populated with equilibria.

Section 7 introduces non-perfectly specialized equilibria. We show that non-specialized equilibria always lie under the upper boundary of the region already determined by the specialized equilibria. Although most non-specialized equilibria are unstable in the presence of economies of scale, we will discuss conditions under which stable non-specialized equilibria can exist. We will also indicate why *stable* non-specialized equilibria will generally either lie inside the region defined by the specialized equilibria, or just below its lower boundary.

Section 8 discusses the underlying economic effects that give the region its characteristic shape. It also includes some significant special cases where an explicit formula for the shape can be given, and where the properties of the various subregions become evident.

Section 9 analyses the effect of changing production functions, the effect of changing country size, and of the structure of demand.

Section 10 develops the economic consequences of the characteristic regional shape and obtains the results 4a-4d about subregions of equilibria.

Section 11 contains a brief summary and remarks on future directions..

Section 1 - Some Basic Properties and the Basic Graph

Equilibrium points in this model, are virtually the same as in the ordinary Walras equilibrium model. At each of our many equilibria there are prices and wages at which supply equals demand for each good. We will use a zero-profit condition for each active producer, so the wage bill for each active producer equals the value of goods produced, while for non-producers the profit for entering into production at these wages and prices, and for low levels of production, is negative or zero.

In a linear model, or one with diseconomies of scale, these conditions would provide a single equilibrium, they lead here, inherently, to many, Economies of scale tend to provide an advantage to those who are actually producing over those who are not, resulting in many stable outcomes. This situation, with its inherent complexity reflected in its many local optima, us an example of the unavoidable differences between local optimization with convexity and local optimization in the presence of non-convexities.

Integer variables enter naturally into this model through a set of 0-1 variables, introduced in Section 3, which determine which country is to be a producer of a given good and which is not. Each pattern of 0's and 1's determines a production pattern which determines one of the many local optima. Finding production patterns whose associated equilibrium points maximize utility then becomes an integer programming problem.

The outcomes from a typical model are illustrated by Figure 1.1 which is based on the data of Table 1.1. Fig.1.1 plots Cobb-Douglas utility on the right vertical axis against normalized national income[4] Z_1 for Country 1 on the horizontal axis. Each dot in the figure represents an equilibrium point. The large dots are outcomes in which only one of the two countries is a producer for each good, these are the perfectly specialized equilibria. The exchange rate w_1/w_2, or The ratio of the wages in the two countries,(exchange rate), corresponding to the

[4] $Z_1 = Y_1/(Y_1+Y_2)$ where Y_1 is the national income of Country 1, and Y_2 is the national income of Country 2.

normalized national incomes is plotted on the top horizontal line. The utility obtained by Country 1 in a state of autarky is marked by the horizontal bar on the right. This example has nine industries.

There are several aspects of Fig.1.1 worth noting. First we see the large number of equilibria that are present even in this nine industry model. Second, the equilibria form an array of points with a rather definite shape which is in fact characteristic of many models. Thirdly, the upper edge of the array of outcomes is rather well defined. In the figure it is marked by a dotted line. The equilibria near this boundary are the ones that do relatively well for Country 1. It is this boundary line, and the equilibria near it, that we will compute by simple and rapid calculations in Section 3. Fourth, there is a lower boundary as well as an upper boundary to the array of perfectly specialized equilibrium points, this lower boundary can also be computed easily.

Each equilibrium point has a utility for Country 2 as well as for Country 1. Figure 1.2 shows the utility of Country 2 on the left vertical axis and the utility of Country 1 on the right as before. The same collection of equilibria is shown as in Fig 1.1 but now the utility for Country 2 is plotted for each equilibrium point instead of the utility for Country 1. Each equilibrium point is represented by a gray dot. Both autarky levels are shown, but only the boundary curves are shown for Country 1. The horizontal axis is still Z_1, and the normalized national income of Country 2 is $Z_2=1-Z_1$. In both figures the utility of each country is normalized separately so that its greatest utility is 1.

The short vertical bar descending from the upper horizontal axis in both figures denotes a connection with the classical linear theory, this connection will be made clearer in Section 3 . There we will introduce the notions of Classical Level and Classical Point. The Classical Point is the equilibrium point (if there is one) at which each good is produced only by its cheaper producer. The Classical Level is the unique exchange rate at which that is possible. There is always a Classical Level but not always a Classical Point in these models. In all our diagrams, as in Figs.1.1 and 1.2, the Classical Level is marked by a vertical bar. The Classical Point is denoted by squares. As there happens to be a Classical Point in this particular 9-industry model, it is indicated by the gray and black squares in Fig. 1.2.

Section 2. Existence of Solutions

This paper emphasizes the array of solutions rather than the existence of any particular one. Nevertheless we need an accurate statement of an existence theorem and of the conditions assumed on the production functions and utility. Then we will be able to demonstrate the existence of a vast array of equilibria, and to count them.

In this model the production functions $f_{i,j}$ for good i in Country j will always have economies of scale. $f_{i,j}(l_{i,j})/l_{i,j}$ will always be a non-decreasing function of the labor input $l_{i,j}$. Also the Cobb-Douglas utility function, or its logarithm, will be used throughout, so for Country j, (j=1,2), we have utility U_j given by

$$U_j=\Pi_i y^{d_{i,j}}{}_{i,j} \text{ and } u_j=lnU_j=\sum_i d_{i,j}\ln y_{i,j}, \quad d_{i,j}>0, \quad \sum_i d_{i,j}=1.$$

with $y_{i,j}$ the quantity of the ith good obtained by Country j. It is a well known consequence of this choice of utility function that Country j spends a constant fraction $d_{i,j}$ of its national income Y_j on good i, for all prices p_i.

For any pattern of production specialization that assigns a set S_j of industries to Country j to produce, a zero-profit pricing equilibrium is a price vector p_i, a set of wage rates w_j, and an allocation $l_{i,j}$ of each country's entire labor supply L_j among the industries in which it participates such that

The supply of the ith good is equal to its demand

$$(2.1) \quad p_i\sum_{j\ (i\in S(j))} f_{i,j}(\ l_{i,j}\ )=\sum_j d_{i,j}Y_j=\sum_j d_{i,j}w_jL_j$$

and each active industry makes a profit of zero. So

$$(2.2) \quad p_i\, f_{i,j}(l_{i,j})=w_jl_{i,j} \quad for\ i\in S_j.$$

Although many papers have been written containing quite general existence theorems for economic models in which production exhibits increasing returns to scale, some of the conditions required by these existence theorems are not satisfied in this model, so we need the theorem that follows. We make two assumptions about the production functions $f_{i,j}$.

A1. Aside from a possible initial interval in which $f_{i,j}(l_{i,j})$ is zero, average productivity $f_{i,j}(l_{i,j})/l_{i,j}$ is continuous and strictly increasing.

A2. Each country in autarky produces a positive quantity of all goods.

Theorem 2.1: Under these assumptions, there will be a zero-profit pricing equilibrium for *any* pattern of specialization in which each of the two countries is the *sole* producer of at least one of the goods in which it specializes. Also at all these equilibria each industry assigned to each country will produce positive quantities of output.

The proof of this theorem is found in Appendix 2-1 .

If we count up the patterns of specialization of production allowed by Theorem 2.1 we will find that this existence theorem provides us with 3^n-$2^{n+1}+1$ equilibria in an n-good model.

While the patterns of production at any one of these many equilibria can not be expected to be stable against large changes that move prices, wages, and output close to another equilibrium point, they can reasonably be expected to be stable against sufficiently small changes. This motivates a further mild restriction on the production functions that is appropriate for economies of scale. We will assume

A3. $\lim l_{i,j}\text{-}>0\ f(l_{i,j})/l_{i,j}=0$.

This zero derivative at the origin ensures that if at a particular equilibrium point Country j is a non-producer of good i, that non-producing industry would earn a negative profit in the immediate neighborhood of the equilibrium. A3 asserts that the non-producer's unit output costs for very low $l_{i,j}$ would be unbounded and so could not compete with those of any producer producing at a fixed finite level. This gives local stability; stability in a much wider sense will be discussed in Section 9.

Condition A3 is satisfied for all production functions of the form $f(l)=el^{\alpha}$ with $\alpha > 1$, as well as by any production function that satisfies the increasing average productivity condition A1, and is zero for an interval to the right of the origin. It does not hold for the Ricardo case el^{α} with $\alpha=1$, but it does hold if el is preceded by an interval of zero output.

Of the $3^n - 2^{n+1} + 1$ equilibria provided by the existence theorem, at least $2^n - 2$, the perfectly specialized ones, are locally stable in the sense just given. On the other hand most, though not necessarily all, of the non-specialized (or intermediate) equilibria are unstable in the sense that a small departure from equilibrium will, under natural dynamics, bring on a larger departure. This will be discussed in Section 7.

In the nine industry model of Figs. 1.1 and 1.2 the array of outcomes contains 510 perfectly specialized equilibria and 19,683 non-specialized ones. The program that computed them gives all the perfectly specialized equilibria and all the intermediates provided by the existence theorem.[5]

We now turn to the analysis of this large array of possible outcomes.

[5] There can always be more intermediate equilibria for sufficiently peculiar production functions.

Section 3. The Array of Solutions

Dealing with the array of equilibria is facilitated by normalized variables that allow us to plot all the equilibria in a finite part of the plane. We also introduce variables $x_{i,j}$ that determine the pattern of production and play a key role in the analysis.

Normalized Variables and the $x_{i,j}$

At any equilibrium point we will have (2.1) and (2.2). Together these imply that, for each good, expenditure equals wages, so

$$d_{i,1}Y_1+d_{i,2}Y_2-w_1l_{i,1}+w_2l_{i,2}.$$

We now define $x_{i,1}$ to be the fraction of the total demand for the ith product that is spent for product made in Country 1. Similarly $x_{i,2}$ is defined to be the fraction of the total demand for the ith product that is spent for product made in Country 2.

(3.0) $x_{i,1}(d_{i,1}Y_1+d_{i,2}Y_2)-w_1l_{i,1}$

(3.1) $x_{i,2}(d_{i,2}Y_1+d_{i,2}Y_2)-w_2l_{i,2}$

From the definition, $0\leq x_{i,j}\leq 1$ and $x_{i,1}+x_{i,2}=1$. From the $x_{i,1}$ we can form the vectors x_1 and from the $x_{i,2}$ the vector x_2, and we will often refer to these two vectors together as the assignment x.

Since there are no profits, the national incomes of the two countries are $Y_j=w_jL_j$. We define the *normalized* national incomes of the two countries to be $Z_1=Y_1/(Y_1+Y_2)$ and $Z_2=Y_2/(Y_1+Y_2)$. Clearly $Z_1+Z_2=1$ and $0\leq Z_i\leq 1$. The ratio $Z_1/Z_2=Y_1/Y_2=(w_1/w_2)(L_1/L_2)$, so Z_1/Z_2 is proportional to the wage ratio for fixed country sizes L_i. We will also use Z without a subscript to denote the two-vector $Z=(Z_1,Z_2)$.

In terms of the normalized national incomes (3.0) and (3.1) become

$$(3.2)\quad x_{i,1}(d_{i,1}Z_1+d_{i,2}Z_2)=l^*_{i,1}Z_1$$

$$(3.3)\quad x_{i,2}(d_{i,2}Z_1+d_{i,2}Z_2)=l^*_{i,2}Z_2.$$

Here the $l^*_{i,j}$ are normalized labor variables, $l^*_{i,j}=l_{i,j}/Lj$ representing the fraction of the labor force in Country j employed in making product i, and the expression in parentheses is the normalized total demand.

In what follows we will also need to refer to the actual labor used in Country j. We denote it by $l_{i,j}$, $l_{i,j}=x_{i,j}(L_j/Z_j)(d_{i,1}Z_1+d_{i,2}Z_2)$.

One of the conditions for equilibrium is that the assignment of labor provided by the $x_{i,j}$ is in fact a partition of the entire labor force, i.e that $\Sigma_i l^*_{i,j}=1$. Since $l^*_{i,j}=(x_{i,j}/Z_j)(d_{i,1}Z_1+d_{i,2}Z_2)$ we can sum this for j=1 and j=2 obtaining the identities

$$(3.4a)\ \sum_i l^*_{i,1} = \sum_i \frac{1}{Z_1}(d_{i,1}\ Z_1+d_{i,2}\ Z_2)x_{i,1}$$

$$(3.5a)\ \sum_i l^*_{i,2} = \sum_i \frac{1}{Z_2}(d_{i,1}\ Z_1+d_{i,2}\ Z_2)x_{i,2}.$$

The n.a.s.c. for the assignment x to provide a partition of the labor force, is that x and the normalized national income Z make $\Sigma_i l^*_{i,j}=1$. So the condition is

$$(3.4)\quad \sum_i (d_{i,1}\ Z_1+d_{i,2}\ Z_2)x_{i,1}=Z_1$$

$$(3.5)\quad \sum_i (d_{i,1}\ Z_1+d_{i,2}\ Z_2)x_{i,2}=Z_2.$$

(3.4) and (3.5) are in fact linearly dependant and therefore we need only one of them. This dependence is a consequence of Walras Law, but it can also be seen directly by adding the two equations [6]. We will use (3.4) and (3.5) interchangeably and refer to either one as the zero excess labor equation.

(3.4), or equivalently (3.5), links *any* assignment x to the Z required to satisfy the excess labor equation. We will refer to that Z as Z(x) In economic terms, since $Z=(Z_1,Z_2)=(w_1L_1,w_2L_2)$, Z(x) gives the wage rates at which the production pattern resulting from x exactly uses the labor of both countries.

Equilibria and Integer x

We now look at the conditions that must be met for x to be an equilibrium point.

For any x, whether it is an equilibrium x or not, the normalized national income Z(x) required to satisfy the zero excess labor equation (3.4) can be calculated from (3.4). This x and Z(x) then determine labor quantities $l^*_{i,j}$ from equations (3.2) and (3.3), whose meaning is that the expenditures must match the wage bills. These labor quantities in turn determine the amounts produced $f_{i,j}(l_{i,j})$.

For x to be an equilibrium point, the zero profit condition (2.2) must also hold for the $f_{i,j}$ and $l_{i,j}$ that have been determined from x. This means that there must be a price p_i for ith good at which, *for producers j who produce at a positive level*, the value of the goods they supply equals their wage bill.

$$(3.P) \quad p f_{i,j} - w_1 l_{i,j} \text{ or equivalently } p f_{i,j} - l^*_{i,j} Z_j.$$

This explicitly determines the price p_i. When there is more than one producer (3.P) requires that both must produce at equal unit cost $1/p_i$. When there is only one, that producer's unit cost determines the price. If condition (3.P) is met by an x and Z(x) satisfying (3.4), supply equals wage bill equals demand, the total labor force is used, and x is an equilibrium point.

While most arbitrarily chosen x do not satisfy (3.P), *all integer* (i.e. 0,1) *x*

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[6] Adding the two equations is also equivalent to adding all the terms in both (3.2) and (3.3) which yields $1=(\Sigma_i l^*_{i,1})Z_1 + (\Sigma_i l^*_{i,2})Z_2$. This implies the useful relations $\Sigma_i l^*_{i,1}=1 \iff \Sigma_i l^*_{i,2}=1$, $\Sigma_i l^*_{i,1}<1 \iff \Sigma_i l^*_{i,2}>1$, and $\Sigma_i l^*_{i,1}>1 \iff \Sigma_i l^*_{i,2}<1$.

do. For integer x there is only one producer of each good who produces at a positive level, so (3.P) is satisfied automatically. Consequently all integer x are equilibria. They are of course the perfectly specialized equilibria.

We will see that the perfectly specialized equilibria are the ones that largely determine the shape of the solution array.

Utility

We will need an expression for utility in terms of x and Z. While we will derive these utility expressions for Country 1 only, the changes for Country 2 are straightforward.

The logarithm of Cobb-Douglas utility is the sum of terms involving the quantity $y_{i,1}$ of the ith good Country 1 receives. The $y_{i,j}$ can be written as the product of two terms $F_{i,1}(Z)Q_i(x,Z)$, where $Q_i(x,Z)$ is the total quantity of the ith good produced in the world and $F_{i,1}(Z)$ is the fraction obtained by Country 1. So the log utility can be written

$$u_1(x,Z)=\ln U_1(x,Z)=\sum_i d_{i,1}\ln F_{i,1}(Z)Q_i(x,Z).$$

Since the goods are all sold at a world price, the fraction going to Country 1 is its expenditure as a fraction of world expenditure so

$$F_{i,1}(Z)=\frac{d_{i,1}Y_1}{d_{i,1}Y_1+d_{i,2}Y_2}=\frac{d_{i,1}Z_1}{d_{i,1}Z_1+d_{i,2}Z_2}.$$

The quantity produced is

$$Q_i(x,Z)=q_{i,1}(x_1,Z)+q_{i,2}(x_2,Z)$$

the sum of the quantities $q_{i,j}$ produced in each country. The $q_{i,j}$ are defined by $q_{i,j}(x_i,Z)=f_{i,j}(l_{i,j})$ where the labor quantity $l_{i,j}$ is determined by x and Z and is found from (3.2) and (3.3). Looking at (3.2) and (3.3) we see that Z determines the world expenditure on good i, and the role of the $x_{i,j}$ is to split that world expenditure between the labor forces of the two countries. In the special case where one country, for example Country 1, is the sole producer, we have $x_{i,1}=1$, $x_{i,2}=0$, and $Q_i(x,Z)=q_{i,1}(1,Z)+q_{i,2}(0,Z)=q_{I,1}(1,Z)$.

The full expression for the utility u_1 is then

$$(3.6)\ u_1(x,Z) = \sum_i d_{i,1} \ln \frac{d_{i,1}Z_1}{d_{i,1}Z_1 + d_{i,2}Z_2} \{q_{i,1}(x_{i,1},Z_1) + q_{i,2}(x_{i,2},Z_2)\}.$$

This expression is complicated both in its dependence on the assignment x and on the normalized national incomes Z. In addition, for equilibrium x, Z and x are linked to each other through (3.4). This makes it difficult to compare the many different equilibria except by fully computing each one. Although useful and suggestive experiments along that line can be done and were done as part of this work[7] we will take a different approach in what follows.

We will emphasize perfectly specialized equilibria and the simplifications that are possible with them. As mentioned earlier, this emphasis will be justified retrospectively as they are the equilibria that determine the shape of the equilibrium region.

Utility for Perfectly Specialized Equilibria

If x_1 and x_2 are any variables constrained to be either 0 or 1, and if $x_1=0$ implies $x_2=1$ and vice versa, then we always have for any function $g(x_1,x_2)$ the tautology $g(x_1,x_2)=x_1g(1,0)+x_2g(0,1)$. The variables x_1 and x_2 act as a switch between the two values of g that are the only ones possible with such restricted variables. The individual terms $x_{i,1}$ and $x_{i,2}$ of an *integer* assignment x are of course variables of this type.

Letting g be successively the individual terms of the sum (3.6), and using the tautology, gives an expression for utility that is valid for integer x. This is the linearized utility Lu_1.

$$(3.L)\ Lu_1(x,Z) = \sum_i x_{i,1} d_{i,1} \ln F_{i,1}(Z) q_{i.1}(1,Z_1) + x_{i,2} d_{i,1} \ln F_{i,1}(Z) q_{i.2}(1,Z_2).$$

For integer x *only* we have $Lu_1(x,Z) = u_1(x,Z)$. The merit of $Lu_1(x,Z)$ is that for fixed Z the expression is now linear in the variables x.

Boundary Calculation Preliminaries

[7]Computer experiments played a significant role in many parts of this paper.

The introduction of the concept of boundary turns out to simplify enormously both the task of searching for high utility equilibria, and of finding the shape of the equilibrium region.

To find the upper boundary of the array of points (Z_1,U_1) corresponding to all perfectly specialized solutions, we might think of defining a function $B_I(Z)$ to be the result of fixing Z and then maximizing $u_1(x,Z)$ over all integer x subject to (3.4), i.e. over all perfectly specialized equilibria having that Z value. Finding this maximal $u_1(x,Z)$, for any given Z, would be an integer programming problem, and the $B_I(Z)$ values obtained this way would, by definition, be equal to or above the utility of any perfectly specialized equilibrium point having that Z value. The collection of points $B_I(Z)$, computed in this way for each Z, would form an upper boundary.

The maximization problem for each fixed Z would have the following economic interpretation: Once Z is fixed, the expenditure in each country for the ith good is completely determined. Therefore the fraction $F_{i,1}$ of the total production of the ith good that goes to Country 1 (or Country 2) is also fixed. The only way to improve the utility that Country 1 gets from the ith good is to increase the quantity of it that is produced. This can only be done, for integer x, by assigning its production entirely to the cheaper of the two possible producers. The labor constraint (3.4) would then prevent this assignment from being made simultaneously for every good, and the maximization problem would be to find the best assignment possible subject to that labor constraint.

While this direction and motivation are fundamentally correct, there is still one difficulty to overcome: precisely as written, equation (3.4) will not usually have *any* solution in integer x for an arbitrary Z, much less many different x to maximize over. This reflects the economic fact that there are equilibria for certain Z only. To deal with this difficulty we need one more concept, the Classical Level, which also turns out to be useful in other ways as well.

<u>The Classical Level</u>[8]

For any Z we can define an assignment $x^C(Z)$, which we will call the classical assignment. The components of $x^C(Z)$ are defined by setting $x_{i,1}=1$ if

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[8]In Gomory[1991] this was referred to as the Ricardo Level.

$q_{i,1}(1,Z) > q_{i,2}(1,Z)$ while otherwise $x_{i,1}=0$. This is simply assigning the production of good i entirely to Country 1 if Country 1 is the cheaper producer at that Z, and otherwise assigning it entirely to Country 2.

For an arbitrary Z and its $x^C(Z)$ we will usually not have equality in (3.4). In fact for Z with very small Z_1, which means low wage in Country 1 since $Z_1=w_1L_1$, the terms on the right in (3.4a), involving as they do Z_2/Z_1 will be very large, and the right side of (3.4a) will be greater than 1. Therefore in (3.4) the left side will be larger than the right. The economic interpretation is simply this: if the wage is very low in Country 1 and production is assigned to the country that is the cheaper producer, the resulting demand for labor In Country 1 will outstrip the supply. Similarly for any $x^C(Z)$ with large Z_1, which means high wage in Country 1, the terms on the right in (3.4a) will be small, their total will be < 1, so demand for Country 1's labor will be less than the supply.

The demand for labor in Country 1 produced by Z and $x^C(Z)$ is easily seen to be monotone decreasing as Z_1 increases, i.e. as wages increase. The individual terms on the right in (3.4a) only decrease while the $x_{i,1}$ switch from 1 to 0, as Country 1 stops being the cheaper producer in industry after industry. It follows that there is a unique transition value of Z_1 which we will call Z_C, the Classical Level, that separates those Z_1 for which demand exceeds the labor supply from those for which the demand is less than the labor supply[9]. More formally we define Z_C the Classical Level to be $Z_C = sup\ Z_1$ *such that if* $x=x^C(Z)$

$$\sum_i (d_{i,1}\ Z_1 + d_{i,2}\ Z_2) x_{i,1} > Z_1.$$

Below the Classical Level, i.e. $Z_1 < Z_C$, if x^C is used, the demand for labor outstrips the supply, and for $Z_1 > Z_C$ the demand is less than the supply.

For Country 2 the situation is reversed. Below the Classical Level the demand for labor is less than the supply, if x^C is used, and *above* Z_C the demand exceeds the supply. The behavior at Z_C itself and the notion of Classical Point, are both explained in Appendix 3-1.

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[9]We could also have defined the Classical Level in terms of the increasing demand for Country 2's labor. The result would, of course, be the same as can be seen from the relationships given in footnote 4.

We are now ready to discuss a complete boundary calculation.

<u>The Boundary $B_I(Z)$</u>

In outlining a boundary calculation that involved maximizing over many equilibria x for a fixed Z, we had encountered the difficulty that, instead of having many specialized equilibria, there were, for most Z, no equilibria at all.

To deal with this difficulty we relax (3.4) to an inequality, which then has many solutions for any Z, and define $B_I(Z)$ by the integer programming problem,

$$(3.7)\quad B_I(Z)=Max_x\ u_1(x,Z)=Max_x Lu_1(x,Z) \qquad x_{i,2}\ integer,$$

$$with \quad \sum_i \{d_{i,1}Z_1+d_{i,2}Z_2\}\ x_{i,2}\leq Z_2 \quad .$$

The inequality assumes the direction shown for Z above the Classical Level ($Z_1 > Z_C$) *and is reversed for Z below the Classical Level*

This relaxation *allows underutilization of labor* in the country whose labor is *scarce*. Consequently maximizing utility for the given Z should push the inequality very close to equality as the attempt is made to use this valuable labor

In (3.7) we have arbitrarily used the inequality form of (3.5) as the constraint. Of course we could just as well have chosen (3.4). Since we will often have occasion to refer to the inequality versions of (3.4) and (3.5) we will refer to these as (3.4i) and (3.5i). It will always be assumed that these inequalities point in the proper directions.[10]

(3.5i) only involves the variables $x_{i,2}$, but the objective function Lu_1 involves

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[10]While we have given an economic meaning to the Classical Level, and used this to choose the proper direction for the inequalities (3.4i) and (3.5i). It is also possible to give a purely mathematical description: Any equation is equivalent to two inequalities, one $\geq$ and one $\leq$. If we consider the maximization problem (for some Z) as a linear (not integer) programming problem, one or the other of the two inequalities will be the binding constraint. This inequality is the one that should be used for the integer maximization problem for that Z.

both $x_{i,1}$ and $x_{i,2}$ If we rewrite Lu_1 in terms of the $x_{i,2}$ only, using $x_{i,1}+x_{i,2}=1$ to eliminate the $x_{i,1}$ we get

$$(3.8)\quad Lu_1(x,Z) = \sum_i d_{i,1}\ln F_{i,1}q_{i,1}(1,Z) + \sum_i x_i d_{i,1}\ln\frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)}$$

so we can put the maximization problem (3.7) in a good computational form involving the $x_{i,2}$ only. If we use $P_1(Z)$ to denote the first sum in (3,8), which is a function of Z but not of the $x_{i,2}$, and use $c_{i,2}(Z)$ to denote $d_{i,1}\ln(q_{i,2}(1,Z)/q_{i,1}(1,Z))$ we obtain

$$(3.9)\quad B_I(Z) = Max_x\ P_1(Z) + \sum_i x_{i,2}c_{i,2}(Z)$$

$$\textit{with}\quad \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}\ x_{i,2} \le Z_2 \quad \textit{and } x_{i,2} \textit{ integer.}$$

(3.9) and variants of it will be our basic tool in dealing with equilibrium regions, so it is worth while to make some observations about its properties. We will use these observations frequently.

(1) Both the objective function and the inequality are linear in x for fixed Z.

(2) While the terms in the inequality are always positive, the $c_{i,2}$ in the objective function can be either positive or negative. The *sign* of $c_{i,2}$ is determined by the ratio $q_{i,2}(Z)/q_{i,1}(Z)$. If $q_{i,2} > q_{i,1}$, or equivalently, if Country 2 is the cheaper producer of the world supply $c_{i,2}$ will be positive. If Country 1 is the cheaper producer $c_{i,2}$ will be negative.

(3) For any Z above the Classical Level:

(a) Those components $x_{i,2}$ of the optimizing x that have $x_{i,2}=1$ must also have $c_{i,2}>0$. For if $c_{i,2}$ were negative, $x_{i,2}$ could be decreased to give a still better solution.

(b) In the optimizing x, or even in any feasible x, not all components $x_{i,2}$, with $c_{i,2}>0$ can have $x_{i,2}=1$. For an x having that property would be $\geq x^C$ in every $x^{i,2}$ component and, above the Classical Level, x^C already violates (3.5i). This is merely a restatement, in terms of (3.9) of one of the properties of the Classical Level, *above* the Classical Level, Country 2 does not have the labor to be the producer of all the goods of which it is the cheaper producer.

The first variant of (3.9) that we will need is the one that uses the $x_{i,1}$ in place of the $x_{i,2}$. This gives, for Z above the Classical Level,

$$(3.10)\quad B_I(Z)+Max_x\ P_2(Z)+\sum_i x_{i,1}c_{i,1}(Z)$$

$$\textit{with}\quad \sum_i \{d_{i,1}Z_1+d_{i,2}Z_2\}x_{i,1}\geq Z_1 \quad \textit{and } x_{i,1} \textit{ integer.}$$

Again the inequality assumes the direction shown for Z above the Classical Level and is reversed for Z below the Classical Level.

The $P_2(Z)$ of (3.10) is identical with the $P_1(Z)$ from (3.9) except for the substitution of $q_{i,2}$ for $q_{i,1}$. Similarly in the remaining part of the objective function, the q's have also been interchanged, so $c_{i,1}=-c_{i,2}=d_{i,1}\ln(q_{i,1}(1,Z)/q_{i,2}(1,Z))$. In economic terms the problem (3.10) is to make the best assignment of producers while being obliged to *over*utilize the labor in the country whose labor is little sought after.

We have for (3.10) the translation of the observations (3a) and (3b) above. For any Z above the Classical Level, the components $x_{i,1}$ of the optimizing x with $x_{i,1}=0$, must also have $c_{i,1}<0$. However not all $x_{i,1}$ with negative $c_{i,1}$ can be 0 in the optimizing solution or even in any feasible x.

<u>Computation of $B_I(Z)$</u>

The $B_I(Z)$ defined by either (3.9) or (3.10) can be computed by any integer programming technique. For a single inequality problem such as this, ordinary dynamic programming is very effective. It allows the computation of a point on the array boundary without examining the 2^n specialized solutions. The dynamic programming calculation itself is spelled out in Appendix 3-2.

Furthermore the dynamic program gives actual integer solutions x so that we can compute Z(x) from (3.4), and go on to compute utility etc. and hence fully describes the maximizing equilibria for each Z_1. As we will see in Section 4, the equilibria so attained will be arbitrarily close to the boundary at Z_1 as the problem size grows. In practice we find these calculations to be rapid and the resulting equilibria to be extremely close to the boundary even for very moderate sized problems, so that we have, in practice, a way of generating almost optimal equilibria for any specified Z_1. We can then spell out the boundary curve by computing a regular grid of different Z_1 from $Z_1=0$ to $Z_1=1$, and we will get a whole series of boundary points and nearby equilibria. Among these points the one with the largest utility is an equilibrium point that either is the utility maximizing equilibrium point for Country 1, or is very close, and approaches the best as the problem size grows. More will be said about the effectiveness of this computation at the end of this section and in Section 5.

The Boundary B(Z)

There is an even easier calculation that gives a slightly weaker but extremely useful boundary curve, which we will call B(Z). To get B(Z) we further relax the problem (3.9) by allowing *continuous* $x_{i,2}$. It is easily seen that with continuous variables the maximizing x will always satisfy the inequality in (3.9) as an equality, so in fact B(Z) is given by maximizing $Lu_1(x,Z)$ subject to (3.5), i.e..

$$(3.9a)\quad Max_x\ Lu_1(x,Z) = P_1(Z)+\sum_i x_{i,2}c_{i,2}$$

$$subject\ to\quad \sum_i (d_{i,1}Z_1+d_{i,2}Z_2)x_{i,2} = Z_2.$$

There is of course a problem equivalent to (3.9a) which uses the $x_{i,1}$ instead of the $x_{i,2}$ and (3.4) instead of (3.5). We will refer to this version of the problem as (3.10a).

In (3.9a) we are looking at a particularly simple linear programming problem with only one equation and upper bounds, $0 \leq x_{i,2} \leq 1$, on the variables $x_{i,2}$. The solution technique for such a special linear programming problem or "continuous knapsack problem" is well known and particularly simple[11]. (3.9a)

[11]Some useful information on this class of problems is contained in Gomory and Gilmore [1966].

can be thought of as filling a space of length Z_2 with amounts $x_{i,2}$ (not necessarily integer) of goods. The ith good has length $d_{i,1}Z_1+d_{i,2}Z_2$ and value $c_{i,2}$; the goal is to fill the space with the most valuable assortment of goods.

The solution to such a problem is to put in goods in succession in the order of their value per unit length, which we will call value density. The densest variable is used first. When its turn comes the amount $x_{i,2}$ of each good is increased from zero until either the amount $x_{i,2}=1$, or the equation (3.5) is satisfied (i.e., the space is used up), whichever occurs first. If the $x_{i,2}$ reaches 1 first, we start again with the next good in order of value density. If for the jth good the equation is satisfied for some value of $x_{j,2} < 1$, the current values for all variables $x_{i,1}$ are the optimizing solution. Note that $x_{j,2}$ is the only variable that is non-integer in this solution. The variables that preceded it in value density are 1, and those after it are 0.

This calculation is then repeated for different Z to get the boundary curve. It is the results of these simple calculations that appear as the dotted lines in our figures. While this already is a rapid and simple calculation it can be further improved. (Appendix 3-3).

This calculation too has an economic interpretation. The length $d_{i,1}Z_1+d_{i,2}Z_2$, is, for any fixed Z_1, proportional to the amount of labor required in Country 2 when Country 2 is the sole producer of the ith good. Similarly the expression for the value in (3.11) represents the *change* in utility for Country 1 resulting from Country 2 becoming the producer instead of Country 1. The industries for which Country 2 is to be the producer will be selected in the order of value density, those which yield the greatest improvement in utility per labor hour are chosen first.

If we consider the case of two countries with identical demand structure, i.e. $d_{i,1}=d_{i,2}$, the length is simply $d_{i,1}$ and the density, or change in utility per labor hour, is $c_{i,2}/d_{i,1}=\ln(q_{i,2}(1,Z)/q_{i,1}(1,Z))$. Industries i will be chosen in the algorithm before industry j if $\ln(q_{i,2}(1,Z)/q_{i,1}(1,Z)) > \ln(q_{j,2}(1,Z)/q_{j,1}(1,Z))$. In other words industry i will be chosen before industry j if it has greater *comparative advantage*.

However when the countries have dissimilar demand structures, the order of choice is determined by $c_{i,2}/(d_{i,1}Z_1+d_{i,2}Z_2)=(d_{i,1}/(d_{i,1}Z_1+d_{i,2}Z_2))\ln(q_{i,2}/q_{i,1})$ which involves q_2/q_1 but also is influenced by the value $d_{i,1}/(d_{i,1}Z_1+d_{i,2}Z_2)$ which measures the relative importance of the ith good to Country 1. Roughly speaking when the ith good is more important to Country 1 than to Country 2 it will tend to be picked earlier as Country 1 maximizes its utility. Thus when the countries have non-identical demand structures, Country 1 will choose a different production pattern to optimize its utility than Country 2 would have chosen to optimize its utility.

It is also important to realize that while the algorithm maximizes Country 1's utility by having goods as much as possible produced by the cheaper producer, all this is within the setting of an assumed *fixed* Z or fixed exchange rate w_1/w_2. As we will see in Section 8, when Z is not fixed, Country 1 can often *gain utility by becoming the producer of goods of which it is the less efficient producer* because acquiring an industry changes Z and improves the exchange rate, giving it a larger slice of all the world's goods.

So far we have discussed the continuous variable method only in terms of obtaining a boundary, not in terms of obtaining high utility equilibrium points. Of course the optimizing x, being non-integer, is usually not itself an equilibrium point. However equilibria can be obtained by rounding the single non-integer variable that will appear in the solution to (3.9a) either up or down. It seems clear that these two equilibria will be close to the boundary for large problems, and in Section 4 we show that they do in fact approach the boundary as the problem size grows. This is dealing with an integer programming problem by the time-honored device of rounding.

Lower Boundaries

While the goal so far has been to find the upper boundary of the array of perfectly specialized equilibria, exactly the same methods will give us the *lower* boundary. If we minimize the objective functions in the problems (3.9) and (3.9a), instead of maximizing, we will get lower boundaries $BL_I(Z)$ and $BL(Z)$ corresponding to the two different relaxations. This approach produces the lower boundaries seen in figures 1.1 and 1.2 and fixes all perfectly specialized equilibria to be somewhere between these curves.

The Two Methods

The two methods of calculation we have been using, one with integer variables and (3.4i) and one with continuous variables and the equality (3.4), are two different relaxations of the original maximization problem described in the subsection entitled Boundary Calculation Preliminaries. Both methods seem to have their advantages in thinking about boundary related problems and both will be used in the rest of this paper.

We can consider either calculation both from the point of view of generating boundary curves and from the point of view of finding actual equilibria near those curves.

Figure 3.1, which is an 8 product model based on Table 3.1, resembles Figures 1.1 and 1.2 except that the upper boundary $B_I(Z)$ has been added for Country 1. This is the jagged black line[12] under the B(Z) curve in Figure 3.1. $B_I(Z)$ does follow the location of the integer points more precisely than does B(Z). However in Fig. 3.2, which is a 17-product model based on Table 3.2 we see that the two boundary curves are much more alike.

Both calculations can also give us points near their respective boundaries, the integer calculation does this automatically while the continuous calculation does this by rounding the non-integer variable up or down. Fig 3.3 is based on Table 3.3 and represents a problem with 27 goods. It shows the B(Z) from the continuous calculation together with the integer points obtained by the integer maximization calculation. From the more than 100 million specialized equilibria in the 27 good model, the calculation has produced the 75 shown in the figure that are sitting virtually on top of B(Z). If we select from these the one that maximizes the utility of Country 1, we will get a utility value that is within 1/6 of one percent of the highest point of B(Z), so this equilibrium point is at least within 1/6 of one percent of the highest utility that can be obtained by Country 1.[13]

It is also worth noting that the *utility to Country 2 is low* for any of the points that are near maximal for Country 1. This is in line with statements 1 and 4b of the introduction.

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[12]B_I was plotted from the data points using a standard plotting routine to create the line. The plotting does introduce some systematic distortion as what appear to be near vertical segments of the boundary in the figure should, in fact, be vertical.

[13]It is conceivable at this point in the paper that there are non-specialized equilibria above B(Z) yielding still higher utilities. However, as mentioned in the introduction, this possibility is ruled out in Section 7.

Both calculations appear to have their advantages and to be very effective in bounding the solution array in actual computation. However the boundary B(Z) is much smoother and, as we will see later, is easier to deal with theoretically. On the other hand the integer calculation is capable of producing very many more actual equilibria near B(Z).

A very good way of combining the strengths of both methods is to use them together as a sort of coarse and fine microscope. First, using the continuous method, we obtain an entire boundary, for example B(Z), the upper boundary of Country 1 in Fig 3.3. Then, in some narrower range of interest, for example near the maximizing hump of B(Z), we compute the nearby integer points by using the integer calculation and a fine grid. The result of doing this appears in Fig 3.4 which represents the hump area of Fig. 3.3, from $Z_1=.55$ to $Z_1=.75$, magnified by a factor of five. There are 119 equilibria computed just in that range by the integer method[14] while rounding the continuous method would produce 11. Using this technique it is possible to isolate the equilibria in a particular area, for example the equilibria between near $Z_1=.625$ and $Z_1=.675$ at the very peak of the hump in figure 3.4 and examine them for their common characteristics.

[14]The additional equilibria that appear in Fig. 3.4 and are not in Fig. 3.3 are the result of using a finer horizontal Z_1 grid over the narrower horizontal range.

Section 4 - Convergence to the Boundary

To motivate the convergence discussion that follows we will now indicate very roughly one interpretation of the results of this section and of Section 6.

In this section, we will see that, under reasonable restrictions, every point of $B_l(Z)$ and of B(Z) is approached by equilibrium points as the number of products in the model increases. Then, in Section 6, we will see that every point that lies between the upper and lower boundary curves is also approached by equilibria. Thus the entire shape between the upper and lower boundary curves eventually fills in with equilibria.

The main assumption behind these results it that as the number of industries increases, any individual industry represents a decreasing fraction of the national income.

Our approach will aim at proving these results with minimal complexity. As a result the estimates will be crude, and the results described here seem to occur in practice in much smaller problems that these estimates would indicate.

Outline of the Proof.

Let us consider any point on B(Z), for example the point (Z', B(Z')). The utility B(Z') is attained by the (generally non-equilibrium) assignment x' that solves the maximization problem (3.9a) for Z=Z'. This x' gives the log utility $u_1(x',Z')=\ln B(Z')$. We will go through three steps to estimate the distance from the boundary point $(Z',U_1(x',Z'))$ to a point $(Z,U_1(x,Z(x))$ associated with an equilibrium x. For convenience in estimating we will work with points (Z,u_1) using the log utility u_1 instead of the (equivalent) points (Z,U_1), using utility, that appear in our diagrams.

Step 1 introduces the concept of near equality (n.e.) equilibrium points and shows that they exist and that their Z-coordinates are always near Z'. For these n.e. x we give an estimate of $|Z'-Z(x)|$. Step 2 shows that n.e. equilibria also have utilities, $u_1(x,Z(x))$ which, while they are of course dependant on the values of x and Z(x), change only slightly if Z' is substituted for Z(x). We will estimate the amount $|u_1(x,Z')-u_1(x,Z(x)|$ which is that change. Step 3 shows that, after making the substitution of Z' for Z(x) in the utility, some of the n.e. equilibria then have utility near B(Z'), i.e. $|(u_1(x',Z')-u_1(x,Z')|$ is small. Steps 2 and 3 together imply $|u_1(x',Z')-u_1(x,Z(x)|$ is small, which means that the utility of the equilibrium point is close to that of $B(Z_1)$. This, and the result of Step 1 establishes nearness in both the Z and U dimensions and completes the proof.

Step 1 - Equilibria With Nearby Z.

Let us assume that we are above the Classical Level, $(Z' > Z_C)$. For this choice of Z' we define a *near equality (n.e.) integer equilibrium point x* to be an integer x satisfying (3.5i) for Z=Z' and such that increasing some component $x_{k,2}$ of x from 0 to 1 would result in a new x that does not satisfy (3.5i).

There are always n.e. equilibria for any choice of Z' because the optimal integer solution to (3.9) is n.e., as is the x obtained from rounding down the non-integer component of the optimizing solution to the continuous problem (3.9a). Both must be n.e. since, as we observed in Section 3, they both have (since we are above the Classical Level) components that are zero but have positive objective function entries. If raising one of these components to 1 did not violate (3.5i) it would produce a still better solutions to (3.9) or to (3.9a) respectively which would be a contradiction. So these optimal solutions must be n.e., as are many others.[15]

For n.e. equilibria we can state:

Lemma 4.1:If x is n.e. then

$$|Z'_1 - Z_1(x)| \le \frac{\delta}{1-g}$$

where δ is the largest of the individual demands $d_{i,j}$ and g, which measures the departure of the demands from identical demands, is defined by

[15]For example:If we follow the procedure for solving the continuous knapsack problem, but do not choose the successive pieces in order of value density, instead choosing at each step *any* of the remaining pieces that have positive density, the resulting x, rounded down, will be n.e.

$$g = \frac{1}{2} \sum_i |d_{i,1} - d_{i,2}|$$

g is 0 for identical demands $d_{i,1}=d_{i,2}$, and g=1 for "orthogonal demands" $d_{i,1}d_{i,2}=0$.

The idea of the proof of Lemma (4.1) is this: Since x is an equilibrium point, x and Z(x) satisfy the equation (3.5). Since x is n.e., x and Z' satisfy the inequality (3.5i). However a change in only one component of x, for instance if $x_{k,2}$ changes from 0 to 1, causes the changed x and Z' to violate (3.5i). Therefore for some value of $x_{k,2} < 1$ the equation (3.5) must be satisfied. So we have a situation where changing x just a little, i.e. in just one component, allows it to satisfy (3.5) with Z' in place of Z(x). This suggests that Z(x) and Z' are also only a little different from one another.

A proof along these lines is given in Appendix 4-1.

Step 2 - Equilibria With u_1(x,Z') Near u_1(x,Z(x))

Since we now know that Z(x) is near Z' for any n.e. equilibrium, we can show that u_1(x,Z(x)) is near u_1(x,Z') by bounding the derivative of u_1 with respect to Z_1. In Appendix 4-2 we show that the derivative at any point Z is bounded by

$$(4.1)\ M(Z') = \frac{1}{Z'_m}\{2 + \frac{\alpha(Z')}{Z'_m}\}.$$

Here $Z'_m=\min(Z'_1,Z'_2)$, and $\alpha(Z')=\max\ \alpha_{i,j}(Z')$, with each $\alpha_{i,j}=f_{i,j}'(l_{i,j})l_{i,j}/f_{i,j}$, which is the ratio of marginal to average productivity evaluated at the labor level required to be sole producer.

If we combine this bound with Lemma 4.1 we have:

Lemma 4.2: If x is n.e. then

$$|\ u_1(x,Z') - u_1(x,Z(x))\ | \le \frac{\delta M(Z^*)}{1-g}.$$

where Z^* lies between Z(x) and Z'. This completes Step 2.

Step 3 - Equilibria With $u_1(x,Z')$ Near $u_1(x',Z')$

Since Z' is above the Classical Level the variables appearing at a positive level in the solution x' to (3.9a) will have positive $c_{i,2}$. If we round the one non-integer variable in x' down to obtain an integer solution x_r we can decrease the log utility at most by an amount equal to the largest $c_{i,2}$. The $c_{i,2}$ are given by

$$(4.2)\quad c_{i,2}=d_{i,1}\ln\frac{q_{i,2}(1,Z(x))}{q_{i,1}(1,Z(x))}.\quad \textit{So if}\quad R(Z)=\max_i\left|\ln\frac{q_{i,2}(1,Z(x))}{q_{i,1}(1,Z(x))}\right|$$

that decrease must be $\leq \delta R(Z')$. This gives

$$0\leq u_1(x',Z')-u_1(x_r,Z')\leq\delta R(Z').$$

R will be large if for the given Z', in some industry, Country 2 can produce a much greater quantity as sole producer than Country 1 as sole producer.

Now consider any integer x that is at least as good as x_r as an integer solution to (3.9). Both the maximizing integer solution and the rounded solution x_r itself are examples of such x. For such an x

(4.3) $\quad u_1(x_r,Z') \leq u_1(x,Z') \leq u_1(x',Z')$ so $\quad 0 \leq u_1(x',Z')-u_1(x,Z') \leq \delta R(Z')$.

This completes the three steps of the proof for Z' above the Classical Level.

Theorem and Corollaries

For Z' below the Classical Level we need only switch to using the $x_{i,1}$ and the argument will proceed in exactly the same way.

If we put together (4.3) and lemmas (4.1) and (4.2) we can state the following theorem:

Theorem 4: If Z',B(Z') is any point of B(Z) then, provided g is not 0, there is a perfectly specialized equilibrium point x with $|Z'-Z_1(x)| \leq \delta/1\text{-}g$ and $|u(x',Z')-u_1(x,Z(x))| \leq \delta\{R(Z')+M(Z')\}$.

Remarks on the theorem:

(1) The theorem can be restated in terms of the utility itself as supplying an equilibrium x with $|Z'-Z_1(x)| \leq \delta/1\text{-}g$ and with $e^{-\delta(R+M)} \leq B(Z')/U_1(x,Z(x)) \leq e^{\delta(R+M)}$.

(2) Since $B_I(Z) \leq B(Z)$ the theorem holds with $B_I(Z')$ in place of $B(Z')$.

(3) since M involves Z_1 or Z_2 in its denominator, and R tends to be large for extreme exchange rates, both M and R will tend to be large and provide weaker bounds as we approach either $Z_1=0$ or $Z_2=0$. Slower convergence in these areas is in fact visible in most of our diagrams.

In the course of our proof we have also proven the following corollary.

Corollary 4.1: The equilibria produced by the two maximization algorithms are among the perfectly specialized equilibria x mentioned in Theorem 4.

The theorem gives us a rough feeling for the distance of our maximizing points from the boundary, however in practice this seems to be a gross overestimate. However our main purpose here is to show convergence rather than estimate its rate, so we will continue in that direction.

Theorem 4 relates any boundary point to a nearby integer solution in terms of the parameters of the given problem. However it can also tell us what happens as problems get large under reasonable circumstances.

Consider a sequence of problems with increasing numbers of goods, each of which absorbs a decreasing fraction of the national income. Let us denote the various parameters appearing in the theorem as it is applied to the mth problem by δ_m, R_m, and g_m. When these are known and Z is specified we have M_m.

Then we have the following Corollary:

Corollary 4.2: Let P_m be a sequence of problems with bounded parameters $R_m, 1/(1-g_m)$, and M_m, and with $\delta_m \to 0$. Then, for any Z', and any ϵ there is an m sufficiently large that the point $Z', B_m(Z')$ on the mth boundary curve will have an integer equilibrium point within ϵ of $(Z', \ln B_m(Z'))$ in both coordinates.

It is also immediate that under these circumstances the equilibria produced by our two maximization methods would also be within ϵ.

Let us discuss briefly whether the parameters of a sequence of reasonable models would remain bounded as assumed in the corollary. We will simply assume that the models do not converge on "orthogonal demands" so that $1/(1-g_m)$ remains bounded. α and R depend on the production functions, and if α and R are bounded, so is M. If we assume that the production functions that appear with increasing m are not radically different from those before them, we would expect the various production ratios that make up these parameters to vary in value but remain bounded unless they are being evaluated at ever increasing labor levels. However the largest possible labor input into any one production function will be bounded if individual industry sizes remain bounded, so under that rather reasonable condition the parameters would remain bounded.

We next discuss the condition δ_m->0. If in the sequence of problems the size of Country 1 remained bounded for all m, the amount of labor in some of the active industries would have to approach zero as m became very large, so we will assume that Country 1's size increases unboundedly with m. Now if the demand for any one good remained above a fixed percentage of the total demand, the labor force of that industry would also increase unboundedly. Therefore it is reasonable to assume that each individual demand decreases toward zero, which motivates our assumption δ^m->0.

For example if the sequence of problems was produced by adding new industries one at a time to an existing model, the new industries being roughly the scale of those that preceded, and also enlarging the labor force at the same time by adding the labor for the new industry, the conditions of the corollary would be met, and we would see integer points approach every point of the boundary as m increased, and we would see the results of our maximization calculations approach the boundary as well.

Section 5 - Some Remarks on Computation

One of the contributions of this paper is to provide methods for determining the boundaries of the region of equilibria for large problems, and also to find from the enormous number of equilibria, those giving high utility to one country or the other. In the course of the paper so far we have used phrases like "simple calculation" or "easy algorithm" and have shown figures displaying the results of various calculations. This has included boundary calculations, calculations to obtain good equilibria, calculations showing all perfectly specialized equilibria, and calculations that obtain, in addition to the perfectly specialized equilibria, the extremely numerous intermediate equilibria provided by the existence theorem. We will pause here for a moment to be more concrete in a practical sense about the amount of computation required to solve these problems.

The various computations referred to in the text up to this point were all run on the author's home computer,[16]. All programs were written in Basic by the author and are very far from optimal.

Typical run times are:

(1) For a boundary curve B(Z) made from 60 grid points:
7 industries 1.68 minutes, 17 industries 1.80 minutes, 27 industries 1.97 minutes, 37 industries 2.16 minutes.

(2) For $B_I(Z)$ and its nearby integer points, 60 grid points:
7 industries 3.8 minutes, 17 industries 9.17 minutes, 27 industries 15.3 minutes, 37 industries 20.8 minutes.

(3) For obtaining all the approximately 8000 perfectly specialized equilibria points in the 13 industry model in Fig. 6.1, 8 minutes.

(4) For the computation with roughly 19000 intermediate equilibria in Fig 1.1 about 5 hours.

Computations (1) and (2) grow slowly (linearly in this range of model sizes)

[16]An IBM PC Model 80

with the number of industries. They are also linear in the number of grid points. Computations (3) and (4) of course grow exponentially with the number of industries, and it is this exponential growth that we have circumvented with our algorithms.

Section 6 - Filling In

We will now turn our attention to the region itself rather than to its boundary. We will show that under the same circumstances as in Section 4, the various perfectly specialized equilibria not only approach the upper boundary but entirely fill out the space between the upper and lower boundaries as the number of goods grows.

We start the proof with the following lemma which is immediate from the linear programming point of view. As usual we will assume $Z'_1 > Z_C$

Lemma 6.1: Let B(Z') and BL(Z') be the values of the upper and lower boundary curves for some Z'. Then for any intermediate value V, $BL(Z') \leq V \leq B(Z')$, there is a feasible (non-maximizing} solution x to (3.9a), with at most two non-integer components, for which the value of the objective function (the log utility) is v=lnV.

Proof: Let us add to the maximization problem (3.9a) the linear constraint $Lu_1(x,Z') \leq v$. The problem now has two constraints, so the x that attains the linear programming maximum value, which is v, will have at most *two* variables that are neither 0 or 1.

Let x' now be that optimizing x with its two non-integer components $x'_{j,2}$ and $x'_{k,2}$. The solution x' satisfies (3.5) as an equality so that the integer point obtained by rounding up both $x'_{j,2}$ and $x'_{k,2}$ to 1 can not satisfy (3.5i), while the x obtained by rounding them both down clearly does. It follows that either x itself or one of the two x's obtained by rounding one component up and one down has the n.e. property, and Lemma(4.1) applies to that x, as does Lemma(4.2)

Consequently we have for this x

$$|Z'_1 - Z_1(x)| \leq \frac{\delta}{1-g} \qquad and$$

$$|u_1(x,Z') - u_1(x,Z(x))| \leq \frac{\delta M}{1-g}.$$

To bound the difference between $v=u_1(x',Z')$ and $u_1(x,Z')$ we simply observe that the change in Lu_1 produced by changing the terms $x'_{j,2}$ and $x'_{k,2}$ can not exceed $2\delta R$, so $|v-u_1(x,Z')| \leq 2\delta R$.

Putting together these three elements we have proved

Theorem 6. If (V,Z') is any point between B(Z') and $B_L(Z')$, then, provided g is not 1, there is an integer equilibrium point x with $|Z_1(x)-Z'_1| \leq \delta/1-g$ and with $|v-u_1(x,Z(x))| \leq \delta(2R+M)$.

And we have a similar corollary:

Corollary: Let P_m be a sequence of problems with bounded parameters R_m,1/(1-g_m), and M_m, and with δ_m->0. Then, for any Z', and any ϵ there is an m sufficiently large that any point (Z',V) between $B_m(Z)$ and $B_{L,m}(Z)$ will have an integer equilibrium point within ϵ of (Z',v) in both coordinates.

The fill in effect is already clearly visible in Fig. 6.1 which plots all the perfectly specialized equilibria from the 13 product model based on Table 6.1.

Because of the fill in effect we can discuss various parts of the region of between the upper and lower boundary curves with confidence that they will be populated with equilibria.

Section 7: Non-Specialized Equilibria[17]

So far we have worked entirely with integer solutions, that is to say with perfectly specialized solutions. Some justification for this approach can be seen from the following theorem.

Theorem 7.1: Let x be any equilibrium solution, whether specialized or not. Let Z(x) be the corresponding Z and $u_1(x,Z)$ the utility of x to Country 1, then

$$U_1(x,Z) \leq B_f(Z) \leq B(Z)$$

So *all* the equilibrium points, not just the specialized ones, lie under the upper boundary curves. Intermediate equilibria, however, can lie below the lower boundary curve as is clear from Figs. 1.1 and 1.2.

We now turn to the proof of Theorem 7.1. The idea of the proof is to compare the utility at x, which for our intermediate x is not the same as the linearized utility, with the linearized utility of rounded versions of x. We will use the following lemma to help us compare the utility at x, to the utility of a more rounded version of x.

Lemma 7.1: Let x be an intermediate equilibrium point with associated national income Z(x). Let $q_{i,1}(x_{i,1},Z(x))$ and $q_{i,2}(x_{i,2},Z(x))$ be the quantities of the ith good produced in the two countries. Then $q_{i,1}(x_{i,1},Z(x)) + q_{i,2}(x_{i,2},Z(x)) \leq \text{Min}(q_{i,1}(1,Z(x)), q_{i,2}(1,Z(x)))$.

This lemma states that *either* country, as the sole producer of good i, *at the demand and wage levels of the equilibrium point x*, will produce more than the two countries together at the equilibrium point x. The lemma does *not* assert that more would be produced if one country were actually the sole producer. For if that were to happen, we would have a normalized national income different from Z(x), with different wages and therefore possibly a different outcome. It also important to realize that the lemma does not assert that for any $0 \leq x_{i,1} \leq 1$ the inequality holds, but only for those $x_{i,1}$ *that are part of an equilibrium* x. Without that restriction the

[17]The results of this section are not needed for the analysis in Sections 8-11 except for the implicit use throughout of Theorem 7.1 which asserts that there are no non-specialized equilibria above the region of perfectly specialized equilibria.

result is not true.

Proof: At the equilibrium x, using (3.P), we have for each i prices and wages such that

(7.1) $p_i f_{i,1} = w_i l_{i,1}$ and $p_i f_{i,2} = w_i l_{i,2}$

If we form the ratio of the two expressions in (7.1) and use the relations $f_{i,1}(l_{i,1}) = q_{i,1}(x_{i,1}, Z(x))$ and $l_{i,1} = x_{i,1}(L_1/Z_1)(d_{i,1}Z_1 + d_{i,2}Z_2)$, together with similar relations for $f_{i,2}$ and $l_{i,2}$, we obtain $q_{i,1}(x_{i,1}, Z(x))/q_{i,2}(x_{i,2}, Z(x)) = x_{i,1}/x_{i,2}$ or equivalently

(7.2) $q_{i,1}(x_{i,1}, Z(x))/x_{i,1} = q_{i,2}(x_{i,2}, Z(x))/x_{i,2} = C.$

Since $x_{i,1} + x_{1,2} = 1$, $\quad q_{i,1}(x_{i,1}, Z(x)) + q_{i,2}(x_{i,2}, Z(x)) = C.$

Since the q's are the quantities produced and the $x_{i,j}$ are proportional to the amounts of labor at the fixed $Z = Z(x)$, the production economies of scale conditions assert that the first ratio in (7.2) grows with $x_{i,1}$ and the second with $x_{i,2}$ so

$$q_{i,1}(1, Z(x))/1 \geq q_{i,1}(x_{i,1}, Z(x))/x_{i,1} = C = q_{i,1}(x_{i,1}, Z(x)) + q_{i,2}(x_{i,2}, Z(x)).$$

This proves the inequality of the Lemma for $q_{i,1}$, and the reasoning for $q_{i,2}$ is the same.

In the remaining part of the proof of Theorem 7.1 we assume, as usual, that Z(x) is above the Classical Level.

Let us consider the integer equilibrium point x' obtained from x by rounding down all the non-integer $x_{i,2}$ to 0. Since all the coefficients in the inequality are non-negative and x satisfies (3.5), x' satisfies (3.5i) and therefore is a feasible solution to (3.9). We will compare $u_1(x', Z(x))$ with the utility of x.

Since the maximum value of the objective function in (3.9) is $\ln B_1(Z(x))$ we already have $\ln B_1(Z(x)) \geq u_1(x', Z(x))$. If we can show that $u_1(x', Z(x))$ is $\geq u_1(x, Z(x))$, we would have $\ln B_1(Z(x)) \geq u_1(x', Z(x)) \geq u_1(x, Z(x))$ which would prove the theorem.

To compare the values of $u_1(x,Z(x))$ and $u_1(x',Z(x))$ we look at the individual terms in the two u_1 expressions. Using z as a dummy variable, the terms are of the form

$$d_{i,1}\ln F_i(Z(x))Q_i(z,Z(x)) \quad with$$

$$F_i(Z(x)) = \frac{d_{i,1}Z(x)}{d_{i,1}Z(x)+d_{i,2}Z(x)} \quad and$$

$$Q_i(z,Z(x)) = q_{i,1}(z_{i,1},Z(x))+q_{i,2}(z_{i,2}Z(x)).$$

The F term is the same whether z=x or z=x'. However if we compare the results of putting the components of x and of x' into the z of the second term we note that whenever the components of x and x' are different, because of the rounding, the x' components are always 0 or 1 while the x components come from an intermediate equilibrium point. So the conditions of Lemma 7.1 are fulfilled and we will always get an equal or larger result from the x' component. This shows that $u_1(x',Z(x))$ is $\geq$ $u_1(x,Z(x))$ which establishes the theorem.

Properties of Non-specialized Equilibria

Non-specialized equilibria are harder to analyze than are the specialized ones. Fortunately they are connected by the theorem we have just stated. In addition, our empirical work, of which Fig (1.1) and Fig (1.2) are examples, shows the generally lower utility of non-specialized equilibria.

However, mixed equilibria exist, they are numerous, and they have their own interesting properties. Mixed equilibria in the presence of economies of scale are usually unstable, in the sense that a departure from such an equilibrium tends to increase under reasonable dynamics. If one country increases its scale of production, it can produce at a lower cost, and therefore tends to produce still more etc. etc. However this is not always so, and it is important to realize that there are stable non-specialized equilibria as well.

There are two reasons why we cannot ignore the possibility of *stable* intermediate solutions.

(1) The classical Ricardo model will often have as part of its sole solution,

one good that is produced in both countries. Therefore we should expect that in the model discussed here we may see stable intermediate equilibria when we have production functions $e_{i,j}l^{\alpha_i}$ with α_i sufficiently close to 1.

(2) While we know that all intermediate equilibria are under the upper boundary of our region, and that some are under the lower boundary, we do not discriminate in these remarks between those that are unstable, as most are, and those that are stable. It would be interesting to know if the stable intermediates are all in or near the region. If this were to be so, we could regard the region as being the locus of *all* stable outcomes, the *stable* intermediates as well as the perfectly specialized equilibria.

We will discuss a few special cases of intermediate equilibria and return to the question just raised at the end of this section.

Loan Curves and the Simplest Case

The simplest case of a mixed equilibrium is the case in which only one good, say x_1 is produced in both countries. Let $x(x_{1,1})=(x_{1,1},x')$ where x' is a fixed n-1 vector of 0's and 1's representing the $x_{i,1}$. If we take $x_{1,1}$ as a parameter which varies from zero to 1, then for each value of $x_{1,1}$ we have an x and therefore can compute the national incomes from (3.4) and then the utilities. The result will be a curve connecting the perfectly specialized points x(1) and x(0). An example is the dashed line in figure (7.1).[18]

With isolated exceptions the points along this curve are *not* equilibrium points since the two producers are not producing at the same unit cost. However the points can be given an economic interpretation. Imagine that output from both countries in industry 1 is sold at the world market price. The lower cost producer (say Country 1) will have an income from sales that exceeds his wage bill, while

[18]This figure is based on Table (7.1). It is a six product model, and the loan curve connects the points $x_1=(1,0,0,1,0,1)$ and $x_1=(1,1,0,1,0,1)$. One effect of using a small model is to make the figures more readable, another is that the small number of goods produces large swings in national income after the shift of a single industry. Note that the production functions used here, and described in Table 7.1, are different from those of previous tables. They have less output for low labor levels.

the higher cost producer (Country 2) will have an income that doesn't cover his wage bill. Country 1 places its profit in an international bank and Country 2 borrows from the same institution to cover its wage bill. This arrangement will correspond to the computed points, and we will therefore refer to them as loan points or loan solutions. If the loan is non-zero these are non-equilibrium points and, are subject to market forces, described below, that will cause movement away from these points.

For each Z along such a loan curve, one country or the other will be producing at a lower unit cost. In figure (7.2) the curves of figure (7.1) have been redrawn with the dark part of each curve showing the cheaper producer. The point at which the curve switches from dark to light is a point of equal unit costs and hence an equilibrium point.

In Fig (7.2) there is a single transition point, and hence a single equilibrium point along each curve. This is what one might expect intuitively since as $x_{1,1}$ increases from 0 toward 1 Country 1 would generally become a cheaper producer of good 1 because of economies of scale, while Country 2 becomes more expensive, and this is in fact the commonest case.

If we define prices that cover costs for each country separately by $p_{i,1}=w_1 l_{i,1}/f_{i,1}(l_{i,1})$ and $p_{i,2}=w_2 l_{i,2}/f_{i,2}(l_{i,2})$ we would expect a plot of the p's versus $x_{1,1}$ to look something like fig (7.3) which is in fact that plot for the loan curve of Fig.(7.1). For $x_{1,1}$ near zero $p_{1,1}$ should be very large, and for $x_{1,1}$ near 1 $p_{1,2}$ will be very large. Also $p_{1,1}$ should generally (but not always) decrease with increasing labor to produce a single crossing point, $p_{i,1}=p_{i,2}$ where we would have equilibrium and a price $p=p_{i,1}=p_{1,2}$

The single equilibrium would in fact be the only case if the quantities produced were given by $q_1(x_{1,1},Z')$ and $q_2(x_{1,2},Z')$ for some fixed Z' as in Lemma 7.1 above. Here however we are dealing, not with a fixed Z', but with a varying $Z(x(x_{1,1}))$. Or equivalently we are dealing with a wage rate that varies as $x_{1,1}$ varies. Because of this the possible outcomes are more complex and multiple intersections and multiple equilibria can occur. Fig.(7.4) and its p plot (7.5) provide an example[19].

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[19]This figure is based on Table 7.2, and the curve connects $x_1=(1,1,0,1,0,1)$ and $x_1=(1,1,1,1,0,1)$. Table 7.2 differs from Table 7.1 only in the data on the

In any case the behavior of the $p_{1,1}$ and $p_{1,2}$ in each coming down from unboundedly large values at one end of the interval to finite ones at the other, does force the number of intersections of the two price curves, and therefore the number of equilibrium points, to be odd in every case.

We can also associate a rough (Marshallian) dynamics with Figs (7.3) and (7.5). To the left of the intersection in figure (7.3) Country 2 is the lower cost producer. Country 2 can therefore cover its labor costs and more, and still sell at a price lower than County 1, which at that price can not cover the wages of all its workers. This creates a situation where Country 2 will be motivated and able to increase production and get a larger share of the demand, (increase $x_{1,2}$, decrease $x_{1,1}$), while Country 1, which can not even pay all its work force, must reduce it and lose share, (decrease $x_{1,1}$,increase $x_{1,2}$). These directions of change are shown by the arrows in Fig.(7.3).

These conventions for dynamics can be applied in the same way to the general situation. We will assume that at each Z the producer j with the lower price curve will increase $x_{1,j}$ while the other producer is obliged to decrease. A handy result of this convention is that direction arrows on a curve reverse as the curve passes through a (simple) equilibrium.

In fig (7.3) these dynamics give the intuitively plausible result. The Country 1 producer will not cover his labor costs until he is operating at the scale that gives the equilibrium point, but thereafter he can profitably increase, The Country 2 producer does well until his production is brought down to the equilibrium point, after which these dynamics would cause collapse. We will call an equilibrium, such as that of figure (7.3), in which all arrows point away from the equilibrium point, an *unstable* equilibrium.

Unstable equilibria such as the one illustrated in (7.2) and (7.3) play a role in measuring how large a scale Country 1 must reach before it can compete with Country 2.

However in Fig (7.5) with its triple of equilibria, the first and third points are unstable but at the second one the arrows all point toward the equilibrium

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third product. There has been a change in demand for this product but, more significantly, the production exponent has been reduced to near 1.

point. We will call such a point *stable*. From these conventions and our previous remarks we have at once the following theorem.

Theorem 7.2: The number of intermediate equilibria is always odd. If the intersections are numbered in order of increasing $x_{1,1}$ the odd numbered ones are unstable and the even numbered ones are stable.

Some light on when the different cases occur is given by the following pair of theorems which are both proved in Appendix 7-1.

For production functions of the form $e_{i,j}l^{\alpha i}$, $\alpha_i > 1$, the following holds.

Theorem 7.3: If $q_{1,1}(1,Z(x)) \geq q_{1,2}(1,Z(x))$ for both x(0) and x(1). or if $q_{1,1}(1,Z(x)) \leq q_{1,2}(1,Z(x))$ for both x(0) and x(1), then there is only one intermediate solution $x(x_1)$, $0 < x_i < 1$, and it is unstable.

In words, the theorem asserts that if one country or the other is the cheaper producer over the range $0 \leq x_{1,1} \leq 1$, we have the simple outcome. There is only one non-specialized equilibrium for the product and that one equilibrium is unstable.

That the condition of Theorem 7.3 can not be wholly dispensed with, and that in its absence there are instances of multiple equilibria, is shown by the following partial converse:

Theorem 7.4: If the condition of Theorem 7.3 is not met, there are always multiple equilibria for values of the production exponent α_i sufficiently close to 1.

This implies that in these cases there are stable equilibria. Here we have returned to our earlier remarks and to the situation which approximates the linear Ricardo model, and we do have the appropriate stable equilibrium.

Stable Equilibria and the Equilibrium Region

If we think directly about this situation, or alternatively examine the proofs in Appendix 7-1, we will realize that the force that stabilizes the inherently unstable intermediate equilibria is wage change. If Country 1 increases output of the ith good from an intermediate equilibrium, its economies of scale increase, but so do the wages in Country 1, and this produces a slight counter-effect to the economies of scale. In a small model, one with only a few industries, this counter effect can be large, but in a large model, the change in one industry can have only a small effect on the wage rate, and therefore *stability will require correspondingly weak economies of scale*, i.e. a production function in that industry that is close to linear. This is reflected in the condition that the α_i in Theorem 7.4 must be near 1.

Although we have seen in our graphs that non-specialized equilibria can be below the lower boundary of the region of specialized equilibria, we can now make a plausible case that *stable* non-specialized equilibria will all lie above or near the lower boundary.

The argument depends on the near-linearity of the production functions required for stability. If x represents any intermediate equilibrium with relative national incomes Z(x), the *linearized utility* for that x, as for any x, will lie between B(Z(x)) and BL(Z(x)), the upper and lower boundary points corresponding to Z(x). With economies of scale, the *true utility* for x, will of course have a different, and generally lower, value. However, for an absolutely linear production function, the true utility is in fact *underestimated* by the linearized utility[20] so its value would be ≥ BL(Z(x)). Therefore it is plausible that for a *nearly linear* production function, the true utility would be either larger than the linearized utility, or at worst, very slightly lower, and therefore be either above or just below BL(Z(x)).

Putting these pieces together, we would expect a *stable* intermediate equilibrium x in a large model to require almost linear production functions in the industries where it has stable intermediate solutions. It would therefore have a utility either above or just below the value obtained by the linearized utility for that x. Therefore the stable intermediates would either be in the equilibrium region or just below its lower boundary. In the latter case they would converge to the lower boundary as the problem size becomes large.

In the extreme, if we imagine a problem where the small deviation in one industry affects the wage rate in the country not at all, we would require truly

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[20]The relevant observation here is that $\log(xK_1+(1-x)K_2) \geq x\log K_1 + (1-x)\log K_2$.

linear production functions for stability, and the resulting equilibria would in fact lie in the equilibrium region.

While this is only a plausible argument, it is one that can be made rigorous for a very large class of production functions, and this is planned for a future paper.

Section 8. The General Shape of the Region and Some Special Cases

General Shape of the Region

The many[21] numerical examples that have been examined all show the same characteristic region shape that is seen in Figures 1.1, 1.2, 3.1, 3.2, 3.3, 6.1, 7.1, 8.1, 8.2, and 9.1. These characteristics are, for Country 1:

(1) a steady rise in the height of the upper boundary over a range from $Z_1=0$ to some Z_1 value to the right of Z_C .

(2) To the right of Z_C a height for the upper boundary that is above the autarky level.

(3) A point of maximum utility to the right of Z_C followed by a descent of the upper boundary to the autarky level

(4) Upper and lower boundaries coinciding at Z_1 =0 and $Z_1=1$.

For Country 2 we have of course an equivalent set of statements.

These characteristics are not accidental but derive from the fundamentals of the economic situation being portrayed. We will now interpret the continuous variable calculation in economic terms to show how the characteristic upper boundary shape reflects the underlying economics.[22]

First consider a very small relative national income Z_1. The maximization problem (3.10a) contains the equation (3.4). Both the maximizing and minimizing solutions to (3.10a) will have all but one variable at 0 or a 1. To satisfy (3.4) for sufficiently small Z_1 almost all $x_{i,1}$ can only be zero. So both B(Z) and BL(Z) will

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[21]At the time of writing more than 80.

[22] In Gomory and Baumol[1992] we prove rigorously that not only all these characteristics but some important additional ones hold for significant classes of models.

be near 0. In economic terms, for small Z_1, Country 1, for very small Z_1, can be the producer of only a very few goods comprising a small fraction of world expenditure. Since Country 1's population is fully employed in making these goods, its wage is determined by the small total world expenditure on these goods spread over its entire work force. This results in a very low wage, or equivalently a very poor exchange rate. With this low wage rate and small Z_1, Country 1 gets a small fraction, $F_{i,1} = d_{i,1}Z_1/d_{i,1}Z_1+d_{i,2}Z_2)$, of any good produced in either country. So Country 1 has very low total utility.

We will next discuss the reasons that this utility increases with increasing Z_1. If we consider the boundary for slightly larger Z_1 we add more of some industry to Country 1, (increase the non-integer $x_{k,1}$ term). Since we are well below the Classical Level Z_C, the industry that is being added by the algorithm will be an industry in which Country 1 is, at Z_1 the lower cost producer. Keeping this in mind we can see that there are three effects on the linearized utility function

$$Lu_1(x,Z)-\sum_i x_{i,1}d_{i,1}\ln F_{i,1}(Z)q_{i.1}(1,Z_1)+x_{i,2}d_{i,1}\ln F_{i,1}(Z)q_{i.2}(1,Z_2).$$

which are produced by increasing Z_1.

(1) Since the industry being acquired (industry k) is now producing goods at a lower cost, $q_{k,1} > q_{k,2}$, Country 1's utility improves.

(2) Since its relative national income has gone up, all the $F_{i,1}$ are larger so Country 1 gets a larger fraction $F_{i,1}$ of everything produced in either country.

(3) The wage in Country 2 has gone down and the wage in Country 1 has gone up, so the $q_{i,2}$ associated with positive $x_{i,2}$ have gotten bigger and the $q_{i,1}$ associated with positive $x_{i,1}$ have gotten smaller. This affects utility by raising the cost of Country 1's products and lowering the cost of Country 2's products. These two effects clearly work against each other, and we will assume here what is both plausible and also proved for a wide class of functions in Gomory and Baumol[1992], that the dominant effects are (1) and (2). Since both (1) and (2) improve utility, utility increases monotonically.

At Z_C the algorithm can for the first time no longer find terms that are positive in the objective function, and therefore Country 1 starts to acquire industries in which it is the higher cost producer, consequently effect (1) changes sign and now has a negative effect on utility, becoming more and more negative as Z_1 increases further. Effect (2) remains positive, but the fraction $F_{i,1}$ increases more slowly with larger Z. Eventually (1) overwhelms (2) and the utility curve turns down.

Finally, for Z_1 near 1, Country 1 will be the producer of most goods. The fraction of world production of these goods that Country 1 gets is near 1. Country 1 is now producing most goods and keeping almost all of the production of those goods. So, with the exception of the few goods being made in Country 2 and consumed in Country 1, Country 2's existence has little impact on Country 1. In particular Country 2 provides an almost negligible market for Country 1. Country 1 has almost "returned to autarky". For this reason both boundary curves approaches the autarky level.[23]

We have given plausible reasons that explain the observed upward slope of Country 1's upper boundary, why it turns down somewhere to the right of Z_C, and why it then descends to the autarky level. So we have explained characteristics (1), (3), and (4). If we knew that the boundary was above the autarky level at Z_C on its way up to its maximum, we would then have a plausible reason why the boundary would be above autarky everywhere to the right of Z_C, which is the observed characteristic (2).

That the utility at Z_C should exceed the utility in autarky is intuitively plausible since at Z_C the optimal x will be able to assign production always to the cheaper producer, while exactly using up the labor supply. All but one of these producers, having an $x_{i,j}=1$, will have the benefits of making the entire world supply. It is in fact easy to prove this result in a number of different ways[24].

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[23]A more careful analysis of this "return to autarky" effect is given in Appendix 9-1.

[24]One proof is based on the Ricardo model. Let us define for a particular model M the associated linear model M_L. M_L is the same as M except that the $f_{i,j}$ are replaced by linear production functions $e_{i,j}l$ with $e_{i,j}=(f_{i,j}(l'_{i,j})/l'_{i,j})l$. The $l'_{i,j}$ entering into the constant $(f_{i,j}(l'_{i,j})/l'_{i,j})$ is the labor level needed to make the world supply of good i in Country j at the Classical Level Z_C. At Z_C (only) the maximization problem (3.10a) is completely unchanged by this change of production functions, so the maximizing utility for the two models is the same at Z_C. But we know that in M_L, the Ricardo model, this utility value is greater than the utility in autarky for M_L. The distribution of labor among the industries in autarky is the same in both models, but the utility in autarky for M_L exceeds the

Very similar arguments can be made to explain the monotone increase of the lower boundary and its eventual turn down to the right of Z_C.

After these rough arguments we next discuss some significant special cases where explicit formulae can be obtained for the boundary curve so that the shape of the equilibrium region is obtained in a very direct way.

Special Case - Consistent Comparative Advantage

We consider identical demands, $d_{i,1}=d_{i,2}$, and production function $p_{i,j}(l)=e_{i,j}\, l^{\alpha}$ for a *fixed* exponent α. The fixed exponent α is needed to give *consistent comparative advantage* between the two countries over a wide range of exchange rates (or equivalently over a wide range of Z). To see this we write the expression for comparative advantage, forming the ratio of output for each country taken as sole producer for goods i and j, at some arbitrary relative national income Z. The expression is $[f_{i,1}(l_{i,1})/f_{i,2}(l_{i,2})]/[f_{j,1}(l_{j,1})/f_{j,2}(l_{j,2})]$. Since the labor used in the ith industry is the world demand divided by the wage rate the term in the first square bracket is $(e_{i,1}/e_{i,2})(w_2/w_1)^{\alpha}$ and since α if fixed the entire expression is $(e_{i,1}/e_{i,2})/(e_{j,1}/e_{j,2})$. As this is independent of Z, comparative advantage does not change with Z.

With consistent comparative advantage, we might reasonably expect some simplification in the maximization algorithm since the various industries might be expected to enter the maximization calculation in some fixed order for all Z. We will see that when we also have identical demands this is indeed the case.

In the maximization problem (3.9a) the utility $u_1(x,Z)$ consists of two parts, one independent of x and one dependent on it. The independent part is, for our special case,

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utility in autarky of M because at the autarky labor level, the production functions in M are *below* the linearized ones, since economies of scale give us $f_{i,j}(l^{*}_{i,j})\leq(f_{i,j}(l'_{i,j})/l'_{i,j})la_{i,j}$. Thus in M Country 1's maximizing utility at Z_C exceeds M_L's utility in autarky at Z_C, and therefore, a fortiori, exceeds its utility in autarky in M. The same argument holds for Country 2.

$$\sum_i d_{i,1}\ln F_i q_{i,1} = \sum_i d_{i,1}\ln Z_1 + \sum_i d_{i,1}\ln e_{i,1}\left(\frac{L_1 d_{i,1}}{Z_1}\right)^\alpha$$

$$= K + (\alpha - 1)\ln\frac{1}{Z_1} \quad where \quad K = \sum_i d_{i,1}\ln\ e_{i,1}(d_{i,j}L_1)^\alpha.$$

We can recognize K as Country 1's utility in autarky U^{A1}.

The x dependant part, over which we maximize, is

$$\sum_i x_{i,2}d_{i,1}\ln\frac{q_{i,2}}{q_{i,1}} = \sum_i x_{i,2}d_{i,1}\ln\frac{e_{i,2}(L_2/Z_2)^\alpha}{e_{i,1}(L_1/Z_1)^\alpha} \quad which\ is$$

$$(8.2)\ \left(\sum_i x_{i,2}d_{i,1}\right)\ \ln\frac{Z_1^\alpha}{Z_2^\alpha} + \sum_i x_{i,2}d_{i,1}\ln\frac{e_{i,2}L_2^\alpha}{e_{i,1}L_1^\alpha}.$$

For symmetric demands and any feasible x, the sum in parentheses is, from (3.5), exactly $Z_2=(1-Z_1)$, so it is only the second sum in (8.2) that depends on x.

$$(8.3)\ \alpha(1-Z_1)\ln\frac{Z_1}{1-Z_1} + \sum_i x_{i,2}d_{i,1}\ln\frac{e_{i,2}L_2^\alpha}{e_{i,1}L_1^\alpha}$$

The maximization problem we are left with is one with overall knapsack length $(1-Z_1)$, objective function coefficients $d_{i,1}\ln(e_{i,2}L^{\alpha 2}/e_{i,1}L^{\alpha 1})$, individual piece lengths $d_{i,1}Z_1+d_{i,2}Z_2=d_{i,1}$, and therefore, value densities $\ln(e_{i,2}L^{\alpha 2}/e_{i,1}L^{\alpha 1})$. Since this is independent of Z (this is the algorithmic result of having consistent comparative advantage), we can solve this *once for all* Z, either in the dynamic programming case or in the continuous knapsack case. If we denote the outcome of this maximization by Knap(Z) we have for the utility in this special case

$$u_1(x,Z)= \{\ln U^A{}_1+(\alpha-1)\ln\frac{1}{Z_1}+ \alpha(1-Z_1)\ln\frac{Z_1}{(1-Z_1)}\} + Knap(Z).$$

Since in Knap(Z) we are solving a knapsack problem of length $(1-Z_1)$ we can express the resulting value as a average value density $d(Z_1)$ times the knapsack length $(1-Z_1)$. $d(Z_1)$ will increase monotonically as Z_1 increases and the length of the knapsack decreases, and will be bounded above and below by the greatest and least possible value densities $\ln(e_{i,1}L^{\alpha 2}/e_{i,2}L^{\alpha 1})$. Upon rearranging terms we get for the utility in the consistent comparative advantage case

$$(8.4) \qquad U_1(Z)=U^A{}_1Z_1\,\{(\frac{1}{Z_1})^{Z_1}(\frac{1}{(1-Z_1)})^{(1-Z_1)}\}^{\alpha}\exp((1-Z_1)d(Z_1)).$$

If $d(Z_1)$ varies slowly with Z_1, as it will unless the value densities are very different from each other, this is close to a simple formula giving the boundary shape. The shape is illustrated by an eight country example in Fig.8.1. The formula will apply *exactly* for competition between identical countries which we discuss next.

Special Case - Identical Countries and Consistent Comparative Advantage

Competition among identical countries is non-trivial in this model, and can produce many different equilibrium outcomes. In the identical country case with production functions $e_{i,1}l^{\alpha}=e_{i,2}l^{\alpha}$ and $L_1=L_2$, the objective function coefficients in $Knap(Z_1)$ will all be 0, so d(Z) will be 0. The resulting boundary curve is then given exactly by the formula

$$U_1(Z)=U^A{}_1Z_1\,\{(\frac{1}{Z_1})^{Z_1}(\frac{1}{(1-Z_1)})^{(1-Z_1)}\}^{\alpha}$$

and is plotted in Figure 8.2 for $\alpha=1.5$.

In this situation the lower boundary curve calculation is the same as the upper; they both have objective function 0. It follows that the upper and lower boundary curves coincide, so the curve of Figure 8.2 must have all the integer equilibria directly on it.

The curve of Figure 8.2, based on the formula, exhibits every aspect of the characteristic boundary shape, a shape from which (8.4) can also deviate only slightly.

Even in this special case with the region collapsed to a single line the various regions mentioned in the introduction (4a-d) are all plainly present in simple form. The subregions (4a) and (4d) that are respectively advantageous and disadvantageous relative to autarky can be read off from the location of the horizontal autarky bars. Beyond the two humps are the two subregions (4c) that are relatively disadvantageous to both countries. In particular the region of opposed interests for Country 1, 4c in the introduction, is the curve from the point below the maximum for Country 2 up to Country 1's maximum, so the interests of the two countries are strictly opposed over that entire range, with the maximum for Country 1 being a rather poor outcome for Country 2.

Section 9. General Properties of the Model

Production Functions and Stability

Economies of scale can be thought of as having two distinguishable effects.

The first is what might be referred to as a "barrier to entry" effect. If substantial sunk costs are required for entry into the industry, this gives a *producing* country an advantage over a *nonproducing* one. In our model this aspect of economies of scale shows itself in the low end of the production functions as little output for the labor input. It is this aspect of the production functions that forces a high level of activity before the non-producer can hope to compete with the producer, and (through the condition A3 of Section 2) eliminates the possibility of incremental entries.

In real life these barriers to entry come from many sources aside from the obvious possibility of economies of scale in manufacturing. Examples are knowledge and expertise in the manufacturing process, the largely experience born ability to design a manufacturable product, knowledge of and experience with marketing channels, knowledge of customer needs, and even knowledge of and being known to particular customers. Much knowledge can only be obtained by doing, and there will be period of doing poorly through inexperience for any new entrant. In addition, especially in the case of industries in different countries, there is the question of infrastructure. If one industry is flourishing in Country 1, and non-existent in Country 2, a large part of the difficulty in entry will be to find the people or companies who can build plants of the proper type, and supply parts, specialized instruments, and specialized support services. While some of this can be imported, some cannot, and working at a distance is often not the same as working close by.

All of these factors and many more can make entry into a new industry a large sunk commitment now in exchange for a return that is both distant and inherently uncertain, And that uncertainty too is part of the barrier to entry. All these factors can be thought of as contributing to the shape of the low end of the production function.

The second aspect of economies of scale is the advantage that *larger* scale may give one producer over another when *both* are active in the industry. In this model this is reflected in the shape of the production functions for larger labor quantities.

The two aspects of economies of scale are quite separable, one can have a strong barrier to entry and weak large scale economies in a single production function, or vice versa, or any other combination. Keeping these points in mind we will state and then interpret the following theorem which at this point is quite straightforward.

Theorem 9.1: Let M_1 and M_2 be two different n-industry models with the same demands $d^1_{i,j}=d^2_{i,j}$ and with production functions $f_{i,j}(l_{i,j})$ that are the same for quantities of labor above the autarky level, i.e. for $l_{i,j}>d_{i,j}L_j$. Then the perfectly specialized equilibria and the boundary curves are the same for both models.

Proof: For any integer x the resulting Z(x) will be the same in both models, since the equations (3.4) or (3.5) only involve the demands and not the production functions. Thus the integer equilibria are the same pairs (x,Z(x)) in both models, and these in turn determine the labor quantities $l_{i,j}$. These labor quantities are above the autarky level, which means that the production functions have the same outputs, so for perfectly specialized equilibria the outputs are also the same in both models. These outputs in turn determine the coefficients in the linearized utility, so these too are the same in both models, and these in turn determine the boundaries. This completes the proof.

Although the points and the boundaries are the same what *does* change as the production functions change from M_1 to M_2 is the impediment to entry. If the change from model 1 is to new production functions that rise sharply near 0, the impediment to entry can be made as feeble as we wish. On the other hand if the new production functions are zero till near the autarky level and then jump rapidly back to the production curves of M_1, we have an extremely strong impediment to entry.

Identical and Non-Identical Demand Structures

Any production assignment x determines a Z(x), and worldwide production quantities Q_i of each good. The utility ratio U_1/U_2 resulting from x is

$$(9.1) \quad U_1/U_2 = \frac{\Pi_i\{F_{i,1}Q_i\}^{d_{i,1}}}{\Pi_i\{F_{i,2}Q_i\}^{d_{i,2}}}$$

If we have identical demands, i.e. $d_{i,1}=d_{i,2}$, we have $F_{i,1}=Z_1$ and $F_{i,2}=Z_2$ so (9.1) becomes $U_1/U_2=Z_1/Z_2$. This means that, *for fixed Z,* the utility of one country for any production pattern is a fixed constant times the utility of the other.

The consequences of this are: The x that maximizes Country 1's utility also maximizes Country 2's utility, so the same assignments yield the upper and lower boundaries in both countries. Equilibria on the upper boundary for Country 1 are on the upper boundary of Country 2. Equilibria near the upper boundary of Country 1 are near the upper boundary of Country 2. More generally, for fixed Z, if any x gives Country 1 p percent of the utility that it would attain at the upper boundary, it also gives Country 2 that same percent of its maximal utility.

In economic terms identical demands mean that the *rivalry between the two countries is confined to the determination of* Z, once Z is fixed a production pattern or equilibrium point that is good for one is good for the other.

This benign property of the identical demands does not carry over to the non-identical demand case. There countries will put different weights on different elements of the objective functions and the production plan (even for fixed Z) that is best for one is generally not best for the other.

This is clearly illustrated in Figures 9-1 and 9-2 both of which are based on a 37 good model. We first show, in Fig 9-1, a symmetrized version of the model, i.e. the $d_{i,1}$ and $d_{i,2}$ of the model have both been replaced by $(d_{i,1} + d_{i,2})/2$ so that it has become a model with identical demands. Using a fine grid we have computed a large number of maximizing equilibria around the hump area of Country 1 using the integer programming method. We see, as we would expect, that the corresponding utilities for Country 2 are on or near its upper boundary. However, if we return to the original unsymmetrized problem and repeat the calculation, we can see in Fig. 9-2 that the utilities for Country 2 have moved down sharply from its upper boundary. This is true even though the g-value of the model, $g=1/2\ \Sigma_i\ |d_{i,1}-d_{i,2}|$, is only .187 on a scale in which identical demands measure 0 and orthogonal demands measure 1.

The Effect of Country Size

In any model, let us substitute for Country 2 a smaller Country 2 with the same demands for each good, and the same production functions, but having a smaller labor force L_2.

The change in L_2 does not affect equations (3.4),(3.2), and (3.3). Any specialized equilibrium point x will yield the same Z from (3.4) and the same $l^*_{i,j}$ from (3.2) and (3.3) as before. However the $l^*_{i,2}$, which are normalized labor variables, will be multiplied by a smaller L_2 to get the actual labor $l_{i,2}$. This means smaller quantities $q_{i,2}$ will be produced of every good made by Country 2, while the fraction $F_{i,1}$ of that good going to Country 1 remains the same since the $F_{i,1}$ depend only on $d_{i,j}$ and Z. Therefore at every equilibrium point Country 1's utility is diminished in every term that represents a good made by Country 2.

The smaller L_2 will cause all specialized solutions for Country 1, *with the exception of autarky*, to decrease in utility, so autarky becomes relatively more attractive for Country 1 as Country 2 gets smaller. These *size effects are quite strong*, for example in the model represented by Figs 1.1 and 1.2, Country 2 is twice the size of Country 1 and it has a very significant region of equilibria below its autarky level, while Country 1 does much better. In Fig. 6.1 the size relations of the countries are reversed and Country 1 has a large region of points below autarky while the below autarky region of the smaller Country 2 is quite limited.

Section 10 - Economic Consequences of the Characteristic Regional Shape

As promised in the introduction we have given algorithms for the selection of good equilibria, and for obtaining the boundaries of the region of equilibria. We have shown that the equilibrium region tends to fill up with equilibrium points as the problem becomes large. It remains to review the conclusions (4a)-(4d) of the introduction about the economic consequences of the characteristic regional shape.

We have done this already for the special case of competition between identical countries. Here we will discuss the general case and the consequences of the characteristic shape. Much of what follows will be evident from our previous discussions but it seems worthwhile to pull the various observations together here. We will use Figures 10.1, 10.2 and 10.3, which are all based on Table 6.1, to illustrate the discussion.

Relations to Autarky

Statement (4a) of the introduction asserts the existence of a subregion of equilibria that are advantageous for Country 1 relative to autarky. In Fig. 10.1 we have drawn a horizontal line at the autarky level creating a subregion ABC above autarky. For Country 2 in Fig 10.2 the subregion ABC covers an even wider range. Since the upper boundary is known to be above the autarky level at Z_C, such subregions will always exist.

Similarly (4d) asserts that there are subregions that are worse than autarky. These subregions are the complementary subregions ABD. In Fig. 10.1, this is large, in Fig.10.2, it is small. These regions show the size effects mentioned in Section 9.

Centered around Z_C there is always a subregion of points that are better than autarky for *both* countries. In Fig 10.1 all of the region to the right of B is above autarky, and in Fig 10.2 all of the region to the left of A. Since the point B In Fig.10.1 lies to the left of the point A in Fig.10.2, there is an interval on the Z-axis which is both to the right of B and to the left of A. All points above this interval belong to a subregion in which *both* countries are above autarky. This is the area $BP_1P_2P_3$ in Fig.1 and the similar area $AP_1P_2P_3$ in Fig. 10.2. However, even in the unlikely event of the B for Country 1 being to the right of the A for

Country 2, there would always be, in large problems, equilibria in the immediate vicinity of Z_C that are above autarky for both countries. This is because the arguments of Section 8 have shown the upper boundaries of both countries to be above the autarky level at Z_C.

Opposed and Not Opposed National Interests

Next we return to the notion of *opposed national interests* as discussed in (4b). In Fig 10.3 we have drawn a vertical line L_1 through the point where the upper boundary of Country 1 ceases its monotone rise. We have a similar line L_2 for Country 2. The two lines cut out a region of generally opposing national interests in the following sense. Let L_3 be a third line between L_1 L_2 at relative national income level Z_1 intersecting the equilibrium region for Country 1 in segment S_1 and the equilibrium region of Country 2 in S_2. If Z_1 increases, which means that Country 1 produces a larger fraction of the world's goods, S_1 will generally move up and S_2 will move down. The utility values for equilibria on S_1 will therefore generally move up and the corresponding utilities for Country 2 will generally move down. So over this large region centered on Z_C an increase in the utility for Country 1 is generally associated with a decrease in utility for Country 2.

If, in Fig.(10.3) we continue to move L_3 beyond L_2 and toward $Z_1=1$, (this is marked by L'_3 in Fig. 10.3), both boundaries of Country 1 will eventually turn down. The two boundaries of Country 2 are also monotone decreasing here so we have entered a subregion of equilibria which are increasingly bad for both countries. Equivalently, in this subregion, it is generally in the *best interest of both countries* for Country 1 to *decrease* its normalized national income. Clearly there is a second subregion of this type the other end of the diagram. This discussion then gives a more precise meaning to statement (4c).

The theory we have developed tells us that the various subregions we have mentioned are in fact all well populated with equilibria for large problems. This is also illustrated in Fig 10.4. This contrasts with Fig 10.5, a two country model in which the two perfectly specialized equilibria would not even suggest the existence of the various regions.

Section 11. Extensions and Summary

In the more policy oriented Gomory and Baumol [1992] we extend this work in several directions. These includes some very strong results on gains from trade, the analysis of the case where some industries have economies and some have diseconomies, and a rigorous treatment of the characteristic shape.

What we have seen here is that there is a well defined equilibrium region and that we can calculate its boundaries, and also obtain equilibria that are close to utility maximizing. We have also seen that for large models the equilibrium region fills in completely with equilibrium points. We have observed that there is a characteristic regional shape that extends over a very large range of models and have given an economic rationale for that characteristic shape and discussed its economic consequences. The picture that emerges is one of *a considerable range of conflict* in the interests of the two trading partners.

References

Ethier, Wilfred J., "Internationally Decreasing Costs and World Trade," *Journal of International Economics*, (9), 1979, 1-24

Ethier, Wilfred J, "Decreasing Costs in International Trade and Frank Graham's Argument for Protection," *Econometrica*, (50), September 1982, 1243-1268

Gilmore, P. C. and Gomory, R. E., "The Theory and Computation of Knapsack Functions", Operations Research, November-December, 1966, Vol. 14, No. 6, pp. 1045-1074.pp. 469-479.

Gomory, Ralph E., "A Ricardo Model with Economies of Scale", Proceedings of the National Academy of Sciences, U.S.A., 1991, Vol. 88, Issue 18, pp. 8267-8271.

Helpman, Elhanan, "Increasing Returns, Imperfect Markets, and Trade Theory," in Jones, Ronald W. and Kenen, Peter B., (eds.) *Handbook of International Economics, Volume I*, Amsterdam: North Holland, 1984, 325-365

Helpman, Elhanan and Krugman, P. R., *Market Structure and Foreign Trade*, Cambridge, Mass.: MIT Press, 1985.

Kemp, M. C., *The Pure Theory of International Trade*, Englewood Cliffs: Prentice Hall, 1969.

Krugman, Paul R., "Increasing Returns, Monopolistic Competition and International Trade," *Journal of International Economics*, (9), 1979, 469-479

Viner, Jacob, *Studies in the Theory of International Trade*, New York: Harper and Brothers, 1937

Fig. 1.1

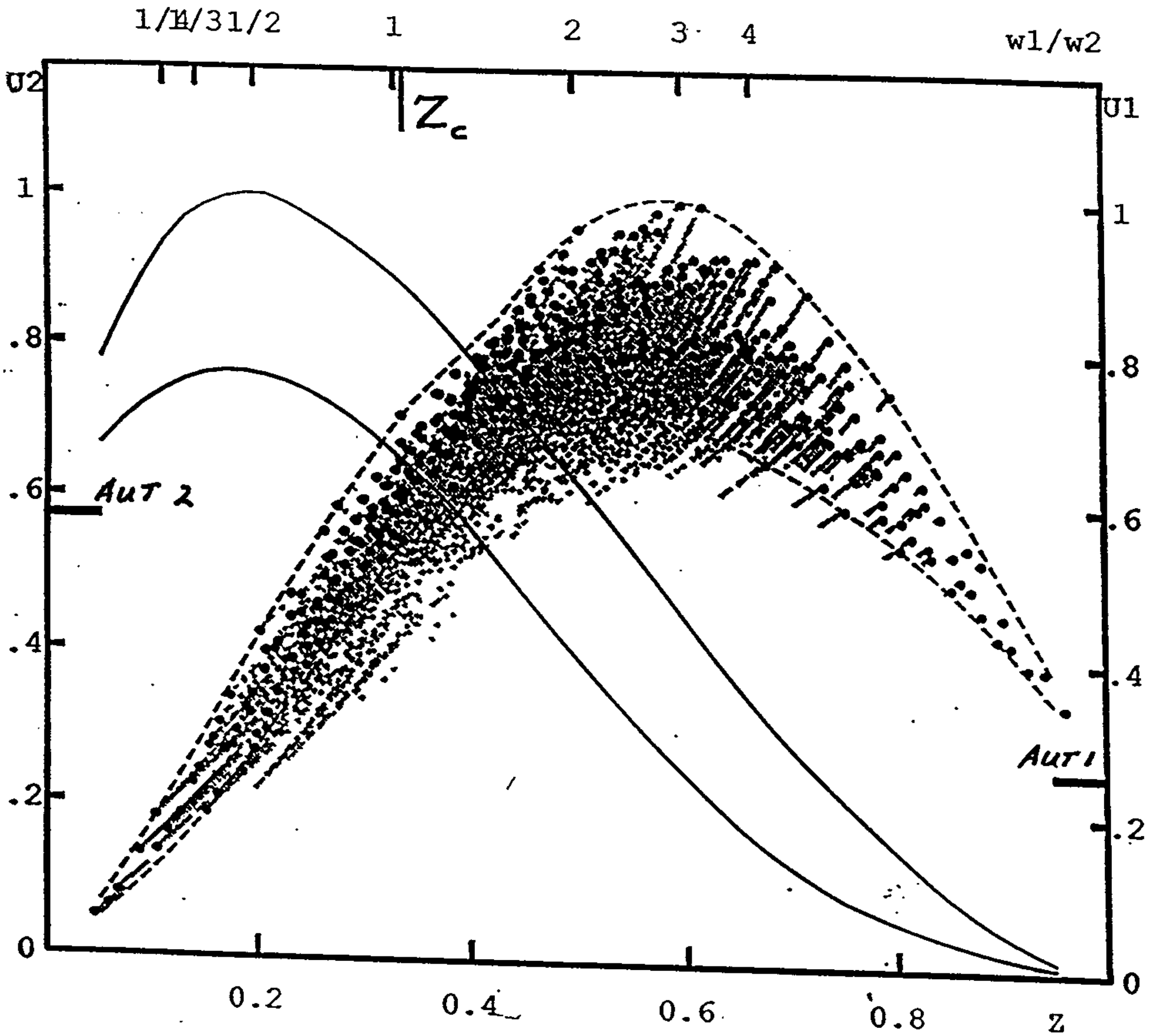

Fig. 1.2

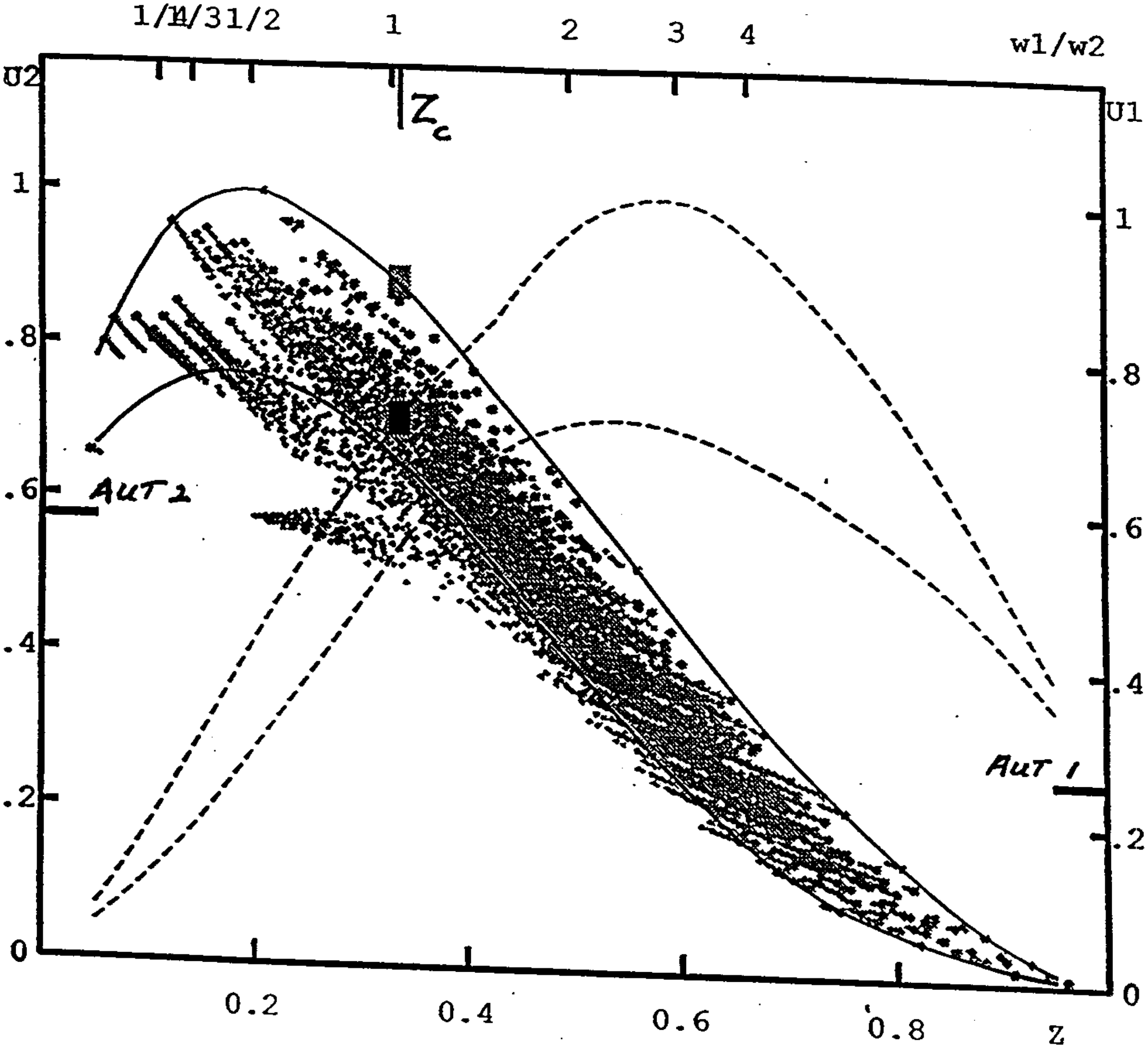

Fig. 3.1

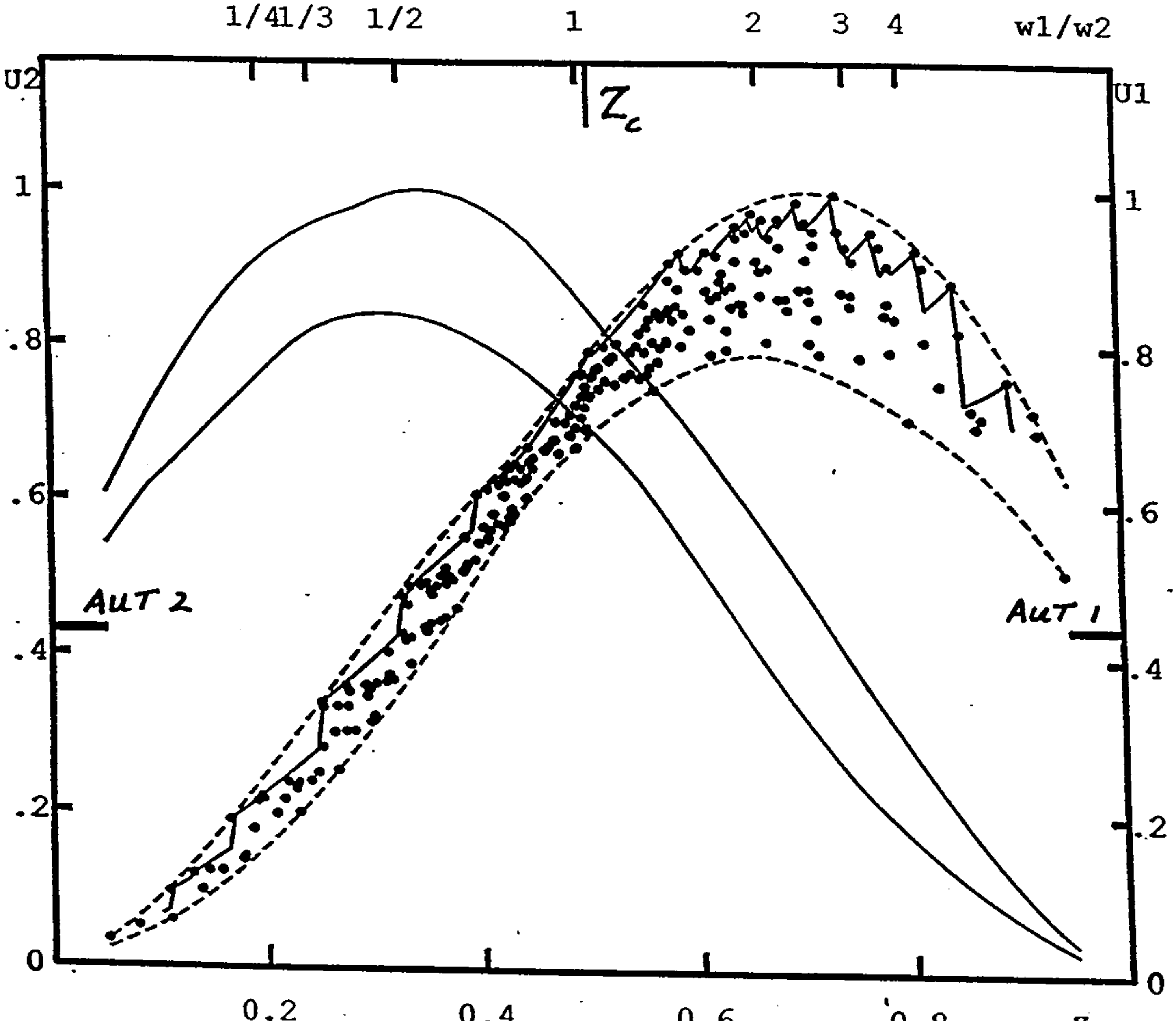

Fig. 3.2

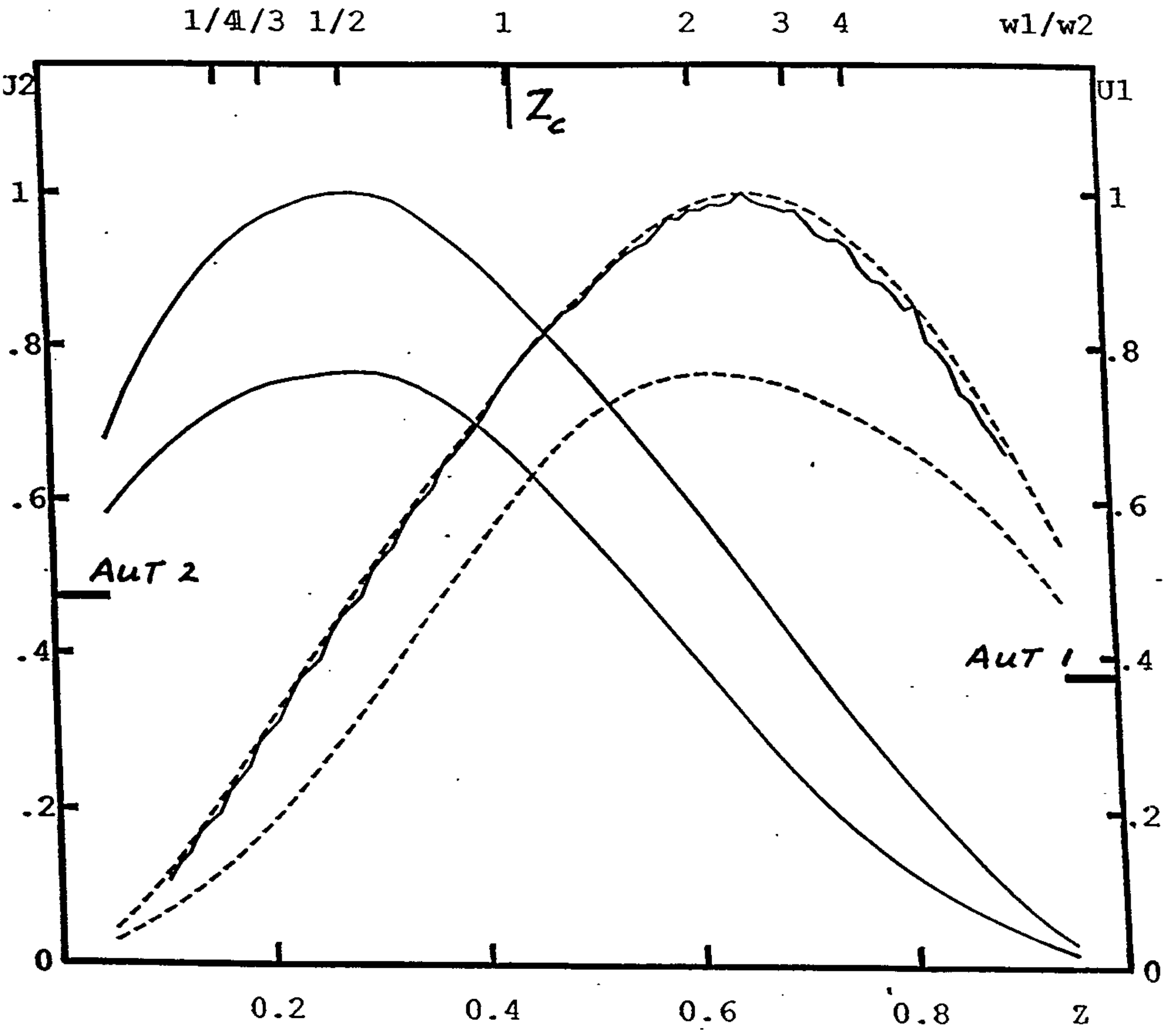

Fig.3.3

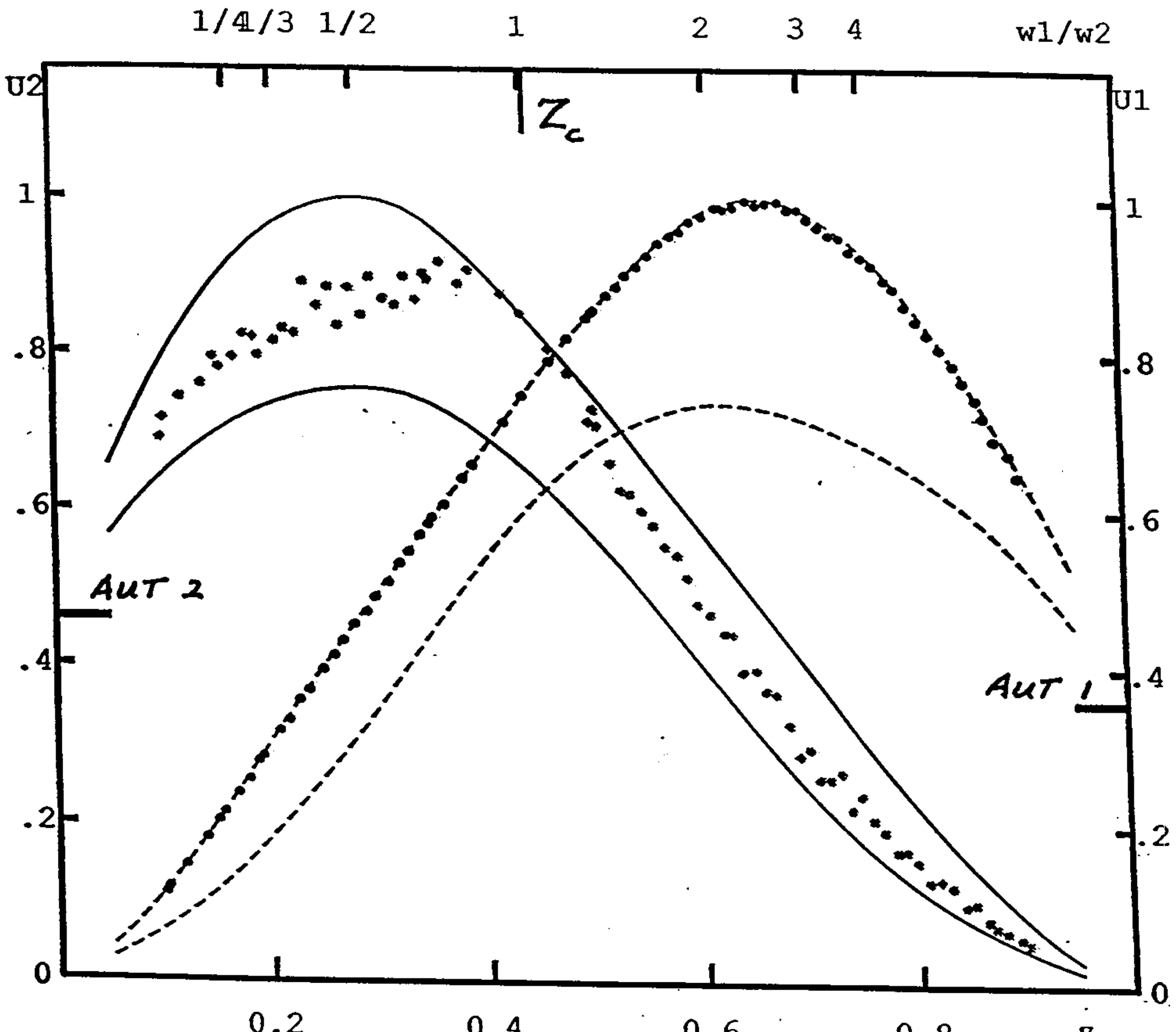

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Fig. 3.4

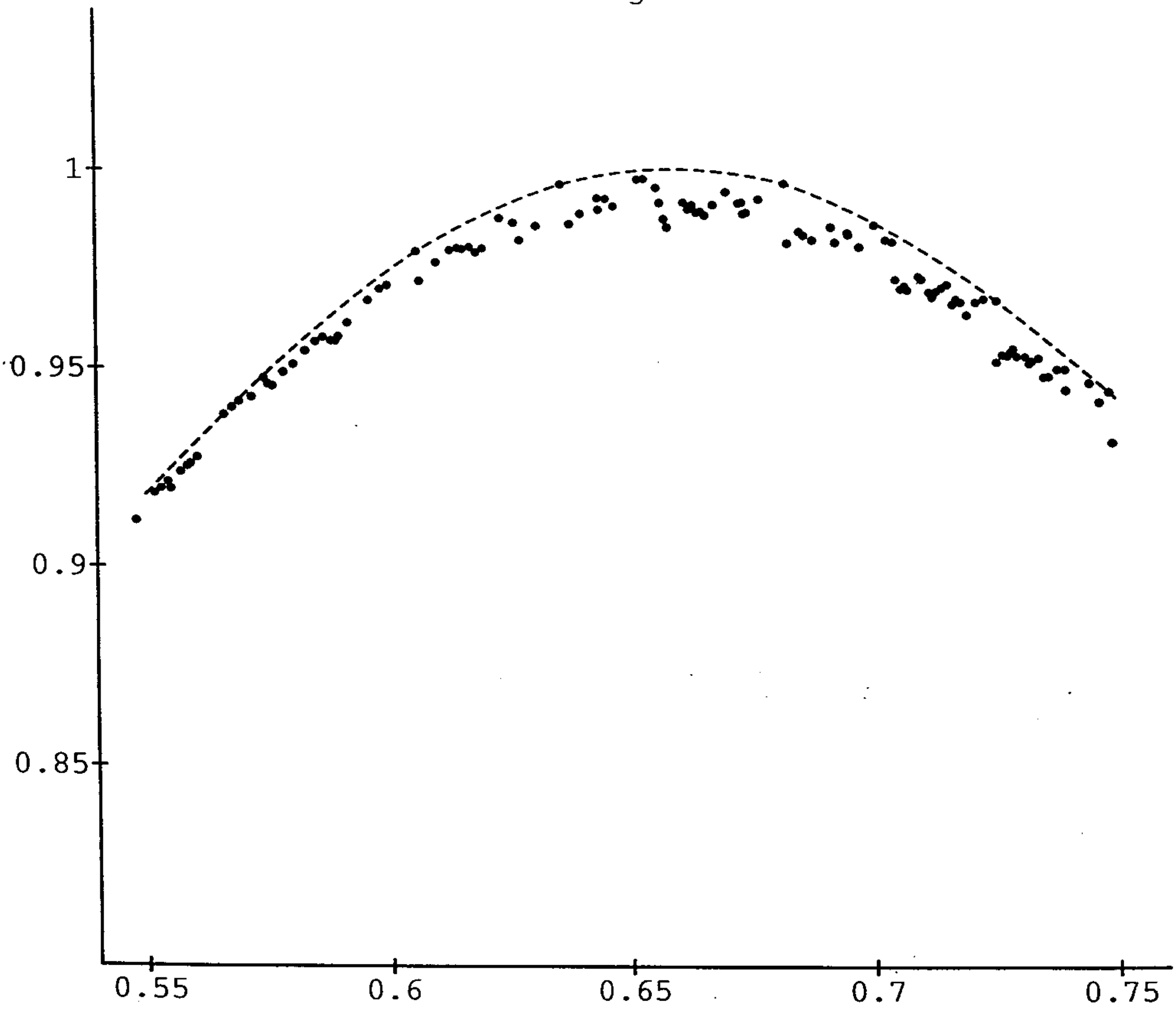


Fig. 6.1

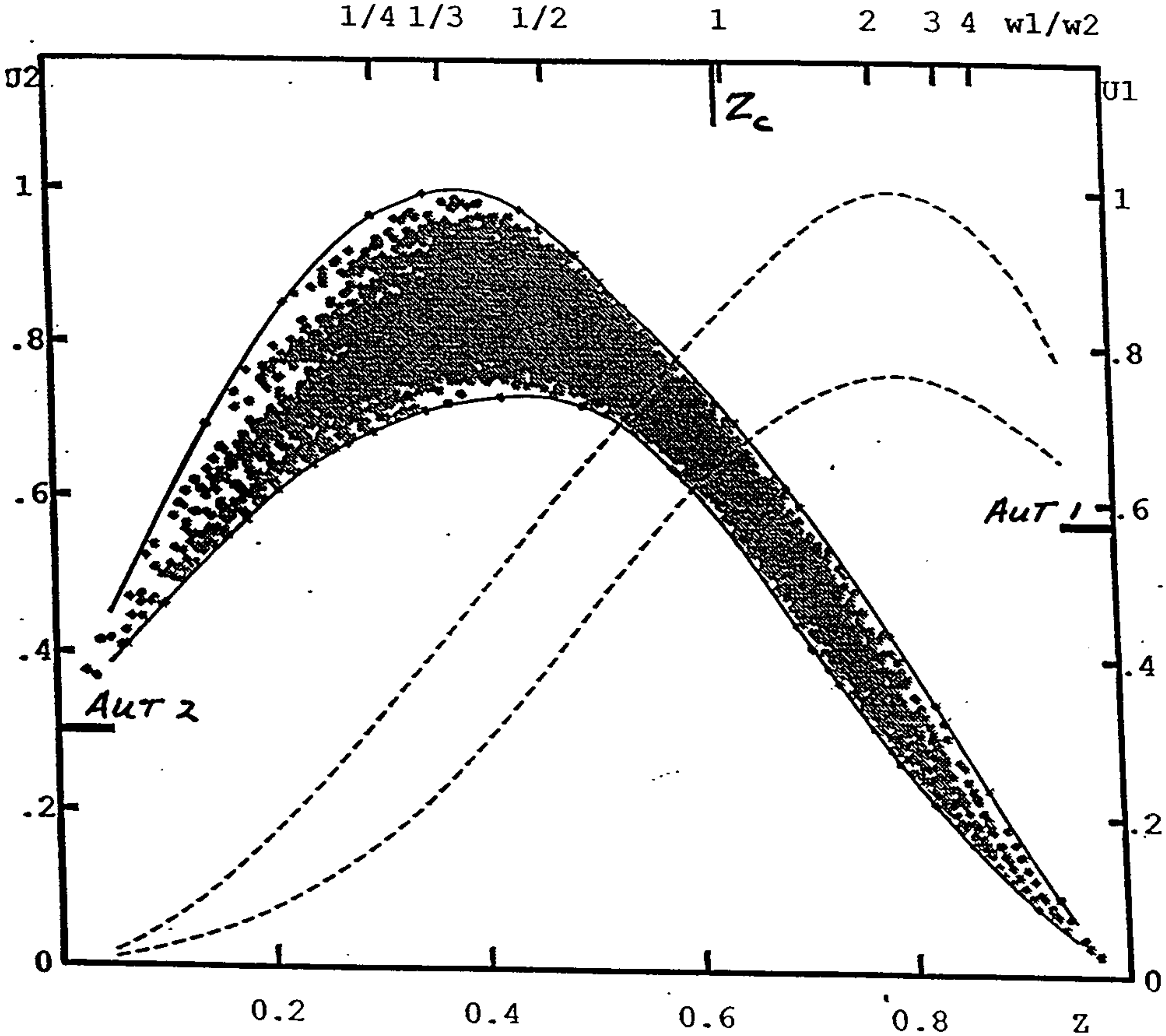


Fig.7.1

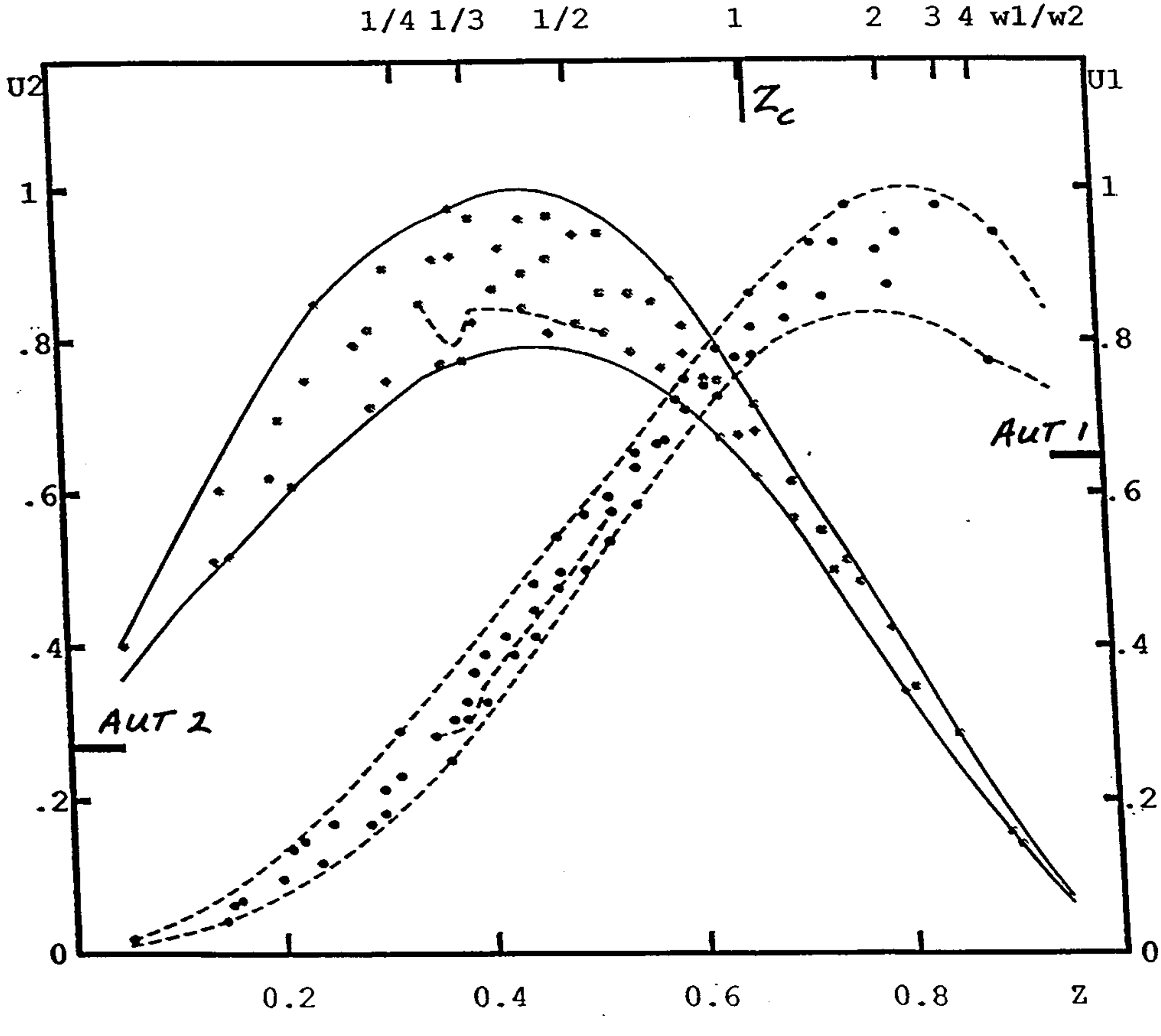


Fig.7.2

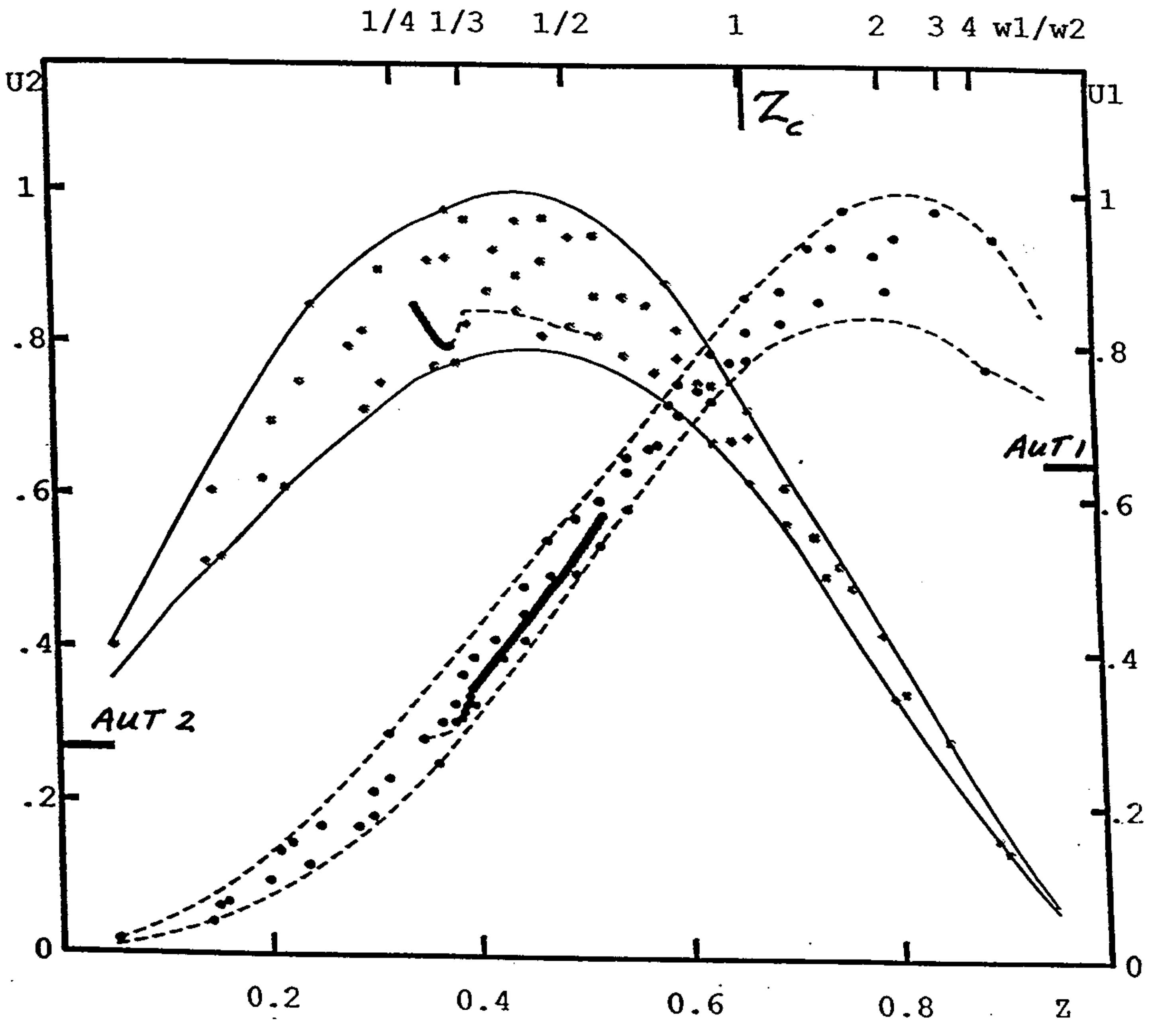


Fig . 7.3

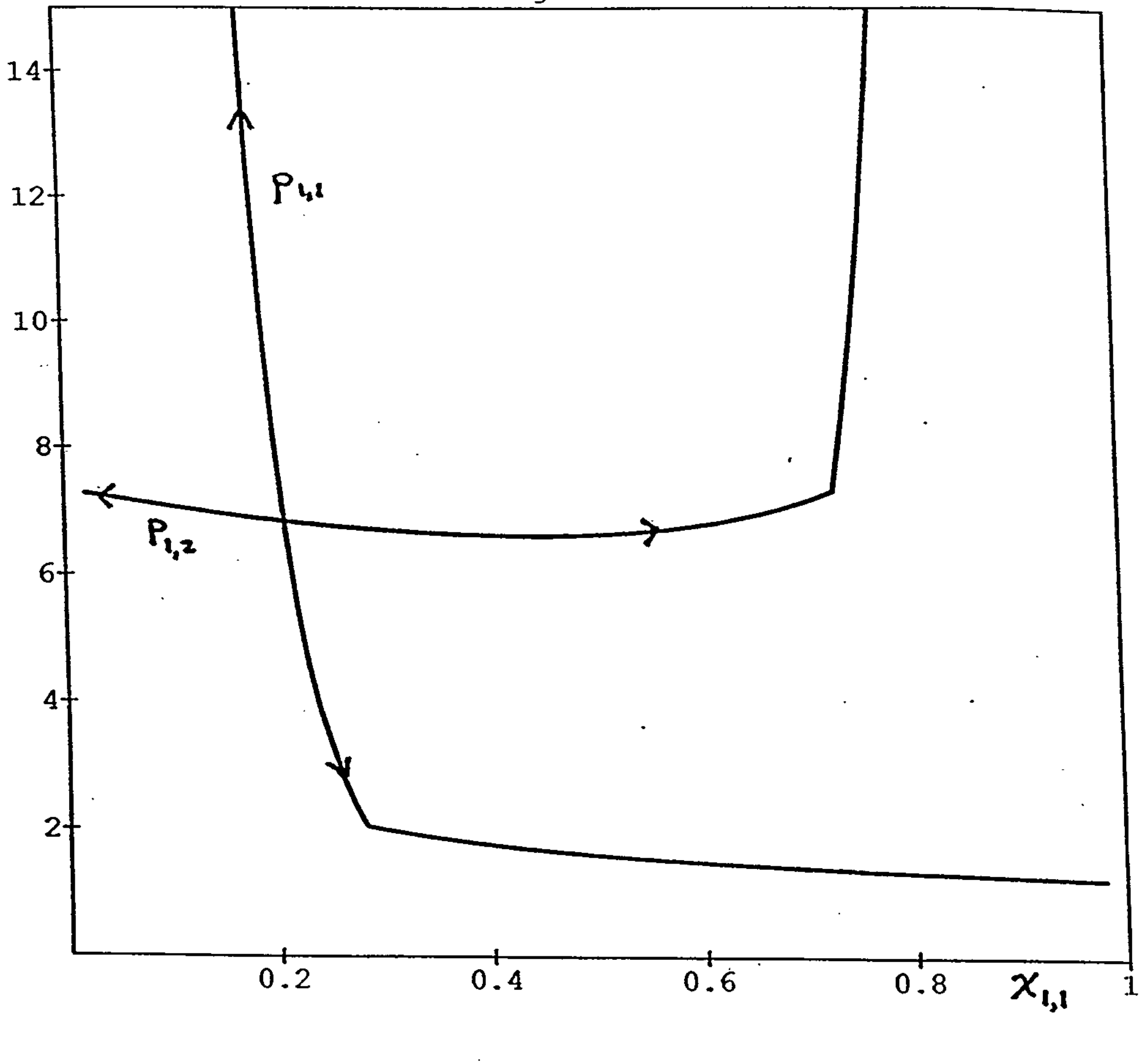


Fig. 7.4

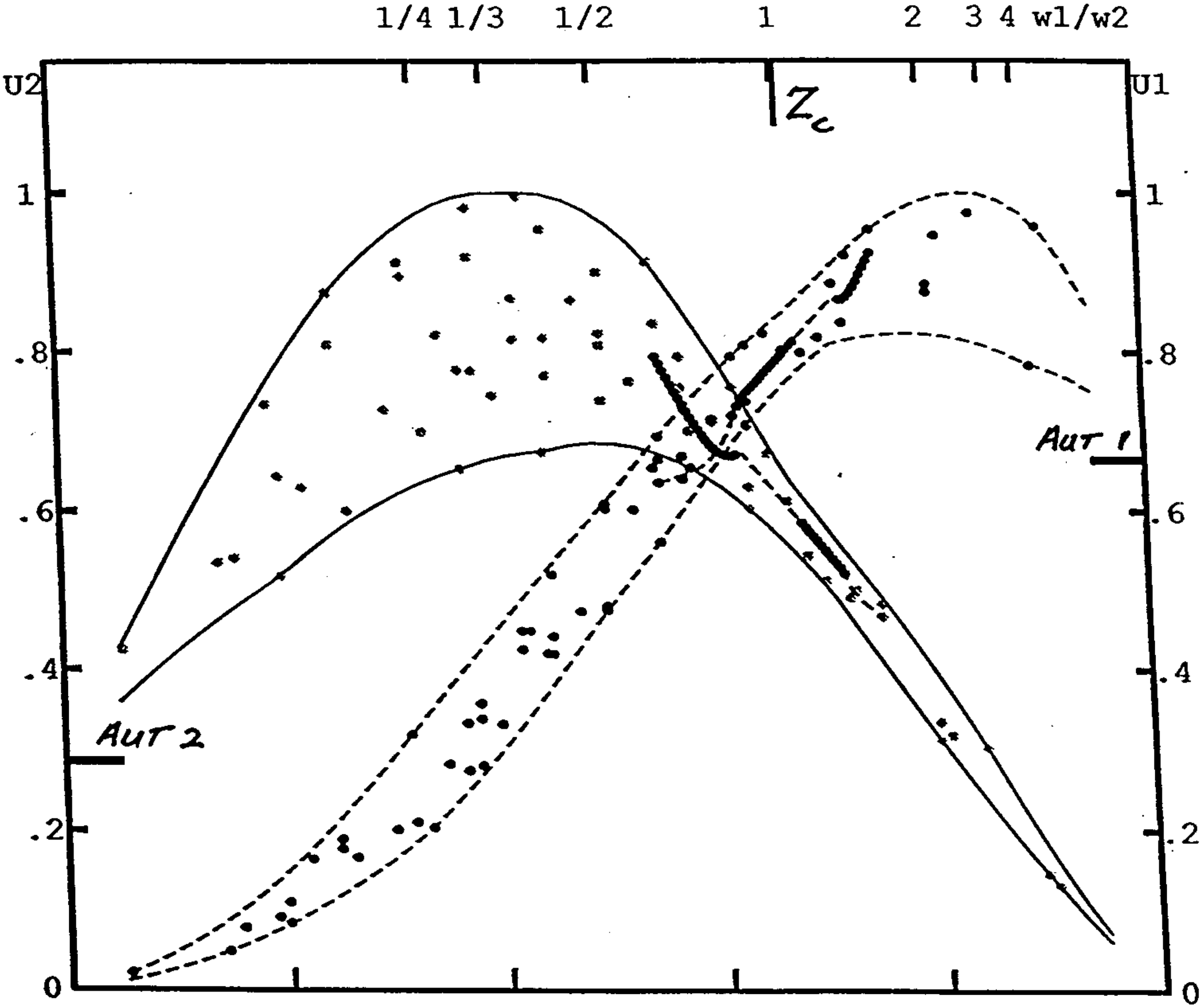


Fig . 7. 5

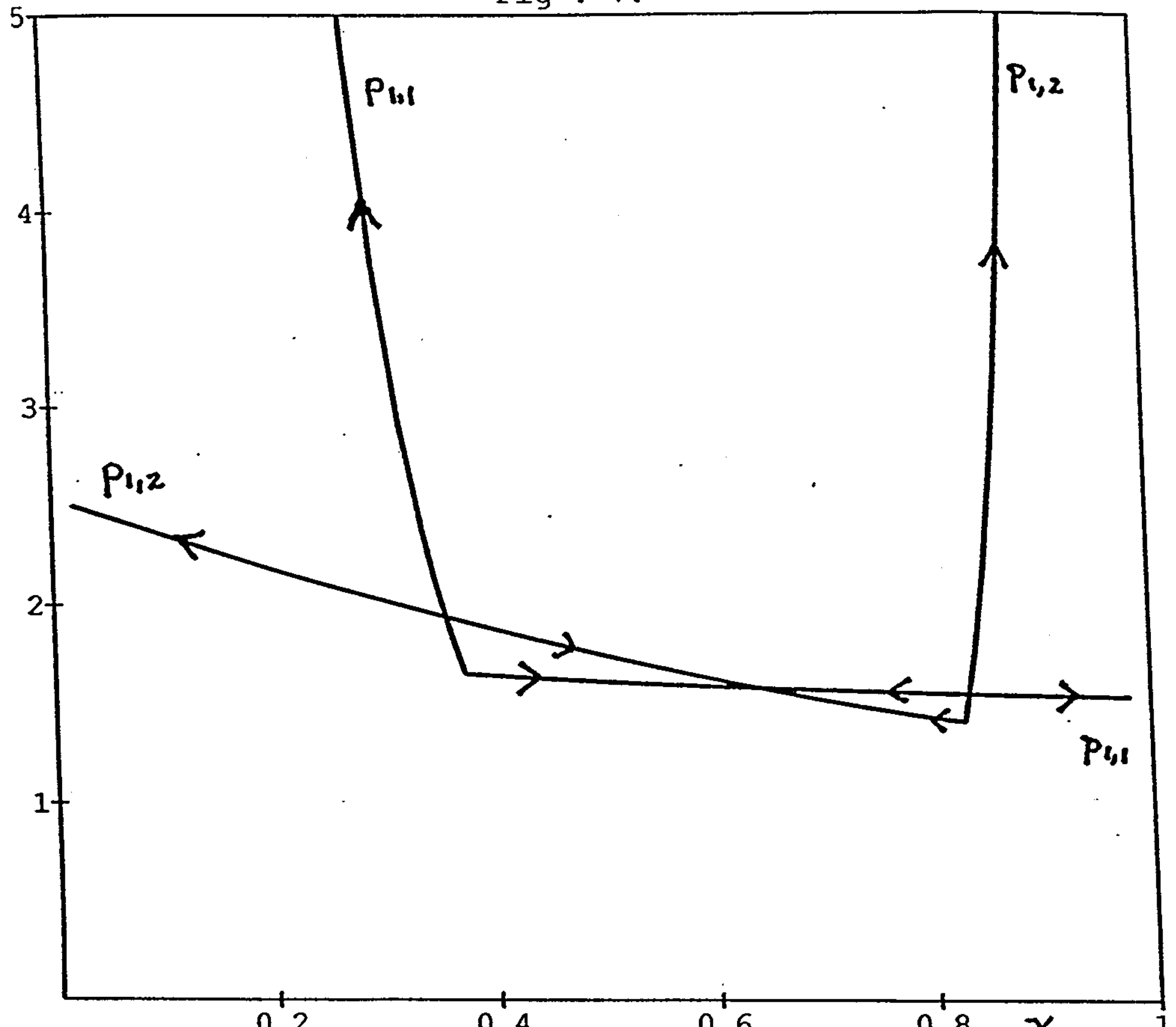


Fig. A7-1.1

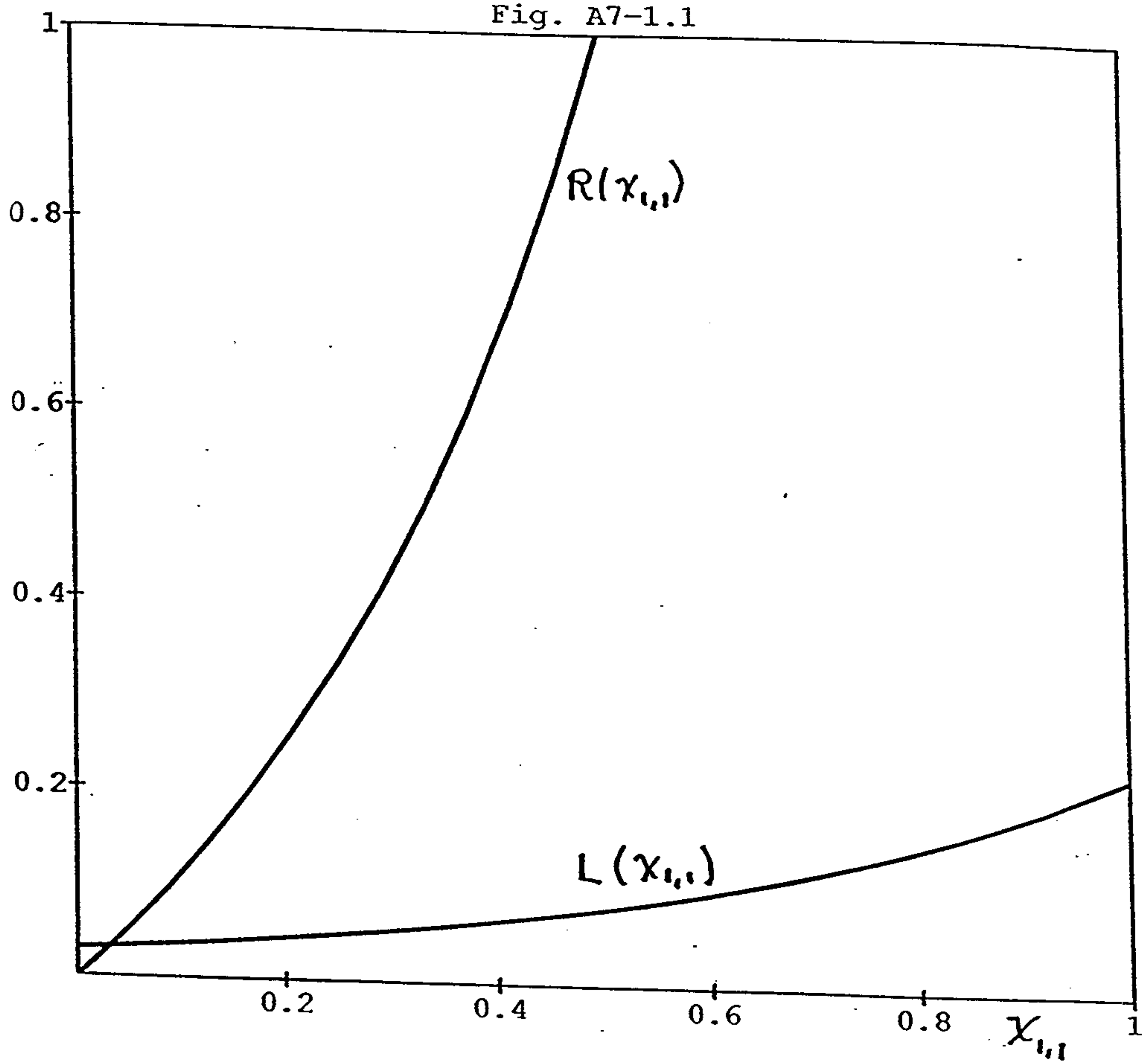
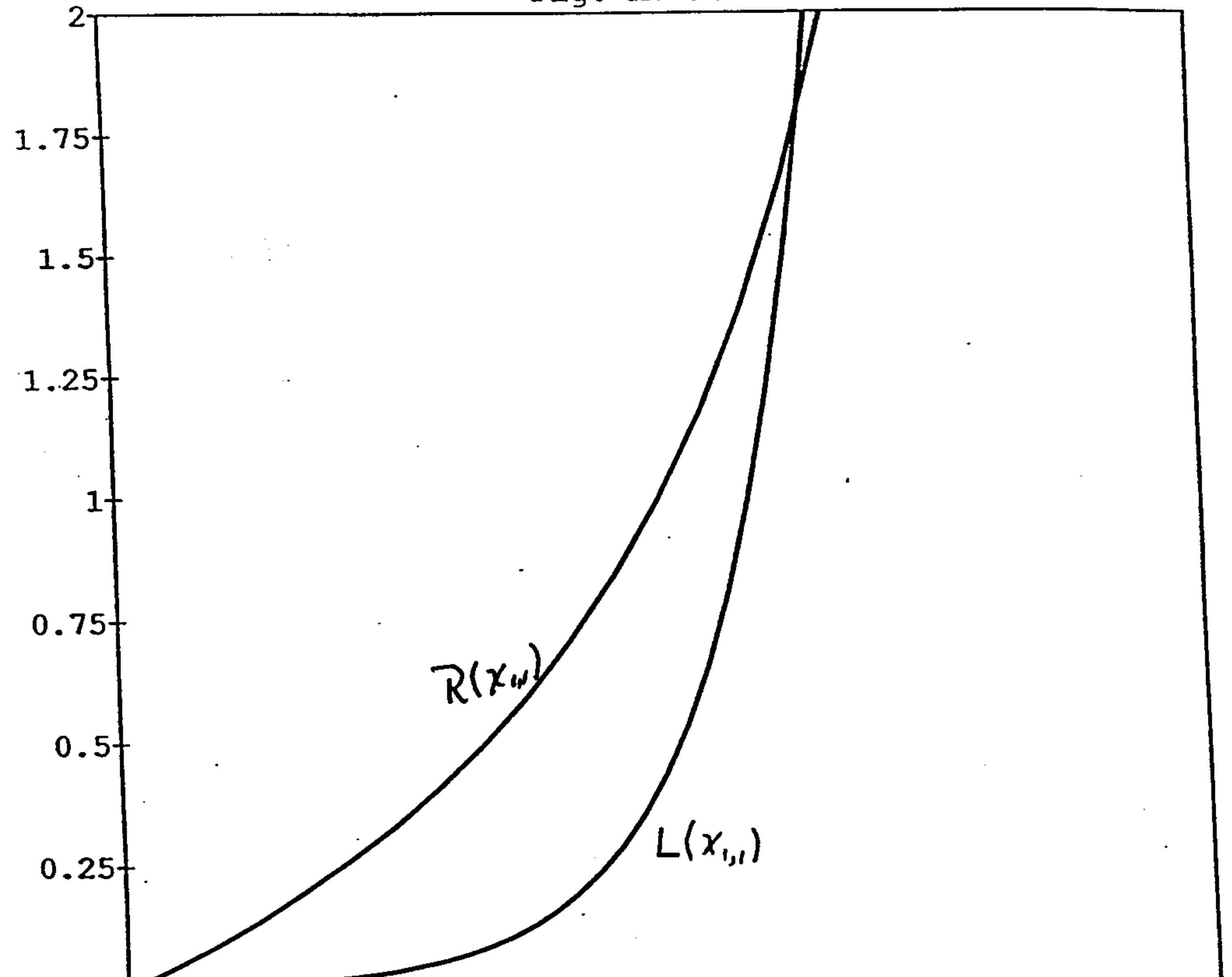
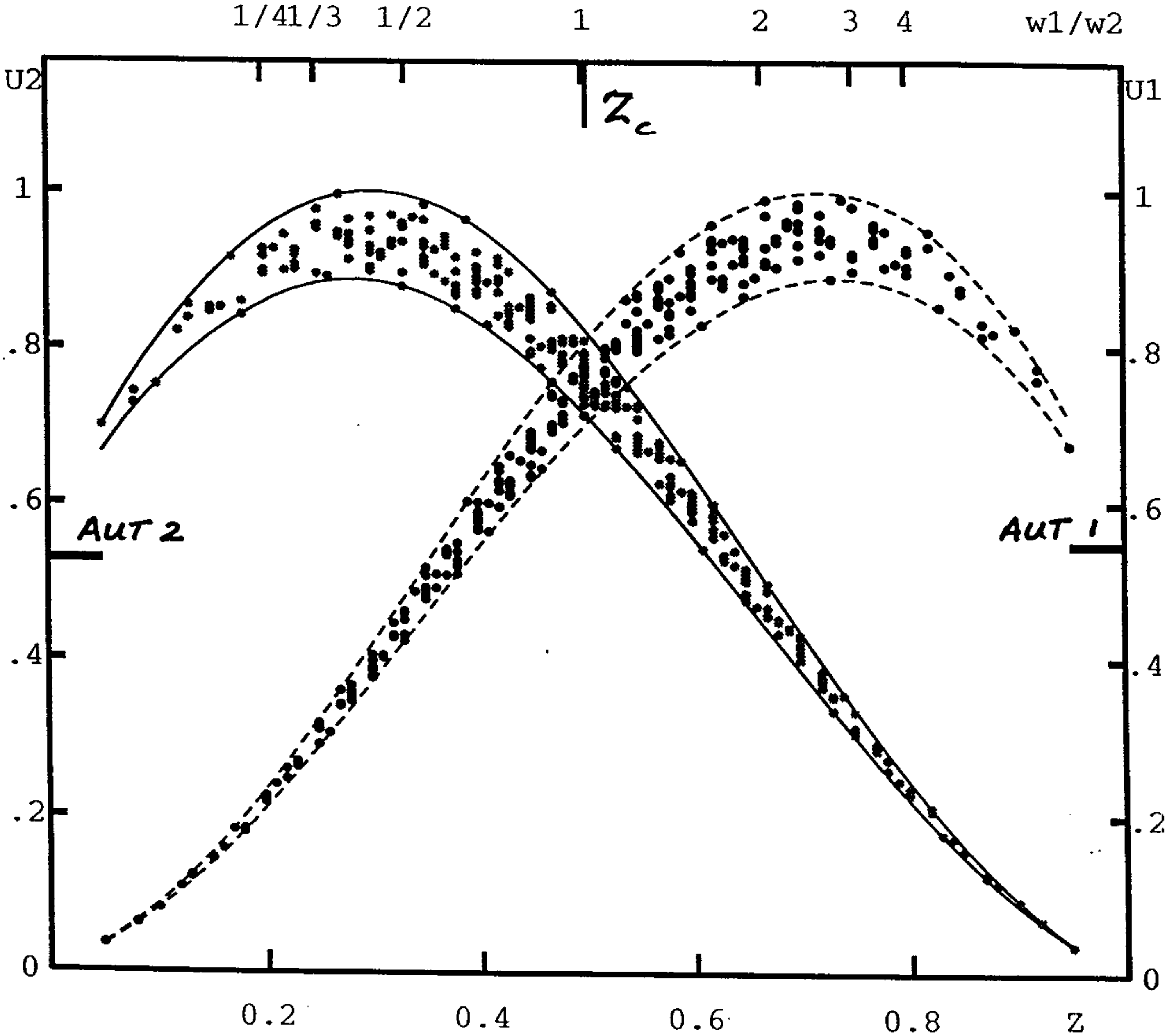


Fig. A7-1.2



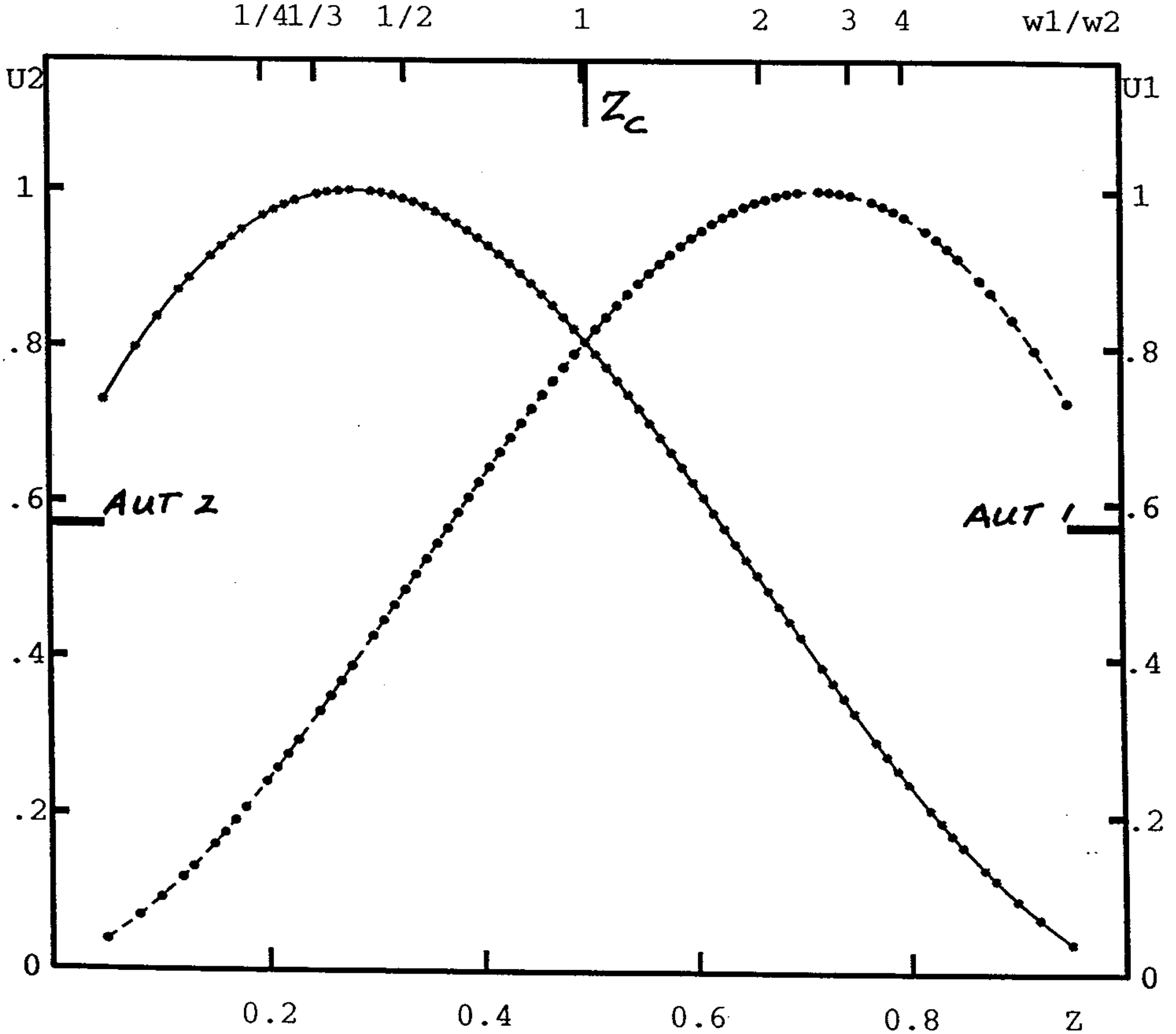
Здреса

Fig. 8.1



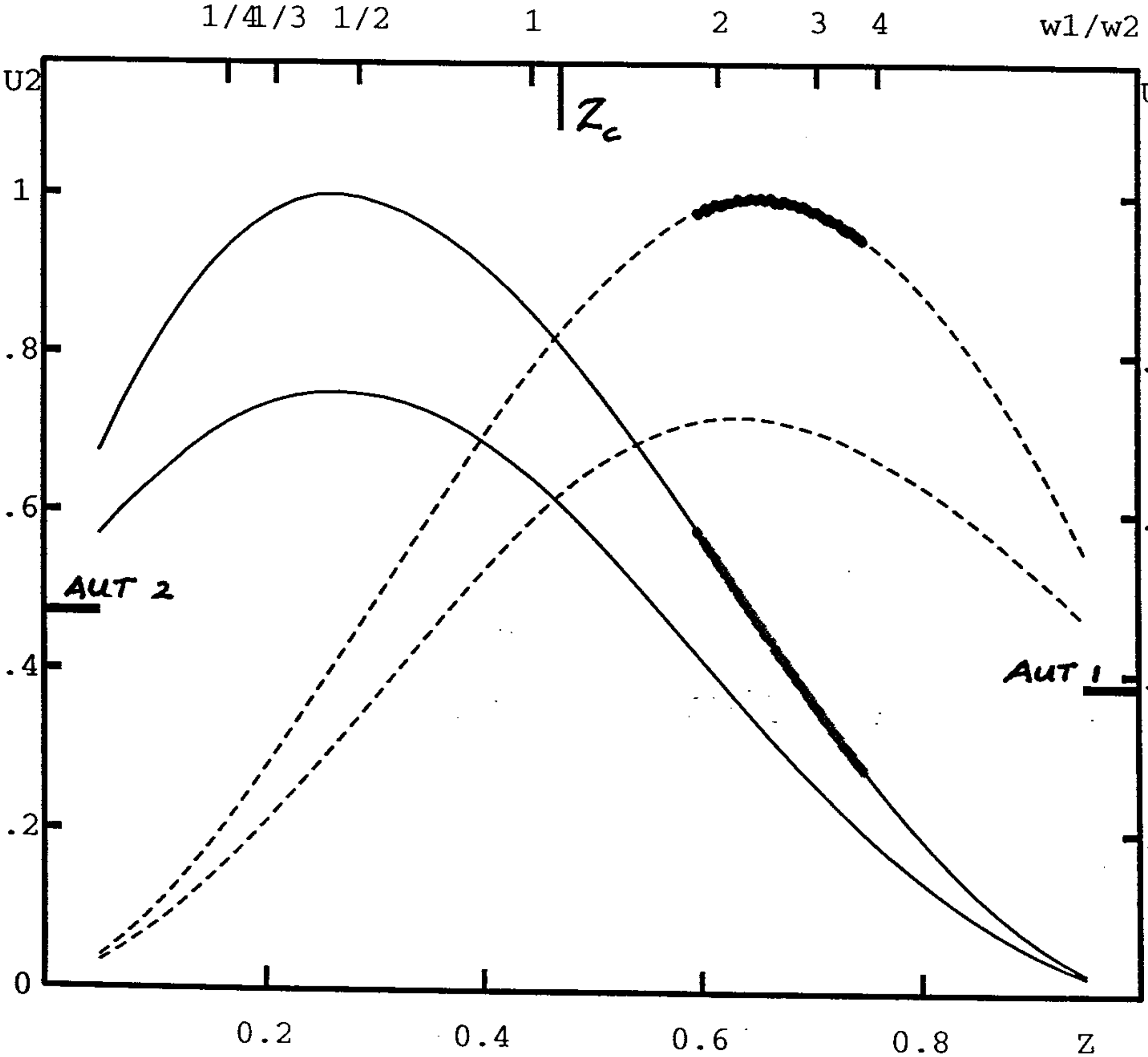
8p same

Fig. 8.2



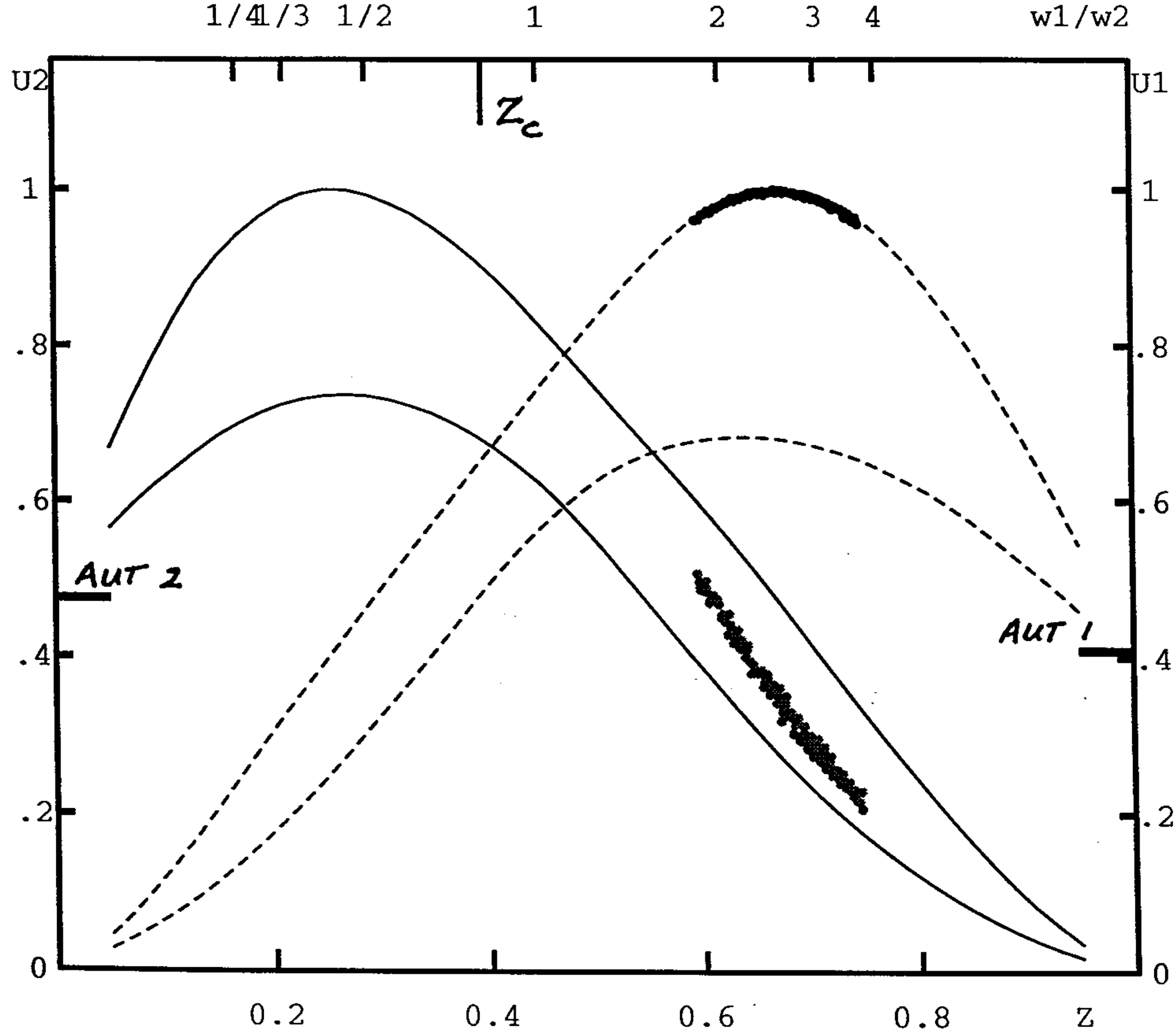
37paug grid .001 .65 to .1 α=0

Fig. 9.1



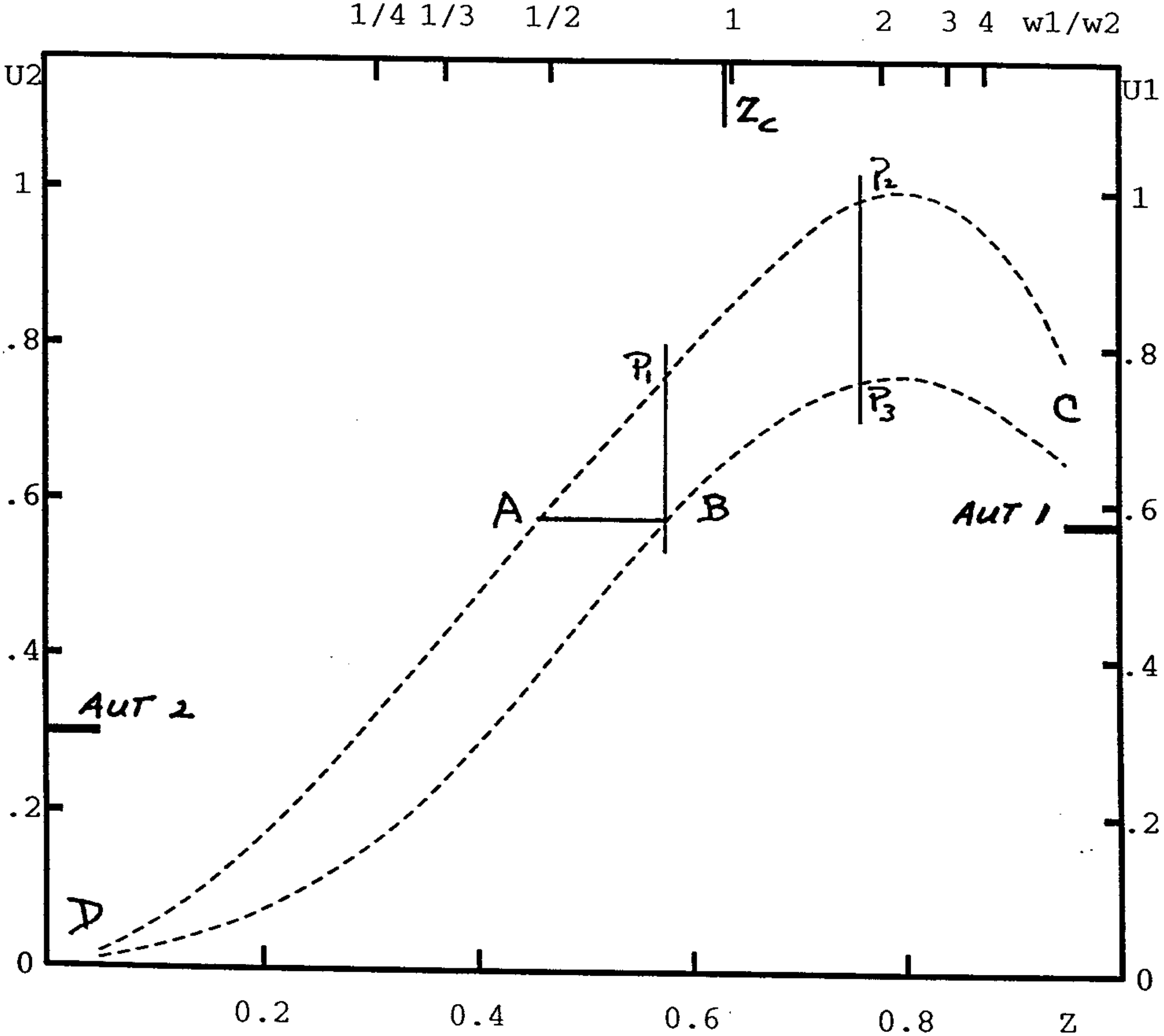
37 p from .6 to .75   α = .187   .6 to .75

Fig. 9.2



13p

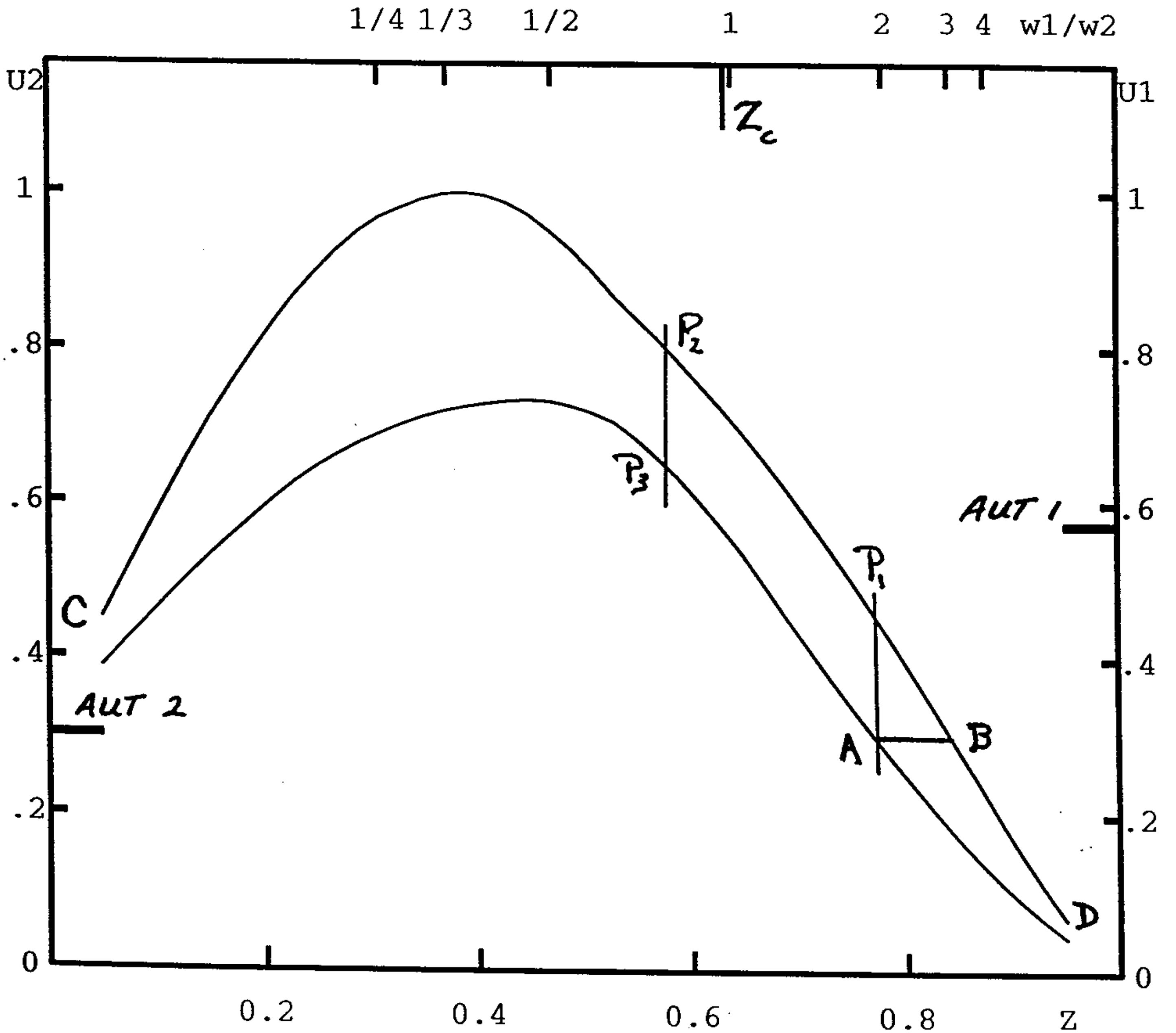
Fig. 10.1



1/4 1/3 1/2 1 2 3 4 w1/w2
U2
U1
Zc
P1
P2
P3
A
B
C
D
AUT 1
AUT 2
0
.2
.4
.6
.8
1
0.2
0.4
0.6
0.8
Z

13p

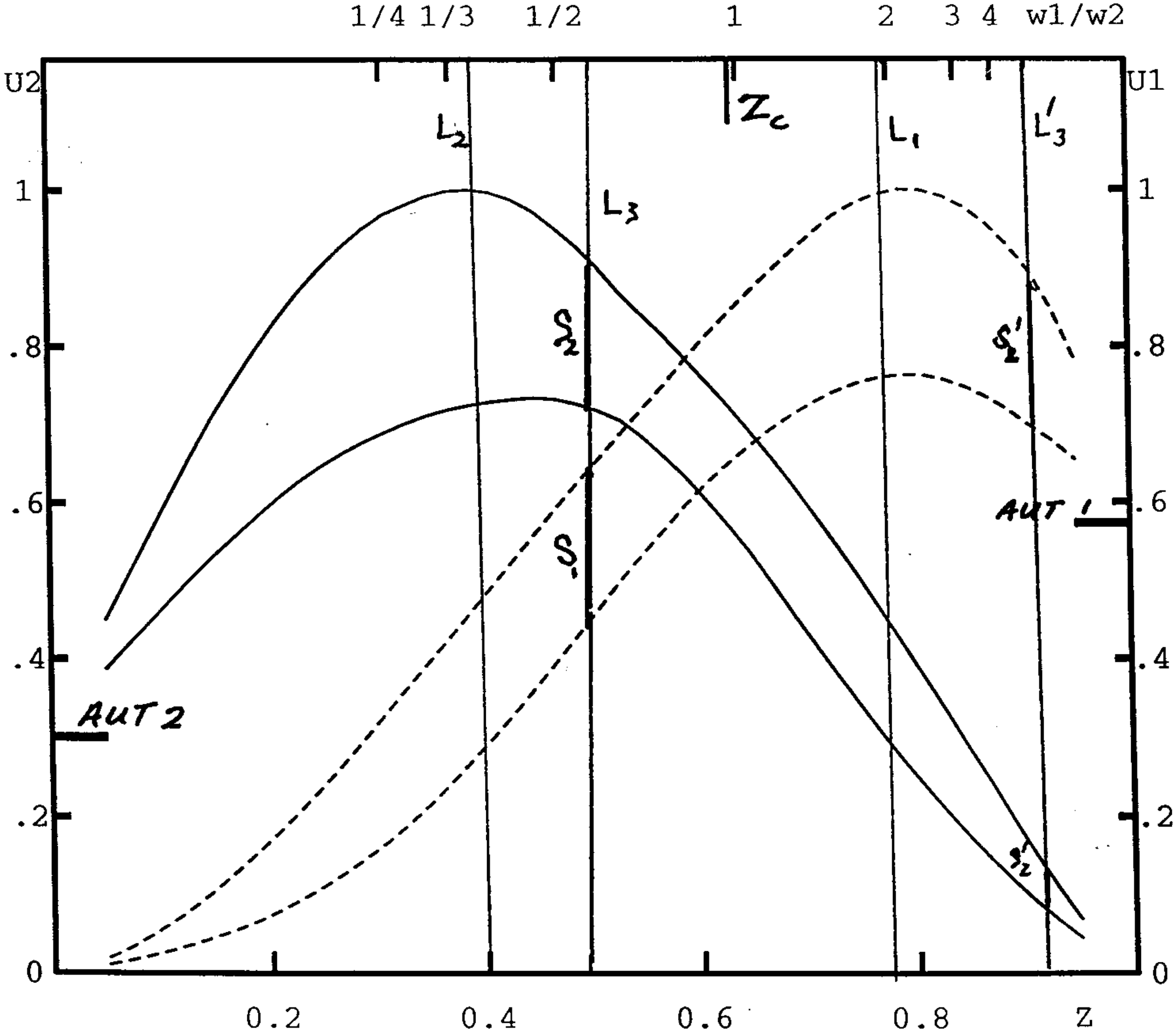
Fig. 10.2



1/4
1/3
1/2
1
2
3
4
w1/w2
U2
U1
Zc
1
.8
.6
.4
.2
0
P2
P3
P1
AUT 1
AUT 2
C
A
B
D
0.2
0.4
0.6
0.8
Z

13 p

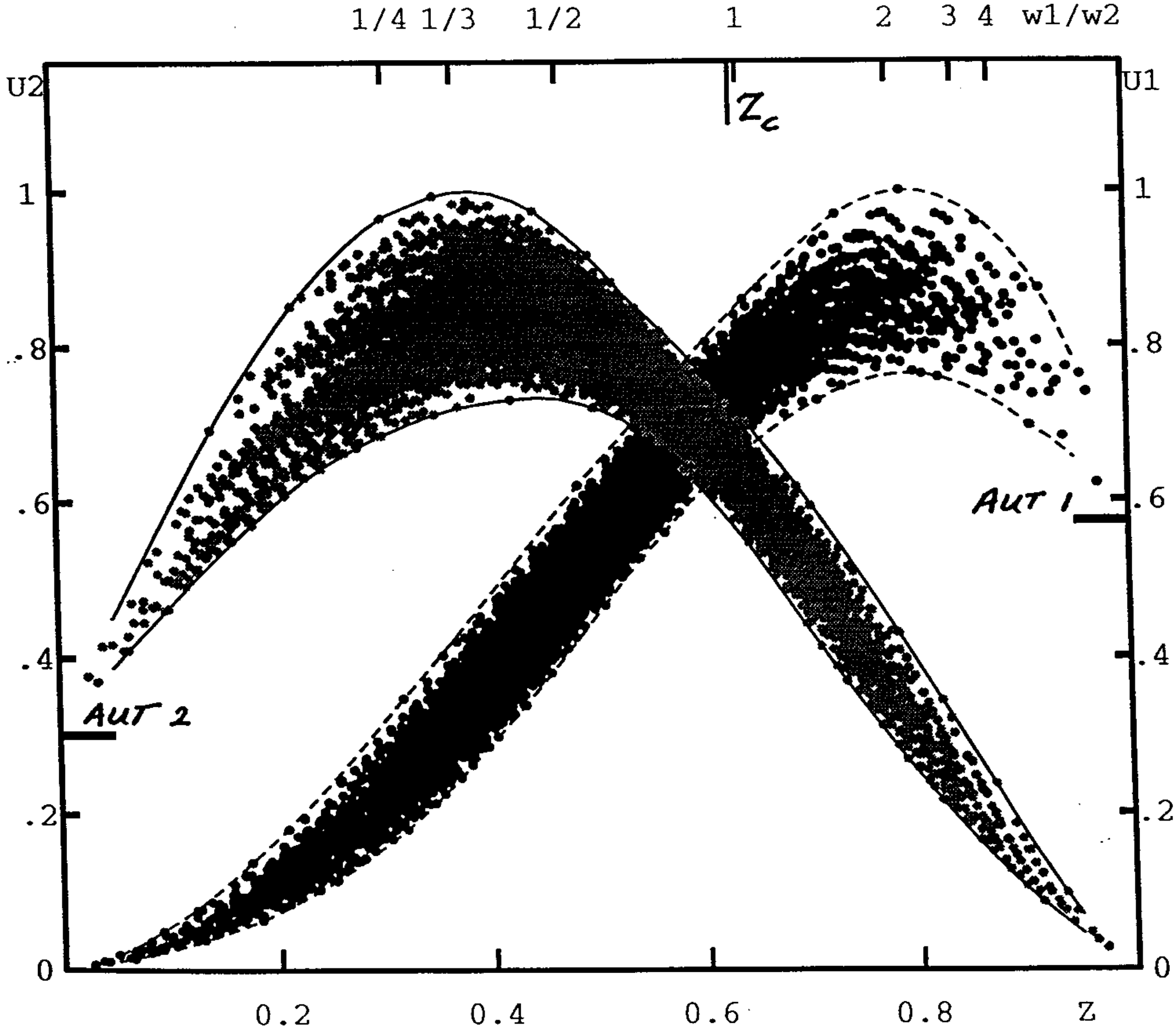
Fig. 10.3



1/4 1/3 1/2 1 2 3 4 w1/w2
U2
U1
Z_c
L_2
L_3
L_1
L_3'
S_2
S_1
S_2'
AUT 1
AUT 2
1
.8
.6
.4
.2
0
0.2
0.4
0.6
0.8
Z

13p

Fig. 10.4



αρ 1

Fig. 10.5

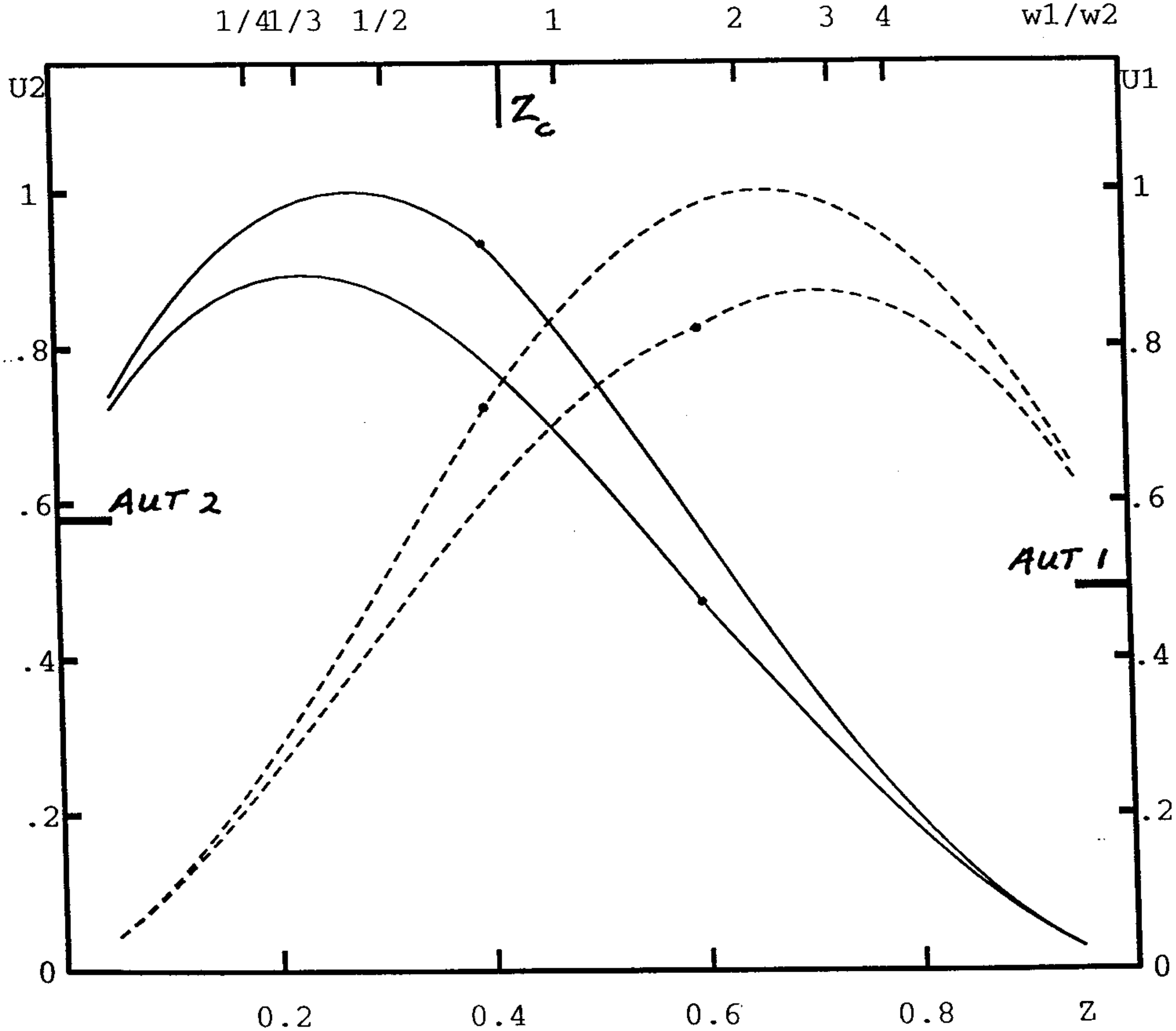


TABLE 1.1[1]

PRODUCTS	1	2	3	4	5	6	7	8	9
C1 Demands	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07
C2 Demands	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20
Production Exponents	1.30	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61
C1 Efficiencies	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30
C2 Efficiencies	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50

C1 - Labor Supply 4   C2 - Labor Supply 8   Production Function $e_{ij}\, l^{\alpha i}$

[1] The demands in all Tables are renormalized to total 1 in actual computation.

TABLE 3.1

PRODUCTS	1	2	3	4	5	6	7	8
C1 Demands	0.05	0.20	0.12	0.15	0.22	0.08	0.08	0.10
C2 Demands	0.12	0.10	0.20	0.15	0.20	0.05	0.08	0.15
Production Exponents	1.00	1.50	1.70	1.90	2.00	2.00	2.10	2.00
C1 Efficiencies	1.00	1.00	0.70	0.90	1.20	1.30	1.10	0.77
C2 Efficiences	0.50	1.00	0.60	0.90	1.00	1.20	1.30	1.02

C1 - Labor Supply 2   C2 - Labor Supply 2   Production Function $e_{ij}\, l^{\alpha i}$

TABLE 3.2

PRODUCTS	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
C1 Demands	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07	0.05	0.20	0.12	0.15	0.22	0.08	0.08	0.10
C2 Demands	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20	0.12	0.10	0.20	0.15	0.20	0.05	0.08	0.15
Production Exponents	1.30	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61	1.10	1.50	1.70	1.90	2.00	2.00	2.10	2.00
C1 Efficiencies	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30	1.00	1.00	0.70	0.90	1.20	1.30	1.10	0.77
C2 Efficiencies	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50	0.50	1.00	0.60	0.90	1.00	1.20	1.30	1.02

C1 - Labor Supply 8 C2 - Labor Supply 10 Production Function $e_{ij}\, l^{\alpha_i}$

TABLE 3.3

PRODUCTS	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
C1 Demands	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07	0.05	0.20	0.12	0.15	0.22	0.08	0.08	0.10	0.12	0.14	0.08	0.19	0.15	0.13	0.14	0.19	0.22	0.11
C2 Demands	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20	0.12	0.10	0.20	0.15	0.20	0.05	0.08	0.15	0.13	0.19	0.06	0.11	0.11	0.17	0.10	0.11	0.11	0.21
Production Exponents	1.30	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61	1.10	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.40	1.70	1.80	2.00	2.10	2.00	1.90	1.80	1.90	1.95
C1 Efficiencies	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30	1.00	1.00	0.70	0.90	1.20	1.30	1.10	0.77	0.77	0.50	0.50	1.00	0.60	0.90	1.00	1.20	1.30	1.02
C2 Efficiencies	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50	0.50	1.00	0.60	0.90	1.00	1.20	1.30	1.02	1.02	0.30	1.00	1.00	0.70	0.90	1.20	1.30	1.10	0.77

Cl - Labor Supply 12   C2 - Labor Supply 15   Production Function $e_{ij}\, l^{ri}$

TABLE 6.1

PRODUCTS	1	2	3	4	5	6	7	8	9	10	11	12	13
C1 Demands	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20	0.13	0.17	0.11	0.25
C2 Demands	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07	0.12	0.05	0.09	0.17
Production Exponents	1.00	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61	1.40	1.30	1.20	2.01
C1 Efficiences	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50	1.10	0.90	1.20	1.01
C2 Efficiences	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30	1.20	0.70	0.82	1.11

C1 - Labor Supply 9   C2 - Labor Supply 5   Production Function $e_{ij}\, l^{ai}$

TABLE 7.1

PRODUCTS	1	2	3	4	5	6
C1 Demands	0.10	0.20	0.20	0.15	0.25	0.10
C2 Demands	0.15	0.15	0.25	0.15	0.25	0.05
Production Exponents	1.15	1.50	1.70	1.90	2.00	2.00
C1 Efficiencies	1.00	1.00	0.70	0.90	1.20	1.30
C2 Efficiencies	0.50	1.00	0.60	0.90	1.00	1.20

C1 - Labor Supply 2   C2 - Labor Supply 1

Production Function $e_{ij} \, l^{xi} f_1(l)$, $f_1(l)=g(x)$, $x=l/(.6\, d_{i,j}\, L_j)$, $g(x)=1$ $x>1$, $g(x)=x^4$ $x<=1$.

TABLE 7.2

PRODUCTS	1	2	3	4	5	6
C1 Demands	0.10	0.20	0.20	0.15	0.25	0.10
C2 Demands	0.15	0.15	0.20	0.20	0.25	0.05
Production Exponents	1.15	1.50	1.10	1.90	2.00	2.00
C1 Efficiencies	1.00	1.00	0.70	0.90	1.20	1.30
C2 Efficiencies	0.50	1.00	0.70	0.90	1.00	1.20

C1 - Labor Supply 2 C2 - Labor Supply 1

Production Function $e_{ij}\, l^{\,xi}\, f_l(l)$, $f_l(l)=g(x)$, $x=l/(.6\ d_{i,j}\ L_j)$, $g(x)=1\ x>1$, $g(x)=x^4\ x<=1$.

Appendix 2-1

Proof of Theorem 2.1

We will use the assumptions on autarky and on the production functions from Section 2. We will start by proving a simpler theorem.

Theorem A1: There is an equilibrium point in which Country 1 is the *sole* producer of any proper non-empty subset of products S_1, with the others, S_2, being *solely* produced by Country 2.

Proof: At a zero profit equilibrium point we must have wages, prices and quantities of labor satisfying the condition that demand equal total wages, so, for any arbitrary choice of $w=(w_1,w_2)$ we can find the corresponding $l_{i,j}$ satisfying that condition. For $i \varepsilon S_1$ we use

(A1.1) $d_{i,1}w_1L_1 + d_{i,2}w_2L_2 = w_1 l_{i,1}$

and for $i \varepsilon S_2$ we use the similar equation

(A1.2) $d_{i,1}w_1L_1 + d_{i,2}\, w_2L2 = w_2 l_{i,2}$.

To satisfy the equilibrium conditions we also need a price p_i. If there is positive output of product i, and only one producer which is the situation here, we can get a price by dividing total demand (or total wages) by the amount produced, provided this is not zero.

However $l_{i,1}$ for either country as sole producer can not be less than the corresponding labor level in autarky as the expression for $l_{i,1}$ is $L_1d_{i,1} + (w_2/w_1)L_2d_{i,2} = l_{i,1}$ and in autarky it is $l^a_{i,1}=L_1d_{i,1}$. Since there was a positive output in autarky (one of the Theorem A1 assumptions) there is some output when either country is the sole producer as well. So we will always be able to compute a price.

However for arbitrary w_1 and w_2 the $l_{i,1}$ resulting from (A1.1) and (A1.2) will not generally add up to L_1, so we will have to show that w can be chosen to do that.

Using (A1.1) we see that if w_2 is sufficiently large relative to w_1, any $l_{i,1}$

alone, $i\varepsilon S_1$, will exceed L_1. Also for very small w_2/w_1, $l_{i,2}$,for any $i\varepsilon S_2$, will exceed L_2.

We need one further observation. Summing over all i gives

(A1.3) $\Sigma_i w_i d_{i,1}L_1+w_2 d_{i,2}L_2=w_1 L^D_1 + w_2 L^D_2$

Where the L^D_i are the total amounts of labor demanded in each country. It follows immediately from (A1.3) that $L^D_1>L_1$ implies $L^D_2<L_2$, $L^D_2>L_2$ implies $L^D_1<L_1$, and $L^D_1=L_1$ implies $L^D_2=L_2$,

Since for w_1 sufficiently small $L^D_1>L_1$, and for w_2 sufficiently small $L^D_1<L_1$ there is some intermediate value of w_2/w_1 with $L^D_1=L_1$. We have just seen that the same equality then holds for Country 2. We have now shown that all the conditions for an equilibrium point can be met, so we have proved Theorem A1.

If we can extend the proof to cover the cases where there are goods produced by both countries, we will have proved Theorem 2.1. In fact after an initial lemma required to show a unique labor level and price when both countries are producers, the proof will proceed in the same way as above.

Let us consider a good i for which, in the specialization being considered, both countries are producers. Let $x_{i,1}$ be the fraction of the total demand that goes as wages to labor in Country 1, while $x_{i,2}$ is the fraction going to country 2. Clearly $x_{i,1}+x_{i,2}=1$. Also, once $w=(w_1, w_2)$ is given, the $x_{i,j}$ uniquely determine the labor levels $l_{i,j}$ in each country. In terms of x then we can state the following lemma:

Lemma A1: For *any* choice of $w=(w_1, w_2)$ there is a unique $x_i(w)=(x_{i,1},x_{i,2})$ and price p_i such that $p_i f_{i,1}=w_1 l_{i,1}$ and $p_i f_{i,2}=w_1 l_{i,2}$. Furthermore $x_i(w)$, and hence the labor levels determined by x, depend continuously on w.

To prove this we need a preliminary remark: The autarky labor levels in both countries are provided by $x^a_{i,1}=d_{i,1}w_1L_1/(d_{i,1}w_1L_1+d_{i,2}w_2L_2)$ and $x^a_{i,2}=d_{i,2}w_2L_2/(d_{i,1}w_1L_1+d_{i,2}w_2L_2)$. Since there is, by assumption, positive output in both countries at these labor levels, it follows that if $x_{i,1}<x^a_{i,1}$ then $x_{i,2}>x^a_{i,2}$ so there is positive output in Country 2 and if $x_{i,2}< x^a_{i,2}$ there is positive output in Country 1.

For any x yielding positive production levels in *both* countries we can obtain different candidate prices for the two producers by defining p_1 by $p_1 f_{i,1}=w_1 l_{i,1} =x_{i,1}(d_{i,1}w_1L_1+d_{i,2}w_2L_2)$ and p_2 by $p_2f_{i,2}=w_1l_{i,2}=x_{i,2}(d_{i,1}w_1L_1+d_{i,2}w_2L_2)$. These prices give zero profit to both producers. However they are not generally equal and we must show that there is an x for which they are equal.

Let $x^s_{i,1}$ be the fraction of total demand that just covers the set up cost of Country 1, i.e.$x^s_{i,1}$=sup $x_{i,1}$ $f_{i,1}(x_{i,1})=0$, and let $x^s_{i,2}$ be the fraction of total demand that just covers the set up cost of Country 2. Let $x_{i,1}$ approach $x^s_{i,1}$ from above. Certainly then p_1 is well defined and in fact as $x_{i,1}$ approaches $x^s_{i,1}$ p_i becomes arbitrarily large. However for these x values p_2 is also well defined because $x_{i,1}$, being near $x^s_{i,1}$ must be below the Country 1 autarky level, and therefore the corresponding $x_{i,2}$ is above the Country 2 autarky level and also provides positive output. Consequently we have well defined p_1 and p_2 with $p_1 > p_2$. If we increase $x_{i,1}$ the labor and output from Country 1 increase continuously and monotonically while labor and output from Country 2 decrease continuously and monotonically. Finally for $x_{i,1}$ such that $x_{i,2}$ approaches $x^s_{i,2}$ from above, we have p_2 becoming arbitrarily large. Hence for some unique in-between x, we must have $p_1=p_2$.

This x is the x of the lemma. Its continuous dependence on w follows directly from the continuity of the production functions, the continuity in the dependence of total demand on w, and the monotone behavior of the p's as functions of x.

With lemma A1 proved we can repeat the reasoning of Theorem A1. Equations A1.1 and A1.2 still hold for the good or goods that are produced by each country alone. A1.3 holds because we can get it by summing the relations

(A1.1') $x_{i,1}\ (w_1L_1d_{i,1}+w_2L_2d_{i,2}) = w_1 l_{i,1}$

(A1.2') $x_{i,2}\ (w_1L_1d_{i,1}+w_2L_2d_{i,2}) = w_2 l_{i,2}$

over all i. Since the demand for labor in each country is continuous, and the labor demanded and since for w_1 sufficiently small $L^D_1>L_1$, and for w_2 sufficiently small $L^D_1<L_1$ there is once again an intermediate value of w_2/w_1 with equality of labor demanded and total labor supply both countries. Since we have established the existence of prices already for each w, this proves the extended theorem.

Appendix 3-1

Classical Point

If we assume that the more efficient producer of each good *is* its producer, i.e. we use the assignment x^C, we have a monotone decrease in the demand for Country 1's labor as Z_1 increases, therefore the left side of (3.4) decreases monotonically. This decrease in continuous except at the points where $q_{i,1}(1,Z)=q_{i,2}(1,Z)$. At these points, given our definition of x_c Country 1 loses an industry and therefore the demand for labor in Country 1 takes a downward jump. At such a point the left side of (3.4) is continuous from the right only. If, in the course of such a downward jump, the left side in (3.4) passes from $> Z_1$ to $\leq Z_1$ the Classical Level, Z_C must be at that Z.

Therefore there are basic possibilities: (1) Z_C is at a point of continuity, or (2) it is at a point of discontinuity.

1) Continuity: In this case the classical assignment x^C exactly uses up the labor forces of the two countries and provides an equilibrium point where each good is being produced by the more efficient producer. We call this equilibrium point the Classical Point.

2) Discontinuity: This case is more complicated. However, except by a rare accident, no equilibrium solution with the properties of the Classical Point is possible. However to allow for such accidents we need to treat two cases:(i) There is only one industry that switches at Z_C and (ii) there are two or more industries switching.

Case (i-a): The jump is to a value strictly $< Z_1$. This is the common case and the case in which there cannot be a Classical Point. For if $x^C(Z)$ satisfied (3.4) we would not have the left hand side in (3.4) $< Z_1$. (i-b). The jump is to a value which is $=Z_1$. Here we have the same conditions as the continuous case, and we have a Classical Point.

Case (ii): This case can be considered as a sequence of switches of type (i). If there is some order in which the downward leaps can be arranged that causes one of the leaps to end at Z_1, this then determines a Classical Point similar to (i-b). The industries that have switched are assigned to Country 2 and the not yet switched industries are assigned to Country 1. If there are different orderings that have this property, there could even be more than one Classical Point. Much more common would be the case where there is no arrangement of leaps that exactly strike Z_1 and in this case there is no Classical Point.

Appendix 3-2

Dynamic Programming

The dynamic programming recursion for an n-piece knapsack problem with total length L, piece lengths l_i and piece values v_i is

$$\phi_0(s)=0$$

$$\phi_m(s)=\mathrm{Max}(\phi_{m-1}(s),\phi_{m-1}(s-l_i) + v_m) \qquad 0\leq s\leq L.$$

If we apply this to (3.9) the l_i for a given Z, would be $(d_{i,1}Z_1+d_{i,2}Z_2)$, total length would be $Z_2=1-Z_1$ and the v_i would be $d_{i,1}\ln(q_{i,2}(1,Z)/q_{i,1}(1,Z)$.

The condition $\phi_0(s)=0$ simply sets the starting values at 0. Each successive ϕ_m gives the best value that can be obtained for length s using only the first m pieces. The ϕ_m are related to the ϕ_{m-1} through the recursion which simply says that the best that can be done at length s with m pieces is either done not using the mth piece, this is the first term after Max, or it is done using it, the second term. $\phi_n(s)$ then gives the maximizing value with n pieces which is the value of the objective function.

To determine the actual x that gives that value requires recording, when calculating $\phi_m(s)$, whether or not the mth piece was used at that s. Then it is possible to backtrack from $\phi_m(L)$ and find out how the value was obtained which gave $\phi_n(L)$. The maximizing x is then given by $x_{i,2} = 0$ if the ith piece was not used, $x_{i,2} =1$ if the ith piece was used.

In actual calculation L is usually divided up into a uniform grid of P points s_i, $s_1=0$ and $s_P=L$. The lengths l_i must then be rounded up or down since for use in the recursion they must fit the grid. Rounding down, which was used in all the calculations of this paper, may introduce some slightly inadmissible combinations, that only fit in because of the rounding, but it will give a $B_I(Z)$ that is too high and therefore is always a valid boundary.

An x that satisfies the inequality (3.5i) in (3.9) will always have its equilibrium value $Z(x) \geq Z$[1], if through rounding down, an optimizing x is obtained that doesn't satisfy (3.5i), x,Z(x) is still a legitimate equilibrium point, but Z(x) may be slightly <Z. However it is still a nearby and valid equilibrium point.

[1]See the italicized remark n Appendix 4-1.

Appendix 3-3

Improved Knapsack Calculation

While the straightforward knapsack calculation for some finite grid of Z's gives a very rapid and simple calculation for each Z value, the calculations can be further refined in the direction of requiring only one calculation per linear programming basis, rather than one calculation per grid point. In visualizing this refinement it is useful to keep in mind a plot of value density versus Z_1 for each of the goods.

The value density is, from 3.11,

$$v_i(Z)=\frac{d_{i,1}\ln\frac{q_{i,2}(1,Z)}{q_{i,1}(1,Z)}}{d_{i,1}Z_1+d_{i,1}Z_2}$$

We can plot the various curves v_i against Z_1. If two curves intersect each other then their density order changes, otherwise it does not.

Let us imagine that for some Z_1 we have the solution and the non-integer variable is the kth one. Then Z_1 can be increased without changing the form of the solution until either $x_{k,2}$ becomes 1 (or 0) or until one of the v_i equals the ith one. For all Z_1 in this range the value of the knapsack problem is obtained with virtually no effort. Until one of these events occurs the $x_{i,2}$ that are 1 remain 1, those that are 0 remain 0, and only $x_{k,2}$ changes to maintain the equality (3.5) in (3.9a). If the event that occurs first is that $x_{k,2}$ becomes 1 (or 0), then we actually have an integer solution, and therefore an equilibrium point, lying on the bounding curve. At this point a new $x_{k,2}$ (the densest of the 0 valued variables) is introduced at a level of zero and the calculation continues with further increases in Z_1. If the first event that occurs is that another density curve crosses the current ith one then there is always one simple choice to be made and after that the calculation continues as before. We will give one illustrative example. Suppose the jth density curve crosses the kth curve in an upward direction. If we set $x_{k,2}$=0 and all the other $x_{i,2}$ as before, we will get a value for $x_{j,2}$ that would enable it to satisfy (3.5). If $x_{j,2}$ is <1 it is the new non-integer variable and the calculation continues. If it is >1, then the calculation continues but now with $x_{j,2}$ =1 and $x_{i,2}$ still the non-integer variable.

By iterating In this way, all Z values can be exhausted using only a finite series of intervals within which the calculation is essentially unchanged.

Appendix 4-1

Proof of Lemma 4,1.

Using first the zero excess labor inequality (3.5i) involving x and Z', and then the n.e. property on the component $x_{k,2}$ gives two inequalities.

$$(\sum_{i \neq k} d_{i,1} x_{i,2})\, Z_1' + (\sum_{i \neq k} d_{i,2} x_{i,2})\, Z_2' \leq Z_2'$$

$$Z_2' \leq (\sum_{i \neq k} d_{i,1} x_{i,2})\, Z_1' + (\sum_{i \neq k} d_{i,2} x_{i,2})\, Z_2' + d_{k,1} Z_1' + d_{k,2} Z_2'$$

Using the equilibrium point x,Z(x) in (3.5) gives the equality

$$(\sum_i d_{i,1} x_{i,2})\, Z_1(x) + (\sum_i d_{i,2} x_{i,2})\, Z_2(x) = Z_2(x).$$

Subtracting the equality from the inequalities, rearranging terms, and using $Z_1+Z_2=1$ gives

$$0 \leq (Z'_1 - Z_1(x))\{-1-(\sum_i d_{i,1} x_{i,2} - \sum_i d_{i,2} x_{i,2})\} \leq Z_1(x) d_{k,1} + Z_2(x) d_{k,2}$$

Since the expression in braces can never be positive *this tells us that $Z_l(x)$ is greater than Z'_l*, which is often a useful thing to know. Using δ to replace $d_{k,1}$ and $d_{k,2}$ and rearranging gives

$$0 \leq Z_1(x) - Z_1' \leq \frac{\delta}{1-\sum_i (d_{i,1}-d_{i,2}) x_{i,2}}$$

The term on the right is maximized when the sum in the denominator is as large as possible. That largest possible value for $x_{i,2}$ that are 0 or 1 is

$$g = \sum_i (d_{i,1}-d_{i,2}) \quad i \text{ such that } (d_{i,1}-d_{i,2}) > 0$$

so that we have

$$0 \leq Z_1(x) - Z_1' \leq \frac{\delta}{1-g}$$

This very nearly establishes the lemma. We need only convert g to the equivalent and more usable form used in the text. If we add up all the terms ($d_{i,1}$-$d_{i,2}$) for which $d_{i,1} < d_{i,2}$, we get a negative sum -g'. However if we add up all the terms both positive and negative we get $\Sigma_i d_{i,1} - \Sigma_i d_{i,2} = 1-1 = 0$. This shows that g-g'=0, and therefore g=g'. It follows that

$$\sum_i |d_{i,1} - d_{i,2}| = g + g = 2g$$

and therefore that g, which we can regard as a measure of the departure from identical demands, is given by

$$g = \frac{1}{2} \sum_i |d_{i,1} - d_{i,2}|.$$

For the interesting special case of two countries with equal demands g is always 0. To get a g of 1 requires "orthogonal demands"

$$\sum_i d_{i,1} d_{i,2} = 0$$

In any other case g is always strictly less than 1.

Appendix 4-2

Proof of Lemma 4.2

To bound the derivative of $u_1(x,Z)$ with respect to Z_1 we differentiate u_1 obtaining

$$(4.1)\quad \frac{du_1}{dZ_1} - \sum_i d_{i,1}\{\frac{d_{i,2}}{Z_1(d_{i,1}Z_1+d_{i,2}Z_2)}\}$$

$$-\sum_i x_{i,1}d_{i,1}\{\frac{d_{i,2}L_1}{Z_1^2}\frac{f'_{i,1}}{f_{i,1}}\}+\sum_i x_{i,2}d_{i,1}\{\frac{d_{i,1}L_2}{Z_2^2}\frac{f'_{i,2}}{f_{i,2}}\} \quad .$$

The first sum consists of $1/Z_1$ multiplied by terms

$$\frac{d_{i,1}d_{i,2}}{d_{i,1}Z_1+d_{i,2}Z_2} \leq Max(d_{i,1}d_{i,2}) \leq d_{i,1}+d_{i,2}$$

so this sum is certainly bounded by $2/Z_1$.

The second part of the expression involves the production functions and their derivatives. These are evaluated at the labor levels $l_{i,j}= l_{i,j}(1,Z_1)$ required when Country j is the sole producer. If we take an individual term from the first sum of the second part

$$\frac{x_{i,j}d_{i,2\ i,1}L_1 f'_{i,1}}{Z_1^2 f_{i,1}}$$

and insert $l_{i,j}$ in the numerator and denominator and introduce the notation $\alpha_{i,1}(l)=f'_{i,1}(l)l/f_{i,1}(l)$ we obtain

$$\frac{x_{i,1}d_{i,2}}{Z_1^2}\frac{d_{i,1}L_1}{l_{i,1}}\frac{f'_{i,1}l_{i,1}}{f_{i,1}} \leq \frac{x_{i,1}d_{i,2}}{Z_1^2}\alpha_{i,1}.$$

Where the inequality is due to the fact that $d_{i,1}L_1$ is the labor level in autarky in

Country 1 and is always ≤ than $l_{i,1}$. There is a similar inequality for each term of the second sum.

Since the two sums in the second part of the expression have opposite signs their total contribution is rather crudely overestimated by $\alpha(Z_1)/(Z_m)^2$, where $\alpha(Z_1) = \max \alpha_{i,j}(Z_1)$ and $Z_m=\min(Z_1,Z_2)$. So we can bound the derivative at any point Z_1 by

$$M(Z_1) = \frac{1}{Z_1}\{2+\frac{\alpha(Z_1)}{Z_1}\}.$$

This bound involves only Z and the $\alpha_{i,j}$. The $\alpha_{i,j}$ are the ratios of marginal to average cost. This ends the proof of Lemma 4.2

Some intuitive feeling for the properties of the α_j comes from the following observation: For production functions $f(l)=kl^{\alpha}$, $f'(l)l/f(l)=\alpha$ for all l. It is also true that if $f(l)$ has increasing returns to scale, $\alpha(l)$ is ≥ 1 for all l. This suggests that we can think of $f'(l)l/f(l)$ as a sort of generalized exponent.

Appendix 7-1

.Proof of Theorems 7.3 and 7.4

Proof of Theorem 7.3

In the notation of section 7 equilibria are the points where

(A4.1) $q_1(x_{1,1}),Z(x))/q_2(x_{1,2},Z(x))= x_{1,1}/x_{1,2}= x_{1,1}/1-x_{1,1}$

and the x referred to in Z(x) is $x(x_{1,1})$.

Since the variable $x_{i,1}$ splits the labor supply for industry i between two countries we always have $l_{i,j}(x_{i,j},Z)=x_{i,j}l_{i,j}(1,Z)$. So for production functions of the form $e_{i,j}l^{\alpha i}$ we have $q_{i,1}(x_{i,1},Z)=(x_{i,1})^{\alpha i}\ q_{i,1}(1,Z)$. So (A4.1) becomes

$$(\frac{x_{1,1}}{1-x_{1,1}})^{\alpha_1}\frac{q_{1,1}(1,Z(x))}{q_{1,2}(1,Z(x))} - \frac{x_{1,1}}{1-x_{1,1}}$$

or equivalently

$$(A4.2)\ \{\frac{q_{1,2}(1,Z(x))}{q_{1,1}(1,Z(x)}\}^{1/(\alpha-1)} - \frac{x_{1,1}}{1-x_{1,1}}$$

We will refer to the left hand side in (A4.2) as $L(x_{1,1})$ and the right hand side as $R(x_{1,1})$ and we will plot L and R versus $x_{1,1,}$ in fig A4.1. We are essentially plotting p_1 and p_2 since if we look back to (1) we can see that L>R is equivalent to $p_1<p_2$ and L<R is equivalent to $p_1>p_2$.

In Fig. A7-1.1 the condition of theorem 7.3 is that $L(x_{1,1})$ should either be below 1 throughout the interval $0\leq x_{1,1}\leq 1$ or always above it. In fig A4.1 we take the first case.

In Fig. A7-1.1 the right hand side starts at 0 with slope 1 and moves up toward infinity. Also since both $R(x_{1,1})$ and its slope are monotone increasing the intersection of the tangent line to this curve with the vertical line $x_{1,1}=1$, which is 1 for $x_{1,1}=0$, always is above 1 for $x_{1,1}>0$.

Next we need some similar statements about $L(x_{1,1})$. Now

$$\{\frac{q_{1,2}(1,Z(x))}{q_{1,1}(1,Z(x))}\}^{1/\alpha-1}=(\frac{e_{1,2}L_2}{e_{1,1}L_1})^{1/\alpha-1}(\frac{Z_1(x)}{1-Z_1(x)})^{\alpha/\alpha-1}$$

Clearly $L(x_{1,1})$ is monotone increasing because $Z_1(x)$ is. To see that the derivative of $L(x_{1,1})$ is monotone increasing as well we will explicitly solve (3.4) or (3.5) for $Z(x)/(1-Z(x))$ and establish that its derivative is monotone. If we use

$$D_{1,1}=\sum_{i>1} x_{i,1}d_{i,1} \textit{ and } D_{1,2}=\sum_{i>1} x_{i,1}d_{1,2}$$

we obtain

$$\frac{Z_1(x)}{1-Z_1(x)}=\frac{D_{1,2}+d_{1,2}x_{1,1}}{(1-D_{1,1})-d_{1,1}x_{1,1}} \quad \textit{with derivative}$$

$$\frac{D_{1,2}d_{1.1}+d_{1,2}(1-D_{1,1})}{((1-D_{1,1})-d_{1,1}x_{1,1})^2}$$

and this last is clearly positive and monotone increasing. Since $L(x_{1,1})$ is a constant times $Z(x)/(1-Z(x))$ raised to a power of one or more, its derivative has the same property.

With this preparation we can assert that only a single intersection of L and R is possible. Suppose otherwise. Then at the second intersection the derivative of $L(x_{1,1})$ must equal or exceed that of $R(x_{1,1})$. Since this derivative is monotone increasing $L(x_{1,1})$ must thereafter lie above the line tangent to $R(x_{1,1})$ at that intersection point. Therefore its intersection with the vertical line $x_{1,1}$ is above that of the tangent line and therefore >1 contradicting the assumption $L(x_{1,1})\leq 1$. This ends the proof of Theorem 7.3

Proof of Theorem 7.4

This proof is quite similar. We now assume $L(0)<1$ and $L(1)>1$, and we abbreviate $\alpha/((\alpha-1)$ to β. Clearly as $\alpha\text{->}1$, β becomes very large. The value of $x_{1,1}$

that gives $L(x_{1,1})=1$ is obtained by finding the $x_{1,1}$ value that makes $L(x_{1,1})^{1/\beta}=1$, and this value c is independent of β. In the diagram fig (A4.2) we see the effect of letting α->1. Since β becomes very large L will be as close to 0 as desired until nearly at $x_{1,1}$=c. As it approaches $x_{1,1}$=c it rapidly rises to 1 and then immediately to very large values. With c fixed and α sufficiently near 1 we can be sure of a first intersection between L and R with a height near 0, and another with its $x_{1,1}$ coordinate near c.This intersection will be just before c (as in the diagram) or just after c depending on whether c is to the left or to the right of 1/2. Of course, since there are an odd total number of intersections there will be still others. This proves Theorem 7.4

Appendix 9-1

Return to Autarchy

We will prove that both the upper and lower boundary curves approaches the autarchy level as Z_1->1.

Equation (3.5) is satisfied by the non-integer x that optimizes (3.9a) for any given Z. As Z_1->1 and Z_2->0, $\Sigma_i d_{i,1}x_{i,2}$, which is the coefficient of Z_1 in (3.5) must approach 0. The optimizing solution x consists of $x_{i,2}$ that are 0 or 1 except for one term the jth. For $\Sigma_i d_{i,1}x_{i,2}$ to approach 0 all the integer terms will have to be 0 and therefore $x_{j,2}$ will be given by

$$x_{j,2}=\frac{Z_2}{d_{j,1}Z_1+d_{j,2}Z_2}$$

which approaches 0 as Z_1 -> 1.

The ith term in the utility is

$$(A9-1.1) \qquad x_{i,1}d_{i,1}F_{i,1}q_{i,1}(1,Z) \ + \ x_{i,2}d_{i,1}\ln F_{i,1}q_{i,2}(1,Z)$$

$$where \ q_{i,1}=f_{i,1}(\frac{L_1(d_{i,1}Z_1+d_{i,2}Z_2)}{Z_1}) \ \ and \ \ q_{i,2}=f_{i,2}(\frac{L_2(d_{i,1}Z_1+d_{i,2}Z_2)}{Z_2}).$$

Clearly the $F_{i,1}$ approach 1 as Z_1->1 and the $q_{i,1}$ term approaches the autarchy quantity as Z_1->1. Since all the $x_{i,1}$ are 1 except $x_{j,1}$ which approaches 1, the $x_{i,1}$ terms alone sum to the utility value in autarky. Also all the $x_{i,2}$=0 except for $x_{,2}$ which approaches zero. It only remains to show that the second term in (A9-1.1) approaches 0 also for i=j.. This actually requires *some* assumption on the production functions because if the production of the jth good grows in some explosive fashion with additional labor, the quantity of goods produced overwhelm their decreasing marginal utility and this alone could boost Country 1's utility to a very high level. However the rate of growth required in to do this is in $f_{j,2}$ is quite extreme. This possibility can be excluded by the assumption that productivity growth is less than exponential i.e. $f(l)/e^l$->0 as l grows very large. This is enough to make the second term approach 0 also for i=j. All this then gives the autarchy value to the utility as Z_1->1.

The reasoning about the lower boundary is almost exactly the same.