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Abstract

We present a general analysis of the war of attrition in discrete time. In contrast to the continuous time formulation, the set of equilibria are sensitive to how the returns to players are treated when they move simultaneously. When the return from moving alone (leading) in any period is greater than or equal to the return from moving simultaneously (tying), we are able to provide a complete characterization of the set of equilibria in both infinite and finite horizon games. In general, such a complete characterization would be quite complicated. However, we illustrate with a number of examples some of the possible equilibrium patterns.

1. Introduction

Many examples of conflict can be modelled as a contest in which the party that prevails is the one who is willing to compete for greatest length of time. The essential characteristics of such contests are that at each instant of time, each party must decide whether to concede its position (move) or to expend resources and continue to compete (wait). At any instant, the payoff to a party if it concedes is less than its payoff if the other party concedes, but if a party is eventually going to be the first to concede, its payoff is lower the longer it waits. Among theoretical biologists, this model of conflict is known as a War of Attrition. It was introduced by Maynard Smith (1974) to study the evolutionary stability of certain patterns of behavior in animal conflicts over territory or mates.[1]

The earliest treatments of the model appeared in the biology literature where it was formulated as a game in continuous time in which the payoffs to the players are symmetric and common knowledge (e.g. Bishop and Cannings (1978)). Later authors have extended the basic model in a number of directions, allowing for various kinds of asymmetries and incomplete information. These include contests in which the asymmetries are observable (e.g. Maynard Smith and Parker (1976), Parker and Rubenstein (1981)) and contests in which the asymmetries are private information (e.g. Bishop, Cannings, and Maynard Smith (1978), Nalebuff and Riley (1985)). The model has also been applied to a number of economic conflicts such as "price wars" (Kreps and Wilson (1982), Fudenberg and Tirole (1983), Benoit (1985)), the

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[1]Following Selten (1980), a strategy r is said to be evolutionary stable if (i) r is a best reply to itself and (ii) for any alternative best reply to r, r is a better reply to r′, the alternative best reply, than r′ is to itself. Condition (i) implies that an evolutionary stable strategy (ESS) is a symmetric Nash equilibrium point while condition (ii) is a stability requirement.

private provision of public goods (Bliss and Nalebuff (1984)), and bargaining (Ordover and Rubinstein (1983), Osborne (1983)). For the most part, these authors have formulated their models in continuous time.[2] A systematic investigation of the War of Attrition in continuous time can be found in Weiss and Wilson (1984) for the case of complete information and in Wilson (1984) for the case of incomplete information.

In this paper we provide a general analysis of the War of Attrition formulated as a game in which (1) the payoffs of both players are common knowledge and (2) time is discrete. We consider both finite and infinite horizon games.

There are at least three reasons for analyzing the game in discrete time. The first and most important reason is to capture the role of ties, that is, the possibility that both players move (concede) at the same time. Ties are irrelevant in the continuous time game since, in equilibrium, they occur with probability zero. In many economic examples, however, the presence of information lags and discreteness in the decision process imply that, as a simple descriptive matter, ties can and do occur. They are particularly likely to occur when the conflict is organized as a sequence of bouts or rounds, as is the case in certain bargaining situations and animal conflicts. We find that even when the length of a period is short and, as a result, the probability of a tie is small, the set of equilibria may be quite sensitive to how the payoffs to ties are treated. The crucial issue is whether a player's return from moving together with the other player in a period is greater than

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[2] There are exceptions. Bishop and Cannings (1978) prove an existence theorem for finite horizon, discrete time models with symmetric payoffs and use the model to examine the dynamic stability of the ESS. Hammerstein and Parker (1982) use a discrete time model to calculate the ESS in a number of numerical examples. Finally, Ordover and Rubinstein (1985) formulate their model in discrete time.

or less than his return from moving alone in that period. We establish that the equilibrium patterns of behavior in games where players try to avoid ties may not be present at all in games where players prefer to move together rather than alone. The converse may also be true.

A little reflection suggests that there are a number of examples where the return from moving alone in a period is not equal to the return from moving together with the other player in that period. The problem of bargaining, for instance, suggests an example of a situation in which the return to a tie is greater than the return to moving alone. If both players concede their positions in the same round, then they generally bargain to a compromise settlement in which each party is better off than it would have been if it had conceded unilaterally, but worse off than it would have been if it had maintained its position. This is also the standard assumption in the models in the biology literature.

The timing of investments in the presence of information externalities (Hendricks (1984)) suggests an example in which the return to moving simultaneously may be less than the return to moving alone. Suppose two firms must decide when to introduce a new product whose profitability is unknown. Each firm has an incentive to delay its commitment of capital and to let the other firm incur the costs of determining the state of demand. Suppose further that there is a slight first mover advantage so that, if both firms begin marketing the product in the same period, each firm earns less than it would have earned had it entered the market by itself. Then, if the information externalities are sufficiently large, each firm prefers to follow rather than to lead and to lead rather than to tie.

A second reason for conducting the analysis in discrete time is that the set of equilibria may not correspond to the equilibria of continuous time games. In Hendricks and Wilson (1985), we demonstrate that the set of limit points of the equilibria of a sequence of discrete time games in which time is partitioned into ever finer intervals is not equivalent to the set of Nash equilibria of the corresponding game formulated in continuous time. In some instances the set of equilibria is larger in the continuous time game and in other instances it is larger in the discrete time game. The differences arise primarily when the time horizon is finite and can be attributed to the fact that in continuous time there is no "next" period or "next to last" period. These results suggest that as long as there is any lag in a player's response to his opponent's move, it may be preferable to analyze the game in discrete time.

Finally, formulating the game in discrete time permits a precise analysis of subgame perfection. This issue is particularly relevant for contests in which the horizon is finite and there is an observable asymmetry between the players. For instance, in certain asymmetric contests[3] analyzed in the biology literature, there are generally two evolutionary stable outcomes, one in which the "strong" animal always wins (the "weak" animal concedes immediately) and one in which the weak animal always wins. When the horizon is finite, the restrictions implied by subgame perfection imply that the second, paradoxical outcome can be eliminated.[4] There are a number of economic applications (e.g. Fudenberg, et. al. (1983)) where the imposition of subgame perfection also eliminates some equilibrium outcomes.

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[3]An asymmetric contest is one in which the animals are in different states or roles on different occasions. An ESS is such contests is essentially a symmetric Bayesian Nash equilibrium point.

[4]For games with an infinite horizon, a more powerful refinement of the equilibrium concept may be required to eliminate such equilibria. For instance, Hammerstein and Parker (1982) assume that the animals make arbitrarily small "mistakes" in identifying their roles.

In our analysis, we distinguish between two types of equilibria. Degenerate equilibria consist of strategy combinations in which one player moves in period 0 with probability 1, while the other player threatens not to move for a sufficiently damaging length of time. Any other equilibrium is called nondegenerate. We first establish that, in any nondegenerate equilibrium, the probabilities of moving, conditional on reaching any period after period 0 and before the terminal period T, are determined by piecing together the solutions to pairs of interrelated difference equations. The precise structure of these equations depend in part upon the relation between the returns to the players when they move simultaneously and their returns when they move alone. In general, the equilibrium distribution functions can be completely characterized only in the case where the return from leading in any period is at least as great as the return from tying in the following period.

We then show that, for a solution to these difference equations to be an equilibrium, it must satisfy a set of terminal conditions. The number of nondegenerate equilibria are then determined by establishing the number of solutions which satisfy these conditions. Normalizing the payoffs so that each player earns 0 in the terminal period, a nondegenerate equilibrium exists in a game with a finite horizon only if the return from leading to both players becomes negative at approximately the same time. When the horizon is infinite, however, the terminal conditions imply essentially no restrictions on the return functions used in most economic applications.

The paper is organized as follows. After introducing our assumptions and establishing some preliminary results in Sections 2 and 3, we begin our characterization of nondegenerate equilibria for specific classes of games. We consider in Section 4 the class of games in which the return from leading in any period for each player is at least as great as his return from tying in the following period. We show that any nondegenerate equilibrium must have one of three possible patterns. Either it is fully mixed, in which case both players move with positive probability conditional on reaching any period before some period t^*; or it is alternating, in which case, conditional on reaching any period before period t^*, one player moves with positive probability if and only if the period is even while the other player moves with positive probability if and only if the period is odd; or it is hybrid, in which case it has a fully mixed pattern up to some period $\tilde{t}$ whereupon it changes to an alternating pattern until some period t^*. In all three cases, once period t^* is reached, both players wait until period T with certainty.

When the horizon is finite, we show that, generically, the terminal conditions for an equilibrium imply the existence of at most one (fully mixed) nondegenerate equilibrium. For symmetric games, this corresponds to a symmetric equilibrium. For a class of (nongeneric) games, the terminal conditions imply not only the existence of a unique fully mixed equilibrium, but also the existence of a one parameter family of alternating equilibria and, for each period $\tilde{t}$ between 0 and t^*, a one parameter family of hybrid equilibria. When the horizon is infinite and the (normalized) return to leading converges to zero, t^* is equal to ∞ and all of these equilibria exist.

In Section 5, we examine the implications for the set of equilibria when the the return from tying in any period t exceeds the return from leading in that period. We show that if the return from tying is sufficiently large, the incentive to wait in order to move together with the other player may eliminate the alternating equilibria of Section 4, and in their place, a class of coordinating equilibria may appear. In these equilibria, both players move with positive probability conditional on reaching some periods and with probability 0 in other periods.

We then turn our attention to the existence and uniqueness of the nondegenerate equilibria for these games. We demonstrate that, if the return to tying is sufficiently high, the difference equations describing a fully mixed equilibrium may be stable. This in turn leads to two possibilities. First, if the horizon is infinite, there may be a continuum of fully mixed equilibria. Second, if the horizon is finite (and the payoffs are not stationary), there may be <u>no</u> fully mixed equilibrium. Finally, we present an example of a finite horizon game with asymmetric payoffs in which there is no nondegenerate equilibrium at all, even though the terminal conditions of Section 3 are satisfied.

One of the implications of the results of Section 3 is that all nondegenerate equilibria are subgame perfect. In Section 6, we turn our attention to the implications of subgame perfection for two <u>degenerate</u> equilibrium outcomes. We show that the terminal conditions which are necessary for the existence of nondegenerate equilibria are both necessary and sufficient for both degenerate equilibrium outcomes to be subgame perfect. If these conditions are not satisfied, there is a period in which the return from leading to one of the players is positive while the return from leading to the other player is negative. In this case, the only outcome which is subgame perfect is for the first player to move immediately.

We conclude with some remarks on the relation between the discrete time and continuous time formulations of these games.

2. <u>The Model</u>

Two players, a and b, must decide when to make a single move in some period t between 0 and T ($0 < T \leq \infty$). Upon reaching any period

$t \in \{t < T: t \in \{0,1,2,\ldots\}\}$, both players must simultaneously decide whether to move or to wait. In what follows, j, k, s, t, t', etc. will refer to nonnegative integers, α to any arbitrary player and β to the other player. The payoffs to the players depend upon which player moves first and the period in which he moves. If player α moves first in period t, he is called the <u>leader</u> and earns a return $A_\alpha(t)$. If player β moves first in period t, player α becomes the <u>follower</u> and earns a return $B_\alpha(t)$. If both players move simultaneously in period t, they are said to tie, and the return to player α is $C_\alpha(t)$. If neither player moves before period T, then player α earns return 0.[5]

The specific class of games we study is an extensive form representation in discrete time of a generalization of the "war of attrition". It is defined by the following assumption.

A1 (a) For $\alpha = a,b$:

(i) $A_\alpha(t) > A_\alpha(t+1)$ for $0 \leq t,\ t+1 < T$;

(ii) $B_\alpha(t+1) > A_\alpha(t)$ for $0 \leq t,\ t+1 < T$;

(iii) $B_\alpha(t) > C_\alpha(t)$ for $t < T$.

(b) If $T = \infty$, then there is a $K < \infty$ such that

$|A_\alpha(t)|, |B_\alpha(t)|, |C_\alpha(t)| > K$ for $0 \leq t < \infty$.

Condition (i) states that the return from leading decreases with time. Condition (ii) states that the return from following in period $t+1$ exceeds the

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[5]The return to player α in period T might reflect his payoff from some continuation game. In this case, we are implicitly assuming that (i) the payoffs have been normalized so that the equilibrium payoff in that game is 0 for each player and (ii) upon reaching period T, both players play their equilibrium strategies. Otherwise, we are simply assuming that payoffs have been normalized so that the return to each player if neither ever moves is 0.

return from leading in period t. Condition (iii) states that the return from tying in period t is less than the return from following in period t. There are no restrictions on the relation between A_α and C_α or on any of the return functions relative to the terminal payoff 0. Together these conditions imply that, in any period, each player prefers to wait for the other player to move, but, if forced to move first, would rather move sooner than later. Condition (b) is required to guarantee that payoffs are well defined in games with an infinite horizon.

The game in extensive form is illustrated in Figure 1. Since the payoffs are determined as soon as one of the players moves, the information sets of each player can be indexed by the period in which a decision is to be made. In each period t, player α must choose between moving M and waiting W. A <u>pure strategy</u> for player α is then a function $m_\alpha:\{0,\ldots,T-1\} \to \{M,W\}$.[6] A <u>behavior strategy</u> for player α is a sequence $\underline{r}_\alpha = \{r_\alpha(t)\}_{t=0}^{T-1}$ of Bernoulli probability distributions over {M,W}, where $r_\alpha(t)$ denotes the probability that player α moves in period t conditional on neither player moving before period t. Finally, for any behavior strategy $\underline{r}_\alpha$ and any period t, we may define $F(t;\underline{r}_\alpha) = 1 - \Pi_{k=0}^{t}(1-r_\alpha(k))$ to be the probability that player α moves in or before period t.[7]

Figure 1 <u>The War of Attrition in Extensive Form</u>

Because of the special structure of the game tree, the payoff to

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[6]We adopt the convention that if $T = \infty$, then $\{0,\ldots,T-1\}$ represents the nonnegative integers.

[7]Note that a distribution function F is associated with a unique $\underline{r}$ only if $F(t) < 1$ for all $t < T$. On the other hand, any distribution function F with support on $\{0,\ldots,T\}$ is generated by some behavior strategy $\underline{r}$.

player α depends only on the combination $(F(\cdot;\underline{r}_a),F(\cdot;\underline{r}_b))$. To ease notation, therefore, we will define the payoff functions directly in terms of the distribution functions, suppressing explicit reference to the underlying behavior strategy. Thus we will express $F(t;\underline{r}_\beta)$ as $F_\beta(t)$ and sometimes refer to F_β as the strategy of player β when in fact we are referring to the underlying family of behavior strategies which generate F_β.

Let $q_\alpha(t) = F_\alpha(t)-F_\alpha(t-1)$ represent the probability that player α moves exactly in period t and let t represent the distribution with $q_\alpha(t) = 1$. Then the payoff to player α from moving in period t, given the strategy F_β, is

$$(2.1)\quad P_\alpha(t,F_\beta) = \textstyle\sum_{j=0}^{t-1} B_\alpha(j)q_\beta(j) + C_\alpha(t)q_\beta(t) + [1-F_\beta(t)]A_\alpha(t).$$

The payoff from waiting until period T is

$$(2.2)\quad P_\alpha(T,F_\beta) = \textstyle\sum_{j=0}^{T-1} q_\beta(j)B_\alpha(j).$$

The expected payoff to player α from the strategy combination (F_a,F_b) is

$$(2.3)\quad P_\alpha(F_a,F_b) = \textstyle\sum_{t=0}^{T} q_\alpha(t)P_\alpha(t,F_\beta).$$

A strategy combination $(\underline{r}_a^*,\underline{r}_b^*)$ is an <u>equilibrium</u> if $P_\alpha(F(\cdot;\underline{r}_\alpha^*),F(\cdot;\underline{r}_\beta^*)) \geq P_\alpha(F(\cdot;\underline{r}_\alpha),F(\cdot;\underline{r}_\beta^*))$ for all strategies $\underline{r}_\alpha$, $\alpha = a,b$ and $\alpha \neq \beta$.

We will also be interested in identifying those strategy combinations which satisfy backwards induction. For any behavior strategy $\underline{r}_\alpha$, let $F(j;t,\underline{r}_\alpha) = 0$ for $j < t$ and $F(j,t;\underline{r}_\alpha) = 1 - \Pi_{k=t}^{j}(1-r_\alpha(k))$ for

$t \leqq j < T$. For $j \geqq t$, $F(j;t,\underline{r}_\alpha)$ is the probability that player α moves in or before period j, conditional on neither player moving before period t. Then a pair of behavior strategies $(\underline{r}_a,\underline{r}_b)$ is a <u>subgame perfect equilibrium</u> if $P_\alpha(F(\cdot;t,\underline{r}_\alpha^*),F(\cdot;t,\underline{r}_\beta^*)) \geqq P_\alpha(F(\cdot;t,\underline{r}_\alpha),F(\cdot;t,\underline{r}_\beta^*))$ for all strategies $\underline{r}_\alpha$, $\alpha = a,b$ and $\alpha \neq \beta$.

3. <u>Characterization of Equilibria: Preliminary Results</u>

In this section we establish some properties of the equilibrium strategy combination for any game which satisfies Assumption A1. We begin by distinguishing between two kinds of equilibria. A <u>degenerate</u> equilibrium is one in which one of the players moves with certainty in period 0. Any other equilibrium is called <u>nondegenerate</u>. We then establish in Lemma 3.2 that if the equilibrium is nondegenerate, there is no period t prior to period T by which either player plans to move with certainty. Since this implies that every information set is reached in a nondegenerate equilibrium, we note that any such equilibrium is necessarily subgame perfect. This result is presented as Theorem 3.1. Finally, we investigate some of the terminal conditions which any nondegenerate equilibrium strategy must satisfy. Defining τ_α to be the earliest period in which $A_\alpha(t) \leqq 0$, we establish that for generic payoffs (i.e. $A_\alpha(\tau_\alpha) \neq 0$), either (a) the horizon is infinite and both players eventually move with probability 1 or (b) there is an interval of periods from τ_α to T in which neither player plans to move. The implications for the support of the strategies when the payoffs are nongeneric are slightly more complicated and are presented in Lemma 3.3.

Throughout this section, $(\underline{r}_a,\underline{r}_b)$ will represent an equilibrium strategy combination and $F_\alpha = F(\cdot,\underline{r}_\alpha)$ the corresponding distribution function for $\alpha = a,b$. For any pair of distributions, (F_a,F_b), define

$$\hat{t} = \hat{t}(r_a,r_b) = \inf\{t: \max(F_a(t),F_b(t)) = 1\}$$

to be either the first period in which one of the players moves with probability 1 or, if no such period exists, period T. If $\hat{t} = 0$, we will say that the equilibrium is <u>degenerate</u>. If $\hat{t} \neq 0$, we will say that the equilibrium is <u>nondegenerate</u>.

<u>Lemma 3.1</u>: For $t < \hat{t}$ and $t+1 < T$, $r_\beta(t) = r_\beta(t+1) = 0$ implies that $r_\alpha(t+1) = 0$.

Suppose the game has reached period $t < \hat{t}$ and the terminal period is at least two periods away. Then Lemma 3.1 says that if player β plans to move with probability 0 in both periods t and t+1, player α never moves in period t+1. The reason is that player α is the leader for sure if he moves in either period t or period t+1. But, since the payoff from leading is declining over time, it follows that he prefers to move in period t rather than period t+1.

<u>Lemma 3.2</u>: If $\hat{t} > 0$, then $\hat{t} = T$.

The argument is as follows. Suppose $0 < \hat{t} < T$. Then one of the players, say player β, moves with probability 1 upon reaching period $\hat{t}$. Now consider the payoff to player α. Conditional on reaching period $\hat{t}$, he earns a return $C_\alpha(\hat{t})$ if he moves in period $\hat{t}$ and a return $B_\alpha(\hat{t})$ if he waits until period $\hat{t}+1$. Since Assumption A1 implies that $B_\alpha(\hat{t}) > C_\alpha(\hat{t})$, it follows that he does better by waiting until period $\hat{t}+1$. Therefore, $r_\alpha(\hat{t}) = 0$.

Likewise, upon reaching period $\hat{t}-1$, his expected return is $r_\beta(\hat{t}-1)C_\alpha(\hat{t}-1) + (1-r_\beta(\hat{t}-1))A_\alpha(\hat{t}-1)$ if he moves in period $\hat{t}-1$ and $r_\beta(\hat{t}-1)B_\alpha(\hat{t}-1) + (1-r_\beta(\hat{t}-1))B_\alpha(\hat{t})$ if he waits until period $\hat{t}+1$. But since Assumption A1 implies that $C_\alpha(\hat{t}-1) < B_\alpha(\hat{t}-1)$ and $A_\alpha(\hat{t}-1) < B_\alpha(\hat{t})$, it follows that player α also does better by waiting until period $\hat{t}+1$ than by moving in period $\hat{t}-1$. Therefore, $r_\alpha(\hat{t}-1) = 0$. But if $r_\alpha(\hat{t}-1) = r_\alpha(\hat{t}) = 0$, then Lemma 3.1 implies that, conditional on reaching period $\hat{t}$, player β moves with probability 0, contradicting our assumption that $r_\beta(\hat{t}) = 1$.

The importance of Lemma 3.2 is that it establishes that, in any nondegenerate equilibrium, there is a positive probability that the game does not end in any finite period t prior to T. This means that there is a positive probability that the equilibrium path reaches any information set. Consequently, any strategy which is optimal starting in period 0 is also optimal starting in any period t greater than 0. Therefore, we may state

<u>Theorem 3.1</u>: If $(\underline{r}_a,\underline{r}_b)$ is a nondegenerate equilibrium, then it is subgame perfect.

We will defer our analysis of subgame perfection for degenerate equilibria until Section 6.

Define

$$t^* = t^*(r_a,r_b) \equiv \inf\{\{t < T: r_a(j) = r_b(j)\} = 0,\ t \leq j < T\} \cup T\}$$

to be the first period such that, upon reaching that period, both players wait until period T with probability 1. For each player α, define

$$\tau_\alpha = \inf\{\{t < T: A_\alpha(t) \leqq 0\} \cup T\}$$

to be either the earliest period in which the return to player α from leading is nonpositive or, if the return from leading is always positive, period T.

The next two lemmata establish the relation between t^* and τ_α.

<u>Lemma 3.3</u>: Suppose $\hat{t} = T$. Then (i) $q_\beta(T) > 0$ implies $t^* \geqq \tau_\alpha \geqq t^*-1$[8]; (ii) $q_\beta(T) = 0$ implies $t^* = \infty$; (iii) $r_\alpha(t^*-1) = 0$ implies $A_\beta(t^*-1) = 0$ and $\tau_\alpha - 1 \leqq \tau_\beta \leqq \tau_\alpha$.

The argument for condition (i) goes as follows. Suppose there is a positive probability that player β waits until period T. Then Lemma 3.1 combined with the definition of t^* implies that either player α moves with positive probability in period t^*-1 or player β moves with positive probability in period t^*-1 and player α moves with positive probability in period t^*-2. If player α moves in period t^*-1, then his return from leading most be nonnegative. If he moves in period t^*-2 and player β moves with positive probability in period t^*-1, then the return from leading for player α must be strictly positive in period t^*-2. In either case, we may conclude that $\tau_\alpha \geqq t^*-1$. On the other hand, suppose that the return from leading in period t^* is strictly positive for player α, that is, $\tau_\alpha > t^*$. Then, conditional on reaching period t^*, player β waits until period T with probability 1. But in this case, the optimal response for player α is to move in period t^*, contradicting the definition. This establishes that $\tau_\alpha \leqq t^*$.

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[8]If $t^* = \infty$, we adopt the convention that $t^*-1 = \infty$.

Condition (ii) is an immediate implication of Lemma 3.2. To establish condition (iii), suppose, conditional on reaching period t^*-1, player β moves with positive probability. Then he must be indifferent between leading in period t^*-1 and earning a return of 0 in period T. Consequently, if player α moves with probability 0 in period t^*-1, $A_\beta(t^*-1)$ must be 0. By definition, then, $t^*-1 = \tau_\beta$, which combined with condition (i), implies that $\tau_\alpha - 1 \leq \tau_\beta \leq \tau_\alpha$.

As we shall see, the characterization of the nondegenerate equilibria may be sensitive to whether or not the return from leading for player α in period τ_α is exactly equal to 0. We will say that the payoff of player α is <u>generic</u> if $A_\alpha(t) \neq 0$, $t \geq 0$.[9] In this case Lemma 3.3 can be strengthened to read:

<u>Lemma 3.4</u>: Suppose $\hat{t} = T$. If the payoffs to both players are generic, then (i) $q_\beta(T) > 0$ implies $\tau_\alpha = t^*$ and (ii) $q_\beta(T) = 0$ implies $t^* = \infty$.

Lemma 3.4 follows from the observation that if the payoffs are generic for both players, then whoever plans to move in period t^*-1 must earn a strictly positive return from leading in period t^*-1. But if the game is to continue with some probability beyond period t^*-1 (as required by Lemma 3.2), then the other player must also plan to move in period t^*-1. The genericity of payoffs then implies that his return from leading in period t^*-1 is strictly positive as well. We conclude therefore, that $\tau_\alpha \geq t^*$, which combined with Lemma 3.3 establishes the result.

Lemmata 3.3 and 3.4 also imply the following necessary conditions for

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[9]Note that Assumption A1 implies that the class of generic payoffs includes the case where $\lim_{t\to\infty} A_\alpha(t) = 0$.

the existence of a nondegenerate equilibrium.

Theorem 3.2: A nondegenerate equilibrium exists only if $\tau_a - 1 \leq \tau_b \leq \tau_a + 1$. If payoffs are generic, a nondegenerate equilibrium exists only if $\tau_a = \tau_b$.

Note that Theorem 3.2 does not imply that these conditions are sufficient for the existence of a nondegenerate equilibrium. As we shall see, the set of conditions under which a nondegenerate equilibrium exists and its characteristics depend critically on the relation between the return from moving simultaneously and the return from leading. We can state the following general result, however, which indicates some of the possibilities for the existence of a continuum of nondegenerate equilibria.

Theorem 3.3: Suppose $(\underline{r}_a, \underline{r}_b)$ is an equilibrium with $r_\beta(0) = 0$ and $r_\alpha(0) > 0$. Then $(\underline{r}^*_a, \underline{r}^*_b)$ is also an equilibrium where (i) $r_\alpha(0) \leq r^*_\alpha(0) \leq 1$, (ii) $r^*_\alpha(t) = r_\alpha(t)$ for $t > t^*$, and (iii) $\underline{r}^*_\beta = \underline{r}_\beta$.

Theorem 3.3 states that if there is an equilibrium with the property that in period 0 player α moves with positive probability $r_\alpha(0)$ and player β moves with probability 0, then increasing the value of $r_\alpha(0)$ and leaving the remainder of the strategy combination unchanged also results in an equilibrium. The reason is simply that an increase in $r_\alpha(0)$ does not affect the optimal response of player β after period 0, and it only reduces the expected return to player β (relative to his return from waiting) from moving in period 0.

We turn now to a detailed analysis of games where the return from moving simultaneously never exceeds the return from leading.

4. Nondegenerate Equilibria: $C_\alpha \leq A_\alpha$

To motivate our analysis of this case, we mention briefly some variations on a model used by Hendricks (1984) to study the problem of the timing of oil exploration. Two firms are assumed to own leases in an area that contains an unknown amount of oil and gas. Their uncertainty about the area's reserves is resolved by drilling exploratory wells, the results of which are public information. Each firm has an incentive to wait and let the other firm incur the costs of finding out whether the area contains sufficient reserves to be worth developing. On the other hand, if a firm waits and ends up drilling anyway because the other firm also waits, it incurs a time cost due to the delay in realizing the expected returns to drilling. Since a firm benefits from the actions of the other firm only if the other firm drills _before_ the first firm commits itself to drilling, it seems reasonable to suppose that the payoff from drilling simultaneously is no greater than the payoff from moving first.

The structure of the exploration game is common to many investment situations in which economic agents have an incentive to delay their commitment of capital and to free ride on the information generated by the investments of other agents. In some of these examples, the payoff to a firm from moving simultaneously with the other firm is likely to be less than its payoff from moving first. For example, suppose two firms market a new product at the same time and demand turns out to be large enough to sustain only one firm. The firms are then forced to engage in a costly war of attrition to determine which firm should exit (see Fudenberg and Tirole (1984) for an analysis of such a model). _Both_ firms would have been better off if one of them had waited, for then the firm which followed could have observed the state of demand and stayed out.

We consider, therefore, the following assumption, which we assume is satisfied throughout this section.

A2 $\quad A_\alpha(t) \geq C_\alpha(t+1) \quad \text{for} \quad t+1 < T.$

Assumption A2 requires that the return to player α from moving alone in period t be greater than or equal to his return from moving together with player β in period t+1.

4.1 The Three Types of Equilibria

The primary role of Assumption A2 is to eliminate any incentive for a player to wait simply to increase his chances of moving simultaneously with the other player. In particular, if player β moves with probability 0 in period t, then player α always prefers to move in period t rather than period t+1. Consequently, we may strengthen Lemma 3.1 to read:

Lemma 4.1: Suppose $\hat{t} = T$. Then if $t+1 < T$, $r_\beta(t) = 0$ implies $r_\alpha(t+1) = 0$.

Define

$$\tilde{t} = \tilde{t}(r_a,r_b) \equiv \inf\{\{t < T: r_a(t)r_b(t) = 0\} \cup T\}$$

to be either the first period such that, upon reaching that period, at least one of the players moves with probability 0 or, if no such period exists, period T. From Lemma 4.1, we may immediately infer

<u>Lemma 4.2</u>: If $\hat{t} = T$, then:

(i) $r_a(t)r_b(t) > 0$ for $t < \tilde{t}$;

(ii) $r_\alpha(t)r_\alpha(t+1) = 0$ and $(1-r_a(t))(1-r_b(t)) < 1$ for $\tilde{t} \leq t < t^*$;

(iii) $q_\alpha(T) = 1 - F_\alpha(t^*)$.

Lemma 4.2 establishes that any nondegenerate equilibrium has the following properties. Both players move with positive probability conditional on reaching any period up to period $\tilde{t}$. Beginning in period $\tilde{t}$ up to t^*, only one of the players moves with positive probability in any period -- one of the players moves with positive probability in the even periods and the other player moves with positive probability in the odd periods. From period t^* up to (but not including) period T, neither player ever moves. If $\tilde{t} = t^*$, we will say that the equilibrium is <u>fully mixed</u>. In this case, both players move with positive probability in every period t up to period t^*, whereupon both players wait until period T with probability 1. If $\tilde{t} = 0$, we will say that the equilibrium is <u>alternating</u>. In this case, the players alternately move with positive probability in every other period up to period t^*, whereupon both players again wait with probability 1 until period T. If $0 < \tilde{t} < t^*$, we will say that the equilibrium is <u>hybrid</u>. In this case, the pattern is fully mixed up to period $\tilde{t}$, alternating from period $\tilde{t}$ to t^*, whereupon both players again wait until period T with probability 1. Finally, we note that if $t^* = T$, then in any period up to period T, there is positive probability that some player moves. (There may also be a positive probability that one or both of the players wait until period T.) The three possible equilibrium patterns are illustrated in Figure 2 for the case where $t^* < T$.

Figure 2. Possible Patterns For Nondegenerate Equilibria

4.2 Properties of the Equilibrium Strategies from 0 to t^*

We turn now to a precise characterization of the strategies in each of the three types of equilibria from period 0 up to period t^*. We show that during those periods in which the strategies are fully mixed (0 to $\tilde{t}-1$), the conditional probabilities are the solution to a first order difference equation, whereas, in those periods t where the strategies are alternating ($\tilde{t}+1$ to t^*-1), the conditional probabilities in period t are uniquely determined by the return functions at t-1, t, and t+1. In the transition periods between fully mixed and alternating strategies (periods $\tilde{t}-1$ and $\tilde{t}$), however, the restrictions implied by equilibrium generally permit a one parameter family of conditional probabilities $(r_\alpha(\tilde{t}-1), r_\alpha(\tilde{t}))$.

Consider first the properties of an equilibrium strategy in the periods where the strategies are fully mixed. Since player α must earn the same expected return from moving in any period $t < \hat{t}$, we obtain immediately the following restriction on the strategy of player β.

Lemma 4.3: The sequence $\{r_\beta(t)\}_{t=0}^{\tilde{t}-2}$ satisfies $r_\beta(t) \in [0,1]$ for $t \leq \tilde{t}-2$ and

$$(4.1)\quad 0 = [B_\alpha(t)-C_\alpha(t)]r_\beta(t) + [C_\alpha(t+1)-A_\alpha(t+1)](1-r_\beta(t))r_\beta(t+1) + [A_\alpha(t+1)-A_\alpha(t)](1-r_\beta(t))$$

Note that (4.1) is bilinear in $r_\beta(t)$ and $r_\beta(t+1)$. It follows, therefore, that $r_\beta(t+1)$ implies a unique $r_\beta(t)$. Moreover, it can be readily verified that $r_\beta(t+1) \in [0,1]$ implies that $r_\beta(t) \in [0,1]$. It then follows by induction that, for any value of $r_\beta(t) \in [0,1]$, there is a unique sequence $\{r_\beta(j)\}_{j=0}^{t}$ satisfying equation (4.1) with $r_\beta(j) \in [0,1]$ for all $j = 0,\ldots,t$. This result is summarized as Lemma 4.4 below.

<u>Lemma 4.4</u>: For any $r_\beta(t) \in [0,1]$, $t < T$, there is a unique sequence $\{r_\beta(j)\}_{j=0}^{t}$ satisfying (4.1) with $r_\beta(j) \in [0,1]$ for all $j < t$.

When $\tilde{t} = \infty$, then equation (4.1) must be satisfied for all t. The existence of a solution with $r_\alpha(t) \in [0,1]$ follows from standard arguments. A more complicated argument is required to show that such a sequence is unique.[10]

Consider next the restrictions implied by the conditions for equilibrium on the conditional probabilities of moving in the periods $\tilde{t}-1$ and $\tilde{t}$ when $\tilde{t}$ is finite. In period $\tilde{t}$ at least one of the players, say player α, plans to move with probability 0. There are essentially three cases to consider. First, if $\tilde{t} = 0$, then we are considering an alternating equilibrium and the only restriction on $r_\beta(0)$ is that it be large enough so that player α prefers to wait until period 1 rather than to move in period 0. If $\tilde{t} > 0$ but $\tilde{t}+1 < t^*$, then we are considering a hybrid equilibrium. In this case $r_\alpha(\tilde{t}-1)$ is uniquely determined by the requirement that player β be indifferent between moving in period $\tilde{t}$ and moving in period $\tilde{t}-1$. However, since player α moves with probability 0 in period $\tilde{t}$, the restrictions on

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[10]The problem arises because there is no terminal condition. Consequently, in order to establish the uniqueness of a solution satisfying Lemma 4.3, it is necessary to establish that the difference equation (4.1) is unstable. See the proof of Theorem 4.3 below.

$r_\beta(\tilde{t}-1)$ and $r_\beta(\tilde{t})$ may be expressed as a <u>single</u> equation plus an inequality. These probabilities must equate the return to player α from moving in periods $\tilde{t}-1$ and $\tilde{t}+1$ subject to the requirement that the return from moving in period $\tilde{t}$ is no greater than his return from moving in period $\tilde{t}-1$. Finally, if $\tilde{t}+1 \geq t^*$, then we are considering a equilibrium which is essentially fully mixed, in which case the conditions on the conditional probabilities at $\hat{t}$ must be considered jointly with the implications of equilibrium for the conditions at t^*. Substituting these restrictions for the first two cases into the payoff functions then yields the following lemma.

<u>Lemma 4.5</u>: Suppose $\tilde{t}+1 < t^*$ and $r_\alpha(\tilde{t}) = 0$. (a) If $\tilde{t} = 0$, then

$$(4.2) \quad r_\beta(0) \geq [A_\alpha(0)-A_\alpha(1)]/[B_\alpha(0)-C_\alpha(0)+A_\alpha(0)-A_\alpha(1)].$$

(b) If $\tilde{t} > 0$, then

$$(4.3) \quad r_\alpha(\tilde{t}-1) = [A_\beta(\tilde{t}-1)-A_\beta(\tilde{t})]/[B_\beta(\tilde{t}-1)-C_\beta(\tilde{t}-1)+A_\beta(\tilde{t}-1)-A_\beta(\tilde{t})]$$

$$(4.4) \quad \frac{[A_\alpha(\tilde{t})-A_\alpha(\tilde{t}+1)]}{[B_\alpha(\tilde{t})-C_\alpha(\tilde{t})+A_\alpha(\tilde{t})-A_\alpha(\tilde{t}+1)]} \leq r_\beta(\tilde{t}) < \frac{[A_\alpha(\tilde{t}-1)-A_\alpha(\tilde{t}+1)]}{[B_\alpha(\tilde{t})-A_\alpha(\tilde{t}+1)]}$$

$$(4.5) \quad r_\beta(\tilde{t}-1) = \frac{[A_\alpha(\tilde{t}-1)-A_\alpha(\tilde{t}+1)] - r_\beta(\tilde{t})[B_\alpha(\tilde{t})-A_\alpha(\tilde{t}+1)]}{[B_\alpha(\tilde{t}-1)-C_\alpha(\tilde{t}-1)+A_\alpha(\tilde{t}-1)-A_\alpha(\tilde{t}+1)] - r_\beta(\tilde{t})[B_\alpha(\tilde{t})-A_\alpha(\tilde{t}+1)]}.$$

Finally, consider the periods between $\tilde{t}$ and t^*. Conditional on reaching these periods, each player alternately moves with positive probability. In those periods when $r_\alpha(t) = 0$, therefore, $r_\beta(t)$ must be chosen so that player α is indifferent between moving in periods t-1 and t+1, given that player β moves with probability 0 in those periods. Consequently, we obtain:

<u>Lemma 4.6</u>: Suppose $\tilde{t} < t$ and $t+1 < t^*$. Then $r_\alpha(t) = 0$ implies

(4.6) $\quad r_\beta(t) = [A_\alpha(t-1)-A_\alpha(t+1)]/[B_\alpha(t)-A_\alpha(t+1)].$

Taken together, Lemmata 4.3, 4.5, and 4.6 summarize the equilibrium restrictions on the behavior strategies in those periods before period t^*-1. Corresponding to each value of t^*, they imply that there is at most one fully mixed equilibrium, a one parameter family of alternating equilibria, and, for each value of $\tilde{t}$, a one parameter family of hybrid equilibria in which the strategies of the players begin to alternate in period $\tilde{t}$. In order to complete the characterization of the set of equilibria, we need to consider the implications for the strategies from period t^*-2 to period T.

4.3 Equilibrium When T (t^*) Is Finite

In this section we consider the set of equilibria in which t^* is finite. We show that, generically, there is a unique nondegenerate equilibrium, which is fully mixed. For a class of "nongeneric" payoffs, however, we show that in addition to a fully mixed equilibrium, there is also a one parameter family of alternating equilibria and, for each $\tilde{t} \leq t^*$, a one parameter family of hybrid equilibria. While it is true that these games constitute a set of measure 0 in an appropriately parameterized space of games, we have chosen to treat this case in detail because, in games with incomplete information, the equilibrium frequently has an alternating structure (see Hendricks (1984)). In these cases, the analysis of the terminal conditions corresponds to our analysis of the nongeneric case below. We consider the generic case first.

4.3.1 <u>Generic Payoffs</u>

If $t^* < \infty$, Lemma 3.2 implies that $q_\alpha(T) > 0$ for $\alpha = a,b$. It then follows from Lemma 3.4 that $t^* = \tau_a = \tau_b$ whenever payoffs are generic. By definition, some player α must plan to move with positive probability in period t^*-1. But this implies that player α be indifferent between moving in period t^*-1 and waiting until period T, which implies in turn that $r_\beta(t^*-1)$ satisfy

$$(4.7) \quad r_\beta(t^*-1) = A_\alpha(t^*-1)/[B_\alpha(t^*-1)-C_\alpha(t^*-1)+A_\alpha(t^*-1)].$$

Then, since $A_\alpha(t^*-1) > 0$, we may conclude that $r_\beta(t^*-1) > 0$. An identical argument then establishes that $r_\alpha(t^*-1)$ must satisfy equation (4.7) and hence that $r_\alpha(t^*-1) > 0$. Finally, since both $r_a(t^*-1)$ and $r_b(t^*-1)$ are strictly positive, Lemma 4.1 implies that the equilibrium must be fully mixed--i.e. $\tilde{t} = t^*$. The uniqueness of the equilibrium then follows from Lemma 4.4 and the uniqueness of the solution to (4.7). It is straightforward to check that such a strategy combination is in fact an equilibrium. Our conclusions may be summarized as:

Theorem 4.1: Suppose payoffs are generic.

(a) There is a nondegenerate equilibrium $(\underline{r}_a,\underline{r}_b)$ with $t^* < \infty$ if and only if $\tau_a = \tau_b < \infty$.

(b) (i) It is unique and fully mixed with $\tilde{t} = t^* = \tau_a = \tau_b$.

(ii) $\{(r_a(t),r_b(t))\}_{t=0}^{t^*-1}$ must satisfy equations (4.1) and (4.7).

If horizon is finite, then by definition t^* must be finite. We therefore obtain the following corollary to Theorem 4.1.

Corollary 4.1: Suppose payoffs are generic and $T < \infty$. Then there is a nondegenerate equilibrium $(\underline{r}_a,\underline{r}_b)$ if and only if $\tau_a = \tau_b$. It is the unique fully mixed equilibrium characterized by part (b) of Theorem 4.1.

4.3.2 Nongeneric Payoffs

We turn next to games with nongeneric payoffs. Since we are supposing that t^* is finite, it follows from Lemma 3.2 that both players wait until period T with positive probability. It then follows from Lemma 3.3 that t^* is equal to either τ_α or $\tau_\alpha+1$.

We consider first the possibility for a fully mixed equilibrium. Suppose that $A_b(\tau_b) = 0$. If $t^* = \tau_b+1$, then, since $A_b(t^*-1) = 0$ and $r_a(t^*-1) > 0$, player b prefers waiting until period T to moving in period t^*-1, contradicting the definition of a fully mixed equilibrium. We conclude, therefore, that $t^* = \tau_b$, which in turn implies, given Lemma 3.3, that τ_a is equal to either τ_b or τ_b-1. But if $\tau_a = \tau_b-1$, then $\tau_a = t^*-1$ in which case player a prefers waiting until period T to moving in period t^*-1, again contradicting the definition of a fully mixed equilibrium. We conclude, therefore, that $t^* = \tau_a = \tau_b$.

As in the case of generic payoffs, equating the payoff from moving in period t^*-1 to the payoff from waiting until period T implies that the probability with which each player moves in period t^*-1 satisfies equation (4.7). If $\tau_a = \tau_b$, then, following the argument of Theorem 4.1, Lemma 4.4 implies the existence of a unique fully mixed equilibrium. We summarize these conclusions in Theorem 4.2.

<u>Theorem 4.2</u>: Suppose $A_\beta(\tau_\beta) = 0$. Then there is a unique fully mixed equilibrium with $\hat{t} = t^* < \infty$ if and only if $\tau_a = \tau_b$. The equilibrium is characterized by the conditions of Theorem 4.1.

Consider next the possibility for alternating and hybrid equilibria. In these equilibria, there is a single last mover, say player β, who moves with positive probability conditional on reaching period t^*-1. In order for such an equilibrium to exist, Lemma 3.3(iii) requires that $A_\beta(t^*-1) = 0$ and $\tau_\alpha - 1 \leqq \tau_\beta \leqq \tau_\alpha$. To establish the sufficiency of these conditions, we need to check that they are consistent with the restriction on the behavior strategies that player α moves with probability 0 in period t^*-1 and that, conditional on reaching period t^*-2, he is indifferent between moving immediately and waiting until period T.

There are two cases to consider. If $\hat{t} < t^*-1$, then, by definition, $r_\beta(t^*-2) = 0$ and $r_\alpha(t^*-2) > 0$. Conditional on reaching period t^*-2, therefore, player α is indifferent between moving immediately and waiting until period T only if

$$(4.8) \quad r_\beta(t^*-1) = A_\alpha(t^*-2)/B_\alpha(t^*-1).$$

If $\tilde{t} = t^*-1$, then, by definition, both $r_\beta(t^*-2)$ and $r_\alpha(t^*-2)$ are positive. Then, conditional on reaching period t^*-2, player α is indifferent between moving immediately and waiting until period T only if

$$(4.9) \quad r_\beta(t^*-2) = \frac{A_\alpha(t^*-2) - r_\beta(t^*-1)B_\alpha(t^*-1)}{[B_\alpha(t^*-2)-C_\alpha(t^*-2)+A_\alpha(t^*-2)] - r_\beta(t^*-1)B_\alpha(t^*-1)} .$$

The restrictions that $r_\beta(t^*-2) \geqq 0$ and that player α prefers not to move in period t^*-1 then imply

$$(4.10) \quad \frac{A_\alpha(t^*-1)}{[B_\alpha(t^*-1)-C_\alpha(t^*-1)+A_\alpha(t^*-1)]} \leqq r_\beta(t^*-1) \leqq \frac{A_\alpha(t^*-2)}{B_\alpha(t^*-1)} .$$

Relations (4.8) to (4.10) together with the relations of Lemmata 4.5 and 4.6 are consistent with well defined behavior strategies if and only if $\tau_\alpha-1 \leqq \tau_\beta \leqq \tau_\alpha$. It is easily verified that any such strategy pair constitutes a pair of best responses. Therefore, we may state

<u>Theorem 4.3</u>: There is a nondegenerate equilibrium with $t^* < \infty$ and $r_\alpha(t^*-1) = 0$ if and only if $A_\beta(\tau_\beta) = 0$ and $\tau_\alpha-1 \leqq \tau_\beta \leqq \tau_\alpha$.[11]

Whenever there is a nondegenerate equilibrium with a single last mover, an argument similar to the proof of Theorem 3.4 implies that there is in fact a continuum of alternating and hybrid equilibria. However, the roles

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[11]Although Theorem 3.1 establishes that these equilibria are subgame perfect, only fully mixed equilibria are "trembling hand" perfect in games with complete information. In general, a nondegenerate equilibrium is trembling hand perfect only if $r_\alpha(t) = 0$ whenever $A_\alpha(t) \leqq 0$.

of α and β in these equilibria may not be interchangeable. Suppose, for example, that τ_b is even and $A_b(\tau_b) = 0$. Then $A_a(\tau_a) \neq 0$ implies that in any alternating or hybrid equilibria, player b must be the last player to move with positive probability. This requirement rules out equilibria in which $\tilde{t}$ is odd and $r_b(\tilde{t}) > 0$ as well as equilibria in which $\tilde{t}$ is even and $r_a(\tilde{t}) > 0$. It is only when the payoffs to both players are nongeneric and $\tau_a = \tau_b$ that all of the possible alternating and hybrid patterns described in Section 4.2 are consistent with equilibria.

We conclude our characterization of the set of nondegenerate equilibria with a finite value of t^* in Theorems 4.4 and 4.5. Theorem 4.4 describes the set of equilibria when the payoffs of only one player is nongeneric. Theorem 4.5 describes the set of equilibria when both players have nongeneric payoffs.

<u>Theorem 4.4</u>: Suppose $A_\beta(\tau_\beta) = 0$ and $A_\alpha(\tau_\alpha) \neq 0$. If $r_\alpha(t^*-1) = 0$, then $t^* = \tau_\beta+1$. For τ_β even (odd) the set of nondegenerate equilibria with $r_\alpha(t^*-1) = 0$ may be indexed as follows:

(i) For each value of $r_\beta(0)$ ($r_\alpha(0)$) satisfying relation (4.2), there is a unique alternating equilibrium with $\tilde{t} = 0$;

(ii) For each even (odd) $\tilde{t}$, $0 < \tilde{t} < t^*-1$, and each value of $r_\beta(\tilde{t})$ ($r_\alpha(\tilde{t})$) satisfying relation (4.4) and for $\tilde{t} = t^*-1$ and each value of $r_\beta(\tilde{t})$ satisfying relation (4.10), there is a unique hybrid equilibrium;

(iii) For each odd (even) $\tilde{t}$, $0 < \tilde{t} < t^*-1$, and $r_\alpha(\tilde{t})$ ($r_\beta(\tilde{t})$) satisfying relation (4.4), there is a unique hybrid equilibrium.

<u>Theorem 4.5</u>: Suppose $A_\beta(\tau_\beta) = A_\alpha(\tau_\alpha) = 0$.

(i) If $\tau_a = \tau_b = \tau$, then the set of nondegenerate equilibria such that $r_a(t^*-1)r_b(t^*-1) = 0$ may be indexed as follows:

(a) For each value of $r_a(0)$ or $r_b(0)$ satisfying relation 4.2, there is a unique alternating equilibrium with $\tilde{t} = 0$.

(b) For each $\tilde{t}$, $0 < \tilde{t} < t^*-1$, and each value $r_a(\tilde{t})$ or $r_b(\tilde{t})$ satisfying relation (4.4) and for $\tilde{t} = t^*-1$ and each value $r_a(\tilde{t})$ or $r_b(\tilde{t})$ satisfying relation (4.10), there is a unique hybrid equilibrium.

(ii) If $\tau_\alpha = \tau_\beta-1$, then $t^* = \tau_\beta$ and $r_\beta(t^*-1) = 0$. The set of equilibria are then defined by conditions (i) to (iii) of Theorem 4.4.

4.4 <u>Equilibrium When $T(t^*)$ Is Finite</u>

We examine next the possibility of nondegenerate equilibria with $t^* = \infty$. The analysis of this case differs from the foregoing analysis in two respects. First, when $t^* = \infty$, there is no "last" period in which the players move. Consequently, the argument which rules out the alternating and hybrid equilibria is no longer valid. This implies in turn that it is no longer directly relevant whether or not the payoffs are generic. Second, there is a possibility that the requirements of Lemmata 4.3, 4.5 and 4.6 imply that a player never waits forever when the horizon is infinite. When this condition is satisfied, the payoffs at the end of the game become irrelevant and all of the strategy pairs consistent with the restrictions of Lemmata 4.3 to 4.6 are equilibria. Otherwise, if player β does plan to wait until the end of the game with positive probability, then the payoff to player α from leading must converge to 0 as time approaches infinity. These conclusions are summarized as Lemma 4.7.

<u>Lemma 4.7</u>: A pair of strategies $(\underline{r}_a, \underline{r}_b)$ with $t^* = \infty$ is a nondegenerate equilibrium if and only if Lemmata 4.3, 4.5 and 4.6 are satisfied and for each α, $q_\alpha(\infty) > 0$ implies either (i) $\lim_{t\to\infty} A_\alpha(t) = 0$ or (ii) $q_\beta(\infty) = 0$ (for $\beta \neq \alpha$).

Although one can construct examples where $\lim_{t\to\infty} A_\alpha(t) \neq 0$ and the conditions of Lemmata 4.2 or 4.6 imply that $q_\beta(\infty) > 0$, it is difficult to find an interesting economic interpretation for these examples. So long as the length of the interval of "real" time between periods is constant and we assume any kind of discounting, the payoff from any action taken far enough into the future must be arbitrarily close to zero. Consequently, the payoff to player α from moving first further and further into the future must converge to his payoff when neither player ever moves. It is only in the case where the interval of "real" time between periods becomes arbitrarily small as the terminal period is reached that it is plausible to assume that the limit of $A_\alpha(t)$ may differ from 0. In this case, however, it may be more appropriate to model the problem in continuous time.

In light of these remarks, we will confine our attention to the case where the following assumption is satisfied.

A3 $\quad \lim_{t\to\infty} A_\alpha(t) = 0.$

An analysis of the class of equilibria when this condition is not satisfied is tedious but essentially follows the arguments of Theorems 4.1 through 4.5.

<u>Theorem 4.6</u>: Suppose that $T = \infty$ and $\lim_{t\to\infty} A_\alpha(t) = 0$ for $\alpha = a,b$. The set of nondegenerate equilibria can be indexed as follows:

(a) For each value of $r_a(0)$ or $r_b(0)$ satisfying relation (4.2), there is a unique (alternating) equilibrium with $\tilde{t} = 0$.

(b) For each $\tilde{t} > 0$ and each $r_a(\tilde{t})$ or $r_b(\tilde{t})$ satisfying relation (4.4), there is a unique (hybrid) equilibrium.

(c) There is a unique (fully mixed) equilibrium with $\tilde{t} = \infty$ determined by equation (4.1).

Theorem 4.6 essentially states that all of the alternating, hybrid, and fully mixed strategy combinations consistent with the restrictions of Section 4.2 are nondegenerate equilibria. As in the finite horizon case with nongeneric payoffs, the value of the conditional probability of moving for the player who moves in the first period of the alternating phase uniquely determines the equilibrium for all of the alternating and hybrid equilibria. The existence of a unique fully mixed equilibrium is equivalent to the existence of a unique solution to equation (4.1) for which $r_\alpha(t)$ stays between 0 and 1 for all t. Given Lemma 4.4, the existence of such a solution follows from standard arguments. The argument for the uniqueness of such a solution is more subtle. The details are presented in the Appendix.

5. <u>Some Examples</u>: $C_\alpha > A_\alpha$

The characterization of the nondegenerate equilibria in the previous section depends critically on the assumption that a player always receives a higher return from moving alone in period t than moving together with the other player in period t+1. There are, however, a number of interesting economic problems where this assumption may be violated. Consider, for example, a model of bargaining in which the issues being negotiated are in some sense indivisible (i.e. the free agency rule in sports). Suppose two parties have a disagreement on such an issue and each party hires an agent to bargain on its behalf. At the beginning of each round of the negotiations, each party instructs its agent either to maintain the party's bargaining position or to concede it. If both parties concede their positions in the same round, then the agents are assumed to bargain to a compromise settlement in which each party is better off than it would have been if it had conceded unilaterally but worse off than it would have been if it had maintained its position. For such a problem, it would be appropriate to replace assumption A2 of Section 4 with the assumption that $C_\alpha(t+1) > A_\alpha(t)$ for all $t+1 < T$.[12]

In this section, we explore some of the implications for the class of nondegenerate equilibria in the case where the return from moving simultaneously exceeds the return from leading. Even when we restrict attention to symmetric games with return functions which decrease exponentially with time, a complete characterization of the class of equilibria appears quite cumbersome. Consequently, we confine most of our analysis to establishing the conditions under which certain patterns of the equilibrium strategies emerge which are not possible under Assumption A2.

Our main results may be summarized as follows. Suppose the return from moving simultaneously is relatively closer to the return from following than it is to the return from leading. Then there are no alternating equilibria. Instead, a class of <u>coordinating</u> equilibria emerges in which both players move with positive probability in some periods and zero probability in

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[12]This assumption is commonly made in the discrete time formulations of animal conflicts in the biology literature (see Bishop and Cannings (1979) and Hammerstein and Parker (1982).

other periods. Second, if the return functions decline exponentially, the difference equation which defines the fully mixed equilibrium may be stable. If the horizon is infinite, this implies that the fully mixed equilibrium may not be unique. If the horizon is finite or the return functions do not decline exponentially throughout the game, it implies that there may be no fully mixed equilibrium. Finally, in an asymmetric game in which one player has a high and the other player a relatively low return from moving simultaneously, these results imply that there may be no nondegenerate equilibrium even if $\tau_a = \tau_b$.

5.1 The Existence of Alternating and Coordinating Equilibria

When $C_\alpha(t+1)$ is less than or equal to $A_\alpha(t)$, Lemma 4.1 requires that any period in which one of the players moves with probability 0 must be followed by a period in which the other player moves with probability 0. In contrast, when we allow $C_\alpha(t+1)$ to exceed $A_\alpha(t)$, a player may have an incentive to wait in order increase his chance of moving together with the other player. As a consequence, there may be periods in which both players move with probability 0 followed by periods in which there is a positive probability that both players move together. Furthermore, the incentive to wait in order to avoid moving alone may be sufficiently strong to actually eliminate the alternating equilibria.

To illustrate this point, consider a game with an infinite horizon in which all of the the return functions decrease exponentially with time. That is, $A_\alpha(t) = A\delta^t$, $B_\alpha(t) = B\delta^t$, and $C_\alpha(t) = C\delta^t$, where $0 < \delta < 1$ and $0 < A < C < B$. Let $(\underline{r}_a,\underline{r}_b)$ be an alternating equilibrium. Then for every period $t > 0$ in which player α chooses $r_\alpha(t) = 0$, both $r_\alpha(t-1)$ and $r_\alpha(t+1)$ must be positive so the value of $r_\beta(t)$ is determined by the relation

$$(5.1)\quad 0 = P_\alpha(t+1,F_\beta) - P_\alpha(t-1,F_\beta)$$
$$= P_\alpha(t+1,F_\beta)-P_\alpha(t,F_\beta) + P_\alpha(t,F_\beta)-P_\alpha(t-1,F_\beta)$$
$$= \delta^t[1-F_\beta(t-1)]\Big[r_\beta(t)[B-C] + (1-r_\beta(t))[\delta A-A]\Big]$$
$$+ \delta^{t-1}[1-F_\beta(t-1)]\Big[r_\beta(t)[\delta C-A] + (1-r_\beta(t))[\delta A-A]\Big].$$

In order for $\underline{r}_\alpha$ to be an optimal response to $\underline{r}_\beta$, it is necessary and sufficient that, for each positive t with $r_\alpha(t) = 0$, the value of $r_\beta(t)$ determined by equation (5.1) satisfy

$$(5.2)\quad P_\alpha(t+1,F_\beta) - P_\alpha(t,F_\beta) \geq 0 \geq P_\alpha(t,F_\beta)-P_\alpha(t-1,F_\beta).$$

Using equation (5.1), it follows that relation (5.2) is satisfied if and only if

$$P_\alpha(t+1,F_\beta) - P_\alpha(t,F_\beta) \geq P_\alpha(t,F_\beta)-P_\alpha(t-1,F_\beta).$$

or

$$r_\beta(t)[B-C] + (1-r_\beta(t))[\delta A-A] \geq r_\beta(t)[\delta C-A] + (1-r_\beta(t))[\delta A-A].$$

Then since equation (5.1) implies $r_\beta(t) > 0$ when $\delta > 0$, we conclude that an alternating equilibrium exists if and only if $B-C \geq \delta C-A$.

To understand this result, one must recognize that $r_\beta(t)$ has to balance two conflicting incentives. On the one hand, it must be small enough to induce player α to move in period t-1 rather than in period t where he may reap the benefits from moving simultaneously with player β. On the other hand, it must be large enough to induce player α to wait until period t+1 in order to receive the possible benefit from following in period t. The larger is the return from moving simultaneously, the smaller must $r_\beta(t)$ be in order to induce player α to move in period t-1, but the larger must $r_\beta(t)$ be in order to induce player to wait until period t+1. When the return from moving simultaneously is sufficiently large, these two conditions are incompatible.

This argument suggests that instead of an alternating equilibrium, we may obtain an equilibrium in which the plan of both players is to periodically move together with positive probability in some periods followed by one or more periods in which both move with probability 0. We will call such an equilibrium a coordinating equilibrium.

Consider then a coordinating equilibrium in which both players move with probability r upon reaching any even period and move with probability 0 in any odd period. Then r is determined by the relation

$$
\begin{aligned}
(5.3)\qquad 0 &= P_\alpha(t+2,F_\beta) - P_\alpha(t,F_\beta) \\
&= P_\alpha(t+2,F_\beta)-P_\alpha(t+1,F_\beta) + P_\alpha(t+1,F_\beta)-P_\alpha(t,F_\beta) \\
&= (1-F_\beta(t))\delta^{t+1}\left[r[\delta C-A]+(1-r)[\delta-1]A\right] \\
&\quad + (1-F_\beta(t-1))\delta^{t}\left[r[B-C]+(1-r)[\delta-1]A\right],
\end{aligned}
$$

where t is an even integer. The strategy determined by r is optimal if and only if

$$
(5.4)\qquad P_\alpha(t+2,F_\beta)-P_\alpha(t+1,F_\beta) \gtrless 0 \gtrless P_\alpha(t+1,F_\beta)-P_\alpha(t,F_\beta)
$$

Using equation (5.3) it then follows that (5.4) is satisfied if and only if

$$P_\alpha(t+2,F_\beta)-P_\alpha(t+1,F_\beta) \gtreqless P_\alpha(t+1,F_\beta)-P_\alpha(t,F_\beta)$$

or

$$r[\delta C-A] + (1-r)(\delta-1)A \gtreqless r[B-C] + (1-r)(\delta-1]A.$$

But, since $\delta < 1$ implies $r > 0$, this relation can be simplified to $\delta C-A \gtreqless B-C$.

We conclude that for the stationary model being considered here, either an alternating equilibrium exists or a two period stationary, two period coordinating equilibrium exists. Both exist only in the special case where $\delta C-A = B-C$. In particular, if $C-A > B-C$, then as δ approaches 1, the alternating equilibria vanishes and the coordinating equilibria appears.

5.2 The Existence and Uniqueness of Fully Mixed Equilibria

When $C_\alpha(t+1) \lesseqgtr A_\alpha(t)$, the proof of the uniqueness of the fully mixed equilibrium in the infinite horizon game depends essentially on the fact the difference equation describing the conditional probabilities across periods is unstable. Consequently, all but one sequence of conditional probabilities must eventually violate the restriction that $r_\beta(t)$ lie between 0 and 1. However, when $C_\alpha(t+1)$ is greater than $A_\alpha(t)$ and sufficiently close to $B_\alpha(t+1)$, the relation between $r_\beta(t+1)$ and $r_\beta(t)$ may actually be a contraction over a range of $r_\beta(t)$. This leads to the possibility of multiple equilibria.

Assuming that the return functions decline exponentially as in Section 4.1, the equation relating $r(t+1)$ and $r(t)$ can be written as[13]

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[13]This assumes that $C \neq A$. If $C = A$, then $r(t) = A(1-\delta)/[B-\delta A]$ for all t.

$$(5.5)\quad r(t+1) \equiv h(r(t)) = \frac{A(1-\delta)}{[\delta(C-A)]} + \frac{(B-C)}{[\delta(A-C)]}\,\frac{r(t)}{[1-r(t)]}$$

Suppose that $\{r^1(t)\}_{t=0}^{\infty}$ and $\{r^2(t)\}_{t=0}^{\infty}$ both satisfy (5.5) with $0 < r^i(t) \leq 1$, for $i = 1,2$. Then

$$(5.6)\quad |r^1(t+1) - r^2(t+1)| = |[B-C]/[\delta(C-A)]|\,|(r^1(t)/(1-r^1(t)) - (r^2(t)/(1-r^2(t))|$$
$$\geq |[B-C]/[\delta(C-A)]|\,|r^1(t) - r^2(t)|.$$

If $A > C$, then $|[B-C]/[\delta(C-A)]| > 1$ which implies that the difference equation (5.5) is unstable. Consequently, the only fully mixed equilibrium is the pair of strategies corresponding to the stationary solution of (5.5) which we will denote by $\bar{r}$.

Now suppose that $C > A$. As illustrated in Figure 3, it may be shown that $h(r)$ is a concave, strictly decreasing function on the unit interval, taking on nonnegative values over the interval $[0,\hat{r}]$, where $\hat{r} = A(1-\delta)/[B-C+A-\delta A]$. The necessary and sufficient condition for $\bar{r}$ to be locally stable is that $|dh(\bar{r})/dr| < 1$. If this condition is satisfied, then there exists an open interval $\Delta = (\bar{r}-\epsilon, \bar{r}+\epsilon)$ such that, for any $r(0) \in \Delta$, $r(t)$ converges to $\bar{r}$. Consequently, when the stationary solution is locally stable and the horizon of the game is infinite, a one-parameter family of fully-mixed, nonstationary equilibria exists in addition to the stationary equilibrium.

Figure 3 <u>A Continuum of Fully Mixed Equilibria</u>

If $|dh(\bar{r})/dr| > 1$, then $\bar{r}$ is locally unstable and, given any

$r(0) \neq \bar{r}$, $r(t)$ either converges to a limit cycle or diverges out of the unit interval. A sufficient condition for the nonexistence of nonstationary fully mixed equilibria is for $|dh(r)/dr| > 1$ for all $r \in [0, \hat{r}]$. Since $h(r)$ is concave, this condition is satisfied if $|dh(0)/dr| > 1$ or, equivalently, $B-C > \delta(C-A)$.

Consider next the implications for the existence of a fully mixed equilibrium for a finite horizon game when the difference equation h is stable. Since $A > 0$, it follows that $A_\alpha(t) = Ae^{-\delta t} > 0$ for all $t < T$ which, combined with Lemma 4.4, implies that $t^* = T$. When $T < \infty$, equation (4.7) implies that $r_\beta(T-1) = A/[B-C+A]$.[14] Now suppose that $[B-C+A(1-\delta)]^2/[B-C]\delta(C-A) < 1$ -- i.e. $|dh(\hat{r})/dr| < 1$. Then any solution to (5.4) with an initial condition $r(0) \leqq \hat{r}$ implies that $r(t)$ converges to $\bar{r}$. Consequently, if $\bar{r} \neq A/[B-C+A]$, then there is no fully mixed equilibrium for T finite but sufficiently large.

A similar argument establishes that, if the return functions are not exponential, there may also be no fully mixed equilibrium even when the horizon is infinite. Choose A, B, and C so that $B-C < C-A$, and suppose that δ is chosen so that equation (5.5) is a contraction for any $r(0) \leqq \hat{r}(\delta)$. Now suppose that in some period $\bar{t}$ the discount factor increases to $\bar{\delta}$ for the remainder of the game (with a corresponding change in A,B, and C so that Assumption A1 is satisfied). Inspection of equation (5.5) reveals that $\hat{r}(\delta)$ converges monotonically to 0 as δ approaches 1. Consequently, we may choose $\bar{\delta}$ so that $\hat{r}(\bar{\delta}) < \bar{r}(\delta)$. But in this case, if $\bar{t}$ is chosen sufficiently large, the value of $r(\bar{t})$ must lie outside the range of $[0,\hat{r}(\bar{\delta})]$ implying that $r(\bar{t}+1) < 0$. Consequently, there is no fully mixed equilibrium. The

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[14]The argument behind equation (4.7) does not require Assumption A2.

problem is illustrated in Figure 4.

Figure 4 No Fully Mixed Equilibrium Exists

We conclude that if the difference equation is stable, then no fully mixed equilibrium generally exists when the horizon is finite and sufficiently large. If the horizon is infinite and the difference equation is stable, a fully mixed equilibrium may not exist when the payoffs are nonstationary, but if it exists, there is generally a one parameter family of such equilibria.

5.3 An Example with No Nondegenerate Equilibrium

As long as the game is symmetric and the horizon is finite, a symmetric fully mixed equilibrium exists.[15] However, if we allow for asymmetries in the returns to moving simultaneously, the arguments of Sections 5.1 and 5.2 can be combined to produce an example where no nondegenerate equilibrium exists, even if the returns to following and leading are the same for the two players.

We assume that the return functions for leading and following are identical for both players and decrease exponentially. That is, for $\alpha = a,b$, $A_\alpha(t) = A\delta^t$ and $B_\alpha(t) = B\delta^t$. If both players move in period t, then we assume that player b earns a return $C_b(t) = A\delta^t$, while player a receives a higher return $C_a(t) = C\delta^t$. Suppose that $0 < \delta < 1$ and $0 < A < C < B$. Then, for T finite but greater than 1, there is no nondegenerate equilibrium.

To see this, note first that in a nondegenerate equilibrium, Lemma 3.2

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[15] If the horizon is infinite, the last example of the previous section demonstrates that no symmetric fully mixed equilibrium may exist. We suspect that it may be possible to construct cases where there is no nondegenerate equilibrium of any kind for this kind of example.

requires that both players plan to wait until period T with some positive probability. Then since $A_\alpha(T-1) > 0$, for $\alpha = a,b$, it follows that player β must move with positive probability in period T-1; otherwise, player α, upon reaching period T-1, moves with certainty. But if player β is to be indifferent between moving in period T-1 and waiting until period T, the probability with which player α moves upon reaching period T-1 must satisfy

$$(5.7) \quad r_\alpha(T-1) = A/[B-C+A].$$

Now let t' be the smallest value of t such that $r_a(j) > 0$ for $j = t,\ldots,T-1$. Since the payoffs to player b satisfy Assumption A2, and equation (5.7) implies that $r_b(T-1) > 0$, it follows from Lemma 4.1 that $r_a(T-2) > 0$. We may therefore conclude that $t' \leq T-2$.

In order for player a to be indifferent between moving in each of the periods, t' to T-1, the sequence $\{r_b(j)\}_{j=t'}^{T-1}$, must satisfy equation (5.5). In particular,

$$r_b(T-1) = h(r_b(T-2)) \leq h(0) = A(1-\delta)/[B-C+A(1-\delta)] < A/[B-C+A],$$

violating equation (5.7).[16]

6. Degenerate Equilibria and Subgame Perfection

In this section we examine the degenerate equilibria for games which satisfy Assumption A1. The degenerate equilibria can be classified into two

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[16]If we suppose that the return to player a upon reaching period T is $A\delta^T$ (rather than 0), then we obtain $r_b(T-1) = A(1-\delta)/[B-C+A(1-\delta)]$. In this case, it appears that there is a nondegenerate equilibrium which is fully mixed up to period T-2.

types depending upon which player moves in period 0. We establish first the conditions under which each type of degenerate equilibrium exists. We show that a degenerate equilibrium in which player α moves first exists if and only if player β has a strategy which implies that moving immediately yields a higher payoff to player α than waiting until period T. We then turn our attention to the implications of subgame perfection. For a game with an infinite horizon satisfying assumption A3, we demonstrate that for each player α there is a subgame perfect equilibrium in which player α moves immediately with probability 1. If the horizon is finite (and payoffs are generic), such an equilibrium exists if and only if (i) $A_\alpha(0) > 0$ and (ii) $\tau_\alpha \geq \tau_\beta$. That is, the return to player α from leading in period 0 must be positive and it must be positive in any period in which the return to player β from moving first is also positive. It then follows from Theorem 4.1 that, generically, both types of degenerate equilibria are subgame perfect whenever a nondegenerate equilibrium exists.

6.1 The Existence of Degenerate Equilibria

We begin with a statement of the conditions which are necessary and sufficient for the existence of a degenerate equilibrium.

Theorem 6.1: Given Assumption A1, there is a degenerate equilibrium with $r_\alpha(0) = 1$ if and only if $A_\alpha(0) \geq \inf_{t<T}\{0, B_\alpha(t)\}$.

As long as player β possesses a strategy which permits a return to player α which is no greater than $A_\alpha(0)$, the best response of player α is to move immediately with probability 1. Since player β is not called upon to carry out his threat, this strategy combination is an equilibrium. Such a threat exists if either the return to player α from moving in period 0 is nonnegative (in which case player β can threaten never to move), or it is less than his return from following in some period (in which case player β threatens to move in that period). Otherwise, it follows from Assumption A1 that it is a dominant strategy for player α to wait until period T in which case no such degenerate equilibrium exists.

Notice that whenever a degenerate equilibrium exists, there is generally a large class of equilibrium strategies which support the same outcome since the specification of the players' strategies in the periods after period 0 is largely indeterminate. The only restriction is that player β adopt a strategy which implies that it is optimal for player α to move in period 0. Some of this indeterminacy can be eliminated by requiring each player to play optimally in every subgame. The remainder of this section is devoted to examining the implications of this restriction.

6.2 The Existence of Subgame Perfect Degenerate Equilibria

For any t, $0 \leq t < T$, let Γ_t denote the subgame starting in period t. To simplify notation, we will say that $(\underline{r}_\alpha, \underline{r}_\beta)$ is an equilibrium for subgame Γ_t if $\{(r_a(j), r_b(j))\}_{j=t}^{T-1}$ is an equilibrium for Γ_t. Our analysis of the subgame perfect equilibria is based on the following Lemma.

<u>Lemma 6.1</u>: A degenerate equilibrium $(\underline{r}_a, \underline{r}_b)$ is subgame perfect if and only if there is a t, $0 < t \leq T$ such that (i) for $0 \leq j < t$, $r_\alpha(j) = 1$ and $r_\beta(j) = 0$; (ii) if $t < T$, then $(\underline{r}_a, \underline{r}_b)$ is a nondegenerate equilibrium for game Γ_t; and (iii) $A_\alpha(j) \geq \sup_{k \geq t} P_\alpha(k, F_\beta)$ for all $j < t$.

Lemma 6.1 states that in any subgame perfect degenerate equilibrium

some player α moves with probability 1 and player β moves with probability 0 conditional on reaching any period j up to some period t. Thereafter, the strategy pair must form a nondegenerate equilibrium. In addition, the payoff to player α from waiting until period t must not exceed his return from leading in any period prior to period t.

The argument goes as follows. Suppose that $(\underline{r}_a,\underline{r}_b)$ is a subgame perfect equilibrium in which player α moves first. Let t be the first period in which player α does not move with probability 1. Assumption A1 then implies that, conditional on reaching any period j, player β prefers to wait until period j+1. It follows, therefore, that player β moves with probability 0 in each period j up to period t. This establishes the necessity of condition (i). Now suppose the game reaches period t-1 and t < T. Then A1 implies that player β cannot move with probability 1, since player α could then earn a higher return from waiting until period t+1 than from moving in period t-1. But this violates the assumption that $r_\alpha(t-1) > 0$. Consequently, any subgame perfect strategy pair must form a nondegenerate equilibrium for the subgame starting in period t. This establishes the necessity of condition (ii). Finally, in order for it to be optimal for player α to move in any period j before period t, the return from leading in period j must be no less than the payoff he receives from waiting until any period k after period t. But since player β never moves in the first t periods, this requirement reduces to the condition that the return from leading in period j be at least as large as his expected payoff from moving in any period after period t. This is condition (iii).

That these three conditions also imply that the strategy pairs are optimal in each subgame follows from Theorem 3.1 which states that all nondegenerate equilibria are subgame perfect.

As in the case of degenerate equilibria, when a degenerate subgame perfect equilibrium exists, there are generally many strategy combinations which support the same outcome. In determining which outcomes can be supported as degenerate subgame perfect equilibria, however, the next lemma permits us to restrict our search to a particularly simple class of strategies. Recall that t^* is the first period in and after which neither player moves with positive probability.

<u>Lemma 6.2</u>: If a subgame perfect equilibrium exists with $r_\alpha(0) = 1$, then there is a subgame perfect equilibrium in which $r_\alpha(t) = 1$ and $r_\beta(t) = 0$ for $t < t^*$. For any such equilibrium, either (i) $t^* = \infty$ or (ii) $t^* \geqq \tau_\alpha$ and $A_\alpha(t^*-1) \geqq 0$.

Lemma 6.2 states that we can always support the outcome of a subgame perfect equilibrium with a pair of strategies in which one player β never moves and the other player α moves with probability 1 conditional on reaching any period up to period t^*, in and after which he never moves. Furthermore, the value of t^* must be chosen so that player α does not prefer to wait in periods before period t^* (implying $A_\alpha(t^*-1) \geqq 0$) and does not prefer to move in periods in and after t^* (implying $t^* \geqq \tau_\alpha$). Note that if the payoff to player α is generic, then these conditions imply that $t^* = \tau_\alpha$.

The argument goes as follows. Choose t^* to be the largest t such that $A_\alpha(t-1) \geqq 0$. If $t^* = T$, then the strategy pair in which player α plans to move immediately upon reaching any period and player β plans to never move is clearly a subgame perfect equilibrium. If $t^* < T$, then Lemma 6.1 implies that the original strategy pair must form a nondegenerate equilibrium for the subgame Γ_{t^*}. Lemma 3.3 then implies that neither player moves with positive probability conditional on reaching any period greater than t^*-1. From Lemma 3.2, we may then conclude that it must be an optimal response for both players to wait until period T. This implies that $A_\beta(t^*) \leqq 0$ and hence that $t^* \geqq \tau_\alpha$. It also implies that $A_\alpha(t^*-1) \geqq 0$. This establishes the Lemma.

We turn now to a characterization of the conditions under which each type of degenerate subgame perfect equilibrium exists. As in Section 4, it will be convenient to distinguish between finite and infinite horizon games. We begin with a statement of the conditions which are necessary and sufficient for the existence of a subgame perfect degenerate equilibrium when the horizon is finite.

<u>Theorem 6.2</u>: Suppose $T < \infty$. There is a subgame perfect equilibrium with $r_\alpha(0) = 1$ if and only if each of the following conditions are satisfied: (i) $A_\alpha(0) \geqq 0$, (ii) $\tau_\beta-1 \leqq \tau_\alpha$, (iii) if $\tau_\alpha < T$ and $A_\alpha(\tau_\alpha) < 0$, then $\tau_\beta \leqq \tau_\alpha$.

So long as the return to player α from moving immediately is greater than or equal to 0 and the return to player β from moving is not strictly positive in any period following a period in which the return to player α is nonpositive, there is a subgame perfect equilibrium in which player α moves immediately. The Theorem is essentially a consequence of Lemma 6.2 and the observation that the optimality of $\underline{r}_\beta$ requires that $A_\beta(t^*) \leqq 0$ if $t^* < T$. The conditions of Theorem 6.2 are also sufficient for the existence of a subgame perfect equilibrium when $T = \infty$. When payoffs are generic, Theorem 6.2 implies

Corollary 6.1: Suppose $T < \infty$ and the payoff to player α is generic. Then there is a subgame perfect equilibrium with $r_\alpha(0) = 1$ if and only if (i) $A_\alpha(0) > 0$ and (ii) $\tau_\beta \leq \tau_\alpha$.[17]

We turn next to the case where the horizon is infinite. As in Section 4, we confine our attention to the case where $\lim_{t\to\infty} A_\alpha(t) = 0$.

Theorem 6.3: If $T = \infty$ and Assumption A3 is satisfied, then for $\alpha = a,b$, there is a subgame perfect equilibrium with $r_\alpha(0) = 1$.

As long as his return from leading is positive in every period, it always pays player α to move if the strategy of player β is always to wait.

6.3 The Relation between Degenerate and Nondegenerate Equilibria

We conclude this section with some remarks on the relation between the existence of subgame perfect degenerate equilibria and the existence of nondegenerate equilibria.

Suppose that $A_\alpha(0) \geq 0$ for each player α. Then Theorem 6.2 implies that both degenerate equilibrium outcomes are subgame perfect if and only if $\tau_a = \tau_b$ when payoffs are generic and $\tau_\alpha - 1 \geq \tau_\beta \geq \tau_\alpha + 1$ when payoffs are nongeneric. In Theorem 3.2 we have established that these same conditions are also necessary for the existence of nondegenerate equilibria. Thus, in general, the existence of two subgame perfect degenerate equilibrium outcomes implies the existence of nondegenerate equilibria. To establish the converse,

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[17] This is also a necessary and sufficient condition for the existence of a degenerate equilibrium which is "trembling hand" perfect (Selten (1975)) even when payoffs are nongeneric. In general an equilibrium is trembling hand perfect only if $r_\alpha(t) = 0$ whenever $A_\alpha(t) \leq 0$.

however, the example of Section 5.3 illustrates the necessity of some additional restrictions on the payoffs. In that example, there is no nondegenerate equilibrium even though a strategy combination in which either of the players moves with probability 1 upon reaching any period and the other player never moves can be shown to be a subgame perfect equilibrium.

Theorems 4.1 through 4.3 do imply, however, that these terminal conditions are sufficient for the existence of a nondegenrate equilibrium if Assumption A2 is satisfied. Hence, in games in which the return from leading is greater than or equal to the return from tying, the existence of both subgame perfect degenerate is generally both necessary and sufficient for the existence of a nondegenerate equilibrium. We summarize these conclusions in the following Theorem.

Theorem 6.4: (i) If $A_\alpha(0) \geqq 0$, then there is a nondegenerate equilibrium only if there is a subgame game perfect degenerate equilibrium in which $r_\alpha(0) = 1$. (ii) If $A_\alpha(0) \geqq 0$ for $\alpha = a,b$, and if Assumption A2 is satisfied, then there is a nondegenerate equilibrium if and only if, for each α, there is a subgame perfect degenerate equilibrium in which $r_\alpha(0) = 1$.

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Appendix

We present here the proofs of the Lemmata and Theorems stated in the text. As in the text, j,k, and t will refer to nonnegative integers. An arbitrary will be denoted by α and the other player by β. We will use repeatedly the fact that if (F_a,F_b) the pair of distribution functions generate by an equilibrium $(\underline{r}_a,\underline{r}_b)$, then $q_\alpha(t) > 0$ implies that $P_\alpha(t,F_\beta) = \sup_j P_\alpha(j,F_\beta)$.

Proof of Lemma 3.1:

Since $q_\beta(t+1) = 0$ and $t < \hat{t}$, it follows that $F_\beta(t+1) < 1$. Therefore $q_\beta(t+1) = q_\beta(t) = 0$ implies

$$P_\alpha(t+1,F_\beta) - P_\alpha(t,F_\beta) = [A_\alpha(t+1)-A_\alpha(t)](1-F_\beta(t+1)) < 0$$

which implies $q_\beta(t+1) = 0$. Q.E.D.

Proof of Lemma 3.2:

Suppose that $0 < \hat{t} < T$. Then for some player β, $q_\beta(\hat{t}) > 0$ and $F_\beta(\hat{t}) = 1$. Consequently, Assumption A1 implies

$$P_\alpha(\hat{t}+1,F_\beta) - P_\alpha(\hat{t}-1,F_\beta) =$$
$$q_\beta(\hat{t}-1)[B_\alpha(\hat{t}-1)-C_\alpha(\hat{t}-1)] + q_\beta(\hat{t})[B_\alpha(\hat{t})-A_\alpha(\hat{t}-1)] > 0.$$

and

$$P_\alpha(\hat{t}+1,F_\beta) - P_\alpha(\hat{t},F_\beta) = q_\beta(\hat{t})[B_\alpha(\hat{t})-C_\alpha(\hat{t})] > 0.$$

Therefore, $q_\alpha(\hat{t}-1) = q_\alpha(\hat{t}) = 0$. But then Lemma 3.1 implies that $q_\beta(\hat{t}) = 0$. A contradiction. Q.E.D.

Proof of Theorem 3.1:

Suppose $(\underline{r}_a,\underline{r}_b)$ is a non-degenerate equilibrium. Then Lemma 3.2 implies that $F(t-1;\underline{r}_\beta) < 1$ for all $t \leq T$. Consequently, $r_\alpha(t) > 0$ implies that $P_\alpha(t,F(\cdot;\underline{r}_\beta)) = \sup_j P_\alpha(j,F(\cdot;\underline{r}_\beta))$. From the definition of $F(j;t,\underline{r}_\beta)$,

$$F(j;t,\underline{r}_\beta) = [F(j;\underline{r}_\beta)-F(t-1;\underline{r}_\beta)]/(1-F(t-1,\underline{r}_\beta))$$

for $0 \leq t \leq j \leq T$. Therefore,

$$P_\alpha(j;F(\cdot;t,\underline{r}_\beta)-P_\alpha(k;F(\cdot;t,\underline{r}_\beta) = [P_\alpha(j;F(\cdot;\underline{r}_\beta)-P_\alpha(k;F(\cdot;\underline{r}_\beta)]/(1-F(t-1;\underline{r}_\beta))$$

for $0 \leq t \leq k,j \leq T$ which implies that for $0 \leq t \leq j \leq T$, $P_\alpha(j,F(\cdot;t,\underline{r}_\beta)) = \sup_{k\geq t} P_\alpha(k,F(\cdot;t,\underline{r}_\beta))$ whenever $r_\alpha(j) > 0$. Q.E.D.

Proof of Lemma 3.3:

(i) Suppose that $q_\alpha(T) > 0$. We show first that $\tau_\alpha \leq t^*$. If $t^* = T$, the result follows by definition. So suppose that $t^* < T$. Then

$$0 \leq P_\alpha(T,F_\beta) - P_\alpha(t^*,F_\beta) = -q_\beta(T)A_\alpha(t^*).$$

This implies that $A_\alpha(t^*) \leqq 0$ from which it follows that $\tau_\alpha \leqq t^*$.

To show that $\tau_\alpha \geqq t^*-1$, suppose first that $t^* = \infty$. Then since $\lim_{t\to\infty}\sum_{j=t}^{\infty} q_\beta(j) = 0$, we have

$$\begin{aligned} 0 &\geqq P_\alpha(T,F_\beta) - \limsup_{t\to\infty} P_\alpha(t,F_\beta) \\ &= \limsup_{t\to\infty}\Big[[B_\alpha(t)-C_\alpha(t)]q_\alpha(t) + \textstyle\sum_{j=t+1}^{\infty}[B_\alpha(j)-A_\alpha(t)]q_\beta(j) \\ &\quad - q_\beta(T)A_\alpha(t)\Big] \\ &\geqq - q_\beta(T)\lim_{t=\infty} A_\alpha(t). \end{aligned}$$

Assumption A1 then implies that $A_\alpha(t) > 0$ for all t which implies that $\tau_\alpha = \infty$.

Next suppose that $t^* < \infty$. Note first that either $q_\alpha(t^*-1) > 0$ or $q_\alpha(t^*-2) > 0$. Otherwise, Lemma 3.1 implies that $q_\alpha(t^*-1) = q_\beta(t^*-1) = 0$, contradicting the definition of t^*. If $q_\alpha(t^*-2) > 0$, then we have

$$\begin{aligned} 0 &= P_\alpha(T,F_\beta) - P_\alpha(t^*-2,F_\beta) \\ &= q_\beta(t^*-2)[B_\alpha(t^*-2)-C_\alpha(t^*-2)] + q_\beta(t^*-1)[B_\alpha(t^*-1)-A_\alpha(t^*-2)] \\ &\quad - q_\beta(T)A_\alpha(t^*-2) \\ &> -q_\beta(T)A_\alpha(t^*-2), \end{aligned}$$

from which it follows that $A_\alpha(t^*-2) > 0$. While if $q_\alpha(t^*-1) > 0$, then

$$\begin{aligned} 0 &= P_\alpha(T,F_\beta) - P_\alpha(t^*-1,F_\beta) \\ &= q_\beta(t^*-1)[B_\alpha(t^*-1)-C_\alpha(t^*-1)] - q_\beta(T)A_\alpha(t^*-1) \\ &\geqq - q_\beta(T)A_\alpha(t^*-1), \end{aligned}$$

which combined with Assumption A1 again implies that $A_\alpha(t^*-2) > A_\alpha(t^*-1) \geqq 0$.

In either case, therefore, $\tau_\alpha \geq t^*-1$.

(ii) Suppose that $q_\beta(T) = 0$. Then Lemma 3.2 implies that $F_\beta(t) < 1$ for all $t < T$ which implies in turn that $t^* = T = \infty$.

(iii) We note first that Lemma 3.2 implies that $q_\alpha(T), q_\beta(T) > 0$ if $t^* < \infty$. Then if $r_\alpha(t^*-1) = 0$, the definition of t^* implies that $r_\beta(t^*-1) > 0$. We may therefore conclude that

$$0 = P_\beta(T,F_\alpha) - P_\beta(t^*-1,F_\alpha) = -q_\alpha(T)A_\beta(t^*-1)$$

which implies $A_\beta(t^*-1) = 0$. By definition, then, $t^*-1 = \tau_\beta$. The condition then follows from condition (i). Q.E.D.

<u>Proof of Lemma 3.4</u>:

Given Lemma 3.3, all that is required is to establish that if $q_\beta(T) > 0$ and $t^* < \infty$, then $\tau_\alpha \geq t^*$. From the proof of Lemma 3.3, we know that $q_\alpha(t^*-1) > 0$ implies $A_\alpha(t^*-1) \geq 0$. If payoffs are generic, then this implies that $A_\alpha(t^*-1) > 0$ which in turn implies that $\tau_\alpha \geq t^*$. The Lemma will follow, therefore, if we can show that $q_\alpha(t^*-1) > 0$.

Suppose not. Then the definition of t^* requires that $q_\beta(t^*-1) > 0$. The assumption that $q_\beta(T) > 0$ then implies that

$$0 = P_\beta(T,F_\beta) - P_\beta(t^*-1,F_\beta) = - q_\alpha(T)A_\beta(t^*-1).$$

From Lemma 3.3(ii) we know that $t^* < \infty$ implies that $q_\alpha(T) > 0$. But this implies that $A_\beta(t^*-1) = 0$, which contradicts the assumption that payoffs are generic. Q.E.D.

Proof of Theorem 3.2:

The theorem follows directly from Lemmata 3.3 and 3.4. Q.E.D.

Proof of Theorem 3.3:

Without loss of generality, we may suppose that $r_\alpha(0) < 1$. Note first that if $\underline{r}_\alpha$ is an optimal response, then $\underline{r}^*_\alpha$ is as well.

Second, note that for all $t > 0$,

$$q^*_\alpha(t) = q_\alpha(t)[(1-r^*_\alpha(0))/(1-r_\alpha(0))]$$

from which it follows that for all $t,j > 0$,

$$P_\beta(t,F^*_\alpha)-P_\beta(j,F^*_\alpha) = [(1-r^*_\alpha(0))/(1-r_\alpha(0))][P_\beta(t,F_\alpha)-P_\beta(j,F_\alpha)].$$

and

$$\begin{aligned}
&P_\beta(t,F^*_\alpha) - P_\beta(0,F^*_\alpha)\\
&= [P_\beta(t,F^*_\alpha)-P_\beta(1,F^*_\alpha)] + [P_\beta(1,F^*_\alpha)-P_\beta(0,F^*_\alpha)]\\
&= [P_\beta(t,F^*_\alpha)-P_\beta(1,F^*_\alpha)] + r^*_\alpha(0)[B_\alpha(0)-C_\alpha(0)]\\
&\quad + [1-r^*_\alpha(0)][A_\alpha(1)-A_\alpha(0)]\\
&= [(1-r^*_\alpha(0))/(1-r_\alpha(0))]\Big[[P_\beta(t,F_\alpha)-P_\beta(1,F_\alpha)]\\
&\quad + [1-r_\alpha(0)][A_\alpha(1)-A_\alpha(0)]\Big] + r^*_\alpha(0)[B_\alpha(0)-C_\alpha(0)]\\
&\geqq [(1-r^*_\alpha(0))/(1-r_\alpha(0))]\Big[[P_\beta(t,F_\alpha)-P_\beta(1,F_\alpha)] + r_\alpha(0)[B_\alpha(0)-C_\alpha(0)]\\
&\quad + [1-r_\alpha(0)][A_\alpha(1)-A_\alpha(0)]\Big]\\
&= [(1-r^*_\alpha(0))/(1-r_\alpha(0))][P_\beta(t,F_\alpha)-P_\beta(0,F_\alpha)].
\end{aligned}$$

We conclude that $\underline{r}_\beta$ is also optimal against $\underline{r}_\alpha^*$. Q.E.D.

Proof of Lemma 4.1:

Since $t+1 < \hat{t}$, it follows that either $F_\beta(t+1) < 1$. Therefore Assumptions A1 and A2 imply that

$$P_\alpha(t+1,F_\beta) - P_\alpha(t,F_\beta) = [C_\alpha(t+1)-A_\alpha(t)]q_\beta(t+1)$$
$$+ [A_\alpha(t+1)-A_\alpha(t)](1-F_\beta(t+1)) < 0$$

which implies $q_\alpha(t+1) = 0$. Q.E.D.

Proof of Lemma 4.2:

Part (i) follows from the definition of $\tilde{t}$. Part (iii) follows from the definition of t^* and induction on Lemma 4.1. To establish part (ii) we may use the definition of $\tilde{t}$ and again argue by induction on Lemma 4.1 to establish that $q_a(t)q_b(t+1) = 0$ for all t, $\tilde{t} \leq t \leq T$. The second half of the condition follows from the definition of t^*. Q.E.D.

Proof of Lemma 4.3:

For any $t < \tilde{t}-1$, the conditions for equilibrium require

$$0 = P_\alpha(t+1,F_\beta) - P_\alpha(t,F_\beta)$$
$$= [B_\alpha(t)-C_\alpha(t)]q_\beta(t) + [C_\alpha(t+1)-A_\alpha(t)]q_\beta(t+1)$$
$$+ [A_\alpha(t+1)-A_\alpha(t)][1-F_\beta(t+1)].$$

The Lemma then follows upon substitution of the expression for $r_\beta(\cdot)$. Q.E.D.

Proof of Lemma 4.4:

Since equation (4.1) is bilinear in $r_\beta(t)$ and $r_\beta(t+1)$, it follows by induction that for any $r_\beta(t)$, $t > 0$, there is a unique sequence $\{r_\beta(j)\}_{j=0}^{t}$ satisfying (4.1). Furthermore, solving for $r_\beta(t)$ as a function of $r_\beta(t+1)$, we obtain

$$r_\beta(t) = g_\beta(r_\beta(t+1);t)$$

$$\text{(A.1)} \qquad \equiv \frac{[A_\alpha(t)-A_\alpha(t+1)] + [A_\alpha(t+1)-C_\alpha(t+1)]r_\beta(t+1)}{[B_\alpha(t)-C_\alpha(t)+A_\alpha(t)-A_\alpha(t+1)] + [A_\alpha(t+1)-C_\alpha(t+1)]r_\beta(t+1)}$$

$$= 1 - \frac{B_\alpha(t)-C_\alpha(t)}{[B_\alpha(t)-C_\alpha(t)+A_\alpha(t)-A_\alpha(t+1)] + [A_\alpha(t+1)-C_\alpha(t+1)]r_\beta(t+1)} .$$

Note that if A2 is satisfied, then $r_\beta(j+1) \in [0,1]$ implies $g_\beta(r_\beta(j+1);j) \in [0,1)$. Consequently, it follows by induction that $r_\beta(t) \in [0,1]$ implies $r_\beta(j) \in [0,1]$ for $j < t$. Q.E.D.

Proof of Lemma 4.5:

Suppose that $\tilde{t} = 0$ and $q_\alpha(0) = 0$. Then since $t^* > 1$, it follows from Lemma 4.2 that $q_\alpha(1) > 0$ and $q_\beta(1) = 0$. Therefore,

$$0 \leqq P_\alpha(1,F_\beta) - P_\alpha(0,F_\beta)$$
$$= q_\beta(0)[B_\alpha(0)-C_\alpha(0)] + (1-q_\beta(0))[A_\alpha(1)-A_\alpha(0)].$$

Noting that $r_\beta(0) = q_\beta(0)$ and rearranging terms yields relation (4.2).

Now suppose that $\tilde{t} > 0$. Then since $q_\alpha(\tilde{t}) = 0$, the definition of $\tilde{t}$ implies that both $r_\beta(\tilde{t}-1) > 0$ and $r_\beta(\tilde{t}) > 0$. Therefore,

$$0 = P_\beta(\tilde{t},F_\alpha) - P_\beta(\tilde{t}-1,F_\alpha)$$
$$= q_\alpha(\tilde{t}-1)[B_\beta(\tilde{t}-1)-C_\beta(\tilde{t}-1)] + (1-F_\alpha(\tilde{t}-1))[A_\beta(\tilde{t})-A_\beta(\tilde{t}-1)].$$

Substituting in conditional probabilities and solving, we then obtain equation (4.3).

To establish relations (4.4) and (4.5), we use the fact that $q_\alpha(\tilde{t}) = 0$ implies $q_\beta(\tilde{t}+1) = 0$, $q_\alpha(\tilde{t}-1) > 0$, and $q_\alpha(\tilde{t}+1) > 0$. Therefore

$$0 = P_\alpha(\tilde{t}+1,F_\beta) - P_\alpha(\tilde{t}-1,F_\beta)$$
$$= q_\beta(\tilde{t}-1)[B_\alpha(\tilde{t}-1)-C_\alpha(\tilde{t}-1)] + (1-F_\beta(\tilde{t}))[A_\alpha(\tilde{t}+1)-A_\alpha(\tilde{t}-1)].$$

Using conditional probabilities and rearranging, we obtain equation (4.5). Also

$$0 \leqq P_\alpha(\tilde{t}+1,F_\beta) - P_\alpha(\tilde{t},F_\beta)$$
$$= q_\beta(\tilde{t})[B_\alpha(\tilde{t})-C_\alpha(\tilde{t})] + (1-F_\beta(\tilde{t}))[A_\alpha(\tilde{t}+1)-A_\alpha(\tilde{t})].$$

Upon substituting in conditional probabilities, we then obtain

$$r_\beta(\tilde{t}) \geq [A_\alpha(\tilde{t})-A_\alpha(\tilde{t}+1)]/[B_\alpha(\tilde{t})-C_\alpha(\tilde{t})+A_\alpha(\tilde{t})-A_\alpha(\tilde{t}+1)].$$

Finally, note that $1 \geq r_\beta(\tilde{t}-1) > 0$ if and only if

$$r_\beta(\tilde{t}) < [A_\alpha(\tilde{t}+1)-A_\alpha(\tilde{t}-1)]/[B_\alpha(\tilde{t})-A_\alpha(\tilde{t}+1)].$$

This establishes relation (4.4). Q.E.D.

Proof of Lemma 4.6:

If $q_\alpha(t) = 0$, it follows from the definition of $\tilde{t}$ and t^* that $q_\beta(t-1) = q_\beta(t+1) = 0$ and both $q_\alpha(t-1)$ and $q_\alpha(t+1)$ are positive. Therefore

$$\begin{aligned} 0 &= P_\alpha(t+1,F_\beta) - P_\alpha(t-1,F_\beta) \\ &= q_\beta(t)[B_\alpha(t)-A_\alpha(t-1)] - (1-F_\beta(t)[A_\alpha(t+1)-A_\alpha(t-1)]. \end{aligned}$$

Substituting in conditional probabilities and solving, we obtain equation (4.6). Q.E.D.

Proof of Theorem 4.1:

We establish first that $r_\alpha(t^*-1) > 0$ implies equation (4.7). Since the definition of t^* implies that $q_\beta(t) = 0$ for $t^* \leq t < T$, it follows that

$$\begin{aligned} 0 &= P_\alpha(T;F_\beta) - P_\alpha(t^*-1,F_\beta) \\ &= q_\beta(t^*-1)[B_\alpha(t^*-1)-C_\alpha(t^*-1)] - q_\beta(T)A_\alpha(t^*-1). \end{aligned}$$

Solving in terms of conditional probabilities then yields equation (4.7).

By definition of t^*, at least one player, say player α, moves with positive probability conditional on reaching period t^*-1. Since the payoffs of player α are generic, it then follows from equation (4.7) that $r_\beta(t^*-1) > 0$. Therefore, we may conclude that $A_\alpha(t^*-1) > 0$ and $r_\alpha(t^*-1) > 0$ for $\alpha = a,b$. These results combined with Lemma 3.3 then imply that (i) $r_\beta(t^*-1)$ satisfies (4.7) for $\beta = a,b$, (ii) $\tau_\alpha > t^*-1$ for $\alpha = a,b$, and hence that (iii) $\tilde{t} = t^* = \tau_a = \tau_b$.

The uniqueness of the equilibrium then follows from Lemma 4.4. Since $A_\alpha(t) < 0$ for $t^* \leqq t < T$, it may be readily verified that this pair of strategies constitute a pair of best responses. Q.E.D.

<u>Proof of Theorem 4.2</u>:

Suppose $A_b(\tau_b) = 0$. If $t^* < \infty$, then for $\alpha = a,b$, Lemma 3.2 implies that $q_\alpha(T) > 0$. If the equilibrium is also fully mixed so that $r_\beta(t^*-1) > 0$ for $\beta = a,b$, then

$$
\begin{aligned}
0 &= P_\alpha(T,F_\beta) - P_\alpha(t^*-1,F_\beta) \\
&= q_\beta(t^*-1)[B_\alpha(t^*-1)-C_\alpha(t^*-1)] - q_\beta(T)A_\alpha(t^*-1) > -q_\beta(T)A_\alpha(t^*-1).
\end{aligned}
$$

which implies that $A_\alpha(t^*-1) > 0$ for $\alpha = a,b$. Then, since Lemma 3.3 implies that $t^*-1 \leqq \tau_\alpha \leqq t^*$ for $\alpha = a,b$, it follows that $t^* = \tau_b$ and hence that $\tau_b-1 \leqq \tau_a \leqq \tau_b$. But if $\tau_a = \tau_b-1$, then $t^* = \tau_a+1$ in which case

$$
\begin{aligned}
0 &= P_\alpha(T,F_\beta) - P_\alpha(\tau_\alpha,F_\beta) \\
&= q_\beta(\tau_\alpha)[B_\alpha(\tau_\alpha)-C_\alpha(\tau_\alpha)] - q_\beta(T)A_\alpha(\tau_\alpha) > - q_\beta(T)A_\alpha(\tau_\alpha),
\end{aligned}
$$

contradicting the definition of τ_α. We conclude that a fully mixed equilibrium exists only if $\tau_a = \tau_b$.

The existence and uniqueness of a fully mixed equilibrium then follows the argument in the proof of Theorem 4.1. Q.E.D.

Proof of Theorem 4.3:

Suppose that $t^* < \infty$ and $r_\alpha(t^*-1) = 0$. Then Lemmata 3.2 and 3.3 imply that $A_\beta(t^*-1) = 0$ and $\tau_\alpha - 1 \leqq \tau_\beta = t^* \leqq \tau_\alpha$. This establishes the "only if" part of the theorem.

To establish the "if" part, we note that these conditions are consistent with well defined behavior strategies satisfying relations (4.2) to (4.6) and (4.8) to (4.10). Furthermore, it may be verified that any pair of strategies satisfying these relations forms a pair of best responses. Q.E.D.

Proof of Theorems 4.4 and 4.5:

Both of these theorems follow upon verifying that relations (4.2) to (4.6) and (4.8) to (4.10) combined with the necessary conditions of Theorem 4.3 are not only necessary but also sufficient for a pair of strategies to form a pair of best responses. Q.E.D.

Proof of Lemma 4.7:

The suffiency of the conditions for a strategy combination with $t^* = \infty$ to be an equilibrium will be left to the reader. We will establish the necessity of the conditions. It is immediate that the conditions of Lemmata 4.3, 4.5 and 4.6 must be satisfied. Suppose that $\lim_{t\to\infty} A_\alpha(t) \neq 0$ and suppose that the requirements of Lemmata 4.3, 4.5 and 4.6 imply that $q_\alpha(\infty) > 0$. Then since $t^* = \infty$, Lemma 4.1 and the conditions for equilibrium imply that there is a a subsequence $\{t^k\}$ such that

$$\begin{aligned}
0 &= P_\alpha(\infty,F_\beta) - \lim_{t^k\to\infty} P_\alpha(t^k,F_\beta) \\
&= \lim_{t^k\to\infty}\Big[[B_\alpha(t^k)-C_\alpha(t^k)]q_\beta(t^k) + \textstyle\sum_{j=t^k}^{\infty}[B_\alpha(j)-A_\alpha(t^k)]q_\beta(j) \\
&\quad - A_\alpha(t^k)q_\beta(\infty)\Big] \\
&= -\lim_{t^k\to\infty} A_\alpha(t^k)q_\beta(\infty) = q_\beta(\infty)\ \lim_{t\to\infty} A_\alpha(t).
\end{aligned}$$

This establishes the Lemma. Q.E.D.

<u>Proof of Theorem 4.6:</u>

Since $\lim_{t\to\infty} A_\alpha(t) = 0$, $\tau_\alpha = \infty$. It then follows from Lemmata 3.2 and 3.3 that $t^* = \infty$ for any non-degenerate equilibrium. To establish conditions (a) and (b), note that Lemmata 4.3, 4.4, and 4.5 imply a unique pair of strategies associated with each $\tilde{t} < \infty$ and each $r_\beta(\tilde{t})$ $(\beta = a,b)$ satisfying the conditions of Lemma 4.5. To establish (c), we need to establish the existence of a unique solution to equation (4.1) with the property that $r_\beta(t) \in [0,1]$ for all t.

We show first there is a sequence $\{r_\beta(t)\}_{t=0}^{\infty}$ which for all $t > 0$ satisfies (4.1) and $r_\beta(t) \in [0,1]$. For $t > 0$, let

$$M_t = \{\{r_\beta(j)\}_{j=0}^{\infty} \in R^\infty\colon (r_\beta(j),r_\beta(j+1)) \text{ solves (4.1) for } j < t, \text{ and } r_\beta(j) \in [0,1] \text{ for } j \leq t\},$$

and for any $k < t$, let $M_t(k) = \{r \in R: r = r_\beta(k), \{r_\beta(j)\}_{j=0}^{\infty} \in M_t\}$. Since $g_\beta(x;j)$ is a continuous and weakly monotonic in x and is contained in [0,1] whenever $x \in [0,1]$, it follows that M_t is a non-empty, compact[1] set and that for each $k \leq t$, $M_t(k)$ is convex. Furthermore, $M_{t+1} \subset M_t$. Let $M_\infty = \cap_{t=0}^{\infty} M_t$. Then M_∞ is a non-empty, compact set with the property that $\{r_\beta(j)\}_{j=0}^{\infty} \in M_\infty$ implies $(r_\beta(t), r_\beta(t+1))$ solves (4.1) and $r_\beta(t) \in [0,1]$ for all $t < \infty$. Therefore, a solution exists.

To prove that the solution is unique, suppose that $\{r_\beta(t)\}_{j=t}^{\infty}$ is a solution to (4.1) for which $r_\beta(t) \in [0,1]$ for $t \geq 0$. Define $\tau(0) \equiv 0$ and for $k > 0$,

$$\tau(k) = \inf\{t > \tau(k-1): A_\alpha(t) = C_\alpha(t)\}.$$

Then for $t+1 = \tau(k)$, equation (4.1) implies:

$$r_\beta(t) = [A_\alpha(t)-A_\alpha(t+1)]/[B_\alpha(t)-C_\alpha(t)+A_\alpha(t)-A_\alpha(t+1)].$$

Suppose that $\{r_\beta^\epsilon(t)\}_{t=0}^{\infty}$ is also a solution to (4.1). Then it follows from part (a) that $\tau(k) < \infty$, implies $r_\beta(t) = r_\beta^\epsilon(t)$ for any $t < \tau(k)$. Let $\bar{k} = \sup\{k: \tau(k) < \infty\}$. Then if $\bar{k} = \infty$, it follows immediately that the solution to (4.1) is unique.

Suppose $\bar{k} < \infty$. Then for $t \geq \bar{k}$, we may invert $g_\beta(.;t)$ to obtain

****

[1]In the product topology.

$$r_\beta(t+1) = h_\beta(r_\beta(t);t)$$

$$\equiv \left(\frac{B_\alpha(t)-C_\alpha(t)}{A_\alpha(t+1)-C_\alpha(t+1)}\right)\left(\frac{r_\beta(t)}{1-r_\beta(t)}\right) - \frac{A_\alpha(t)-A_\alpha(t+1)}{A_\alpha(t+1)-C_\alpha(t+1)} .$$

Now suppose that M_∞ contains more than one sequence. Then it follows from Lemma 4.4 that $M_\infty(k)$ contains more than one element. Furthermore, since $M_\infty(k)$ is convex, we may suppose that there are three such elements and hence three sequences, $\{r_\beta^i(j)\}_{j=0}^\infty$, i = 1,2,3, contained in M_∞. Furthermore, we may choose $\epsilon > 0$ such that $r_\beta^1(\bar{k}) + \epsilon < r_\beta^2(\bar{k}) < r_\beta^3(\bar{k}) - \epsilon$. We will show that $\{r_\beta^3(j)\}_{j=0}^\infty$ cannot belong to M_∞.

Note first that for each $t \geq \bar{k}$, $h_\beta(.;t)$ is an increasing convex function. Then since

$$\frac{dh_\beta(r_\beta(t);t)}{dr_\beta(t)} = \left(\frac{B_\alpha(t)-C_\alpha(t)}{A_\alpha(t+1)-C_\alpha(t+1)}\right)\left(\frac{1}{1-r_\beta(t)}\right)^2$$

it follows upon application of the chain rule that for $t \geq \bar{k}$ and $i = 1,2$:

$$\text{(A.2)} \quad r_\beta^{i+1}(t)-r_\beta^i(t) \geq \left(\frac{A_\alpha(\bar{k})-C_\alpha(\bar{k})}{A_\alpha(t)-C_\alpha(t)}\right) \Pi_{j=\bar{k}}^{t-1}\left(\frac{B_\alpha(j)-C_\alpha(j)}{[A_\alpha(j)-C_\alpha(j)][1-r_\beta^i(j)]^2}\right) \epsilon$$

But if $A_\alpha(t) \geq C_\alpha(t)$, then Assumption A1 implies that $[B_\alpha(t)-C_\alpha(t)]/[A_\alpha(t)-C_\alpha(t)] > 1$ for all t. Therefore Assumption A1(b) and relation (A.2) imply

$$r_\beta^{i+1}(t) - r_\beta^i(t) \geq \epsilon\,[A_\alpha(\bar{k})-C_\alpha(\bar{k})]/2K \equiv \epsilon_1 \quad \text{for all } t \geq \bar{k}.$$

Then since $r_\beta^1(t) \geq 0$ for all $t \geq \bar{k}$, it follows that $(1-r_\beta^2(t)) \leq (1-\epsilon_1)$ for all $t \geq \bar{k}$. Substituting back into inequality

(A.2), we then obtain

$$r_\beta^3(t) \geq \epsilon_1 [1/(1-\epsilon_1)]^{2(t-\bar{k})}.$$

Therefore, for t sufficiently large, $r_\beta^3(t) > 1$ which implies that $(r_\beta^3(t))_{t=0}^\infty \notin M_\infty$. Q.E.D.

Proof of Theorem 6.1:

To establish the necessity of this condition, suppose $A_\alpha(0) < \inf_{t<T}\{0,B_\alpha(t)\}$. Then Assumption A1 implies that

$$\begin{aligned} P_\alpha(T,F_\beta) - P_\alpha(0,F_\beta) &= [B_\alpha(0)-C_\alpha(0)]q_\beta(0) \\ &\quad + \textstyle\sum_{j=1}^{T-1}[B_\alpha(j)-A_\alpha(0)]q_\beta(j) - A_\alpha(0)q_\beta(T) \\ &> 0, \end{aligned}$$

from which it follows that $r_\alpha(0) = 0$.

To establish the sufficiency of this condition, suppose first that $A_\alpha(0) \geq B_\alpha(t)$. Consider any $(\underline{r}_\alpha,\underline{r}_\beta)$ such that $r_\alpha(0) = 1$, $r_\beta(j) = 0$ for $j < t$, and $r_\beta(t) = 1$. Then

$$P_\beta(j,F_\alpha) - P_\beta(0,F_\alpha) = [B_\beta(0)-C_\beta(0)] > 0$$

which implies that $\underline{r}_\beta$ is an optimal response. And

$$\begin{aligned} P_\alpha(j,F_\beta) - P_\alpha(0,F_\beta) &= [A_\alpha(j)-A_\alpha(0)] < 0 && \text{for } j < t \\ &= [C_\alpha(j)-A_\alpha(0)] < [B_\alpha(j)-A_\alpha(0)] \leqq 0 && \text{for } j = t \\ &= [B_\alpha(t)-A_\alpha(0)] \leqq 0 && \text{for } j > t \end{aligned}$$

which implies that $\underline{r}_\alpha$ is an optimal response.

Similarly, if $A_\alpha(0) \geqq 0$, then for any $(\underline{r}_\alpha,\underline{r}_\beta)$ such that $r_\alpha(0) = 1$, and $r_\beta(j) = 0$ for $j < T$, we have again that

$$P_\beta(j,F_\alpha) - P_\beta(0,F_\alpha) = [B_\beta(0)-C_\beta(0)] > 0$$

which implies that $\underline{r}_\beta$ is an optimal response. And

$$\begin{aligned} P_\alpha(j,F_\beta) - P_\alpha(0,F_\beta) &= [A_\alpha(j)-A_\alpha(0)] < 0 && \text{for } j < t \\ &= -A_\alpha(0) \leqq 0 && \text{for } j = T, \end{aligned}$$

which implies that $\underline{r}_\alpha$ is an optimal response. Q.E.D.

<u>Proof of Lemma 6.1:</u>

Suppose that $(\underline{r}_a,\underline{r}_b)$ is a subgame perfect degenerate equilibrium and let $t = \sup\{j: r_\alpha(k) = 1 \text{ for all } k < j\}$. We show first that if $t < T$, then $(\underline{r}_a,\underline{r}_b)$ must be a non-degenerate equilibrium for the subgame Γ_t. Suppose not. Then the definition of subgame perfection implies that $(\underline{r}_a,\underline{r}_b)$ form a degenerate equilibrium for Γ_t. Since $r_\alpha(t) < 1$, by definition, it then follows that $r_\beta(t) = 1$. Consequently,

$$0 = P_\alpha(t+1, F_\beta(\cdot;k,\underline{r}_\beta)) - P_\alpha(t-1, F_\beta(\cdot;k,\underline{r}_\beta)$$
$$= B_\alpha(t) - A_\alpha(t-1) > 0.$$

But this implies that $r_\alpha(t-1) = 0$ which contradicts the definition of t. This establishes the necessity of (ii).

To establish the necessity of condition (i), it is enough to show that $r_\beta(k) = 0$ for all $k < t$. But this follows immediately, since for any $j > k$,

$$P_\beta(j, F_\alpha(\cdot;k,\underline{r}_\alpha)) - P_\beta(k, F_\alpha(\cdot;k,\underline{r}_\alpha) = B_\beta(k) - C_\beta(k) > 0.$$

To establish the necessity of condition (iii), we note that $r_\alpha(j) = 1$ implies that

$$0 \geqq P_\alpha(k, F_\beta(\cdot;j,\underline{r}_\beta)) - P_\alpha(j, F_\beta(\cdot;j,\underline{r}_\beta)$$
$$= P_\alpha(k, F_\beta) - A_\alpha(j)$$

for $t \leqq j \leqq T$.

To establish the suffciency of these conditions, it is enough to show that they imply $P_\alpha(j;j,F_\beta(\cdot;j,\underline{r}_\alpha) \geqq \sup_{k \geqq j} P_\alpha(k;j,F_\beta(\cdot;j,\underline{r}_\alpha)$ whenever $r_\alpha(j) > 0$ and $P_\beta(j;j,F_\alpha(\cdot;j,\underline{r}_\beta) \geqq \sup_{k \geqq j} P_\beta(k;j,F_\alpha(\cdot;j,\underline{r}_\beta)$ whenever $r_\beta(j) > 0$. Since $(\underline{r}_a, \underline{r}_b)$ constitute a non-degenerate equilibrium for the subgame Γ_t, these conditions are automatically satisfied for $j \geqq t$. All that remains is to establish these conditions for player α when $j < t$. By assumption,

$$0 \geq P_\alpha(k,F_\beta) - A_\alpha(t-1) \geq P_\alpha(k,F_\beta(\cdot;j,\underline{r}_\beta)) - A_\alpha(j)$$
$$= P_\alpha(k,F_\beta(\cdot;j,\underline{r}_\beta)) - P_\alpha(j,F_\beta(\cdot;j,\underline{r}_\beta)$$

for all $k \geq t$. For $j < k < t$

$$P_\alpha(k,F_\beta(\cdot;j,\underline{r}_\beta)) - P_\alpha(j,F_\beta(\cdot;j,\underline{r}_\beta)) = A_\alpha(k) - A_\alpha(j) < 0.$$

Q.E.D.

<u>Proof of Lemma 6.2</u>:

Consider any degenerate subgame perfect equilibrium $(\underline{r}_a,\underline{r}_b)$ with $r_\alpha(0) = 1$. Let $t^{**} = \sup\{t \geq 1: A_\alpha(t-1) \geq 0\}$ (From Lemma 6.1, we know that $A_\alpha(0) \geq 0$ and hence that $t^{**} > 0$.). Define $r^*_\alpha(t) = 1$ and $r^*_\beta(t) = 0$ for $t < t^{**}$ and $r^*_\alpha(t) = r^*_\beta(t) = 0$ for $t^{**} \leq t < T$. To establish that $(\underline{r}^*_a,\underline{r}^*_b)$ is subgame perfect, we need only establish that conditions (ii) and (iii) of Lemma 6.1 are satisfied.

If $t^{**} = T$, then these conditions are satisfied by definition of t^{**}. Suppose $t^{**} < T$. Then Lemma 6.1 implies that for any t with $r_\beta(t) > 0$, $(\underline{r}_a,\underline{r}_b)$ forms a nondegenerate equilibrium for the subgame Γ_t. Lemma 3.2 implies that $q_\beta(T;t^{**}) > 0$ and Lemma 3.3 implies that $t^*(\underline{r}_a,\underline{r}_b) \leq t^{**}+1$. Therefore,

$$P_\alpha(T;F_\beta(\cdot;t^{**})) - P_\alpha(t^{**};F_\beta(\cdot;t^{**}))$$
$$= q_\beta(t^{**};t^{**})[B_\alpha(t^*)-C_\alpha(t^*)] - q_\beta(T;t^{**})A_\alpha(t^*) > 0.$$

This in turn implies that $q_\alpha(T;t^{**}) = 1$ which implies that

$$0 \leqq P_\beta(T;F_\alpha(\cdot;t^{**})) - P_\beta(t^{**};F_\alpha(\cdot;t^{**})) = -q_\alpha(T;t^{**})A_\beta(t^{**})$$

Since $A_\alpha(t^{**}) < 0$, by definition, we may then conclude that $(\underline{r}_a^*,\underline{r}_b^*)$ is a equilibrium for Γ_{t*}. Finally, since $P_\alpha(t,F_\beta^*) = A_\alpha(t)$ for all $t < T$, it follows that condition (iii) is satisfied as well.

To establish the last statement of the lemma, we simply note that for all t,

$$P_\alpha(T,F_\beta^*) - P_\alpha(t,F_\beta^*) = -A_\alpha(t).$$

Consequently, if $(\underline{r}_a^*,\underline{r}_b^*)$ is to be an equilibrium, we must have $A_\alpha(t^*) \leqq 0$ whenever $t^* < T$, and $A_\alpha(t^*-1) \geqq 0$. Q.E.D.

Proof of Theorem 6.2:

To establish the necessity of these conditions, suppose $(\underline{r}_a,\underline{r}_b)$ is a subgame perfect equilibrium with $r_\alpha(0) = 1$. Lemma 6.2 implies that there is a $t^* > 0$ such that $r_\alpha(t) = 1$ for $t < t^*$ and $r_\beta(t) = r_\alpha(t) = 0$ for $t^* \leqq t < T$. But then Lemma 6.1(iii) implies that $A_\alpha(0) \geqq P_\alpha(T;F_\beta) = 0$. This establishes condition (i).

To establish the necessity of conditions (ii) and (iii), consider a subgame perfect degenerate equilibrium satisfying the conditions of Lemma 6.2. Note first that Lemma 6.2 implies $t^* \leqq \tau_\alpha+1$. Then if $t^* = T$, conditions (ii) and (iii) follow immediately. If $t^* < T$, then since $(\underline{r}_a,\underline{r}_b)$ is an equilibrium for Γ_{t*}, we must have

$$0 \leq P_\beta(T,F_\alpha(\cdot;t^*)) \geq P_\beta(t^*,F_\alpha(\cdot;t^*)) = A_\beta(t^*) \leq 0$$

which again implies $\tau_\beta \leq t^* \leq \tau_\alpha+1$. Finally, if $A_\alpha(\tau_\alpha) < 0$, then Lemma 6.2 implies that $t^* = \tau_\alpha$ in which case, we obtain $\tau_\beta \leq \tau_\alpha$. Q.E.D.

Proof of Theorem 6.3:

It is sufficient to check that the strategy pair $(\underline{r}_a,\underline{r}_b)$ defined by $r_\alpha(t) = 1$ and $r_\beta(t) = 0$ for all t is an equilibrium for all subgames Γ_t, $t \geq 0$. Q.E.D.

Proof of Theorem 6.4:

If $T = \infty$ and $A_\alpha(0) \geq 0$, then Theorems 4.6 and 6.3 imply that both a nondegenerate equilibrium and a subgame perfect equilibrium with $r_\alpha(0) = 0$ exists.

Suppose $T < \infty$.

(i) Theorems 4.1 through 4.3 imply that a nondegenerate equilibrium exists if and only if $\tau_\beta-1 \leq \tau_\alpha \leq \tau_\beta+1$. If $A_\alpha(0) \geq 0$, Theorem 6.2 implies that these conditions are sufficient for the existence of a subgame perfect equilibrium with $r_\alpha(0) = 1$.

(ii) If $A_\alpha(0) \geq 0$ for $\alpha = a,b$, then Theorem 6.2 implies that these conditions are both necessary and sufficient for the existence of a subgame perfect equilibrium with $r_a(0) = 1$ and a subgame perfect equilibrium with $r_b(0) = 1$.

Q.E.D.

Figure 1. The War of Attrition in Extensive Form

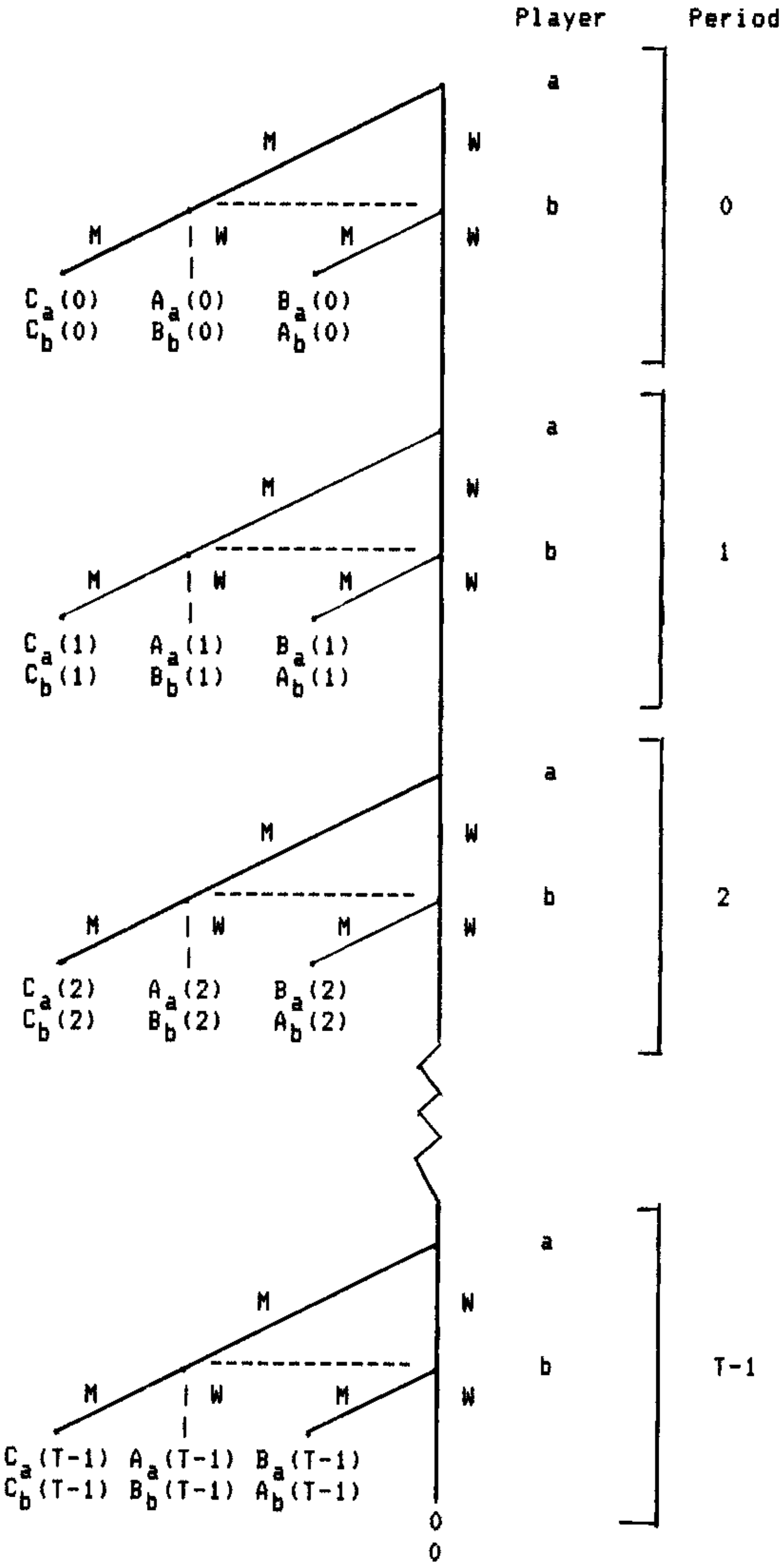

Figure 2. Possible Patterns For Non-degenerate Equilibria

Fully Mixed Equilibria

time:	0	1	2	3	t^*-2	t^*-1	t^* ...	$T-1$	T
r_α:	x	x	x	x	x	x	0 ...	0	?
r_β:	x	x	x	x	x	x	0 ...	0	?

Alternating Equilibria

time:	0	1	2	3	t^*-2	t^*-1	t^* ...	$T-1$	T
r_α:	0	x	0	x	0	x	0 ...	0	?
r_β:	x	0	x	0	x	0	0 ...	0	?

Hybrid Equilbria

time:	0	1	$\tilde{t}-1$	$\tilde{t}$	$\tilde{t}+1$ ...	t^*-2	t^*-1	t^*	$T-1$	T
r_α:	x	x	x	0	x	0	x	0 ...	0	?
r_β:	x	x	x	x	0	x	0	0 ...	0	?

x means $r_\alpha(t) > 0$.

Figure 3

A Continuum of Fully Mixed Equilibria

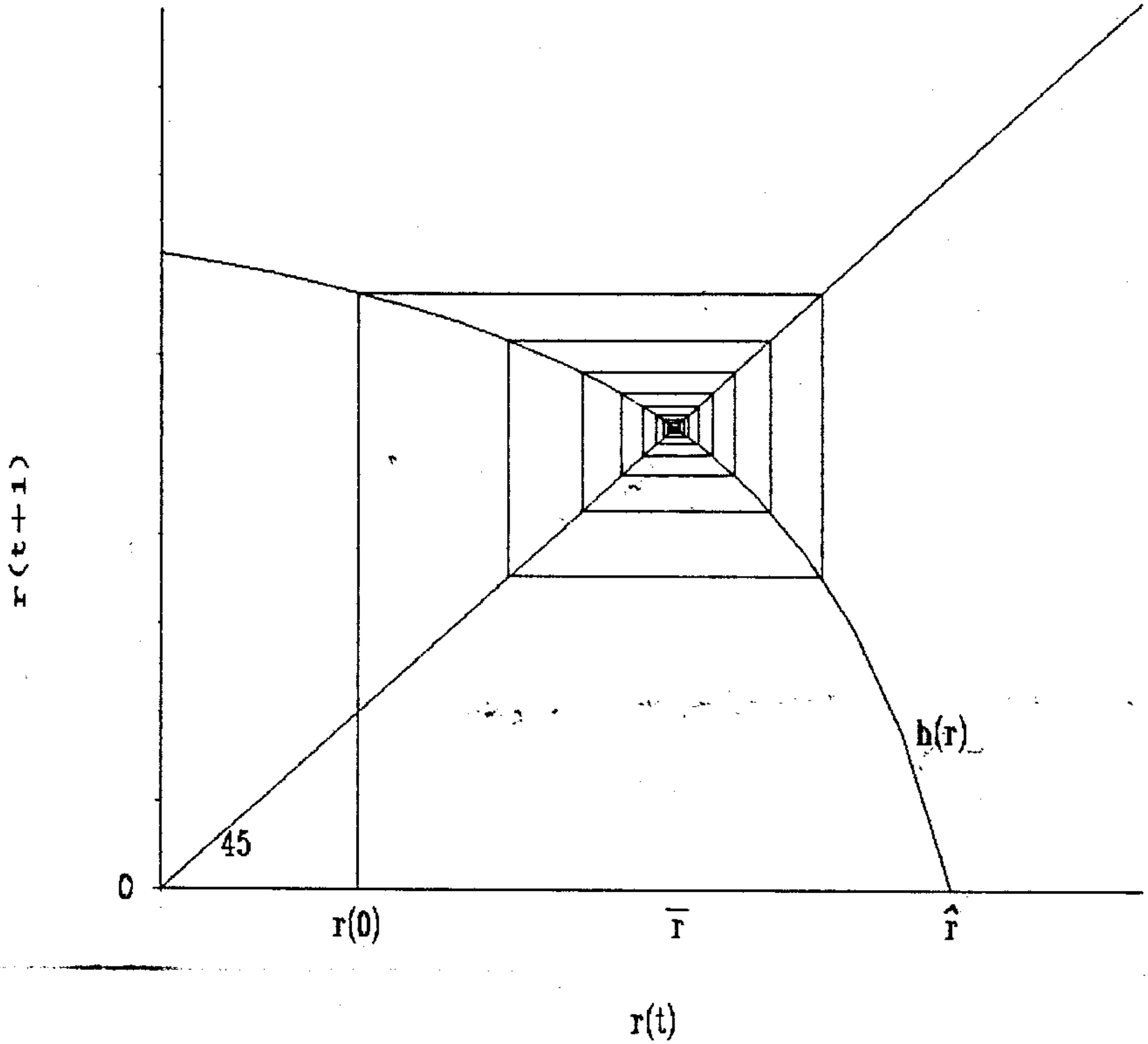

Figure 4

No Fully Mixed Equilibrium Exists