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Abstract: The minimal number of parallel cuts required to divide a cake into *n* pieces is *n*-1. A new 3-person procedure, requiring 2 parallel cuts, is given that produces an envy-free division, whereby each person thinks he or she receives at least a tied-for-largest piece. An extension of this procedure leads to a 4-person division, using 3 parallel cuts, that makes at most one player envious. Finally, a 4-person envy-free procedure is given, but it requires up to 5 parallel cuts, and some pieces may be disconnected. All these procedures improve on extant procedures by using fewer moving knives, making fewer people envious, or using fewer cuts. While the 4-person, 5-cut procedure is complex, endowing people with more information about others' preferences, or allowing them to do things beyond stopping moving knives, may yield simpler procedures for making envy-free divisions with minimal cuts, which are known always to exist.

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1. Introduction

The literature on fair division has burgeoned in recent years, with two academic books (Brams and Taylor, 1996; Robertson and Webb, 1998) and one popular book (Brams and Taylor, 1999) providing overviews. There is also a more specific literature on cake-cutting—our focus here—which concerns the fair division of a divisible heterogeneous good over which different people may have different preferences.

Some of the cake-cutting procedures that have been proposed are discrete, whereby players make cuts with a knife—usually in a sequence of steps—but the knife is not allowed to move continuously over the cake. Moving-knife procedures, on the other hand, permit such continuous movement and allow players to call "stop" at any point at

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which they want to make a cut or mark. While there are now about a dozen such procedures for dividing a cake among three players such that each player is assured of getting a largest or tied-for-largest piece (Brams, Taylor, and Zwicker, 1995)—and so will not envy another player (resulting in an *envy-free division*)—only one procedure (Stromquist, 1980) makes the envy-free division with only two cuts. This is the minimal number for three players; in general n-1 cuts is the minimum number of cuts required to divide a cake into n pieces.

For two players, the well-known procedure of “I cut, you choose” leads to an envy-free division if the cutter divides the cake 50-50 in terms of his or her preferences; by taking the piece he or she considers larger and leaving the other piece for the cutter (or choosing randomly if the two pieces are tied in his or her view), the chooser ensures that the division is envy-free.

The moving-knife equivalent of this procedure is for a knife to move continuously across the cake, say from left to right. Assume the cake is cut when one player calls "stop." If each of the players calls "stop" when he or she perceives the knife to be at a 50-50 point, then the first player to call "stop" will produce an envy-free division if he or she gets the left piece and the other player gets the right piece. (If both players call "stop" at the same time, the pieces can be assigned to the two players randomly.) Surprisingly, to go from two players making one cut to three players making two cuts cannot be done by a discrete procedure if the division is to be envy-free (Robertson and Webb, 1998, pp. 28-29; additional information on the minimum numbers of cuts required to give envy-freeness is given in Shishido and Zeng, 1999). In fact, the 3-person discrete procedure that makes the fewest cuts is one discovered independently by John L.

Selfridge and John H. Conway about 1960; it is described in, among other places, Brams and Taylor (1996) and Robertson and Webb (1998) and requires up to five cuts. There is no known discrete 4-person envy-free procedure that uses a bounded number of cuts, but Brams, Taylor, and Zwicker (1997) give a moving-knife 4-person procedure that requires up to 11 cuts. Peterson and Su (2000) give an analogous 4-person envy-free moving-knife procedure for chore division, whereby each player thinks he or she receives the smallest (or tied-for-smallest) piece of an undesirable item.

In this paper, we will show that (i) Stomquist's 3-person envy-free moving-knife procedure and (ii) Brams, Taylor, and Zwicker's 4-person envy-free moving-knife procedure can be improved on, but in two different senses. In the case of (i), its two cuts are already minimal; however, we will give another 2-cut procedure that requires only two *simultaneously* moving knives, not the four that Stromquist's procedure requires. In the case of (ii), we will, like Brams, Taylor, and Zwicker, require more than one simultaneously moving knife (in some cases, we require five) but show that their 11-cut maximum can be reduced to a 5-cut maximum.

Our 3-person, 2-cut procedure is simpler than Stromquist's, and will serve to introduce the notion of "squeezing," which will be used repeatedly in our 4-person, 5-cut procedure. This 4-person, 5-cut procedure is arguably no simpler than that of Brams, Taylor, and Zwicker: while it reduces the maximum number of cuts needed to produce an envy-free division by more than half, it requires more stages and finer distinctions to implement than that of Brams, Taylor, and Zwicker.

We pave the way for introducing the 4-person, 5-cut envy-free procedure by describing a simple 4-person, 3-cut procedure that gives each player a *proportional* *piece*--one that he or she thinks is at least 1/*n* of the cake if there are *n* players. (If all players receive what they believe to be proportional pieces, the division is said to be *proportional*.) But more than giving a proportional division, the 4-person, 3-cut procedure makes at most one player envious, which we characterize as *almost envy-freeness*.

Our 4-person, 5-cut procedure is not as complex as Brams and Taylor's (1995) general *n*-person discrete procedure. Their procedure illustrates the price one must one pay for a procedure that works for all *n*: not only is it more complex than any bounded procedure we know of, but it also places no upper bound on the number of cuts that are required to produce an envy-free division; this is also true of other n-person envy-free procedures (Robertson and Webb, 1997; Pikhurko, 2000). The number of cuts needed will depend on the players' preferences over the cake.

The paper proceeds as follows. In section 2 we give the 3-person, 2-cut envy-free procedure that uses only two simultaneously moving knives. In section 3, we build on this procedure to present the almost envy-free 4-person, 3-cut procedure, which also uses only two simultaneously moving knives.

In section 4, we give the 4-person envy-free procedure that uses at most 5 cuts. Unlike the preceding procedures, in which the pieces assigned to the players are connected, some of the four pieces that constitute the envy-free division produced by this procedure may be the union of two or three non-adjacent pieces. Moreover, the 4-person, 5-cut procedure is far more complicated than either the 3-person, 2-cut envy-free procedure or the 4-person, 3-cut almost envy-free procedure.

Curiously, while we know that there exists a 4-person, 3-cut envy-free division (more on the existence question later), we know of no procedure that implements it. In section 5 we speculate on how such a procedure might work. We also discuss the possibility of finding bounded procedures that yield envy-free divisions for more than four persons. We conclude that if they exist, they may be of mathematical interest but are likely to be quite complicated and of little or no practical value. Accordingly, we suggest new directions in cake-cutting research.

2. A 3-Person, 2-Cut Envy-Free Procedure

To begin the analysis, we make the following assumptions that will be used throughout the paper:

1. The goal of each player is to maximize the minimum-size piece he or she can guarantee for himself or herself, regardless of what the other players do. To be sure, a player might do better by not following such a *maximin strategy*; this will depend on the strategy choices of the other players. In the subsequent analysis, however, we assume that all players are *risk-averse*: they never choose strategies that might yield them larger pieces if they entail the possibility of giving them less than their maximin pieces.

2. The preferences of the players over the cake are continuous, enabling us to invoke the intermediate-value theorem. Suppose, for example, that a knife moves across a cake from left to right and, at any moment, the piece of the cake to the left of the knife is A and the piece to the right is B. If, for some position of the knife, a player views piece A as being larger than piece B, and for some other position he or she views piece B as being larger than piece A, then there must be some intermediate position such that the player values the two pieces the same.

3. The cuts of the cake are parallel to each other. Although the shape of the cake is not important, this assumption allows us to view the cake as a line segment, which will simplify our discussion.

4. Let A be the piece of a cake between two given knives, and suppose that the left knife is moved rightward while the right knife is kept stationary. Then we want the movement of the left knife to be such that every player sees piece A as converging to size 0 as this process continues. To ensure convergence, we assume that the knife is moved at a constant speed by a neutral party, whom we call a *referee*. We will also allow players to move knives—sometimes, two at once—to change the sizes of pieces. In this case, the speeds of these knives may vary in a manner that will depend on the situation.

The notion of "speed" makes sense, because we can imagine that the cake is located on a segment of the real line on which there is a unit of length. The assumption of constant speed avoids a situation in which the piece is seen as decreasing in size but not converging to 0. To show how the latter situation can arise, fix some point x strictly between the position of the left and right knives. If the left knife is moving in such a way that, in each second that passes, its distance to point x is halved (and thus the speed of the knife is decreasing), the players will not view the size of piece A as converging to 0. The assumption of constant speed, however, ensures convergence to 0.

Throughout the paper, we will refer to players by number, i.e., player 1, player 2, etc. We will call odd-numbered players "she" and even-numbered players "he."

We next describe the 3-person, 2-cut envy-free procedure and show that it gives an envy-free solution. While the cuts are made by two knives in the end, initially one player makes “marks,” or virtual cuts, on the line segment defining the cake; these marks may subsequently be changed by another player before the real cuts are made.

Theorem 1. *There is a moving-knife procedure for three players that yields an envy-free division of a cake using two cuts.*

Proof. Assume a referee moves a knife from left to right across a cake. The players are instructed to call "stop" when the knife reaches the 1/3 point for each. Let the *first* player to call "stop" be player 1. (If two players call "stop" at the same time, randomly choose one.) Have player 1 place a mark at the point where she calls "stop" (the right boundary of piece A in the diagram below), and a second mark to the right that bisects the remainder of the cake (the right boundary of piece B below). Thereby player 1 indicates the two points that, for her, trisect the cake into pieces A, B, and C:

```
      A          B          C
/-----------|-----------|-----------/
            1           1
```

Because neither player 2 nor player 3 called "stop" before player 1 did, each of players 2 and 3 thinks that piece A is at most 1/3. They are then asked whether they prefer piece B or piece C. There are three cases to consider:

1. If players 2 and 3 each prefer a different piece—one player prefers piece B and the other piece C—we are done: players 1, 2, and 3 can each be assigned what they consider to be at least a tied-for-largest piece.

2. Assume players 2 and 3 both prefer piece B. A referee places a knife at the right boundary of B and moves it to the left. Meanwhile, player 1 places a knife at the left boundary of B and moves it to the right in such a way that the amounts of cake traversed on the left and right are equal for player 1. Thereby pieces A and C increase

equally in player 1’s eyes. At some point, piece B will be diminished sufficiently to B'—in either player 2 of player 3’s eyes—to tie with either piece A' or C', the enlarged A and C pieces. Assume player 2 is the first, or tied for the first, to call "stop" when this happens; then give player 3 piece B', which she still thinks is the largest or the tied-for-largest piece. Give player 2 the piece he thinks ties for largest with piece B' (say, piece A'), and give player 1 the remaining piece (piece C'), which she thinks ties for largest with the other enlarged piece (A'). Clearly, each player will think he or she got at least a tied-for-largest piece.

3. Assume players 2 and 3 both prefer piece C. A referee places a knife at the right boundary of B and moves it to the right. Meanwhile, player 1 places a knife at the left boundary of B and moves it to the right in such a way as to maintain the equality, in her view, of pieces A and B. At some point, piece C will be diminished sufficiently to C'—in either player 2 or player 3’s eyes—to tie with either piece A' or B', the enlarged A and B pieces. Assume player 2 is the first, or tied for the first, to call "stop" when this happens; then give player 3 piece C', which she still thinks is the largest or the tied-for-largest piece. Give player 2 the piece he thinks ties for largest with piece C' (say, piece A'), and give player 1 the remaining piece (piece B'), which she thinks ties for largest with the other enlarged piece (A'). Clearly, each player will think he or she got at least a tied-for-largest piece. Q.E.D.

Note that who moves a knife or knives varies, depending on what stage is reached in the procedure. In the beginning, we assumed a referee moves a single knife, and the first player to call "stop" (player 1) then trisects the cake. But in cases 2 and 3, at the next stage of the procedure, it is a referee and player 1 that move two knives simultaneously, “squeezing” what players 2 and 3 consider to be the largest piece until it eventually ties, for one of them, with one of the two other pieces.

3. An Almost Envy-Free 4-Person, 3-Cut Procedure

Squeezing can also be used to produce an almost envy-free 4-person, 3-cut division by applying the procedure we describe next. This procedure, like the 3-person, 2-cut envy-free procedure, is relatively simple. Like this procedure, too, all pieces are connected since only the minimal number of cuts is used.

Theorem 2. *There is a moving-knife procedure for four players that yields an almost envy-free division of the cake— it is proportional and at most one player is envious—using three cuts.*

Proof. Assume a referee moves a knife from left to right across a cake. The players are instructed to call "stop" when the knife reaches the 1/4 point for each. Call the first player to call "stop" player 1, and the second player to call "stop" player 2. (As in the previous section, a tie can be broken randomly.) Have players 1 and 2 puts marks at the points where they call "stop" (see the adjacent numbers, 1 and 2, below the line in the diagram below). Then have player 2 trisect the remainder of the cake, so the initial division will be a quadrisection of the cake for player 2 into pieces A, B, C, and D:

```
      A           B           C           D
/-----------|-|-----------|------------|------------/
           1 2           2            2
```

Because neither player 3 nor player 4 called "stop" before player 2 did at the beginning, players 3 and 4 think that piece A is at most 1/4. They are then asked whether they most prefer piece B, C, or D.

We can treat players 2, 3, and 4 as if they were players 1, 2, and 3 in the proof of Theorem 1: they are dividing a cake into three pieces, which are called initially B, C, and D (instead of A, B, and C) and which player 2 (rather than player 1) thinks are all the same size. Theorem 1 shows that a division can be made such that—after the squeezing of one piece if players 2, 3, and 4 do not each prefer different pieces initially—every player thinks that a different one of the expanded or contracted pieces B', C', and D' is largest or tied for largest.

So far this procedure has led to an division of *part* of the cake—whose left edge is defined by the first 2 on the left and whose right edge is the right boundary of the cake—into three pieces such that players 2, 3, and 4 do not envy each other. Neither do they envy player 1 if player 1 is given piece A, which goes from the left boundary of the cake to the first 2 mark, since players 2, 3, and 4 all believe that piece A is at most 1/4 of the cake. However, player 1, even though she gets a proportional piece (i.e., piece A is at least 1/4 for her), may still envy either one or two of the other players. (Player 1 cannot envy all three of the other players, because her proportional piece A rules out the possibility that all three remaining pieces are greater than 1/4.)

Thus, the division is proportional and at most one of the four players (i.e., player 1) is envious, making the procedure almost envy-free. Q.E.D.

In general, a different almost envy-free division of the cake will result if the knife of the referee moves from right to left instead of from left to right. In this case, the possibly envious player will be the one that is the first to call "stop" from the right and who therefore gets the piece defined by the right boundary of the cake and the mark placed by the second player to call "stop" from the right.

Although we have not succeeded in finding a 4-person envy-free procedure that uses only 3 cuts, the almost envy-free 3-person procedure just described is better at reducing envy than the well-known moving-knife procedure of Dubins and Spanier (1961). Under the Dubins-Spanier procedure, a referee moves a knife from left to right across a cake. The first player to call "stop" gets the piece to his or her left of the point where the knife stops, the next player to call "stop" gets the next piece to his or her left, and so on.

A maximin strategy for this procedure is for each player to call "stop" when he or she perceives the knife to have traversed $1/m$ of the cake not already allocated, where m is the number of players that have not yet called "stop". Thereby each player ends up with a proportional piece. In particular, the first player to call "stop" will get what he or she believes to be $1/n$. But, if no other player called "stop" at the same time as this player, all the other players will obtain pieces they believe to be greater than $1/n$, because they perceive the first piece to be less than $1/n$ of the cake and therefore have more than $(n-1)/n$ of the cake to divide.

Suppose the Dubins-Spanier procedure is used by four players, and player 1 is the first to call "stop" when she perceives 1/4 of the cake to have been traversed. She is now out of the picture, so to speak, and will envy at least one of the other players unless she thinks player 2 and player 3, the second and third players to call “stop,” did so exactly at the two points where she (player 1) would have trisected the remainder of the cake. Likewise, player 2 will be envious unless he thinks player 3 called "stop" exactly at the point where he (player 2) would have bisected the remainder.

Note that while player 3 thinks she creates a two-way tie for largest in making the last cut, player 4, who never called "stop" before any other player (but perhaps called "stop" at the same time as another player), will get what he thinks is the single largest piece (on the right), unless he called "stop" at the same times as players 1, 2, and 3. Although neither player 3 nor player 4 will be envious, players 1 and 2 may be. Thus, the Dubins-Spanier procedure, because it can make as many as two players envious, is not almost envy-free.

4. A 4-Person, 5-Cut Envy-Free Procedure

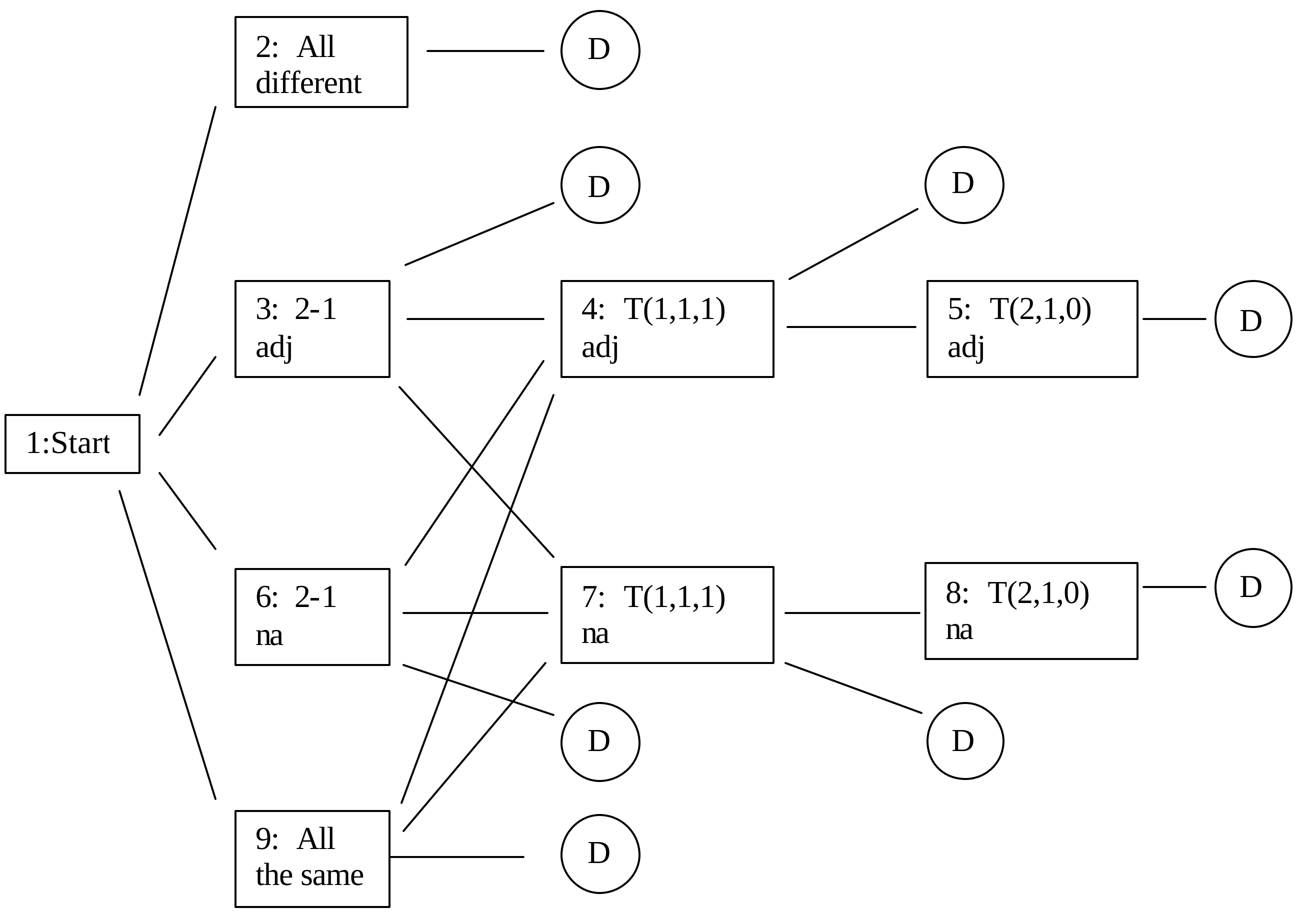
In this section, we show how to use the notion of squeezing to produce an envy-free division among four people using at most five cuts. It will be convenient to do most of the analysis in the context of pie division, rather than cake division. We will prove a theorem on envy-free pie division from which the cake-division theorem will easily follow.

What is the difference between pie division and cake division? When cutting a cake, our convention is that any two cuts are parallel, and this justified our perspective that our cake can be viewed as a line segment. When cutting a pie, by contrast, we assume that the pie is a disk and that all cuts are between the center and a point on the circumference (as we would cut a real pie). Then, just as our parallel-cut assumption for a cake justified our viewing the cake as a line segment, our present assumption justifies our viewing the pie as a circle. Finally, we randomly choose a point on the circle, break the circle at this point, and view the pie as a line segment with the endpoints identified.

Theorem 3. *There is a moving knife procedure for four players that yields an envy-free division of a pie using at most five cuts. Three of the four players will each receive a connected piece and the other player will receive either a connected piece or else a union of two such pieces.*

In the proof, we shall frequently refer to Figure 1, so we first discuss the figure and then begin giving the details of the proof. The figure provides a kind of flow chart for the procedure we are about to describe.

Figure 1



In the figure, each box or circle represents a state in the process. An arrow from state *i* to state *j* indicates that, in following the procedure to be described, moving from state *i* to state *j* is a possibility. If there is only one arrow leaving state *i*, and that arrow goes to state *j*, then going to state *j* is the only possibility upon leaving state *i*.

The D's in circles stand for "done." When we arrive at such a state, we will have produced an envy-free division of the pie using the required number of cuts.

We must explain the *T(p,q,r)* notation in the figure. At each stage in the process after the Start state, there will be a temporary assignment of pieces of pie to each of the four players. Thus, at any point in the process, we may ask questions such as, "Which piece does player 1 think is the largest piece," or "Does player 2 think that there are two pieces that are tied for largest?" We define *T(p,q,r)*, where *T* denotes "tie," as follows:

T(p,q,r) means that:

1. There are *p* players that believe there is a (two-way) tie for largest piece. Say that this tie is between pieces A and B.
2. Besides these *p* players, *q* players believe piece A is largest.
3. Besides these *p* players, *r* players believe piece B is largest.
4. Any players not among these $p + q + r$ players believe that the other two pieces (i.e., C and D) are tied for largest. (It will turn out, in every case we consider, that $p + q + r = 3$, so that there is only one "other" player.)

An example of *T*(1,1,1) is given by

```
     A           B           C           D
/-----------|-----------|-----------|-----------/
    1,2         1,3          4           4
```

where a player's number under a piece of pie indicates that the given player views that piece as at least tied for largest. (Notice that this type of diagram is similar, but not identical, to that used previously. In sections 2 and 3, the numbers indicated marks put by players on the cake, whereas in this section, the numbers will be used to keep track of the largest and tied-for-largest pieces of the players.)

An example of $T(2,1,0)$ is given by

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
   1,2,3        4         1,2         4
```

When we write "*T(p,q,r)*," we do *not* exclude the possibility of other ties. So, for example,

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
    1,2        1,3        1,4         4
```

is still an illustration of $T(1,1,1)$.

We observe that in the situations just considered, there is a natural distinction to be made, depending on whether the two tied pieces are adjacent or non-adjacent. In the figure, "adj" denotes "adjacent" and "na" denotes "non-adjacent." So, in our examples above, the first is "$T(1,1,1)$ adj" and the second is "$T(2,1,0)$ na."

Notation for states 2, 3, 6, and 9 in Figure 1 will be explained in the proof.

Throughout the proof, we shall refer to pieces A, B, C, and D of pie. These are the pieces of the division shown in our diagrams above; when the process is complete, each player will be given exactly one of these pieces. However (in contrast with our usage in sections 2 and 3 when A, B, C, and D denoted the initial pieces and A', B', C', and D' denoted the pieces at the end of the process), the pieces are not fixed but change throughout the procedure.

We will refer to the knife between pieces A and B as knife A/B, the knife between pieces B and C as knife B/C, and so on. We remind the reader that, because we are presently considering a pie rather than a cake, the left and right endpoints are identified. Hence, pieces A and D are adjacent and are separated by knife D/A.

Proof of Theorem 3. We first note that in order to divide a pie into four pieces, one for each player, at least four cuts are required. If we divide the pie using four cuts, each player will receive a connected piece. If we use five cuts, then three players receive a connected piece, and the fourth player receives either a connected piece, or else a union of two such pieces. Hence, the second sentence of the theorem follows easily from the first.

To prove the first sentence, refer to Figure 1. It is clear in the figure that all paths lead to state D. Thus, we must show that the figure is correct and the number of cuts is as claimed. We examine each state in the figure.

Assume that our players have been arbitrarily named players 1, 2, 3, and 4.

<u>State</u> <u>1</u>: This is where we begin. Player 1 positions knives so as to divide the pie into four equal pieces, in her view. Assume that the pie is labeled as follows:

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
     1          1          1          1
```

Each of players 2, 3, 4 picks which piece each thinks is largest or tied for largest. (Although ties will be central later in the proof, we ignore ties at this stage and have players select just one piece, breaking a tie randomly if necessary.) Without loss of generality, we may assume that this selection leads to one of the following situations:

```
       A          B          C          D
/-----------|-----------|-----------|-----------/
       1         1,2        1,3        1,4
```

or

```
       A          B          C          D
/-----------|-----------|-----------|-----------/
       1          1         1,2       1,3,4
```

or

```
       A          B          C          D
/-----------|-----------|-----------|-----------/
       1         1,2         1        1,3,4
```

or

```
       A          B          C          D
/-----------|-----------|-----------|-----------/
       1          1          1       1,2,3,4
```

These are states 2, 3, 6, and 9, respectively.

State 2: Assume, without loss of generality, that the situation is as follows:

```
       A          B          C          D
/-----------|-----------|-----------|-----------/
       1         1,2        1,3        1,4
```

In this case, the division obtained by giving piece A to player 1, piece B to player 2, piece C to player 3, and piece D to player 4, is envy-free, and we are done.

State 3: Assume, without loss of generality, that the situation is as follows:

```
       A          B          C          D
/-----------|-----------|-----------|-----------/
       1          1         1,2       1,3,4
```

With knife C/D kept stationary, squeeze piece D by having a referee move knife D/A to the left. (There will now be some of piece A on the right end of the diagram.) Player 1 controls knives A/B and B/C and moves these knives in such a way so as to maintain, in her view, the equality of pieces A, B, and C.

Only player 3 or 4 can call "stop." One of them will do so when she or he believes that piece D shrinks, and piece A, B, or C expands, to the point that piece A, B, or C is now tied for largest with piece D.

We make the following observations:

a. From player 1's perspective, piece D is shrinking and pieces A, B, and C are getting larger. Hence, she will not think that (the new) piece D is the largest piece.

b. Since knife C/D is not moving, knife B/C is moving to the left. Hence, piece C is going through superset changes and piece D is going through subset changes. Hence, player 2 will not think that (the new) piece D is largest or tied for largest.

c. Player 2 may think that (the new) piece A or (the new) piece B is now the largest piece.

d. Player 3 or 4 must eventually call "stop," because each believes that piece D is tending toward size 0.

Without loss of generality, we may assume that this procedure leads to one of the following situations:

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1         1,3        1,2        3,4
```

or

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1          1        1,2,3       3,4
```

or

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1        1,2,3        1         3,4
```

or

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1         1,2        1,3        3,4
```

In the first case, give piece A to player 1, piece B to player 3, piece C to player 2, and piece D to player 4, and we are done. The second and third cases are states 4 and 7, respectively. In the fourth case, give piece A to player 1, piece B to player 2, piece C to player 3, and piece D to player 4, and we are done. In analyzing states 4 and 7, we omit the "1" that appears in the second and third diagrams under C and B, respectively, because it is not needed.

State 4: Assume, without loss of generality, that the situation is as follows:

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1          1         2,3        3,4
```

We wish to squeeze pieces C and D. With knife C/D stationary, a referee moves knife D/A to the left. Player 3 controls knife B/C and moves it to the right so as to maintain, in her view, the equality of pieces C and D. Player 1 controls knife A/B and moves it so as to maintain, in her view, the equality of pieces A and B.

Only players 2, 3, or 4 can call “stop.” Player 3 calls “stop” if piece A or B becomes tied for largest. Player 2 calls “stop” if piece A, B, or D becomes tied for largest. Player 4 calls “stop” if piece A, B, or C becomes tied for largest.

We make the following observations:

a. From player 1’s perspective, pieces C and D are shrinking and pieces A and B are getting larger. Hence, she will not think that (the new) piece C or (the new) piece D is the largest piece.

b. Player 2, 3, or 4 must eventually call “stop,” because each believes that pieces C and D are tending toward size 0.

Without loss of generality, we may assume that this procedure leads to one of the following situations:

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1         1,3        2,3        3,4
```

or

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1         1,2        2,3        3,4
```

or

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1          1        2,3,4       3,4
```

In the first case, give piece A to player 1, piece B to player 3, piece C to player 2, and piece D to player 4, and we are done. In the second case, give piece A to player 1, piece B to player 2, piece C to player 3, and piece D to player 4, and we are done. In the third case, we are in state 5.

<u>State</u> <u>5</u>: Assume, without loss of generality, that the situation is as follows.

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
     1          1        2,3,4       3,4
```

We wish to squeeze pieces C and D. With knife C/D kept stationary, a referee moves knife D/A to the left. Players 2, 3, and 4 each has a knife. Call these knives B/C-2, B/C-3, and B/C-4, respectively, because these knives will be taking the place of knife B/C. Each of the three players moves his or her knife so as to maintain (in each's own view) the equality of pieces C and D. Then knives B/C-3 and B/C-4 begin where knife B/C was. Since player 2 initially thinks that piece C is larger than piece D, he will begin by placing his knife (i.e., B/C-2) to the right of where knife B/C was. Notice that the (left-to-right) order of B/C-2, B/C-3, B/C-4 can change during the process.

Meanwhile, player 1 controls knife A/B. She moves knife A/B so as to maintain, in her view, the equality of pieces A and B, where "B" refers to the piece between knife A/B and whichever of the knives B/C-2, B/C-3, and B/C-4 is the middle knife. (As noted, which is the middle knife can change along the way. Player 1 just focuses on whichever is the middle knife at any given time.)

Only players 2, 3 or 4 can call "stop." In determining when to call "stop," each player looks only at the middle knife of the knives B/C-2, B/C-3, and B/C-4. (A tie for middle knife will not present any problem.) Player 2, 3, or 4 calls "stop" when piece A or B becomes tied for largest, in his, her, or his own view. Assume, without loss of generality, that when player 2, 3, or 4 calls "stop," the left-to-right order of the three new B/C knives is B/C-4, B/C-3, B/C-2. When we refer in what follows to "B" or to "C," we mean the B and C determined by B/C-3, the middle knife.

We make the following observations:

a. From player 1's perspective, pieces C and D are shrinking and pieces A and B are getting larger. Hence, she will not think that (the new) piece C or (the new) piece D is the largest piece.

b. Player 2, 3, or 4 must eventually call "stop," because each believes that pieces C and D are tending toward size 0.

From now on, we refer to knife B/C-3 as knife B/C, since this is the knife that will make the cut. Let C4 be the piece between knives B/C-4 and C/D, and let C2 be the piece between knives B/C-2 and C/D.

If player 2 called "stop," then we may assume, without loss of generality, that the procedure led to the following situation:

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1        1,(2)    (2),3,(4)    3,(4)
```

The preferences of players 2 and 4 need some explaining, and so we put them in brackets.

Since player 2 called "stop," he thinks that piece A or B, say B, is tied for largest. But, since knife B/C-2 is to the right of knife B/C, player 2 thinks that piece C is larger than piece C2. And he thinks that pieces C2 and D are tied. So, therefore, player 2 thinks that piece C is larger than piece D. Hence, player 2 thinks that pieces B and C are tied for largest.

Because player 4 did not call "stop," he thinks that one of pieces C and D is largest. But since knife B/C-4 is to the left of knife B/C, player 4 thinks that piece C is smaller than piece C4. And he thinks that pieces C4 and D are tied. So, therefore, player 4 thinks that D is the largest piece. Hence the correct diagram is as follows:

```
      A         B         C         D
/-----------|-----------|-----------|-----------/
      1        1,2       2,3       3,4
```

In this case, give piece A to player 1, piece B to player 2, piece C to player 3, and piece D to player 4, and we are done.

Because the analysis of when player 3 or player 4 calls "stop" is similar to the above, we omit most of the details.

If player 3 called “stop,” then we may assume, without loss of generality, that the procedure leads to the following situation:

```
      A         B         C         D
/-----------|-----------|-----------|-----------/
      1        1,3       2,3       3,4
```

In this case, give piece A to player 1, piece B to player 3, piece C to player 2, and piece D to player 4, and we are done.

If player 4 called “stop,” then we may assume, without loss of generality, that the procedure leads to the following situation:

```
      A         B         C         D
/-----------|-----------|-----------|-----------/
      1        1,4       2,3       3,4
```

In this case, give piece A to player 1, piece B to player 4, piece C to player 2, and piece D to player 3, and we are done.

<u>State</u> <u>6</u>: This is similar to state 3. Assume, without loss of generality, that the situation is as follows:

```
      A         B         C         D
/-----------|-----------|-----------|-----------/
      1        1,2        1       1,3,4
```

With knives B/C and C/D kept stationary, squeeze piece D by having a referee move knife D/A to the left. Player 1 controls knife A/B and an additional knife that we will call knife X. Knife X starts at the same place at knife A/B. As we proceed, knife X will be to the left of knife A/B. The piece between knives X and A/B is now a part of piece C.

Notice that because knives B/C and C/D do not move, the piece between these two knives obviously does not change in size. Player 1 moves knives A/B and X so as to maintain, in her view, the equality of pieces A, B, and C, where piece A is the piece between knives D/A and X, and piece C now consists of two parts, the old part and the new part, which is the piece between knives X and A/B.

Only player 3 or 4 can call "stop." One of these players will do this when she or he believes that piece D shrinks, and piece A, B, or C expands, to the point that piece A, B, or C is now tied for largest with piece D.

We make the following observations:

a. From player 1's perspective, piece D is shrinking and pieces A, B, and C are each getting larger. Hence, she will not think that (the new) piece D is the largest piece.
b. Since knife B/C is stationary and player 1 sees pieces A, B, and C as all getting larger, it follows that knife A/B is moving to the left. Thus, piece B is going through superset changes. Since piece D is going through subset changes, player 2 will not think that (the new) piece D is the largest piece.
c. Player 2 may think that (the new) piece A or (the new) piece C is now the largest piece.

d. Player 3 or 4 must eventually call “stop,” because both believe that piece D is tending toward size 0.

Without loss of generality, we may assume that this procedure leads to one of the following situations:

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
     1        1,2,3        1         3,4
```

or

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
     1         1,2        1,3        3,4
```

or

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
     1         1,3        1,2        3,4
```

or

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
     1          1        1,2,3      3,4
```

For clarity, we have not shown knife X or the new part of piece C (between knives X and A/B) in the diagrams above.

The first case is state 7. In the second case, give piece A to player 1, piece B to player 2, piece C to player 3, and piece D to player 4, and we are done. In the third case, give piece A to player 1, piece B to player 3, piece C to player 2, and piece D to player 4, and we are done. The fourth case is state 4. In our analysis of states 7 and 4, we omit the “1” that appears in the first and fourth diagrams above under B and C, respectively, because it is not needed.

State 7: This is similar to state 4 and also includes ideas introduced in our study of state 6. Assume, without loss of generality, that the situation is as follows.

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1         2,3         1         3,4
```

We wish to squeeze pieces B and D. With knives B/C and C/D stationary, a referee moves knife D/A to the left. Player 3 controls knife A/B and moves it to the right so as to maintain, in her view, the equality of pieces B and D.

Player 1 controls a new knife, knife X. As in state 6, knife X starts at the same place as knife A/B. As we proceed, knife X will be to the left of knife A/B. The piece between knives X and A/B is now a part of piece C. Player 1 moves knife X so as to maintain, in her view, the equality of pieces A and C, where piece A is the piece between knives D/A and X, and piece C now consists of two parts, the old part and the new part, which is the piece between knives X and A/B.

Only players 2, 3 or 4 can call "stop." Player 2 calls "stop" if piece A, C, or D becomes tied for largest in his view. Player 3 calls "stop" if piece A or C becomes tied for largest in her view. Player 4 calls "stop" if piece A, B, or C becomes tied for largest in his view.

We make the following observations:

a. From player 1's perspective, pieces B and D are shrinking and pieces A and C are getting larger. Hence, she will not think that (the new) piece B or (the new) piece D is the largest piece.

b. Player 2, 3, or 4 must eventually call "stop," because each believes that pieces C and D are tending toward size 0.

c. Player 4 views pieces B and D as getting smaller (as does everyone), because each is going through subset changes. However, player 4 may think that piece B is getting smaller at a slower rate than is piece D and so, at some point, he may think that piece B is tied for largest. The same is true for player 2, with the roles of B and D reversed.

Without loss of generality, we may assume that the procedure leads to one of the following situations:

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1         2,3        1,3        3,4
```

or

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1         2,3        1,4        3,4
```

or

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1        2,3,4        1         3,4
```

As in our study of state 6, we have not shown knife X or the new part of piece C (between knives X and A/B) in the diagrams above.

In the first case, give piece A to player 1, piece B to player 2, piece C to player 3, and piece D to player 4, and we are done. In the second case, give piece A to player 1, piece B to player 2, piece C to player 4, and piece D to player 3, and we are done. In the third case, we are in state 8.

<u>State</u> <u>8</u>: This is similar to state 5 and also includes ideas introduced in our study of state 6. We assume, without loss of generality, that the situation is as follows:

```
       A          B          C          D
/-----------|-----------|-----------|-----------/
       1       2,3,4         1         3,4
```

We wish to squeeze pieces B and D. With knives B/C and C/D kept stationary, a referee moves knife D/A to the left. Players 2, 3, and 4 each has a knife. Call these knives A/B-2, A/B-3, and A/B-4, respectively. These knives take the place of knife A/B. Each of these three players moves his or her or his knife so as to maintain (in each's own view) the equality of pieces B and D. Note that knives A/B-3 and A/B-4 begin where knife A/B was. Since player 2 initially thinks that piece B is larger than piece D, he will begin by placing his knife (i.e., A/B-2) to the right of where knife A/B was.

Player 1 controls a new knife, knife X. As in states 6 and 7, knife X starts at the same place as (the original) knife A/B. The piece between knife X and the middle knife of knives A/B-2, A/B-3, and A/B-4 is now a part of piece C. Player 1 moves knife X so as to maintain, in her view, the equality of pieces A and C, where piece A is the piece between knives D/A and X, and piece C is the old piece C together with the piece between knife X and whichever of the knives A/B-2, A/B-3, and A/B-4 is the middle knife.

Only players 2, 3, or 4 can call "stop." In determining when to call "stop," each player looks only at the middle knife of the knives A/B-2, A/B-3, and A/B-4. Player 2, 3, or 4 calls "stop" when piece A or C becomes tied for largest, in his, her, or his own view. Assume, without loss of generality, that when player 2, 3, or 4 calls "stop," the left-to-right order of the three new A/B knives is A/B-4, A/B-3, A/B-2. When we refer in what follows to "B," we mean the B determined by A/B-3, the middle knife. Similarly, "C" refers to the old piece C together with the piece between knives X and A/B-3.

We make the following observations:

a. From player 1's perspective, pieces B and D are shrinking and pieces A and C are getting larger. Hence, she will not think that (the new) piece B or (the new) piece D is the largest piece.

b. Player 2, 3, or 4 must eventually call "stop," because each believes that pieces B and D are tending toward size 0.

From now on, we refer to knife A/B-3 as knife A/B, because this is the knife that will make the cut. Let B4 be the piece between knives A/B-4 and B/C, and let B2 be the piece between knives A/B-2 and B/C. As in states 6 and 7, we have not shown knife X or the new part of piece C (between knives X and A/B) in the diagrams below.

The next part of the analysis is similar to that used in state 5, so we omit the details.

If player 2 called "stop," then we may assume, without loss of generality, that this procedure leads to the following situation:

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
     1         2,3        1,2        3,4
```

In this case, give piece A to player 1, piece B to player 3, piece C to player 2, and piece D to player 4, and we are done.

If player 3 called "stop," then we may assume, without loss of generality, that this procedure leads to the following situation:

```
     A          B          C          D
/-----------|-----------|-----------|-----------/
     1         2,3        1,3        3,4
```

In this case, give piece A to player 1, piece B to player 2, piece C to player 3, and piece D to player 4, and we are done.

If player 4 called “stop,” then we may assume, without loss of generality, that this procedure leads to the following situation:

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1         2,3        1,4        3,4
```

In this case, give piece A to player 1, piece B to player 2, piece C to player 4, and piece D to player 3, and we are done.

State 9: We assume, without loss of generality, that the situation is as follows.

```
      A          B          C          D
/-----------|-----------|-----------|-----------/
      1          1          1       1,2,3,4
```

We squeeze piece D by keeping knife C/D fixed and having a referee move knife D/A to the left. Player 1 controls knives A/B and B/C and moves them so as to maintain, in her view, the equality of pieces A, B, and C.

Only player 2, 3, or 4 can call “stop.” We do not actually stop the process until the moment when the second player has called “stop.” Each player calls "stop" when piece A, B, or C is, in his or her view, tied for largest.

An issue arises in our analysis of this state that did not arise in any other state. Notice that it need not be the case that players 2, 3, and 4 all view pieces A and B as increasing. (They will view piece C as increasing, because it is going through superset changes.) Therefore, as the process goes on, we must allow a player who has called "stop" to take it back. For example, say that player 2 decides, at some point, that piece A is tied for largest with piece D and calls "stop." But, before a second player calls "stop," player 2 might decide that piece D is now the largest (not tied with anyone). In this case, we allow player 2 to take back his "stop."

We make the following observations:

a. From player 1's perspective, piece D is shrinking and pieces A, B, and C are getting larger. Hence, she will not think that (the new) piece D is the largest piece.

b. We will eventually have a second player that calls "stop," because players 2, 3, and 4 each believes that piece D is tending toward size 0.

Without loss of generality, we may assume that this procedure leads to one of the following situations:

```
        A          B          C          D
   /-----------|-----------|-----------|-----------/
        1        1,2,3        1         3,4
```

or

```
        A          B          C          D
   /-----------|-----------|-----------|-----------/
        1         1,2        1,3        3,4
```

or

```
        A          B          C          D
   /-----------|-----------|-----------|-----------/
       1,3         1         1,2        3,4
```

or

```
        A          B          C          D
   /-----------|-----------|-----------|-----------/
        1          1        1,2,3       3,4
```

The first case is state 7. In the second case, give piece A to player 1, piece B to player 2, piece C to player 3, and piece D to player 4, and we are done. In the third case, give piece A to player 3, piece B to player 1, piece C to player 2, and piece D to player 4, and we are done. The fourth case is state 4. In our analysis of states 7 and 4, we omit the "1" that appears in the first and fourth diagrams under B and C, respectively, because it is not needed.

This concludes our analysis of the nine states. We have shown that we always complete the procedure and arrive at an envy-free division. Concerning the number of cuts, we need only observe that in every case, we made cuts using knives A/B, B/C, C/D, D/A and sometimes X. Hence, we have used at most 5 cuts. Q.E.D.

Theorem 4. *There is a moving knife procedure for four players that yields an envy-free division of a cake using at most five cuts. Also, either*

a. *two of the four players each receives a connected piece, and each of the other two players receives either a connected piece or else a union of two such pieces, or*

b. *three of the four players each receives a connected piece, and the other player either receives a connected piece or a union of two such pieces or a union of three such pieces.*

Proof. Theorem 4 follows easily from Theorem 3. Given a cake, we temporarily pretend that it is a pie by identifying the endpoints. We then apply Theorem 3 to obtain an envy-free division of the pie, using at most 5 cuts, such that three of the four players each has a connected piece and the other player either has a connected piece or else a union of two such pieces. Then we return to our original cake by breaking the identification of the endpoints. Clearly, the number of cuts is still at most five. The various possibilities listed in the theorem correspond to whether breaking the identification of the endpoints causes no new disconnection, or causes a disconnection in a previously connected piece and, if it does cause a new disconnection, whether this new disconnection occurs in a piece that was already disconnected (and so now is the union of three pieces). Q.E.D.

5. Conclusions

It would be wonderful if we could somehow eradicate envy entirely for four or more players with a procedure that requires only the minimal $n - 1$ parallel cuts. In principle, this is possible. Stromquist (1980) and Woodall (1980) proved that there exists an n-person envy-free division of a cake, using only $n - 1$ parallel cuts (for recent extensions, see Ichiishi and Idzik (1999)). But how to achieve such a lovely division is by no means evident.

The squeezing operation that we successfully used for three persons seems only capable of giving almost envy-freeness for four persons, if we insist on only three cuts. To guarantee envy-freeness, we showed that two additional cuts beyond the minimal three suffice for four persons, which implies that the pieces some players receive may be disconnected. This is not appealing if it is land that is being divided and all the players want connected pieces.

The problem with finding an envy-free solution, using only $n - 1$ cuts, seems to be that the operations for moving knives that we allow, as well as the information that the players have, is insufficient to give such a solution. While the procedures put a great deal of weight on creating ties, it seems that the players need to be able to make cuts that take into account more information about the valuations of the other players to effect an envy-free division with $n - 1$ cuts. Just as trisecting an angle with only a straightedge and a compass is impossible, we suspect that a 4-person, 3-cut procedure is also impossible unless new operations are allowed or new information about the relative valuations of the pieces by different players is introduced.

Consider the possibility of giving the players more information. Assume they know not only their own valuations of the cake but also are told the other players' valuations. Then it should be possible for them to calculate an envy-free solution that uses only $n - 1$ cuts, because we know such a solution exists.

But this calculation introduces two problems. First, there may be many solutions. Indeed, because such an envy-free solution is efficient (or Pareto-optimal) among the set of solutions using $n - 1$ parallel cuts (Brams and Taylor, 1996, pp. 149-151), different solutions will favor different players. (It is not known whether envy-free pie division with radial cuts is efficient; see Gale, 1993, p. 51.) Which of a possible infinity of solutions is fairest?

Even if a unique solution is agreed upon, the second problem is finding rules of a game that would enable the players to implement such a solution as an equilibrium outcome. It should be an equilibrium so that the players, once they reach it, will have no reason to depart from it. But the rules should also give the players an incentive to choose it, especially if there are other equilbria, by making the desired equilibrium dynamically stable in the sense that the players' optimal strategies in a multi-stage game would lead them to select it.

Alternatively, an arbitrator might be asked to calculate such a solution from the players' preferences. In that case, however, the players may not have an incentive to be truthful in revealing their preferences. Creating "incentive compatibility"—by making it in the interest of the players to be truthful—is also a problem in designing the rules of a game without an arbitrator if the players can indeed benefit from not being truthful.

The procedures we have described are not incentive compatible—they can be manipulated by wily players. As we indicated earlier, however, any attempt by a player to gain a larger piece of cake (e.g., by not calling "stop" when there is a tie but waiting a bit longer) carries the risk of that player's getting less. In effect, the strategies our procedures prescribe ensure the maximin outcomes of envy-freeness and almost envy-freeness, but players willing to take chances may, on occasion, do better by departing from these strategies.

Patently, challenges remain for finding better cake-cutting procedures. Our 4-person, 5-cut envy-free procedure is hardly one we would expect players to use; the situation surely gets worse for five or more players if one makes envy-freeness the *sine quo non* of a cake-cutting solution. Almost envy-free procedures, or those that give approximate envy-free solutions (Brams and Kilgour, 1996, pp. 130-133; Su, 1999; Zeng, 2000) or invoke other criteria of fairness like the amount of competition for a good (Brams and Kilgour, 2001), seem fruitful ways to go. Another promising direction is to change the rules of the cake-cutting game along the lines mentioned earlier by (i) putting more information at the disposal of the players and (ii) giving them more opportunities to make adjustments in boundaries in a manner that facilitates the selection of fair outcomes.

We encourage thinking hard about these alternatives to expand the storehouse of simple and practicable procedures. Ultimately, we hope, they would be applicable to the settlement of real-life disputes of the kind discussed in Brams and Taylor (1999).

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