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Paradoxes of Fair Division

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Abstract

Two or more players are required to divide up a set of indivisible items that they can rank from best to worst. They may, as well, be able to indicate preferences over subsets, or packages, of items. The main criteria used to assess the fairness of a division are efficiency (Pareto-optimality) and envy-freeness. Other criteria are also suggested, including a Rawlsian criterion that the worst-off player be made as well off as possible and a scoring procedure, based on the Borda count, that helps to render allocations as equal as possible.

Eight paradoxes, all of which involve unexpected conflicts among the criteria, are described and classified into three categories, reflecting (1) incompatibilities between efficiency and envy-freeness, (2) the failure of a unique efficient and envy-free division to satisfy other criteria, and (3) the desirability, on occasion, of dividing up items unequally. While troublesome, the paradoxes also indicate opportunities for achieving fair division, which will depend on the fairness criteria one deems important and the trade-offs one considers acceptable.

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1. INTRODUCTION

Paradoxes, if they do not define a field, render its problems intriguing and often perplexing, especially insofar as the paradoxes remain unresolved. Voting theory, for example, has been greatly stimulated by the Condorcet paradox, which is the discovery by the Marquis de Condorcet (1785) that there may be no alternative that is preferred by a majority to every other alternative, producing so-called cyclical majorities. Its modern extension and generalization is Arrow's (1951) theorem, which says, roughly speaking, that a certain set of reasonable conditions for aggregating individual preferences into some social choice are inconsistent.

In the last fifty years, hundreds of books and thousands of articles have been written about these and related social-choice paradoxes and theorems, as well as their ramifications for voting and democracy. Nurmi (1999) provides a good survey and classification of voting paradoxes, as well as offering advice on "how to deal with them."

There is also an enormous literature on fairness, justice, and equality, and numerous suggestions on how to rectify the absence of these properties or attenuate their erosion. But paradoxes do not frame the study of fairness in the same way that they have inspired social-choice theory.

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To be sure, the notion that justice and order may be incompatible, or that maximin justice à la Rawls (1971) undercuts the motivation of individuals to strive to do their best, underscores the possible trade-offs in making societies more free and egalitarian. For example, an egalitarian society may require strictures on free choice to ward off anarchy; rewarding the worst-off members of a society may deaden competition among the most able if their added value is siphoned off to others.

Obstacles like these that stand in the way of creating a just society are hardly surprising. They are not paradoxes in the strong sense of constituting a logical contradiction between equally valid principles. Here we use paradox in a weaker sense—as a conflict among fairness conditions that one might expect to be compatible. Because we are surprised to discover this conflict, it is "nonobvious," as one of us labeled a collection of paradoxes he assembled about politics (Brams, 1976; see also Fishburn, 1974, and Fishburn and Brams, 1983).

The fair-division paradoxes we present here all concern how to divide up a set of indivisible items among two or more players. In some paradoxes, we assume the players can do no more than rank the items from best to worst; in others, we assume they can, in addition, indicate preferences over subsets, or packages, of items.

The main criteria we invoke in assessing the fairness of a division of the items are *efficiency* (there is no other division better for everybody, or better for some players and not worse for the others) and *envy-freeness* (each player likes its allocation at least as much as those that the other players receive, so it does not envy anybody else), but we also consider other properties in evaluating the fairness of a division of the items. Because the items are indivisible, no splitting of them between players is allowed.[2]

Our paradoxes demonstrate the opportunities as well as the limitations of fair division. Thus, for example, while the only division of items in which one player never envies the allocation of another may be nonexistent or inefficient, we note that there is always an efficient and envy-free division for two players—even when they value all items the same—as long as they do not rank all subsets of items the same. We also show that fair division may entail an unequal division of the items.

We divide the paradoxes into three categories:

1. The conflict between efficiency and envy-freeness (paradoxes 1 and 2);
2. The failure of a unique efficient and envy-free division to satisfy other fair-division criteria (paradoxes 3 and 4);
3. The desirability, on occasion, of dividing items unequally (paradoxes 5, 6, 7, 8).

While the paradoxes highlight difficulties in creating "fair shares" for everybody, they by no means render the task impossible. Rather, they show how dependent fair division is on the fairness criteria one deems important and the trade-offs one considers acceptable. Put another way, achieving fairness requires some consensus on the ground rules (i.e., criteria) and some delicacy in applying them (to facilitate trade-offs when the criteria conflict).

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[2] Fair-division procedures that allow for the splitting or sharing of (divisible) goods are discussed in, among other places, Young (1994), Moulin (1995), Robertson and Webb (1998), and Brams and Taylor (1996, 1999).

We mention three technical points before we proceed to specific examples. First, we assume that players cannot compensate each other with side payments—the division is only of the indivisible items. Second, all players have positive values for every item. Third, a player prefers one set *S* of items to a different set *T* if (i) *S* has as many items as *T* and (ii) for every item *t* in *T* and not in *S*, there is a distinct item *s* in *S* and not *T* that the player prefers to *t*. For example, if a player ranks items 1 through 4 in order of decreasing preference 1234, we assume that he or she prefers the set {1,2} to {2,3}, and {1,3} to {2,4}, whereas the comparison between {1,4} and {2,3} could go either way.

2. EFFICIENCY AND ENVY-FREENESS: THEY MAY BE INCOMPATIBLE

1. A unique envy-free division may be inefficient. Suppose there is a set of three players, {A, B, C}, that must divide a set of six indivisible items, {1, 2, 3, 4, 5, 6}. Assume the players strictly rank the items from best to worst as follows:

Example I

A: 1 2 3 4 5 6

B: 4 3 2 1 5 6

C: 5 1 2 6 3 4

The unique envy-free allocation to (A, B, C) is ({1,3}, {2,4}, {5,6}), or for simplicity (13, 24, 56), whereby A and B get their 1st and 3rd-best items, and C gets its 1st and 4th-best items. Clearly, A prefers its allocation to that of B (which are A's 2nd and 4th-best items) and that of C (A's two worst items). Likewise, B and C prefer their allocations to those of the other two players. Consequently, the division (13, 24, 56) is envy-free: all players prefer their allocations to those of the other two players, so no player is envious of any other.

Compare this division with (12, 34, 56), whereby A and B receive their two best items, and C receives, as before, its 1st and 4th-best items. This division Pareto-dominates (13, 24, 56), because two of the three players (A and B) prefer the former allocation, whereas both allocations give player C the same two items (56).

It is easy to see that (12, 34, 56) is Pareto-optimal, or efficient: no player can do better with some other division without some other player or players doing worse, or at least not better. This is apparent from the fact that the only way A or B, which get their two best items, can do better is to receive an additional item from one of the two other players— assuming all items have some positive value for the players—but this will necessarily hurt the player who then receives fewer than its present two items. Whereas C can do better without receiving a third item if it receives item 1 or 2 in place of item 6, this substitution would necessarily hurt A, which will do worse if it receives item 6 for item 1 or 2.

The problem with efficient allocation (12, 34, 56) is that it is not *assuredly* envy-free. In particular, C will envy A's allocation of 12 (2nd and 3rd-best items for C) if it prefers these two items to its present allocation of 56 (1st and 4th-best items for C). In the absence of information about C's preferences for subsets of items, therefore, we cannot say that efficient allocation (12, 34, 56) is envy-free.[3]

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[3] Henceforth we will mean by "envy-free" a division such that, no matter how the players value subsets of items consistent with their rankings, no player prefers any other player's allocation to its own. If a division is not envy-free, we call it "envy-possible" if a player's allocation *may* make it envious of another player, depending on how it values

But the real bite of paradox 1 stems from the fact that not only is inefficient division (13, 24, 56) envy-free, but it is uniquely so—there is no other division, including an efficient one, that guarantees envy-freeness. To show this in Example I, note first that an envy-free division must give each player its best item; if not, then a player might prefer a division, like envy-free division (13, 24, 56) or efficient division (12, 34, 56), that does give each player its best item, rendering the division that does not envy-possible or envy-ensuring. Second, even if each player receives its best item, this allocation cannot be the only item it receives, because then the player might envy any player that receives two or more items, *whatever* these items are.

By this reasoning, then, the only possible envy-free divisions in Example I are those in which each player receives two items, including its top choice. It is easy to check that no efficient division is envy-free.[4] Similarly, one can check that no inefficient division, except (13, 24, 56) that gives each player two items—including its best—is envy-free, making this division uniquely envy-free.

2. There may be no envy-free division, even when all players have different preference rankings. While it is bad enough when the only envy-free division is inefficient (paradox 1), it seems even worse when there is no envy-free division. This is

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subsets of items (illustrated by the example in the text). It is "envy-ensuring" if it causes envy, independent of how the players value subsets of items. In effect, a division that is envy-possible has the potential to cause envy. By comparison, an envy-ensuring division always causes envy, and an envy-free division never causes envy.

[4] We previously showed that division (12, 34, 56) is not envy-free. As another example, consider efficient division (16, 34, 25). Whereas neither B nor C envies each other or A, A might envy either B's 34 or C's 25 allocations, making this division envy-possible.

trivial to show when players rank items the same. For example, if two players both prefer item 1 to item 2, then the player that gets item 2 will envy the player that gets item 1.

In the following example, each of three players has a different ranking of three items:

Example II

A: 1 2 3

B: 1 3 2

C: 2 1 3

There are three divisions of (A, B, C) that are efficient—(1, 3, 2), (2, 1, 3), and (3, 1, 2)—in which at least one player gets its best item. It is evident that none is envy-free, because the player that gets item 1 in each (A or B) will be envied by at least one of the other two players. For instance, in the case of the division (2, 1, 3), both A and C will envy B.

Can an inefficient division be envy-free, as was the case in Example I? It is not hard to see that this situation cannot occur in Example II for the reason given above: the player that gets item 1 will be envied. But in the case of an inefficient division, "trading up to efficiency" reduces the *amount* of envy. For example, consider inefficient division (2, 3, 1), in which each player receives its 2nd choice. Because A envies C, B envies C, and C envies A, a trade of items 1 and 2 between A and C is possible. It yields efficient division (1, 3, 2), in which only B envies A.

Besides (2, 3, 1), the other two inefficient divisions—(1, 2, 3) and (3, 2, 1)—also allow for "trading up to efficiency." In the first, a trade of items 2 and 3 between B and

C yields efficient division (1, 3, 2); in the second, a trade of items 1 and 2 between B and C yields efficient division (3, 1, 2). Three-way trades are also possible. For instance, starting from inefficient division (3, 2, 1), a three-way trade, whereby A sends item 3 to B, B sends item 2 to C, and C sends item 1 to A, yields efficient division (1, 3, 2).

Trading up to efficiency is also possible in Example I: by exchanging items 2 and 3, A and B can turn inefficient division (13, 24, 56) into efficient division (12, 34, 56). As in Example II, however, no efficient division is envy-free. The difference between Examples I and II is that Example II does not admit even an inefficient envy-free division.

3. UNIQUE EFFICIENT AND ENVY-FREE DIVISIONS: THEIR INCOMPATIBILITY WITH OTHER CRITERIA

3. A unique efficient and envy-free division may lose in voting to an efficient and envy-possible division. So far we have shown that efficiency and envy-freeness may part company either by there being no envy-free division that is also efficient (Example I), or no envy-free division at all (Example II). But when these properties coincide, and there is both an efficient and an envy-free division, it may not be the choice of a majority of players, as illustrated by the following example:

Example III

A: 1 2 3 4 5 6

B: 5 6 2 1 4 3

C: 3 6 5 4 1 2

There are three efficient divisions in which (A, B, C) each get two items: (12, 56, 34); (12, 45, 36); and (14, 25, 36). However, only the third division, (14, 25, 36), is envy-free. Whereas C might envy B's 56 allocation in the first division, and B might envy A's 12 allocation in the second division, no player envies another player's allocation in (14, 25, 36).

But observe that both A and B prefer the first division, (12, 56, 34), to the envy-free third division, (14, 25, 36), because they get their top two items in the first division; only C gets its top two items in (14, 25, 36). Hence, the first division would defeat the envy-free third division, (14, 25, 36), by simple majority rule.

The situation is not so clear-cut when we compare the second division, (12, 45, 36), with the envy-free (14, 25, 36). In fact, there would be a tie vote: C would be indifferent, because it gets its top two items, 36, in each division; A would prefer the second division (top two items versus 1st and 4th-best items); and B would prefer the envy-free division, (14, 25, 36) (1st and 3rd-best items versus 1st and 5th-best items).

Thus, if there were a vote, the unique envy-free division, (14, 25, 36), would lose to the envy-possible division, (12, 56, 34), and it would tie with the other envy-possible division, (12, 45, 36). If there were approval voting (Brams and Fishburn, 1983), and A, B, and C voted only for the divisions that give each player its two best items, then the envy-free division, (14, 25, 36), would get 1 vote, compared to 2 votes each for both of the envy-possible divisions, (12, 56, 34) and (12, 45, 36). In sum, players will choose an envy-possible over the unique envy-free division, (14, 25, 36), in either pairwise comparisons or approval voting.

4. Neither a Rawlsian criterion nor the Borda count may choose a unique efficient and envy-free division.

Besides using voting to select an efficient division, consider the following Rawlsian (maximin) criterion to distinguish among efficient divisions: choose the division that maximizes the minimum rank of items that players receive, making a worst-off player as well off as possible. To illustrate in Example III, envy-possible division (12, 45, 36) gives a 5th-best item to B, whereas each of the two other divisions gives a player, at worst, a 4th-best item. Between the latter two divisions, the envy-possible division, (12, 56, 34) is, arguably, better than the envy-free division, (14, 25, 36), because it gives the other two players—those that do not get a 4th-best item—their two best items, whereas envy-free division (14, 25, 36) does not give B its two best items.[5]

The Borda count would also give the nod to the envy-possible division, (12, 56, 34), compared not only with the envy-free division, (14, 25, 36), but also with the other envy-possible division, (12, 45, 36). Awarding 6 points for obtaining a best item, 5 points for obtaining a 2nd-best item, . . ., 1 point for obtaining a worst item in Example III, the latter two divisions give the players a total of 30 points, whereas envy-possible division (12, 56, 34) gives the players a total of 31 points.[6] Hence, an envy-possible division beats the unique envy-free division, based on both the Rawlsian criterion and the Borda count.

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[5] This might be considered a second-order application of the Rawlsian criterion: if, for two divisions, players rank the worst items a player receives the same, choose the division that gives them the best of their next-worst items, etc. This is an example of a lexicographic decision rule, whereby outcomes are ordered on the basis of a most important criterion; if that is not determinative, use a next-most important criterion, and so on.

[6] The standard scoring rules for the Borda count in this 6-item example would give 5 points to a best item, 4 points to a 2nd best item, . . ., 0 points to a worst item. We depart

4. THE DESIRABILITY OF UNEQUAL DIVISIONS (SOMETIMES)

5. An unequal division of items may be preferred by all players to an equal division. In section 3 we showed that neither (i) pairwise comparison voting or approval voting (paradox 3), nor (ii) a Rawlsian criterion or the Borda count (paradox 4), always selects a unique efficient and envy-free division. In the following example, there is also a unique efficient and envy-free division—involving an equal division of the items—but there may be grounds for choosing an efficient but unequal envy-possible division:

Example IV

A: 1 2 3 4

B: 2 3 4 1

It is not difficult to show that (13, 24) is the only efficient and envy-free division. Two other equal divisions, (12, 34) and (14, 23), while better for one player and worse for the other, are envy-possible and therefore not envy-free.

The aforementioned three equal divisions all give total Borda scores of 12 to their players. If we eliminate the envy-possible division, (14, 23), on the grounds that it fails the Rawlsian criterion by giving A its worst item (item 4), then the comparison reduces to that between envy-free division, (13, 24), and envy-possible division, (12, 34).

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slightly from this standard scoring rule to ensure that each player obtains some positive value for all items, including its worst choice, as assumed earlier.

Curiously, it is possible that *both* A and B prefer the unequal envy-possible division, (134, 2), to the equal envy-possible division, (12, 34).[7] Thus, unequal divisions might actually be better for all players than equal divisions.

Ruling out equal division (12, 34) in such a situation, let us compare (134, 2) with the envy-free (equal) division (13, 24). Clearly, (134, 2) is better than (13, 24) for A, but it is worse for B.

This leaves open the question of which of these two divisions, involving an equal and an unequal division of the items, comes closer to giving the two players "fair shares." As the next paradox shows, an unequal division may actually be more egalitarian—as measured by Borda scores—than an equal division.

6. An unequal division of items may (i) maximize the minimum Borda scores of players and (ii) maximize the sum of Borda scores. In paradox 5, we showed that an unequal but envy-possible division of items may compare favorably with an equal and envy-free division. To make this kind of comparison more precise, consider the following example:

Example V

A: 1 2 3 4 5 6 7 8 9

B: 3 1 2 4 5 6 7 8 9

C: 4 1 2 3 6 5 7 8 9

[7] This is true if A prefers 34 to 2, and B prefers 2 to 34.

There are exactly two unequal divisions, (12, 357, 4689) and (12, 3589, 467), that maximize the minimum Borda scores of players, which are (17, 17, 17) for both divisions. On the other hand, there are two equal divisions, (129, 357, 468) and (129, 358, 467), that maximize the minimum Borda scores of players, which are (18, 17, 16) for the first division and (18, 16, 17) for the second division. The sum of the Borda scores in each case is 51, which, it can be shown, is the maximal sum among all possible divisions (equal or unequal).

Notice that the worst-off player in the two unequal divisions garners 17 points (so does the best-off player, because the Borda scores of all players are the same), whereas the worst-off player in the two equal divisions receives fewer points (16). By the Rawlsian criterion, based on Borda scores, therefore, the unequal divisions are more egalitarian.

None of the four equal or unequal divisions is envy-free—all are envy-possible or envy-ensuring. Likewise, all four divisions are "efficient-possible" in the sense that there may be a more efficient division, but this is not guaranteed. Take, for example, the unequal division (12, 357, 4689). B or C might prefer A's 12 allocation, just as A might prefer B's or C's allocation, so a trade could make two, or even all three, players better off. Unlike our previous examples, in which divisions called "efficient" were all "efficient-ensuring" (i.e., there were no trades that could improve the lot of all traders, however players valued subsets of items), this is not the case in Example V.

The Rawlsian criterion, based on Borda scores, seems a reasonable one to distinguish among all efficient-possible and envy-possible divisions. In Example V, it is not only unequal divisions that do best on this criterion, but these divisions also maximize the sum of Borda scores, which might be considered a measure of the overall utility or welfare of the players.

7. An unequal division of items may (i) maximize the sum of Borda scores (maxsum) but not (ii) maximize the minimum Borda score (maximin). There was no conflict between maxsum and maximin in Example V—two unequal divisions satisfied both of these properties. But as the next example illustrates, this need not be the case:

Example VI

A: 1 2 6 4 3 5

B: 3 4 5 1 2 6

C: 6 2 1 4 3 5

There are two unequal maxsum divisions, (12, 345, 6) and (1, 345, 26), whose Borda scores are, respectively, (11, 15, 6) and (6, 15, 11). Each gives a total Borda score of 32, and a minimum score for a player of 6.

By contrast, there are two equal maximin divisions, (12, 35, 46) and (14, 35, 26), whose Borda scores are, respectively, (11, 10, 9) and (9, 10, 11). Each gives a total Borda score of 30, and a minimum score for a player of 9.

Presumably, the egalitarian would choose one of the two (equal) maximin divisions to ensure players of a minimum score of 9. The utilitarian would choose one of the two (unequal) maxsum divisions to ensure the greatest total score of 32.

Unfortunately, the total Borda scores of maxsum and maximin divisions may be arbitrarily far apart (Brams, Edelman, and Fishburn, 2000). We believe that when there is a difference, as in Example VI, maximin generally gives the fairer division by guaranteeing that the Borda score of the worst-off player is as great as possible.[8]

Although maximin may require an unequal division of the items (Example V), the opposite can also be true (Example VI). By the same token, maxsum divisions can be the product of either equal or unequal divisions (both kinds give the same maxsum in Example V).

Incidentally, it is easily seen that the smallest example for which there is a divergence between maximin and maxsum is that in which two players must divide three items, with the players having preference rankings 123 and 132. Maxsum is 7, given by divisions (12, 3) and (2, 13), but the maximin divisions, (1, 23) and (23, 1), which each give Borda scores of (3, 3), have a lower total score of 6.

The maximin divisions, nevertheless, would seem to be fairer than the maxsum divisions, in which the player that gets a single item receives a Borda score of only 2. As we will show in our final paradox, however, the maximin division may be quite implausible, depending on how players value subsets.

8. If There Are Envy-Free Divisions, None May Be Maximin. In the following example, there are two players but an odd number of items, so no equal division of the items is possible:

[8] To be sure, assuming that the differences in ranks are all equal, as Borda scoring does, is a simplification. If cardinal utilities could be elicited that reflect the players' intensities of preference, then they—instead of rank scores—could be used to equalize, insofar as possible, players' satisfaction with a division of the items. For fair-division bidding schemes that incorporate cardinal information, see Brams and Taylor (1996, 1999), Brams and Kilgour (2000), and Haake, Raith, and Su (2000).

Example VII

A: 1 2 3 4 5

B: 1 2 3 4 5

Because the players rank the items exactly the same, all divisions are efficient, making the choice of a fairest one appear to be difficult.

Only six divisions, however, are what Brams and Fishburn (1999) call *undominated splits*:

(1, 2345); (12, 345); (13, 245); (14, 235); (15, 234); (145, 23).

These divisions are those in which, in the absence of information about preferences over subsets, either of the two allocations might be preferred by a player, making each undominated. All these divisions, therefore, are envy-possible.

The maximin divisions are (13, 245) and (14, 235), which give Borda scores of, respectively, (8, 7) and (7, 8) to the players. But neither division might be envy-free if, say, *both* players prefer 13 to 245 and 14 to 235. These preferences imply that both players prefer 12 to 345 in the second division, (12, 345), and 145 to 23 in the sixth division, (145, 23), precluding these divisions, as well, from being envy-free.

But given these preferences of A and B, it is possible that each player would prefer a different allocation in the other two divisions, making them envy-free. For example, A might prefer 1 in the first division and 15 in the fifth, whereas B might prefer the complements: 2345 in the first, and 234 in the fifth.

In none of our previous examples with envy-free divisions was such a division not maximin. But as we have just illustrated, there may be several envy-free divisions, none of which is maximin. This divergence points to the limitation of maximin as a criterion for choosing divisions, because Borda scoring may not reflect the intensity of player preferences that can be better gleaned from player preferences over subsets.

5. CONCLUSIONS

The eight paradoxes pinpoint difficulties in dividing up indivisible items so that each player feels satisfied, in some sense, with its allocation. The first two paradoxes illustrated that efficient and envy-free divisions may be incompatible because the only envy-free division may be inefficient, or there may be no envy-free division at all.

Both of these paradoxes require at least three players (Edelman and Fishburn, 2000). When there are only two players, even when they rank items exactly the same, it turns out that efficient and envy-free divisions can always be found, except when the players have the same preferences over all subsets of items (Brams and Fishburn, 1999).

But the existence of even a unique efficient and envy-free division may not be chosen by the players for other reasons. In particular, such a division will not necessarily be selected when players vote for the division or divisions that they prefer. Also, a unique efficient and envy-free division will not necessarily be the division that maximizes the minimum rank of items that players receive, so the Rawlsian criterion of making the worst-off player as well off as possible may not single it out.

As a way of measuring the value of allocations to find those divisions that are most egalitarian, we used Borda scoring, based on player rankings of the items. We showed that the division that maximizes the minimum Borda scores of players (the maximin division) may not necessarily be the one that maximizes the total Borda score of all players (the maxsum division), indicating the possible conflict between egalitarian and utilitarian outcomes.

This difference may show up when there are as few as two players dividing up three items, making it impossible to divide the items equally between the players. But even when this is possible, unequal divisions of items may be the only ones that satisfy the maximin criterion. While indicating a preference for this criterion over the maxsum criterion when the two clash, we illustrated how maximin divisions may fail badly in finding envy-free divisions. Indeed, there may be no overlap between maximin and envy-free divisions.

Our purpose is not just to indicate the pitfalls of fair division by exhibiting paradoxes that can occur. There are also opportunities, but these depend on the judicious application of selection criteria when not all criteria can be satisfied simultaneously.

Several recent papers have suggested constructive procedures for finding the most plausible candidates for fair division of a set of indivisible items (Brams and Fishburn, 1999; Fishburn and Edelman, 2000; Brams, Edelman, and Fishburn, 2000; Herreiner and Puppe, 2000). We find this direction promising, because it is potentially applicable to ameliorating, if not solving, practical problems of fair division—ranging from the splitting of the marital property in a divorce to the determination of the boundaries in an international dispute. But some trade-offs are ineradicable, as the paradoxes demonstrate, and how best to handle them is by no means evident.

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