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ABSTRACT

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This paper draws the tentative conclusion that a single class of nonlinear, damped, forced, oscillator with delay can be used to describe the growth rates for both the indices of consumer durable and nondurable goods production. These data are monthly data from 1919 to 1988. The same class of model fits the entire period, although with parameter drift. The model is prescribed to track the seasonal components of the time series. However, the degree of fit as measured by R^2 has a low value of about 79% during the war years and is often in excess of 96%. Variations in the series at business cycle frequencies are re-expressed by this model in terms of drift in the values of the parameters. Examination of the Laplace transform of the linear approximation indicates that there has been a qualitative change in the dynamical properties of both series and that the two series also differ qualitatively; these conclusions are drawn based on an examination of the differences in the parameter values within the same class of model.

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INTRODUCTION

In a previous paper, Ramsey and Lian(1992), and in a prior most innovative paper by Barsky and Miron(1989), developed and illustrated the idea that the well known seasonality of some economic time series could be usefully exploited. Barsky and Miron examined quarterly dummies. Ramsey and Lian examined the growth rates of monthly data. This renewed interest in seasonal components for their own sake is a welcome return to the earlier concerns of the NBER.

The major post War emphasis for empirical macro-economic analysis has been on the examination of economic behavior at intermediate length scales; that is, on the "business cycle," see for example, Durlauf(1990), Falk(1986), and Gabisch and Hans-Walter(1987). Consequently, the time path of economic variables at shorter length scales, such as seasonal variation, and the path of economic variables at very long time scales, usually designated "drift", or "trend" were removed with little attention paid to the details of these components, except in so far as those details might affect the analysis of business cycles at intermediate length scales.

However, this myopic attention on the intermediate term business cycle by eliminating seasonal components and "trend", may well have compounded the difficulties involved in that analysis. A major tenet of this paper is that the business cycle of most interest to economists is not understandable until the higher frequency components are themselves well understood and accurately modeled. The reasons for believing this are many, but an obvious first point is the recognition that the lower frequency components of a time series are often the sub-harmonics of higher frequencies and of their differences. The second obvious point to make in this context is that the use of an inaccurate "model" of the seasonal components to remove seasonality may well obscure the observation of the intermediate frequencies.

Seasonal variation is the closest that economic data come to exhibiting steady state oscillatory behavior. Now that interest in the dynamics of economic relationships has been renewed, it is opportune to reopen the analysis of the seasonal components of economic time series. The paper by Ramsey and Lian showed graphically and by other more traditional means that the indices of monthly production of consumer durable and nondurable goods have a well pronounced cyclic pattern and that that pattern is remarkably robust to war and depression. The work in that paper generated the idea that the seasonal components might be modeled by a single class of models that would differ over time only by parameter drift. The drift in the parameters of the seasonal model would then become the inputs to an analysis of the business cycle.

The objective of this research was to attempt to find a single model for the seasonal components of each economic index that exhibits seasonal variation. Attention was focused on just two indices; the indices of production for durable and nondurable consumer goods. It was anticipated that the parameters of such a model would be subject to drift and to occasional shocks. The work of both Barsky and Miron and of Ramsey and Lian indicated that a high proportion of the total variation of the series would be explained by the seasonal model.

MODEL SELECTION

The variables of interest in the previous paper by Ramsey and Lian were the growth rates, that is, the relative first differences, for some indices of production. Growth rates were taken as the variable of interest for several reasons. First of all, the growth rates are of interest in their own right. Secondly, it is well known that the statistical properties are seemingly simple. And finally, growth rates are closer to exhibiting "steady state" stochastic behavior than is true for the levels that are dominated by "drift."

The phase space diagrams produced in the Ramsey and Lian paper indicated that the seasonal components of some production indices are oscillators; that is, the second derivative is a function of the growth rate and of the first derivative of the growth rate. The work reported in Ramsey and Lian (1992) indicated that the differential equation describing the oscillator seems to be forced by a term that can be approximated by a Fourier series, see Equation [1].

$$\ddot{u}_t = g(\dot{u}_t,\ u_t,\ f(t))$$
$$\text{where:}$$
$$f(t) = \text{forcing term}$$
$$u_t = \text{growth rate} \tag{1}$$

Because there is no theoretical model commonly accepted as a useful approximation to the seasonal components of the growth rates of the indices of interest, the research in this paper is unabashedly "data driven." That is to say that as a first step in the understanding of these types of data, the attempt was made to discover from analysis of the data themselves the probable form of the underlying model. However, the approach is far from a naive "maximization of R^2" by choice of variables.

First of all the choice of model was guided by the analysis of the plots and other evidences of periodicity that were examined in the Ramsey and Lian paper; a copy of one such plot, Figure 1, is included from that paper to illustrate the idea. Spectral analysis indicated that there were four seasonal frequencies, the corresponding periods of which are; one half, one third, one fourth, and one fifth of a year.

The chosen model had to meet two sets of informal criteria; one local and one global. The local criteria were the usual ones in that a reasonable level of R^2 was sought, very highly significant " t" statistics, and very highly significant F ratios for the extra regression sums of squares for the potential inclusion or exclusion of candidate variables. Some analysis of the properties of the residuals was carried out, but as it was anticipated that there would be observable systematic effects at intermediate and lower time scales, there was no attempt to achieve white noise residuals. However, every effort was made to ensure that in examining the properties of the residuals that there was no evidence of systematic behavior at seasonal time scales.

The global criteria, even though informally carried out, were very important in trying to achieve a single stable model of the data. Various subsets of the data were fitted and the results compared to each other. The underlying concept was that a single class of model should be relevant for all sub-periods, even though each period might be characterized by different parameter values.

Therefore, the idea of symmetry in the results was an important component. For example, in the approximation to the "forcing term" in the differential equation for the growth rate, both the cos and sin terms should be included in each sub-period, even though for some one sub-period the recorded "t" statistic for one of the spectral coefficient pairs might not indicate a reasonable level of statistical significance. Another example is that if the model is an oscillator, then some form of dampening, whether linear or nonlinear, is required; consequently the sought differential equation must contain some function of the first derivative. However, symmetry in the response was not imposed; that is, the assumption that acceleration would be affected in the same way for both positive and negative levels of growth was not imposed.

While variation in the parameters of the model over sub-periods is to be expected, the presumption was that such variation would be slow relative to the relevant periods of seasonal oscillation. Consequently, the approximate stability, subject to observational error, of the estimated coefficients across sub-periods was an important consideration. This requirement is, of course, a special case of the more general requirement that the same variables and functional form apply for all sub-periods. While functional uniformity over time was an important criterion, such conformity was not imposed across indices.

Once a model was chosen for each index, the usefulness of the model was tested by "forecasting", or by extrapolation of the estimated model. More precisely, the empirically discovered equation was fitted to the post war data up to, but not including, the last five years of observations. The estimated model was used to "forecast" out of sample the dependant variable, which is the second derivative of the growth rate. However, the observed values for the growth rate and the first derivative of the growth rate were used as the "explanatory" variables. The idea for this comparison is merely to see whether, notwithstanding the presumed drift in some coefficients, the estimated model provides evidence for its continued relevance.

Finally, using the estimated coefficients the model was simulated by the Runga-Kutta method both in sample and out of sample for the last sixty months. The objective, given that there is recognized parameter drift, was to explore the robustness of the results and to examine the extent to which the simulations over intermediate length scales would diverge from the observed data.

While the simulation "forecast" is potentially a stringent test, the coefficient estimates obtained from a simple least squares fit of the differential equation are not accurate enough to produce truly useful results. Nevertheless, the exercise is informative in that non-divergence of the simulated data from the path of the actual data over reasonably long periods of time provides reassurance as to the reliability of the model, even at this first crude stage of the analysis. This part of the analysis is to be regarded as merely indicative and preliminary.

ANALYTICAL SOLUTION OF THE "LINEARIZED" VERSION OF THE OSCILLATOR

The model to be discussed below was obtained after a very considerable amount of empirical research. The idea for the chosen model was suggested by some practical work on analyzing the periodic behavior of tankers attached to fuel mooring buoys; for a most enlightening explanation of this example see Thompson and Stewart (1986). Some of the models that were rejected in favor of that shown below are various versions of the Duffing, Mathieu-Hill, and Van der Pol equations.

While the complete nonlinear differential equation that seems to apply to both the data sets has not yet been solved analytically, a linearized version has. The full model is shown in

Equation 2 and the linearized version in Equation 3.

$$\ddot{u}_t + \alpha \dot{u}_{t-1} + [\sum_{i=1}^{4} \beta_i \hat{u}_{it}] = \sum_{i=1}^{4} [a_i \cos(\omega_i t) + b_i \sin(\omega_i t)]$$

where:

$$\begin{aligned} \hat{u}_{it} &= [\gamma_{1t}u^+, \gamma_{1t}u^-, \gamma_{2t}u^+, \gamma_{2t}u^-] \\ \gamma_{1t} &= 1;\ \textit{if mth} = \textit{Sep-Feb};\ 0, \textit{otherwise}; \\ \gamma_{2t} &= 1;\ \textit{if mth} = \textit{Mar-Aug};\ 0, \textit{otherwise}; \\ u^+ &= pos(u_t) \\ u^- &= neg(u_t) \end{aligned} \quad (2)$$

$$\ddot{u}_t + \alpha \dot{u}_{t-1} + \beta u_t = \sum_{i=1}^{4} [a_i \cos(\omega_i t) + b_i \sin(\omega_i t)] \quad (3)$$

While there are a total of thirteen coefficients that were estimated, the model is in fact quite simple. The basic structure of the model is that of a damped, forced, oscillator, where the oscillator component allows for nonlinearity and time irreversibility and the dampening term is subject to a delay.

The Fourier series approximation to the forcing term involves eight terms, four cos terms and four sin terms. There are four fundamental frequencies corresponding to the periods of: one half, one third, one fourth, and one fifth of a year; and in the empirically observed data one must allow for four arbitrary phase terms.

The dampening term enters the equation with a delay of one month, which in itself introduces an element of nonlinearity into the equation.

The four oscillator terms involving the growth rate are a first approximation to the actual, as yet, unknown nonlinear term.

The essential idea that is captured by this formulation is that the change in acceleration, that is, the change in the value of the second derivative depends on both the sign of the growth rate and on the period of the year in which the effect occurs. The two selected periods of the year are September through March and April through August. Clearly, this portion of the equation requires further insight and more analysis to perceive what this crude approximation represents. In any event, the specification allows the acceleration reaction to differ depending on whether growth rates are positive or negative, so that if they differ, then one can conclude that the process is time irreversible, even in the absence of the linear dampening term that itself produces time irreversibility.

The linearized version that is shown in Equation (3) is easily solved by the use of Laplace transforms, see for example, Strang(1986). However, this can only be done by using an approximation to the exponential term that is introduced by the delay in Du_t. It is useful to view the homogeneous and particular solutions separately.

Consider first the homogeneous version of Equation (3). Let U_0 and $\dot{U}_0$ be the initial conditions that hold in period t = 0. The Laplace transform of the homogeneous equation is:

$$U(s)\,(s^2 + \alpha s e^{-s} + \beta) - U_0(s + \alpha e^{-s}) - \dot{U}_0 = 0 \qquad \textbf{(4)}$$

By solving for U(s) in this equation, we get:

$$U(s) = \frac{U_0(s + \alpha e^{-s}) + \dot{U}_0}{(s^2 + \alpha s e^{-s} + \beta)} \tag{5}$$

The denominator in this equation can be factored by its roots, $\{\lambda_i\}$, so that Equation (5) can be solved by partial fractions to yield the solution:

$$\begin{aligned} u_t &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} + Ce^{\lambda_3 t} \\ A &= \frac{U_0(\lambda_1 + \alpha e^{-\lambda_1}) + \dot{U}_0}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \\ B &= \frac{U_0(\lambda_2 + \alpha e^{-\lambda_2}) + \dot{U}_0}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \\ C &= \frac{U_0(\lambda_3 + \alpha e^{-\lambda_3}) + \dot{U}_0}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \end{aligned} \tag{6}$$

This result is obtained by using a quadratic approximation to the exponential; that is, the term e^{-s} is replaced by $(1 - s + s^2/2)$. This approximation varies from reasonable to excellent for values of s below 1 in absolute value. Consequently, the major effect on the solutions is that the single real root is over estimated in absolute value, especially when the absolute value of the actual real root is greater than one.

The Laplace transform of the forcing term is given by:

$$V(s) = \sum_{i=1}^{4} a_i\left(\frac{s}{s^2 + \omega^2{}_i}\right) + b_i\left(\frac{\omega_i}{s^2 + \omega^2{}_i}\right) \tag{7}$$

The particular solution is given by solving the equation:

G(s) is the Laplace transform of the homogeneous equation. This

$$U(s) = \frac{V(s)}{(s^2 + \alpha s e^{-s} + \beta)} = V(s)\,G(s), \tag{8}$$

form of the Laplace transform indicates that the solution is the convolution sum of the inverse transforms of V(s) and of G(s). The particular solution to the linearized delay version of the differential equation is:

$$\begin{aligned} u_t &= \int_0^t (Ae^{\lambda_1(t-s)} + Be^{\lambda_2(t-s)} + Ce^{\lambda_3(t-s)})\, f(s)\, ds \\ f(s) &= \sum_{i=1}^{4} a_i \cos(\omega_i s) + b_i \sin(\omega_i s) \end{aligned} \tag{9}$$

The coefficients A,B, and C in the last equation are the same coefficients that were obtained in the homogeneous solution above. The complete solution is the sum of the homogeneous and the particular solutions as given by Equations 6 and 9.

DETAILED DISCUSSION OF THE EMPIRICAL RESULTS

Only two indices have been examined, the production indices for consumer durables and for consumer nondurables. The data are monthly from January 1919 to April 1988, but three observations are lost at the beginning and two at the end of the series because of the calculation of the derivatives of the growth rates; details of these calculations and of the data sources are in Ramsey and Lian(1992).

This section is in four parts: a very brief review of the empirical findings from the previous paper, Ramsey and Lian(1992);

a detailed discussion of each of the two indices; and finally a comparison of the results for the two indices.

SUMMARY OF CURRENT INFORMATION ON THE TWO INDICES

Both series are well known to contain seasonal components, Barsky and Miron(1991) and the references contained therein.

The nondurable series have significant power at the "seasonal" frequencies that correspond to the periods one half, one third, one quarter, and one fifth of a year. In addition, the Ramsey and Lian (1992) paper, hereafter RL, also showed that the nondurable series have power at the frequencies that correspond to the periods of one year and 243.5 months, the latter having by far the larger power.

Figures 1 and 1A show in a striking manner the "shape" of the seasonal periodicity of the nondurable series. Figure 1 plots the phase space diagram for the growth rate of nondurable goods; that is, Figure 1 plots the first two derivatives of the growth rate, u_t. Figure 1 was obtained by seasonally smoothing the observations on Du_t and D^2u_t with a seven point smooth at a twelve month interval. The accompanying Figure 1A shows that with respect to the smoothed data during the post war period that the variation in the phase space diagrams is due more to phase shifting than it is to random noise.

The amplitude of the phase space orbit is the sum of squares of the components Du and D^2u. Figure 2 shows the time series plot of the amplitudes for the entire history of the data. From this plot we see that the amplitudes have varied enormously over the whole period; the Depression years had the biggest amplitude and the War itself the smallest, the ratio of smallest to largest being about nine to one. The late post war period has been characterized by a steady decline in the range of amplitude variation.

The durable consumer goods index was analyzed in a similar manner. The durable goods index has a strong set of seasonal frequencies that match those of nondurable goods, as well as the annual and the 243.5 month cycles. In addition, the durable goods index has power at the frequencies that correspond to the periods of 18.2 and 34.7 months respectively. The plot of the squared amplitudes for durable goods has the same general shape as that for nondurable goods that is shown in Figure 2, except that the late post war decline in the range of amplitude variation has been far more extensive, see Figure 6.

SPECIFICATION OF NONDURABLE GOODS PRODUCTION INDEX AS AN OSCILLATOR

The nonlinear model that was discussed in detail in the previous section was arrived at in accordance with the procedures that were outlined above. Because considerable experimentation was involved in the process, notwithstanding the global constraints that were imposed on the choice of model; the danger of over fitting remains high. If a model is seriously over fitted, that is, the parameter estimates and the choice of the model itself are mainly functions of noise, then its forecasts can be expected to be both unreliable and the paths of the actual and the forecast series should in general be expected to diverge strongly and quickly.

Consequently, the various forecast checks that were implemented are even more important than usual. The outcomes are correspondingly even more interesting.

The final model that was fitted to all the data was that shown in Equation [2]; Equation [10] shows the stochastic version in which an error term, ϵ_t, has been added to represent observational errors; the definitions of the other variables and the coefficients are the same as those given in Equation [2]. The chosen parametrization is such that the theoretically stable signs for the coefficient α and the sum of the coefficients { β_i } are positive; there are no a priori constraints on the signs of the coefficients

$$\ddot{u}_t = -[\sum_{i=1}^{4} \beta_i \hat{u}_{it}] - \alpha \dot{u}_{t-1} + \sum_{i=1}^{4} [a_i \cos(\omega_i t) + b_i \sin(\omega_i t)] + \epsilon_t \tag{10}$$

{ a_i, b_i }.

The procedure used to estimate the equation was ordinary least squares, even though the presumption of an additive error term for D^2u implies that both Du and u also contain errors of observation. Consequently, the ordinary least squares estimates are biased.

However, given the realized small error variances and the reasonable assumption that the sizes of the observational errors in Du and u are nearly equal to that for D^2u, the bias effect will be small. Further, the ordinary least squares estimates obtained below are to be regarded as providing initial conditions for a more rigorous analysis that will combine Runga-Kutta solutions to the differential equation with fitting the data. The preferred method will, once implemented, provide simultaneous estimates for all three variables, u_t, Du_t, and $D^2{}_t$ and will be able to allow for the joint inaccuracy in all three variables in the equation.

Tables 1 to 4 summarize the regression results. Tables 1 to 3 present the estimates for separate regressions on three separate periods; the middle part of the pre-war period, 1926 to 1937, the war period including the immediate post war recovery, 1937 to 1962, and the later post war period, 1962 to 1987. Table 4 presents three overlapping sets of estimates for the late post war period, 1962 to 1975, 1969 to 1981, 1975 to 1987. The major point of interest is the extent to which the coefficient estimates are constant, or indicate a relatively slow drift over time. The drift in coefficient values potentially provides a succinct statement of the intermediate term effects that require further study. For example, the differences in the coefficient estimates for the war period, or for the depression, relative to the values for the late post war period enable one to characterize the effects of the war, or of the depression, in terms of those differences in parameter values.

The overall impression from examining these Tables is that the effect of Du_t, α in the Tables, in dampening the oscillations increased during the war, but that the post war period is characterized by a weak dampening term. For the β_i terms, only the first and the third varied to a significant extent; the first declined during the war and recovered later, whereas the third coefficient either declined to very small values, or even became locally unstable; the evidence from Table 4 lends some credence to the latter notion. The effect of the one half year period forcing term declined during the war, but returned to pre-war levels later. The one fourth year period forcing term increased steadily.

Comparing the variation in the coefficient estimates for the late post war period is more revealing as these data are less noisy. The α and β_2 coefficients increase monotonically over this period, the coefficients for the first two forcing frequencies decline, and the rest remain constant, some remarkably so. Plots for the variation of α and for the first two forcing terms during this period are shown in Figures 3(a) to 3(c).

Figure 3(d) shows a similar plot for the α coefficient variation during the pre war period. The remaining coefficient estimates for the pre war period exhibited little systematic variation relative to the relatively large variances of estimate that prevailed during this period.

Overall, the variance estimates for all coefficients are remarkably stable over very long intervals of time.

These numerical results become interesting mainly in the context of the underlying differential equation. Unfortunately, because the oscillator term is nonlinear this is not an easy task. However, regression and simulation experiments have indicated that the linearized version as shown in Equation (3) is a reasonable approximation to the major movements of the series. Consequently, we can consider as a first approximation a linearized version obtained by substituting u_t for the $\{\hat{u}_{it}\}$ and using the mean of the $\{\beta_i\}$. The homogeneous portion of the differential equation including the delay in Du_t is then easily obtained by the application of the Laplace transform. If u_0 and $\dot{u}_0$ are the initial conditions for the differential equation, then the Laplace transform of the linearized equation is:

$$U(s) = \frac{U_0(s + \alpha\exp(-s)) + \dot{U}_0}{s^2 + \alpha s\exp(-s) + \beta} \tag{11}$$

The term $(s^2 + \alpha s\exp(-s) + \beta)$ in equation 5 has three roots; two that derive from the basic damped oscillator and a third that is generated by the presence of the exponential term that itself arises from the delay in Du_t.

Table 5 shows for each of the sub-periods discussed in Tables 1 to 4 the roots for this equation. The roots were calculated by using the quadratic approximation to the exponential term. Two other calculations of potential economic interest were carried out. The Maximum Delay Period, (MDP), is the maximum over the three roots of the delay period, that is , of the time required to reduce the term, $e^{(-\rho t)}$ to $e^{-1} = 0.368$, where the root $\lambda_i = \rho_i + i\delta_i$. The MDP provides a measure for the longest time duration of any shock to the system, the larger MDP, the longer the effects of any given shock will be felt.

The period that is shown in the last column of Table 5 is the period in months that corresponds to the natural frequency of the oscillator as given by δ_i, $P = 2\pi/\delta$. The natural frequency of the homogeneous part of the equation is important for analyzing the dynamical path of the system both in the short run and in the longer term. This is especially true when the natural frequency and the forcing frequencies are incommensurate, or have small differences. In these cases, the "short run" properties of the system determine the behavior of the system at intermediate, or even, at long time scales.

Figures 4(a) to 4(f) are plots of the Laplace transform function $s^2 + \alpha s e^{-s} + \beta$ for each of the main sub-periods and the three sub-periods of the last period that were used for the estimation of the coefficients in Table 5. The "generic" shape for this equation is shown in Figures 4(c) to 4(f). Because e^{-s} will dominate the other terms eventually, the equation always has at least one real root. As the parameters α and β vary, so does the degree of curvature of the equation and whether there are one or three real roots. This equation provides a practical example of critical phase transition in that, looking only at the three major periods, one sees that the pattern of roots is critically dependent on the position of the relative minimum of the curve that lies between the real root and zero. If the coefficients of Equation 3 change in such a manner that the relative minimum is increased from a negative value, the solution space will pass from three real roots with no natural harmonic, through a single repeated root at the critical point, to a pair of complex roots with a natural frequency.

Except for the curve in Figure 4(c) representing the "average effect" during '62 to '87, the equation has complex roots for all other periods and even for the sub-periods during '62 to '87; this last result is an example of the limitations of using the linear approximation. However, the curves in Figures 4(d) to 4(f) indicate that each of the sub-periods is close to being "forced" into the "three real root" phase, that is, a relatively minor change in the average value of $\{\beta_i\}$, or in α, will change the dynamical system from one having a pair of complex roots to one having three real roots.

The salient conclusions from these results for the homogeneous part of the differential equation are that the late post war period has a declining Maximum Delay Period, MDP, that is, a declining period for the duration of any shock to the system and that the period of the natural frequency, while substantially greater than in the prewar period, has itself been declining rapidly. This conclusion, together with the observed shifts in the parameters for the forcing terms that were noted above, characterize the change in the dynamics that are occurring in the nondurable sector of the economy.

FORECASTS AND SIMULATIONS

While an effort was made to derive an equation that would, up to parameter drift, be suitable for all periods of time, the key question is whether the model that has been obtained is robust to simulation and whether forecasts based on these results are useful, even without allowing for parameter drift. The purpose of this section is to document the extent to which this is true.

To this end, the last three hundred months were selected from the series and the model was re-estimated using only the first two hundred and forty observations. Two sets of tests were performed.

The first test was to see whether the regression equation itself, notwithstanding its potential regression estimate bias, could be usefully extrapolated to the remaining sixty months of observations without any alteration in the equation, or refitting of the parameters in light of the above documented changes in the parameter values. In this check of the results, the observed values of u_t and of Du_t were used. The statistic that was used in this test was the "relative mean squared error", where the relative mean squared error, RMSE, is defined by the ratio of the mean squared error of forecast, $\sum (D^2 u_t - D^2 \hat{u}_t)^2$, where $D^2 \hat{u}_t$ is the least squares forecast of D^2u_t, to the estimated variance of the variable D^2u_t. The result is an average over the sixty forecast months.

The nondurable goods results can be easily summarized. The RMSE overall for the sixty month forecasts is 7.3% and the five year average for each month of the year varies from a low of 0.1% in October to a high of 6.6% in February; most of the RMSE's are about 2%. A plot of the predicted values against the observed for the entire three hundred observations is shown in Figure 5. There is no indication that there is an increase over time in the RMSE over the time horizon of the forecast.

The second test is potentially more stringent in that the estimated parameter values together with three initial conditions would be used to simulate simultaneously by use of a Runga-Kutta expansion the entire time path of the triple (u_t, Du_t, D^2u_t). Unfortunately, while the least squares regression estimates are useful initial conditions for a full analysis, they are unlikely to provide results of sufficient accuracy to be able to track the finer details of the series. Further, given the known drift in the parameter values, it is most unlikely that a simulation based only on the regression estimates will provide useful forecasts of the variables. This is especially true for u_t. Nevertheless, a simulation of the system based on the regression estimates can be helpful in determining the level of the general degree of fit and whether the results seriously degrade over time. The simulation test is a tougher test in that there is no reliance on the observed values of the growth rate itself and of its derivative.

Using the same coefficient estimates as were used in the last set of comparisons together with the appropriate initial conditions, Runga-Kutta methods were used to simulate without added noise the triple $(u, Du, D^2u)_t$ over the last three hundred month period.

The spectra of the simulated series were compared to those of the actual series for the same period; the former set of spectral plots were essentially smoothed versions of the latter. The range and variance of each of the simulated variables was less than the corresponding value for the actual series over the simulated period. Numerous plots of both sets of series indicated that there was broad agreement between the two sets of series. However, the details were often missed as were the exceptionally large values for the observed variables, and the timing of changes in direction were often off by one or two time points. There was no indication that the simulated series were diverging over time from the actual series, either by exploding or collapsing.

Initial estimates of the RMSE between the crude simulated series and the actual over the forecast period were not impressive as was to be expected. The RMSE over sixty months for D^2u was 19.9%, for Du was 44% and for the growth rate, u_t itself, 68%. The progression from low to high values for the RMSE going from D^2u_t to u_t is to be expected, since the regression model minimizes the error sum of squares for D^2u_t only. In order to put these results into better perspective, a series of sub-sets of sixty observations on u_t were selected from the fitted data, that is, the data that provided the coefficient estimates, and the RMSE's were calculated for these "in sample" forecasts. The results were 46%,102%,54% and 53% for a median value of about 54%.

The main lesson to be gathered from these results is that the out of sample results are as good, or as bad, as the in sample results, that the model, even without adjustment for the known drift in the parameters, maintains an approximation that does not degrade over time out of sample.

SUMMARY OF DURABLE GOODS PRODUCTION INDEX AS AN OSCILLATOR

The research strategy did not impose the constraint that the chosen functions should be the same for both indices, although that was the outcome. This result is fortunate in that having the same functional form facilitates the comparison of the dynamical properties of the two series. Tables 6 to 9 summarize the least squares regression fits in the same format as was used for the nondurable goods index.

The seasonally smoothed amplitude plot for the durable goods index is shown in Figure 6. One of the more striking features of this plot is the strong decline during the late post war period in the maximum amplitude of the series; this condition will have important implications for the subsequent analysis.

Comparing coefficients across time periods in Tables 6 to 9 and as summarized in Table 10, one notices first that α was very low in the thirties, and somewhat high during the War and its immediate recovery period relative to the late post war period. The average β coefficient was high during the depression, declined substantially during the war and then recovered in the post war period. Similar comments apply to the first two coefficients in the set of forcing terms; the remainder, at least relative to the observed standard errors, were constant.

The post war period when divided into three sub-periods indicates a monotonic drift up for α and down for β, see Table 10.

A plot of the observed values for D^2u against the predicted for the post war period is shown in Figure 7. The next figure shows the increase in the value of the forcing term corresponding to the period of one half year for a sequence of "rolling fits"over the entire period 1962 to 1987, this is the equivalent plot to those shown in Figures 3(a) to 3(d) for nondurable goods.

As was the case with nondurable goods index, the estimates of the variances of all the coefficients are remarkably constant as a review of Tables 6 to 9 will confirm.

Table 10 summarizes the coefficient values and the derived values for the roots of the Laplace transform of the homogeneous form of the delay differential equation. Figure 9 presents graphs for each of the three major sub-periods of the function $((\alpha/2)s^3 + (1-\alpha)s^2 + \alpha s + \beta)$, which is the quadratic approximation for the denominator in Equation (5); the plots for the sub-periods of the late post war period are not included since they are qualitatively the same as that shown in Figure 9(c).

During the war period the natural period of the homogeneous equation was about five months and during the late post war period the natural period was constant at about six months. The MDP was about three months during the war, but since the war MDP increased from about two to three months.

The model was refitted to the first three hundred observations of the last three hundred and sixty observations. The estimated coefficients were used to forecast the next sixty months as was done for the nondurable goods index. Recalling Figure 6, it is to be anticipated that the RMSE test will not be very impressive because of the shift in the roots of the differential equation. Nevertheless, the experiment was run and the forecasts are plotted with the observed values in Figure 10. The within sample forecast error was 4.6%, whereas the out of sample error was 44%. This difference arose in large part because the variance of D^2u fell from 2.88 E-4 within sample to 1.43 E-4 out of sample.

Because of the substantial change in the roots the Runga-Kutta simulations were not run.

A COMPARISON OF RESULTS FOR THE TWO INDICES

A review of Tables 5 and 10 will highlight the comparisons between the two indices. One of the striking characteristics is that the Laplace transform roots for the two indices during the war are remarkably similar. The late post war results are however quite different. The natural frequency for durable goods is about six months, whereas that for nondurable goods is about seventeen months, but declining fast. The MDP for durable goods has risen from two to three months, whereas the MDP for nondurable goods is falling from five to two and a half months. A comparison of Figures 4 and 9 shows that the war and post war plots for the durable goods index are qualitatively the same as the pre-war and war plots for the nondurable goods index. In the pre war and post war periods the two series are qualitatively different as the graphs in the two figures show so clearly.

SUMMARY, CONCLUSIONS, AND FURTHER WORK

The analysis above has demonstrated that a single class of model can be used to describe the short term time path of two indices of production and that that single class of model holds over the entire recorded history of the two series.

Variation in these indices at "business cycle time scales" is represented by the slow variation in the parameter values of the model of the dynamical system. The major difference between this approach and the usual business cycle approach is that in this analysis the economic effects of the two time scales are interdependent and are not simply additive; business cycle characteristics can only be examined in conjunction with the analysis of the so called "seasonal components."

The model class that seems to fit these data is a nonlinear, damped, forced oscillator with delay. A large and varied class of alternatives were considered and rejected in favor of the model presented above. The models that were rejected included numerous versions of the Duffing, Mathieu-Hill, and Van der Pol equations.

The linearized version captures most of the coarse grain analysis, but misses the finer details. A detailed analysis of the properties of the solution to the linearized version indicated a large number of interesting results. While the same functional form for the differential equation holds approximately everywhere the estimated models are qualitatively different across indices and over time. This conclusion was documented in Tables 5 and 10 and more strikingly in Figures 4 and 9.

The overall impression of the statistical results, notwithstanding the obvious drift in parameter values, is one of stability of the coefficient estimates. Some of the coefficient estimates exhibited remarkable stability over a sixty year time span; although the quality of this impression depends on the prejudices of the individual reviewer.

A comparison of the estimates of the standard errors indicates that the two sets of estimates are very similar. The stability of the variance estimates for each model over time is impressive.

This research has already raised several questions and is likely to raise more. The forcing term involves four specific frequencies, only two of which might be intuitively plausible. In particular, the strong appearance of the periods represented by one third and one fifth of a year pose interesting theoretical questions about the dynamics of industrial production. Of greater weight is the question as to why the dynamics of both series involve the precise periodic sequence of one half, one third, one quarter, and one fifth of a year.

A greater challenge is to discover the nature of the mechanism that is currently being crudely approximated by the$\{\hat{u}_{it}\}$. This should be the subject of both theoretical and empirical research. Also the theoretical reasons for the appearance of the delay in the effect of Du_t is a potential area of useful research.

A considerable amount of further work is obviously needed. The first task will be to improve the formulation of the nonlinear terms involving the growth rate. This will pave the way for the development of a model for the variation of the coefficients in the two indices. The modeling of the drift in the parameters is, of course, the subject of interest of business cycle research, so that these models should stimulate alternative ways of thinking about and modeling the business cycle.

Finally, a revised model that includes terms for what is now termed drift, will inevitably require simulation in order to predict jointly all three elements of the differential equation, u_t, Du_t, and D^2u_t. At such a time a useful model for the analysis of growth rates will have been obtained. In the process it is to be hoped that these results will stimulate a substantial effort in developing the underlying economic theory.

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Table 1
Estimation Results for NonDurable Goods
for Period March 1926 to November 1937

Coefficient	Estimate	Std. Error	t-Value	Sig. Level	F-Ratio ΔRSS	P-Value
β_1	0.17529	0.053465	3.2786	0.0013	92.16	.0000
β_2	0.114463	0.124058	0.9227	0.3579	7.13	.0086
β_3	0.130298	0.105694	1.2328	0.2199	41.23	.0000
β_4	0.134197	0.061778	2.1722	0.0317	102.71	.0000
α	0.556307	0.046008	12.0914	0.0000	404.96	.0000
a_1	-0.007293	0.001081	-6.7480	0.0000	52.57	.0000
a_2	0.005105	0.001237	4.1284	0.0001	12.75	.0005
a_3	0.001428	0.001044	1.3679	0.1737	1.37	.2433
a_4	-0.001173	0.001118	-1.0492	0.2961	.75	.3985
b_1	0.00297	0.001309	2.2688	0.0250	3.28	.0724
b_2	-0.004001	0.001136	-3.5207	0.0006	12.47	.0006
b_3	-0.000268	0.001046	-0.2563	0.7982	.07	.8008
b_4	0.000017	0.001037	0.0164	0.9869	.00	.9871

R-SQ. (ADJ.) = 0.8381 SE = 0.008549 Durbin Watson = 0.998
Number of Observations = 140
Coefficient of Skewness = 1.44166 Standardized Value = 6.96387
Coefficient of Kurtosis = 7.44361 Standardized Value = 17.978
$\Sigma_i \beta_i$ = 0.552 $< \beta_i >$ = 0.138

Table 2
Estimation Results for NonDurable Goods
for Period Novemeber 1937 to November 1962

Coefficient	Estimate	Std. Error	t-Value	Sig. Level	F-Ratio ΔRSS	P-Value
β_1	0.040648	0.033368	1.2182	0.2242	85.01	.0000
β_2	0.300605	0.086008	3.4951	0.0005	1.53	.2166
β_3	0.229795	0.048228	4.7648	0.0000	168.86	.0000
β_4	0.168767	0.055452	3.0435	0.0026	307.84	.0000
α	0.649262	0.027105	23.9535	0.0000	1630.60	.0000
a_1	-0.004904	0.00044	-11.1400	0.0000	155.93	.0000
a_2	0.002469	0.000518	4.7693	0.0000	25.56	.0000
a_3	0.003093	0.000428	7.2294	0.0000	44.70	.0000
a_4	-0.001054	0.000448	-2.3549	0.0192	3.72	.0548
b_1	0.000676	0.000543	.12445	0.2143	1.47	.2269
b_2	-0.001903	0.00042	-4.5329	0.0000	23.38	.0000
b_3	-0.000942	0.000422	-2.2343	0.0262	5.38	.0210
b_4	0.00149	0.000405	3.6809	0.0003	13.55	.0003

R-SQ. (ADJ.) = 0.8915 SE = 0.004821 Durbin Watson = 1.183
Number of Observations = 300
Coefficient of Skewness = 0.032366 Standardized Value = 0.228865
Coefficient of Kurtosis = 1.0607 Standardized Value = 3.75015
$\Sigma_i \beta_i = 0.740$ $< \beta_i > = 0.185$

Table 3
Estimation Results for NonDurable Goods
for Period December 1962 to November 1987

Coefficient	Estimate	Std. Error	t-Value	Sig. Level	F-Ratio ΔRSS	P-Value
β_1	0.187271	0.025647	7.3020	0.0000	37.59	.0000
β_2	0.248262	0.189251	1.3118	0.1906	42.44	.0000
β_3	-0.11995	0.063662	-1.8842	0.0606	853.51	.0000
β_4	0.101403	0.032096	3.1594	0.0017	1696.76	.0000
α	0.439896	0.028833	15.2565	0.0000	2719.69	.0000
a_1	-0.007653	0.000363	-21.1060	0.0000	641.82	.0000
a_2	0.008259	0.000515	16.0408	0.0000	158.15	.0000
a_3	0.004257	0.000328	12.9775	0.0000	129.98	.0000
a_4	-0.000401	0.000397	-1.0101	0.3133	.55	.4663
b_1	0.005088	0.000565	9.0019	0.0000	70.56	.0000
b_2	-0.002116	0.000353	-5.9942	0.0000	47.27	.0000
b_3	0.001208	0.000372	3.2497	0.0013	9.21	.0026
b_4	0.001519	0.000308	4.9305	0.0000	24.31	.0000

R-SQ. (ADJ.) = 0.9555 SE = 0.003427 Durbin Watson = 1.225
Number of Observations = 300
Coefficient of Skewness = -0.17989 Standardized Value = -1.27201
Coefficient of Kurtosis = 0.173621 Standardized Value = 0.613844
$\Sigma_i \beta_i = 0.416$ $< \beta_i > = 0.104$

Table 4
Comparison of Coefficient Estimates for Three Subperiods of the Late Post-War Period: NonDurable Goods

	Period C_1*		Period C_2*		Period C_3*	
Coefficient	Estimate	Std. Error	Estimate	Std. Error	Estimate	Std. Error
β_1	0.185548	0.036141	0.198563	0.036731	0.192308	0.029989
β_2	0.115989	0.234148	0.27992	0.239425	0.455476	0.258894
β_3	-0.148704	0.072246	-0.124729	0.093599	-0.173686	0.096491
β_4	0.119086	0.03457	0.117551	0.040248	0.117316	0.048389
α	0.29469	0.039573	0.370355	0.041026	0.405241	0.040573
a_1	-0.009666	0.000445	-0.007885	0.000505	-0.006148	0.000494
a_2	0.01154	0.000781	0.009555	0.000756	0.007661	0.000601
a_3	0.004736	0.000402	0.004697	0.000448	0.004516	0.000428
a_4	-0.000571	0.000503	-0.000594	0.00055	-0.000318	0.000499
b_1	0.007732	0.000844	0.006136	0.000809	0.00599	0.00068
b_2	-0.004139	0.000463	-0.002144	0.000476	-0.001011	0.000442
b_3	0.001789	0.000486	0.001896	0.000532	0.001844	0.000477
b_4	0.001565	0.000368	0.001316	0.000421	0.001106	0.000404
	$\Sigma_I \beta_I = 0.276$ $\langle \beta_I \rangle = 0.069$		$\Sigma_I \beta_I = 0.472$ $\langle \beta_I \rangle = 0.118$		$\Sigma_I \beta_I = 0.592$ $\langle \beta_I \rangle = 0.148$	

* Period C_1: December 1962 to May 1975
Period C_2: February 1969 to August 1981
Period C_3: June 1975 to November 1987

Table 5
Coefficient Summary and Roots
By Sub-Period: NonDurable Goods Index

Period Index*	α	$\langle\beta_i\rangle$	λ_1	λ_2	λ_3	MDP**	P**
A	0.56	0.138	-0.31	-0.63±1.10i		3.2	5.7
B	0.65	0.185	-0.32	-0.38±1.27i		3.2	4.9
C	0.44	0.104	-1.24	-0.86	-0.44	2.3	None
C_1	0.29	0.069	-4.47	-0.212±0.248i		4.7	25.3
C_2	0.37	0.118	-2.77	-0.320±0.358i		3.1	17.6
C_3	0.405	0.148	-2.17	-0.380±0.436i		2.6	14.4

* A → 1926 to 1937; B → 1937 to 1962; C → 1962 to 1987; C_1 → 1962 to 1975; C_2 → 1969 to 1981; C_3 → 1975 to 1987

** MDP → Maximum Delay Period (in months) - maximum $\{t_i\} \ni e^{-\rho_i t_i} = e^{-1}$
P → Period in Months of the natural frequency of the oscillation
$P = (2\pi)/\delta$
$\lambda_i t = (\rho_i + i\delta_i)t$

Table 6
Estimation Results for Durable Goods
for Period March 1926 to November 1937

Coefficient	Estimate	Std. Error	t-Value	Sig. Level	F-Ratio ΔRSS	P-Value
β_1	0.223322	0.033191	6.7283	0.0000	30.07	.0000
β_2	0.167693	0.053827	3.1154	0.0023	8.53	.0041
β_3	0.258707	0.04391	5.8918	0.0000	28.06	.0000
β_4	0.294539	0.04521	6.5149	0.0000	53.14	.0000
α	0.298896	0.042628	7.0117	0.0000	384.85	.0000
a_1	-0.008045	0.001915	-4.2009	0.0000	18.14	.0000
a_2	0.005032	0.001909	2.6367	0.0094	6.60	.0113
a_3	0.001024	0.001843	0.5557	0.5794	.34	.5651
a_4	0.000092	0.001838	0.0501	0.9601	.02	.8848
b_1	0.001061	0.00214	0.4961	0.6207	.27	.6094
b_2	-0.003192	0.001896	-1.6840	0.0946	2.76	.0989
b_3	-0.001865	0.001807	-1.0318	0.3041	1.07	.3040
b_4	0.000029	0.001815	0.0162	0.9871	.00	.9872

R-SQ. (ADJ.) = 0.7897 SE = 0.015140 Durbin Watson = 0.800
Number of Observations = 140
Coefficient of Skewness = 1.38874 Standardized Value = 6.70826
Coefficient of Kurtosis = 4.88335 Standardized Value = 11.7944

$\sum \beta_i = 0.9442$ $\langle\beta_i\rangle = 0.236$

Table 7
Estimation Results for Durable Goods
for Period November 1937 to November 1962

Coefficient	Estimate	Std. Error	t-Value	Sig. Level	F-Ratio ΔRSS	P-Value
β_1	0.060579	0.027885	2.1724	0.0306	.14	.7088
β_2	0.181645	0.029331	6.1929	0.0000	36.95	.0000
β_3	0.181624	0.028134	6.4556	0.0000	45.33	.0000
β_4	0.291969	0.029772	9.8070	0.0000	112.04	.0000
α	0.60913	0.026884	22.6573	0.0000	1563.49	.0000
a_1	-0.002603	0.000683	-3.8081	0.0002	15.16	.0001
a_2	0.000705	0.000694	1.0160	0.3105	1.81	.1793
a_3	0.002028	0.000666	3.0469	0.0025	8.42	.0040
a_4	-0.000147	0.000697	-0.2103	0.8336	.00	.9751
b_1	-0.002061	0.000689	-2.9935	0.0030	9.53	.0022
b_2	-0.00147	0.0006869	-2.1420	0.0330	4.50	.0347
b_3	0.0007	0.000669	1.0458	0.2965	1.11	.2919
b_4	0.000573	0.000661	0.8676	0.3864	.75	.3956

R-SQ. (ADJ.) = 0.8567 SE = 0.008018 Durbin Watson = 0.906
Number of Observations = 300
Coefficient of Skewness = -0.68207 Standardized Value = -4.82298
Coefficient of Kurtosis = 6.40854 Standardized Value = 22.6576

$\sum \beta_i = 0.7158$ $\langle\beta_i\rangle = 0.179$

Table 8
Estimation Results for Durable Goods
for Period December 1962 to November 1987

Coefficient	Estimate	Std. Error	t-Value	Sig. Level	F-Ratio ΔRSS	P-Value
β_1	0.192828	0.029206	6.6024	0.0000	41.09	.0000
β_2	0.269129	0.029228	9.2079	0.0000	369.12	.0000
β_3	0.132346	0.022338	5.9247	0.0000	113.44	.0000
β_4	0.194262	0.025034	7.7598	0.0000	222.61	.0000
α	0.540277	0.028608	18.8858	0.0000	7281.22	.0000
a_1	-0.004381	0.000342	-12.8245	0.0000	96.25	.0000
a_2	0.00417	0.000421	9.9088	0.0000	224.09	.0000
a_3	0.002449	0.000288	8.4945	0.0000	69.30	.0000
a_4	0.000581	0.000354	1.6383	0.1025	12.03	.0006
b_1	-0.001741	0.000425	-4.0936	0.0001	15.74	.0001
b_2	-0.000531	0.000356	-1.4922	0.1367	2.51	.1140
b_3	0.000111	0.000366	0.3045	0.7610	.30	.5920
b_4	0.000682	0.000259	2.6294	0.0090	6.91	.0090

R-SQ. (ADJ.) = 0.9658   SE = 0.003129   Durbin Watson = 0.924
Number of Observations = 300
Coefficient of Skewness =-0.330294   Standardized Value = -2.33553
Coefficient of Kurtosis = 0.97914   Standardized Value =3.46178

$\sum \beta_i = 0.789$ $\langle\beta_i\rangle = 0.197$

Table 9
Comparison of Coefficient Estimates for Three Subperiods of the Late Post-War Period: Durable Goods

	Period C_1*		Period C_2*		Period C_3*	
Coefficient	Estimate	Std. Error	Estimate	Std. Error	Estimate	Std. Error
β_1	0.196893	0.054845	0.178292	0.042939	0.156025	0.030427
β_2	.0252742	0.046409	0.198388	0.044432	0.164665	0.041983
β_3	0.136819	0.029216	0.143051	0.034296	0.075621	0.03203
β_4	0.171165	0.033195	0.157445	0.034239	0.187585	0.035575
α	0.534207	0.041977	0.557535	0.044669	0.568335	0.042523
a_1	-0.004688	0.000675	-0.005172	0.000533	-0.004787	0.000365
a_2	0.004913	0.00072	0.004507	0.000698	0.003921	0.00054
a_3	0.002543	0.000482	0.002801	0.00044	0.002647	0.00032
a_4	0.000527	0.000636	0.000212	0.000537	0.000528	0.000382
b_1	-0.003089	0.000638	-0.002094	0.000705	-0.00032	0.000584
b_2	-0.00103	0.000678	-0.000794	0.000532	-0.000081	0.000383
b_3	0.000439	0.000675	0.000137	0.000554	0.000139	0.000429
b_4	0.000784	0.000385	0.000743	0.0003386	0.000615	0.000303

	Period C_1	Period C_2	Period C_3
R-SQ. (ADJ.)	= 0.9715	= 0.9667	= 0.9680
$\sum \beta_i$	= 0.7576	= 0.6772	= 0.5839
$\langle\beta_i\rangle$	= 0.189	= 0.169	= 0.146

* Period C_1: December 1962 to May 1975
Period C_2: February 1969 to August 1981
Period C_3: June 1975 to November 1987

Table 10
Coefficient Summary and Roots
By Sub-Period: Durable Goods Index

Period* Index	α	$<\beta_i>$	λ_1	λ_2	λ_3	MDP**	P**
A	0.299	0.236	-4.3	-0.190±0.575i		5.26	10.9
B	0.609	0.179	-0.351	-0.466±1.206i		2.85	5.2
C	0.540	0.197	-0.529	-0.587±1.017i		1.89	6.2
C_1	0.534	0.189	-0.519	-0.613±0.994i		1.93	6.7
C_2	0.558	0.169	-0.396	-0.594±1.085i		2.53	5.8
C_3	0.568	0.146	-0.318	-0.602±1.121i		3.14	5.6

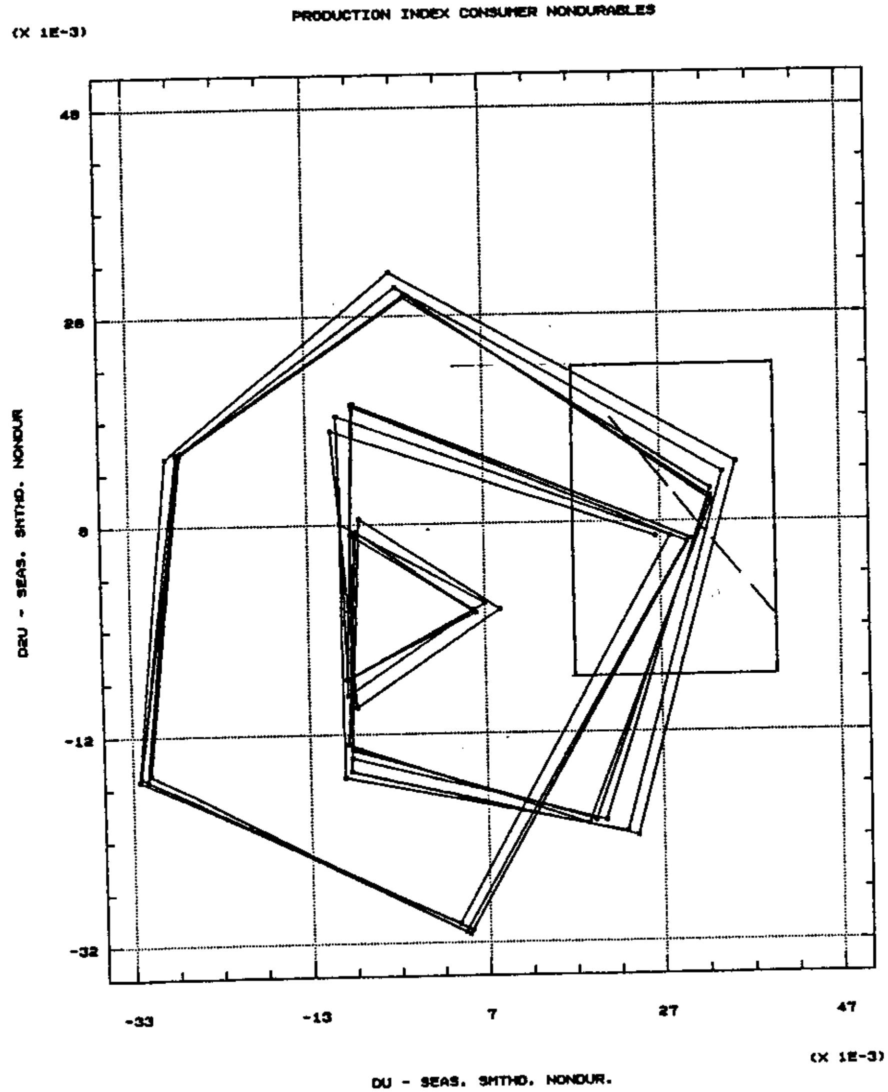
* A → 1926 to 1937; B → 1937 to 1962; C → 1962 to 1987; C_1 → 1962 to 1975; C_2 → 1969 to 1981; C_3 → 1975 to 1987

** MDP → Maximum Delay Period (in months) - maximum $\{t_i\} \ni e^{-\rho_i t_i} = e^{-1}$
P → Period in Months of the natural frequency of the oscillation
$P = (2\pi)/\delta$
$\lambda_i t = (\rho_i + i\delta_i)t$

Figure 1

PHASE SPACE DIAGRAM: SEASONALLY SMOOTHED

PRODUCTION INDEX CONSUMER NONDURABLES

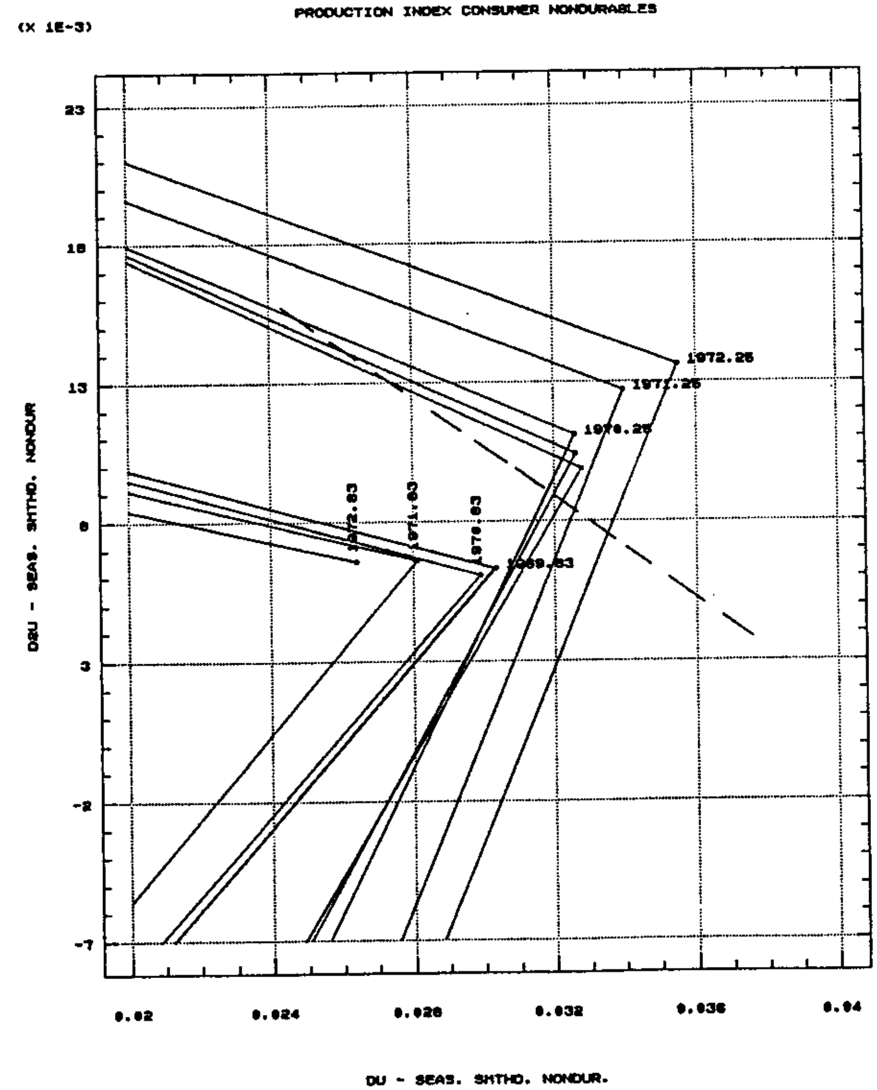


NOV. '67 TO NOV. '72

Figure 1a

PHASE SPACE DIAGRAM: SEASONALLY SMOOTHED

PRODUCTION INDEX CONSUMER NONDURABLES



BLOW UP OF SECTION INDICATED IN ORIGINAL

Figure 2

PLOT OF THE AMPLITUDES FOR NONDURABLE

GOODS INDEX SEASONALLY SMOOTHED(12 mths)

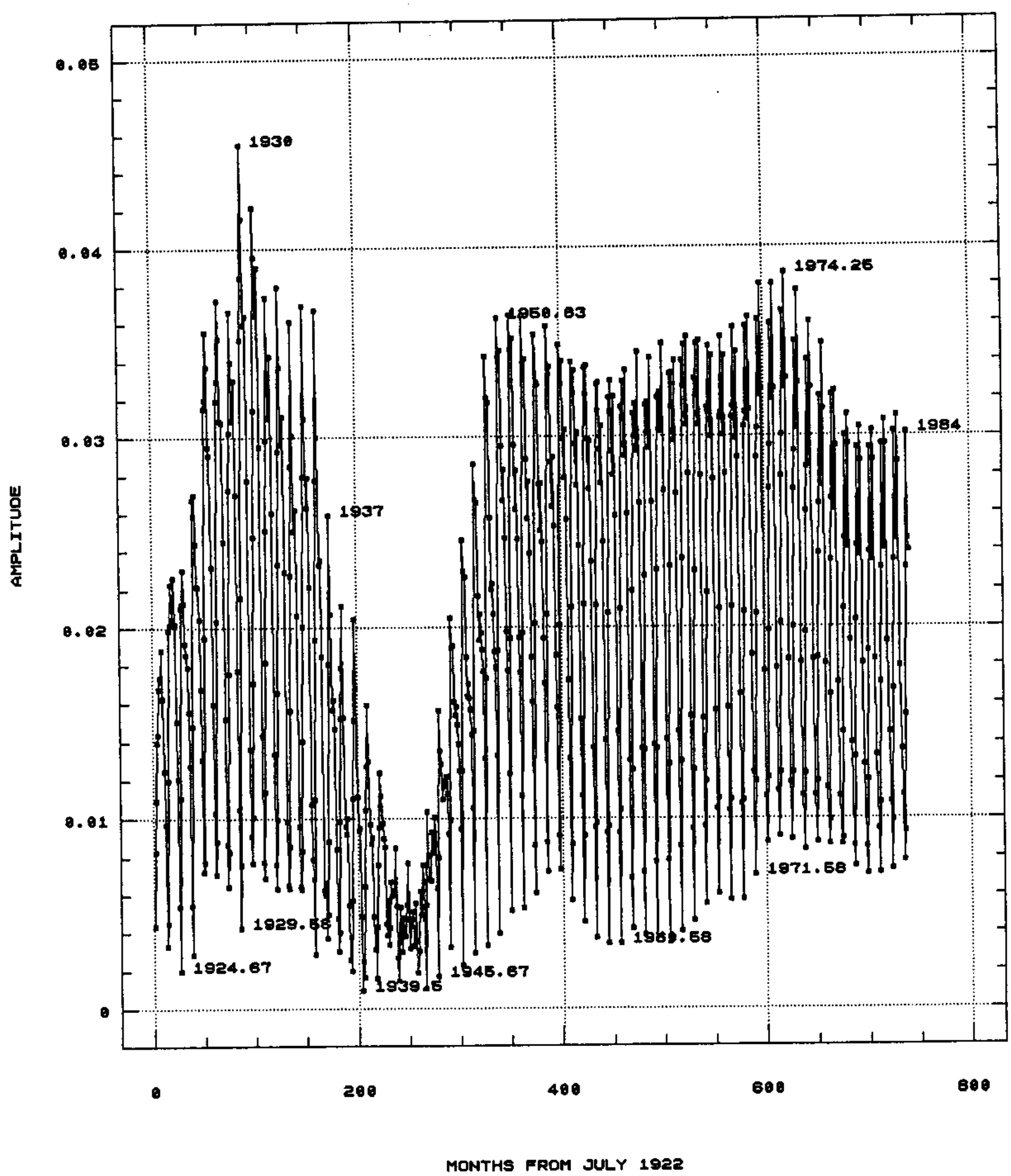


Figure 3a

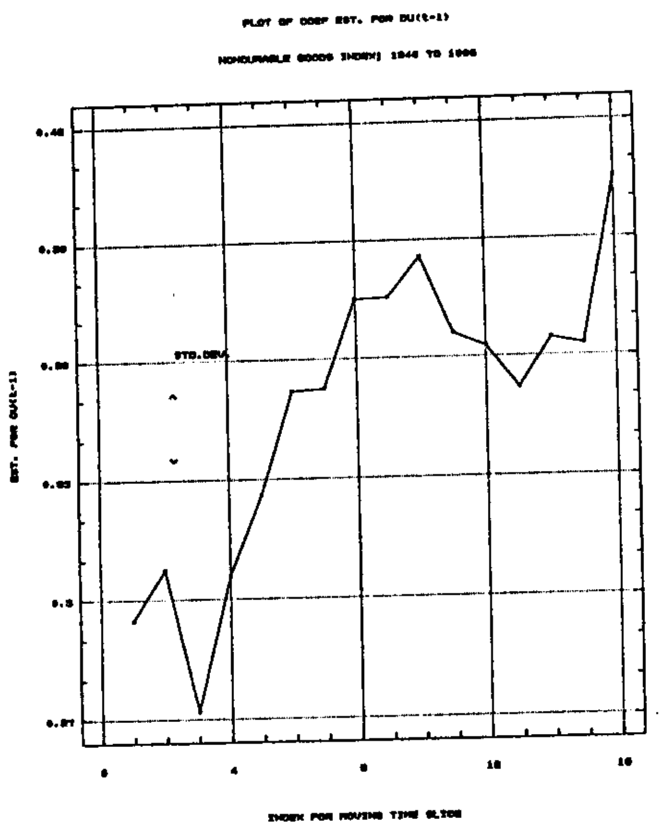


Figure 3c

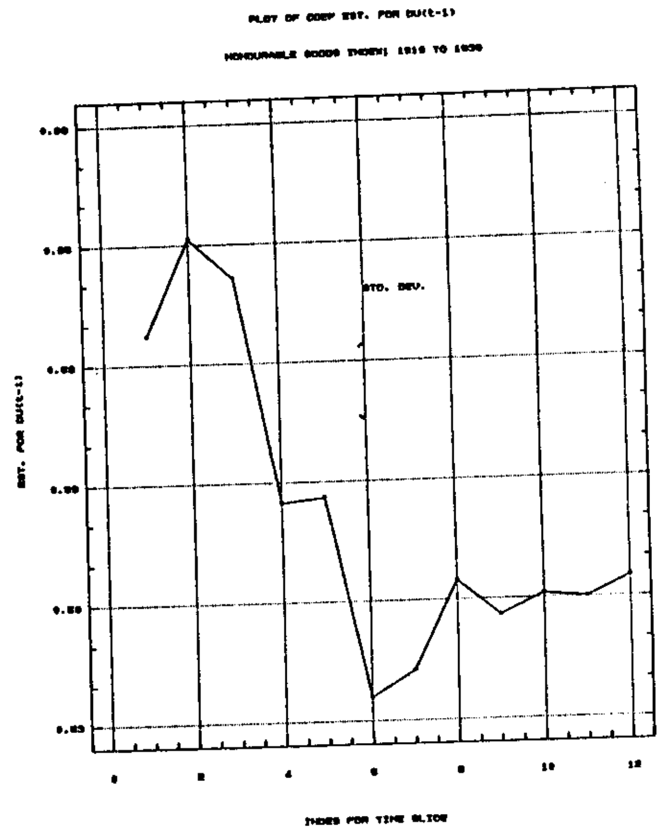


Figure 3b

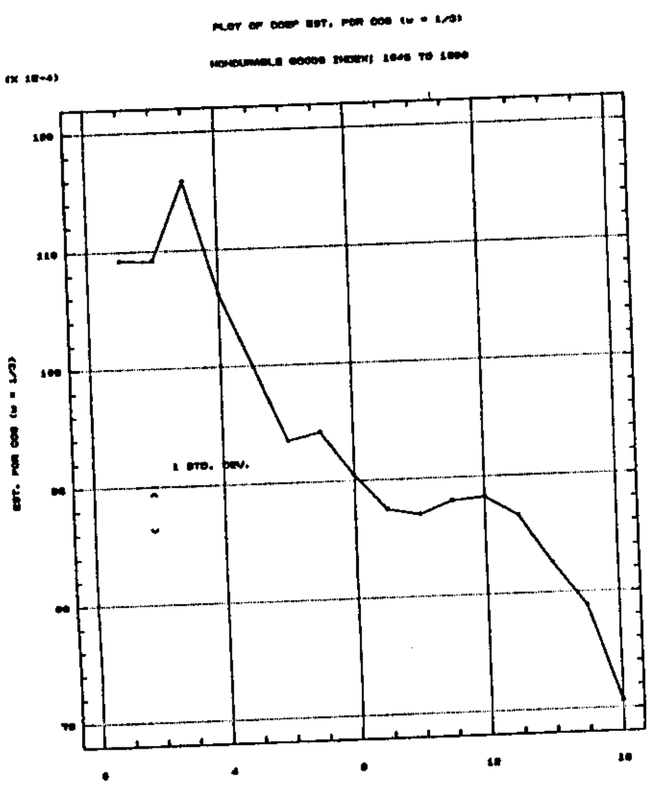
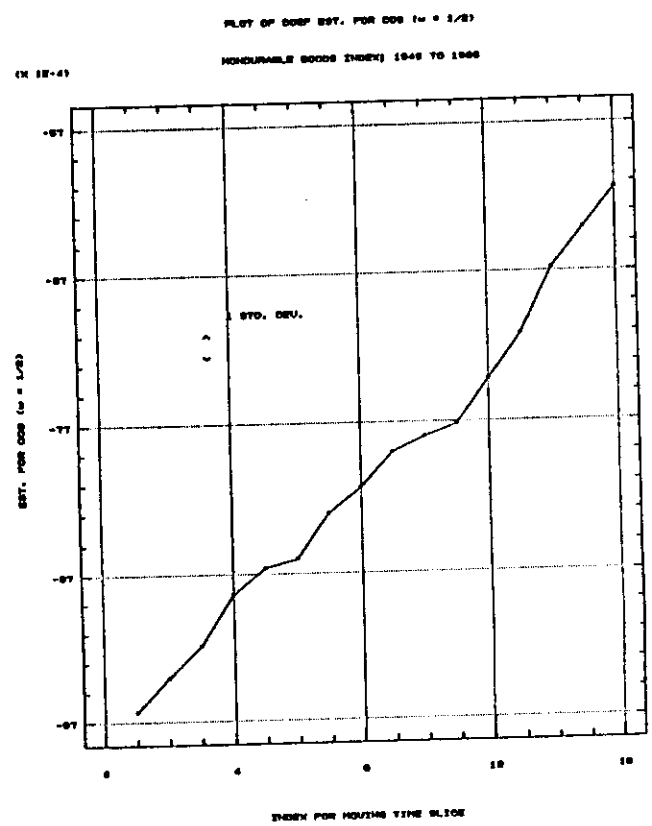


Figure 3d



Figures 4a-4f: Plots of the Function $[(\alpha/2)s^3 + (1-\alpha)s^2 + \alpha s + \beta]$ for Various (α, β): NonDurable Goods

Figure 4a

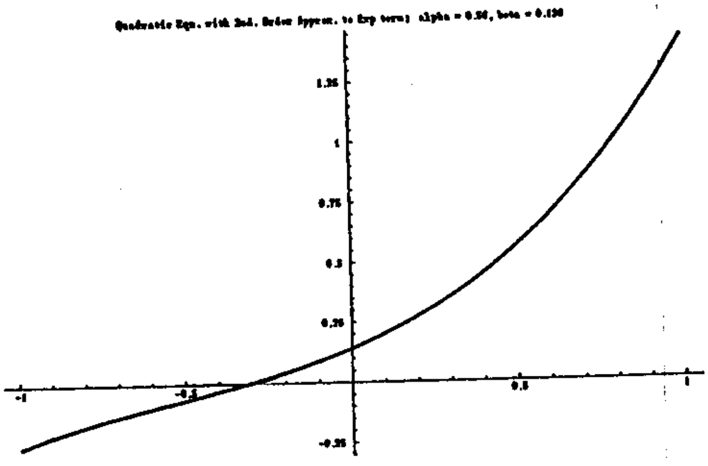


Figure 4b

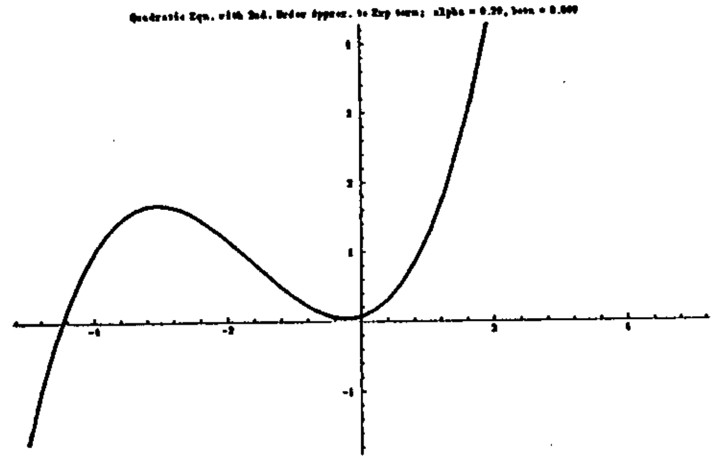


Figure 4c

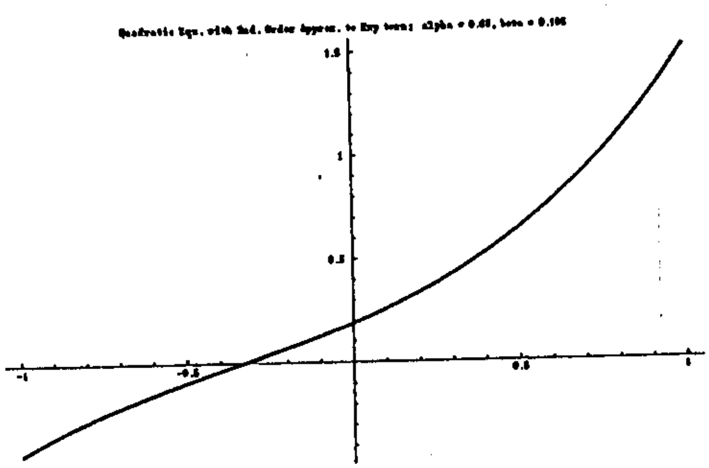


Figure 4d

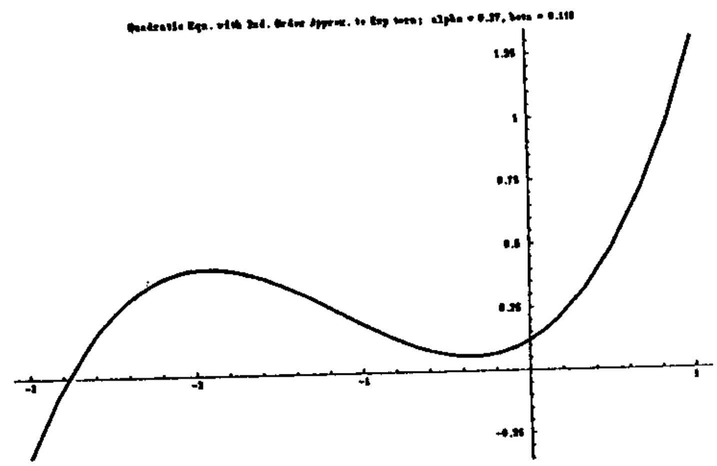


Figure 4e

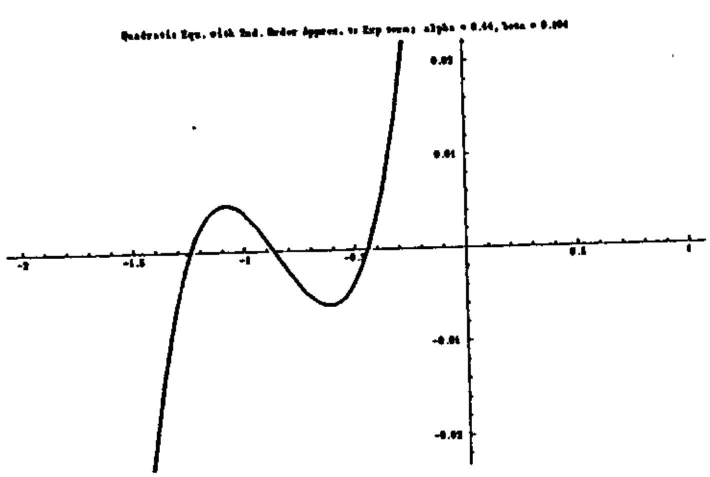


Figure 4f

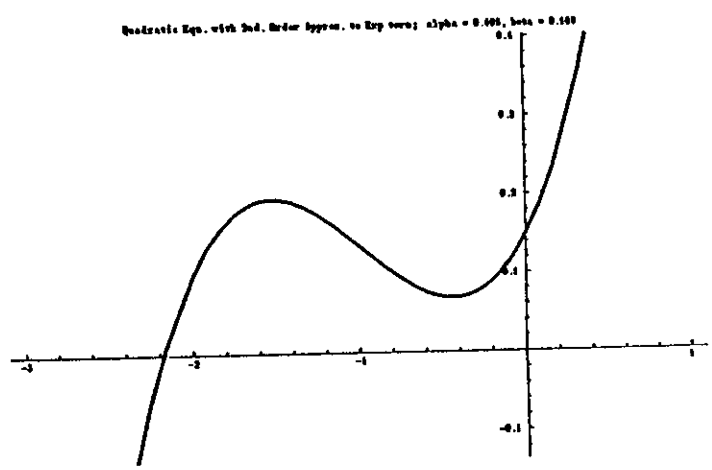


Figure 5

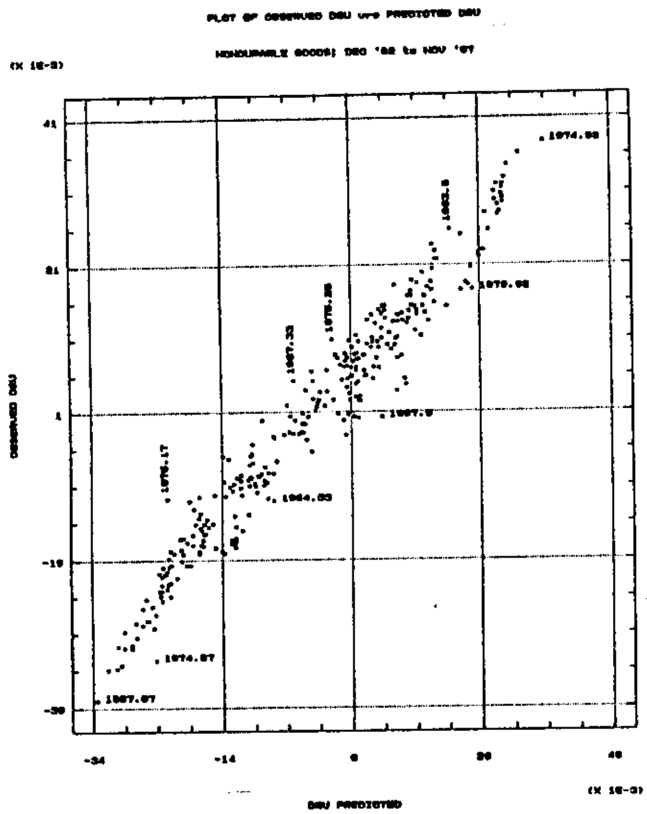


Figure 6

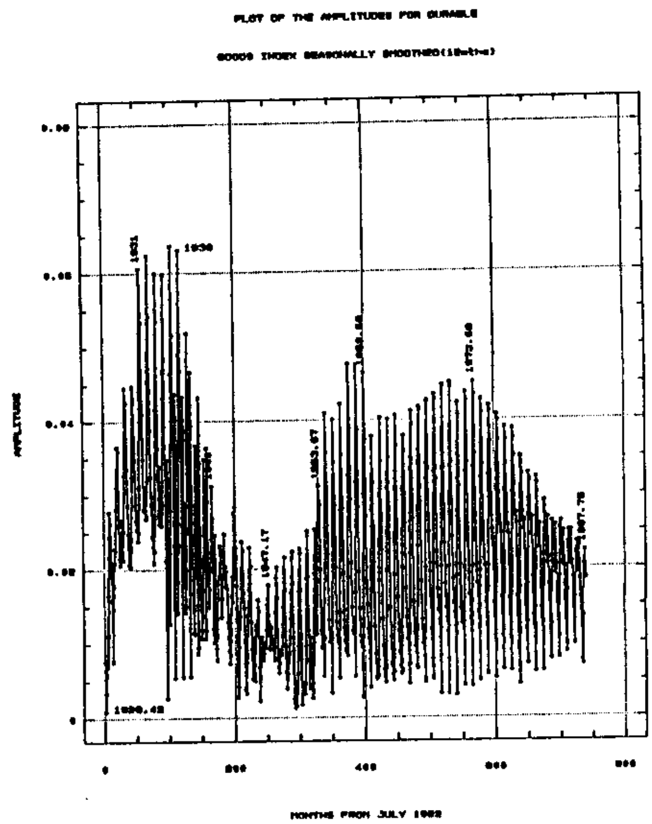


Figure 7

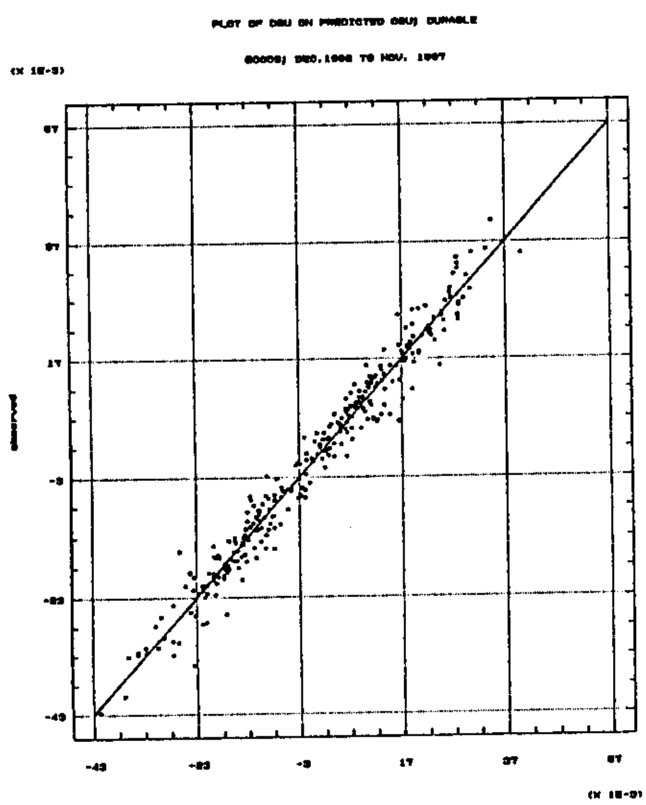
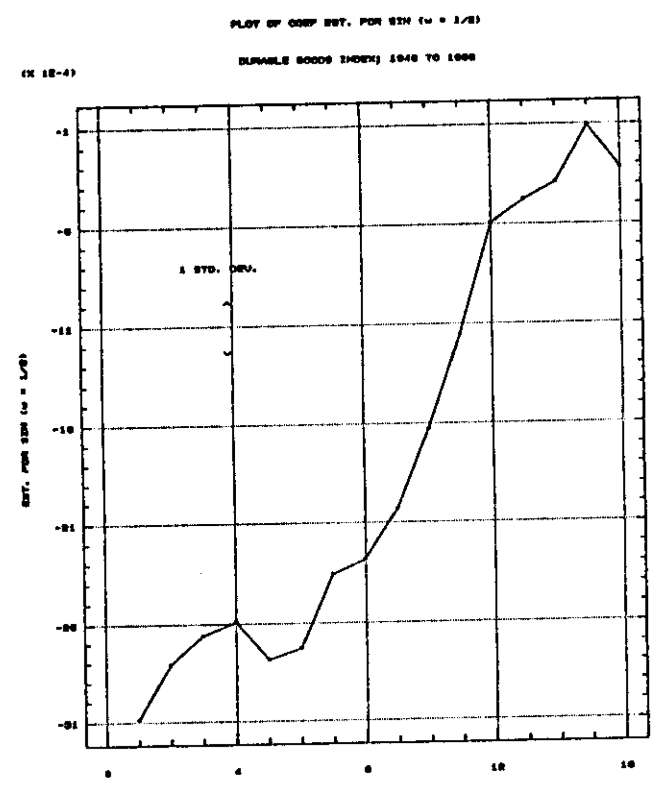


Figure 8



Figures 9(a)-(c): Plots of the Function $[(\alpha/2)s^3 + (1-\alpha)s^2 + \alpha s + \beta]$ for Various (α, β): Durable Goods

Figure 9(a)

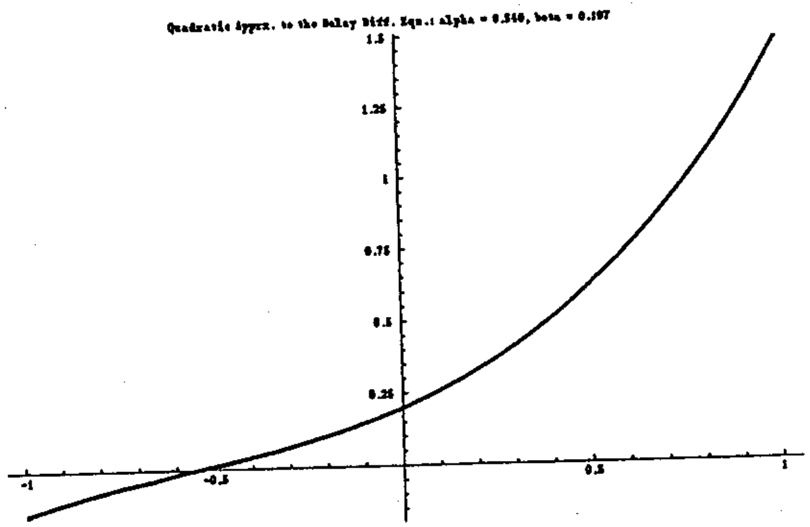


Figure 9(b)

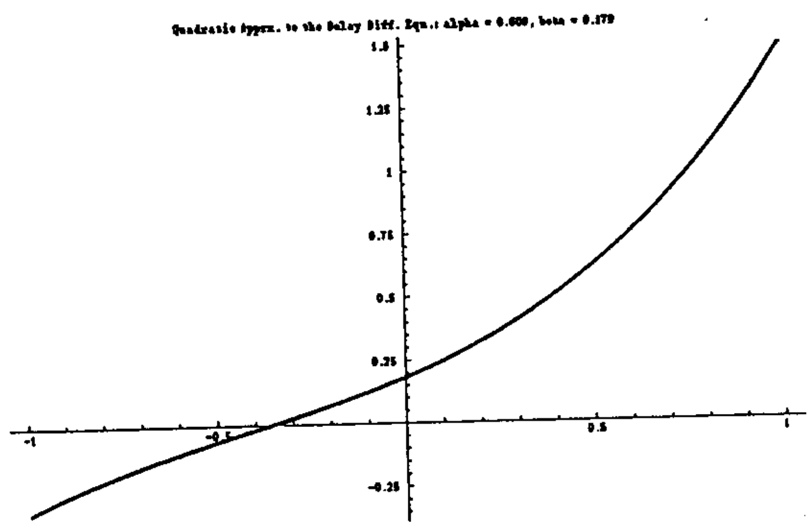


Figure 9(c)

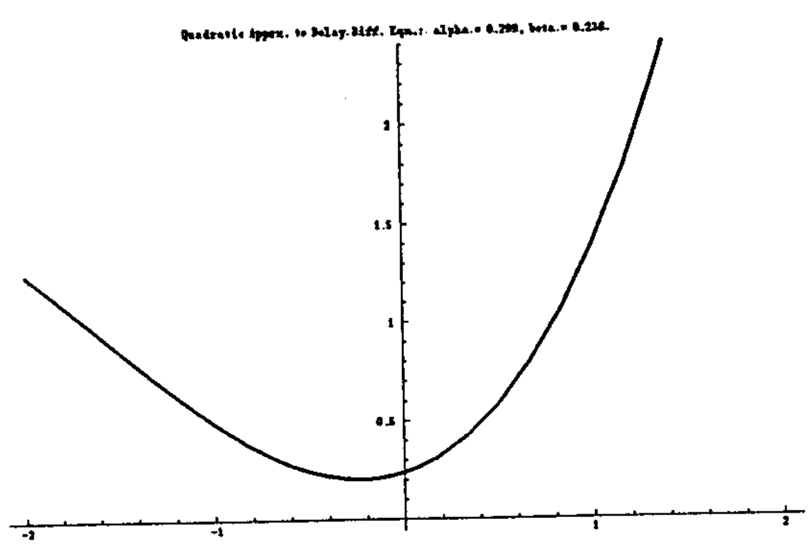
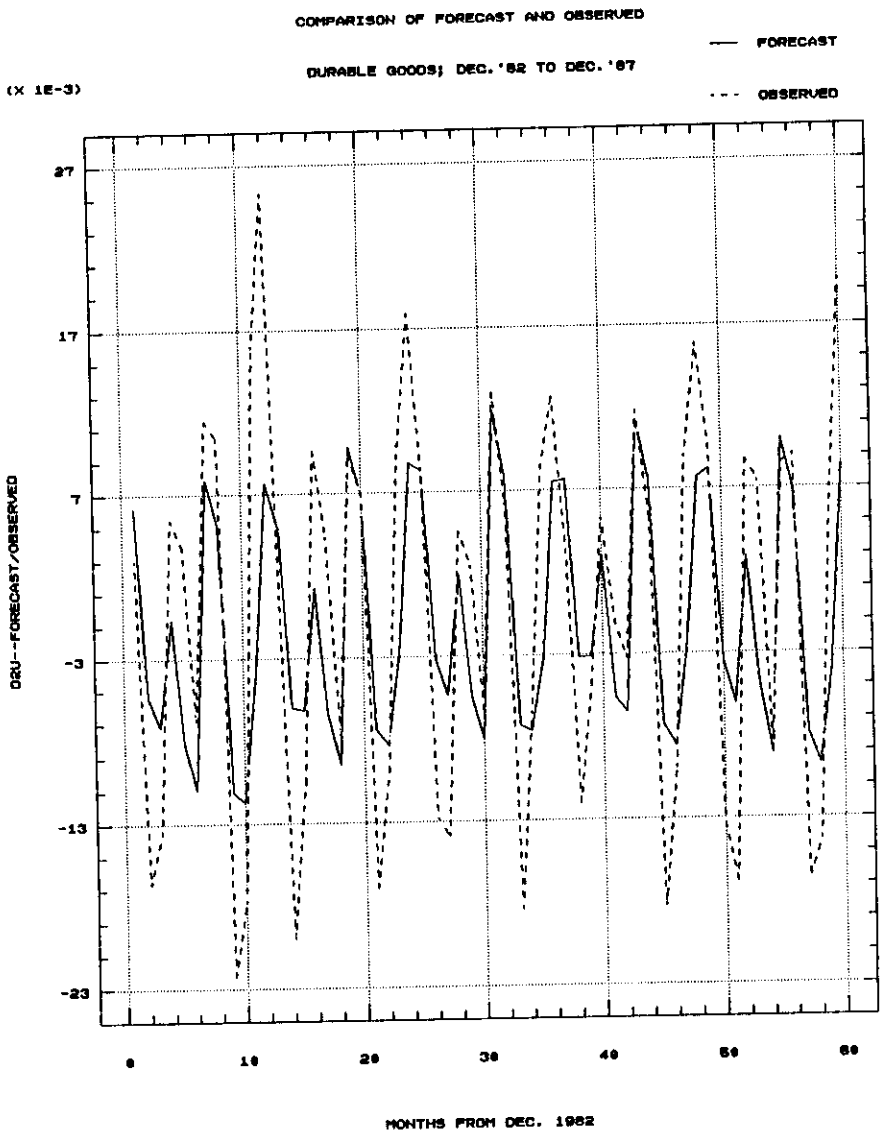


Figure 10



COMPARISON OF FORECAST AND OBSERVED
DURABLE GOODS; DEC.'82 TO DEC.'87
— FORECAST
--- OBSERVED
(X 1E-3)
ORU--FORECAST/OBSERVED
27
17
7
-3
-13
-23
0
10
20
30
40
50
60
MONTHS FROM DEC. 1982