ECONOMIC RESEARCH REPORTS

# THE PARADOX OF MULTIPLE ELECTIONS

by **Steven J. Brams,**
**D. Marc Kilgour,**
**and William S. Zwicker**

**RR # 96-09** **March 1996**

**C.V. STARR CENTER FOR APPLIED ECONOMICS**

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, NY 10003-6687

# The Paradox of Multiple Elections

Steven J. Brams
Department of Politics
New York University
New York, NY 10003

D. Marc Kilgour
Department of Mathematics
Wilfrid Laurier University
Waterloo, Ontario N2L 3C5
CANADA

William S. Zwicker
Department of Mathematics
Union College
Schenectady, NY 12308

## *ABSTRACT*

## The Paradox of Multiple Elections

Assume that voters must choose between voting yes (Y) and voting no (N) on three propositions on a referendum. If the winning combination is NYY on the first, second, and third propositions, respectively, the *paradox of multiple elections* is that NYY can receive the fewest votes of the $2^3 = 8$ combinations. Several examples of this paradox are illustrated, and necessary and sufficient conditions for its occurrence, related to the "incoherence" of support, are given.

The paradox is shown, via an isomorphism, to be a generalization of the well-known paradox of voting. One real-life example of the paradox involving voting on propositions in California, in which not a single voter voted on the winning side of all the propositions, is given. Several empirical examples of variants of the paradox that manifested themselves in federal elections—one of which led to divided government—and legislative votes in the House of Representatives, are also analyzed. Possible normative implications of the paradox, such as allowing voters to vote directly for combinations using approval voting or the Borda count, are discussed.

*JEL Classification*: D71. *Keywords*: Aggregation paradoxes; paradox of voting; electoral systems; legislatures; referenda; divided government.

# The Paradox of Multiple Elections[1]

## 1. Introduction

Aggregation paradoxes abound in the literature of statistics and social choice. Roughly speaking, they describe situations in which the sum of the parts is, in some sense, not equal to the whole, such as an election outcome that fails to mirror the "will" of the electorate. They include the paradox of voting and Arrow's impossibility theorem (Arrow, 1963), Anscombe's paradox (Anscombe, 1976, Wagner, 1983, 1984), Ostrogorski's paradox (Daudt and Rae, 1976, Deb and Kelsey, 1987; Kelly, 1989), and Simpson's paradox (Simpson, 1951, Gardner, 1976, Wagner, 1982). An interesting attempt to "understand, classify, and find new properties" of such paradoxes is given in Saari (1995); more generally, see Saari (1994), Lagerspetz (1995), and Nurmi (1995).

The paradox we analyze here is an aggregation paradox, but it occurs in a particular context: multiple elections, in which voters may not know the results of one election before they vote in another. Such voting is commonplace, as when a voter votes, on the same day, for president, senator, or representative in a presidential election year, or for or against several propositions in a referendum. The paradox is also applicable to multiple votes on a bill and its amendments in a legislature, whereby voting is sequential so the voter acquires some information, as votes are taken, but does not know the results of future votes.

We call the set of winners in each of the individual elections the *winning combination*. Surprisingly, as we will show, few if any voters may actually have voted

[1]We thank Aaron Epstein, Jean-Pierre Benoit, Morris P. Fiorina, Richard F. Potthoff, and Alan D. Taylor for valuable discussion and communications, and Gary King and Kenneth F. McCue for providing the Los Angeles county ballot data, analyzed later, as well as helpful comments. The research assistance of Sophia Anninos and, especially, Katri K. Saari is gratefully acknowledged. Steven J. Brams gratefully acknowledges the support of the C.V. Starr Center for Applied Economics at New York Univeristy. This is a substantially revised version of our earlier paper entitled "A New Paradox of Vote Aggregation."

for this combination. Indeed, if there are as few as three propositions in a referendum—on which voters can vote either yes (Y) or no (N)—the outcome of the election may be YYY (i.e., all propositions pass), even though not a single voter voted for this winning combination.

We call this phenomenon *the paradox of multiple elections* and give an example, and several variants of this example, in section 2.[2] In section 3 we offer a theoretical analysis of the paradox, showing conditions necessary and sufficient for a combination to win and for the paradox to occur. We also show how the occurrence of the paradox depends on the "incoherence" of support for the winning combination.

In section 4 we give a real-life example of the paradox, based on the choices made by 1.8 million Los Angeles county voters choosing among the 28 propositions on the 1990 California ballot. In addition, we discuss other empirical examples that are not full-blown paradoxes but, nonetheless, indicate a discrepancy between the most popular parts of a combination and the less popular whole.

*Divided government,* whereby the president is of one party and one or both houses of Congress is controlled by the other party, may be interpreted as a manifestation of this discrepancy.[3] In 1980, for example, the Republicans won control of the presidency and the Senate, while the Democrats retained control of the House, which we indicate by

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[2]Some readers will not view this result as paradoxical because, once illustrated and explained, the apparent contradiction disappears. Nonetheless, the idea that a winning combination can receive zero votes seems surprising and counterintuitive to most people on first hearing, which conforms with the informal sense in which "paradox" is used in political science (Brams, 1976; Maoz, 1990).

[3]Sometimes "divided government" is used to mean that only the House of Representatives is controlled by a different party from that of the president, but the exact definition is not important. For a discussion of the merits and demerits of divided government, see "Symposium: Divided Government and the Politics of Constitutional Reform" (1991), Brady (1993), and McKay (1994). Of course, because control of the House and Senate depends on which party wins a majority of seats, which does not necessarily mean having a majority of votes across all House and Senate elections, there is some ambiguity about any purported discrepancy between the winning combination and the number who supported it.

RRD for control of the presidency, Senate, and House, respectively. Based on the outcomes in congressional districts, however, this winning combination came in only fourth out of the eight possible combinations. This outcome was decidedly more paradoxical than the DDD outcome of the previous presidential election (1976), which was also the most popular combination.

In section 5 we consider two legislative examples from the House of Representatives in which there was an apparent paradox of voting (the Wilmot Proviso of 1846 and the Revenue Act of 1932). We explicate the sense in which this paradox is a special case of the more general multiple-election paradox.

In section 6 we consider certain normative and social-choice consequences of the paradox. For example, should voters be presented with the opportunity to choose combinations on ballots? If so, should they be allowed to vote for more than one combination under approval voting, or to rank the combinations under the Borda count? Answers to these questions, and their implications for making coherent social choices, are explored in the context of democratic political theory.

## 2. Referendum Voting: An Illustration of the Paradox

Consider a referendum in which voters can vote either Y or N on each proposition on the ballot. The paradox of multiple elections occurs when the combination of propositions that wins receives the fewest votes, or is tied for the fewest. We first illustrate the *basic paradox*, without ties for fewest, after which we illustrate both stronger and weaker versions:

**Example 1 (basic paradox, without ties for fewest: 3 propositions).** If there are 3 propositions, there are are $2^3 = 8$ combinations, because each voter can make one of two choices (i.e., Y or N) on each proposition. (We consider the possibilitity of abstention later.) Suppose 13 voters cast the following numbers of votes for each of the eight combinations:

YYY: 1   YYN: 1   YNY: 1   NYY: 1   YNN: 3   NYN: 3   NNY: 3   NNN: 0.

On each proposition, *i*, we indicate the total numbers of voters voting for Y or N by Y(*i*) and N(*i*), where $i = 1, 2$, or 3. The election results for each proposition are

$$N(1) > Y(1), \qquad N(2) > Y(2), \qquad N(3) > Y(3),$$

each by 7 votes to 6.

Thus, when votes are aggregated separately for each proposition, which we call *proposition aggregation*, the winning combination is NNN. However, when votes are aggregated by combination, which we call *combination aggregation*, this combination (i.e., NNN) comes in last, because it is the only one that receives 0 votes.

This example is *minimal*, like those that follow, in the sense that no example with fewer voters can meet the stated conditions of the paradox. The construction depends on assigning the fewest votes (i.e., 0) to the paradoxical winner NNN, the next-fewest (i.e., 1) to combinations that agree in one proposition, and then finding the smallest number for the combinations that agree in two propositions (i.e., 3) so as to create the paradox—that is, so that NNN (barely) wins under proposition aggregation.

The paradox vividly illustrates the difference that may arise from aggregating votes by proposition and by combination. It also illustrates how proposition aggregation may leave no voter completely satisfied with the outcome: NNN is not supported by any of the 13 voters.

More generally, we say that a *paradox of multiple elections* occurs when the winning combination under proposition aggregation receives the fewest, but not necessarily zero, votes (as in Example 1). Of course, if the winning combination receives some support, there *will* be some voters completely satisfied with the outcome.

Note in Example 1 that YNN, NYN, and NNY are tied for first place under combination aggregation. We stress, however, that this is not to say that most voters would prefer one of these combinations to the proposition-aggregation winner, NNN—only that NNN is not the first choice of any voters if they are sincere in their voting.

The paradox, which describes a conflict between two different aggregation procedures, does not depend on either sincere or strategic voting: voters may be perfectly sincere in voting for their preferred position on every proposition, or they may be strategic (in some sense). The paradox says only that majority choices by proposition aggregation may receive the fewest votes when votes are aggregated by combination.

Example 1 is not the minimal example of the basic paradox if we allow ties for fewest votes. Example 2 shows that the paradox can occur with only 3 voters:

**Example 2 (basic paradox, with ties for fewest: 3 propositions).** Suppose there are 3 voters who cast the following numbers of votes for the three combinations

| | | |
|---|---|---|
| YYN: 1 | YNY: 1 | NYY: 1. |

YYY is the winning combination, according to proposition aggregation, by 2 votes to 1 for each proposition; yet it receives 0 votes as a combination. But so do the other four combinations (YNN, NYN, NNY, and NNN), so in this case we can say only that the winning combination ties for fewest, not—as in Example 1—that it is the *only* combination with the fewest (0 in this case) votes.

A more pathological form of the paradox can occur if there are four propositions, which gives $2^4 = 16$ possible voting combinations. Then we may get what we call a *complete-reversal paradox*:

**Example 3 (complete-reversal paradox, without ties for fewest: 4 propositions).** Suppose there are 31 voters who cast the following numbers of votes for the sixteen different combinations:

| | | | | | |
|---|---|---|---|---|---|
| YYYY: 0 | YYYN: 4 | YYNY: 4 | YNYY: 4 | NYYY: 4 | |
| YYNN: 1 | YNYN: 1 | YNNY: 1 | NYYN: 1 | NYNY: 1 | NNYY: 1 |
| YNNN: 1 | NYNN: 1 | NNYN: 1 | NNNY: 1 | NNNN: 5. | |

Then it is easy to show that Y beats N for each of the four propositions by 16 votes to 15, but YYYY is the only combination to receive 0 votes. The new wrinkle here is that NNNN, the "opposite" of the proposition-aggregation winner, YYYY, has the most votes (i.e., 5) and therefore wins under combination aggregation.

The complete-reversal paradox also occurs in the following simpler example, wherein several combinations tie for fewest votes:

**Example 4 (complete-reversal paradox, with ties for fewest: 4 offices).** Suppose there are 11 voters who cast the following numbers of votes for five combinations:

YYYN: 2 YYNY: 2 YNYY: 2 NYYY: 2 NNNN: 3.

There is a complete-reversal paradox, because YYYY ties eleven other combinations for the fewest votes (0) but, nevertheless, wins according to proposition aggregation by 6 votes to 5 against its opposite, NNNN, the combination winner with 3 votes.

It is not difficult to show that the basic paradox cannot occur if there are only two offices, but a milder version of this paradox can arise—namely, that the winner by proposition aggregation can come in as low as third (out of four) by combination aggregation. We call this the *two-proposition paradox*:

**Example 5 (two-proposition paradox, without ties for fewest).** Suppose there are 5 voters who cast the following numbers of votes for the four combinations

YY: 1 YN: 2 NY: 2 NN: 0.

While YY (1 vote) wins under proposition voting by 3 votes to 2 on each of the two propositions, it is behind both YN and NY (2 votes each) under combination aggregation; it is ahead only of NN (0 votes), putting it in next-to-last place.

This relatively low rank for the proposition-aggregation winner in the two-proposition case may still describe a possibly serious discrepancy between the two aggregation procedures. Indeed, there is no theoretical limit on *how far* behind the proposition-aggregation winner can be from the first-place and second-place combination winners (though it is easy to demonstrate that it will always place above the fourth-place combination).

If there are only two propositions, but we permit voters a third option of abstaining (A), then the basic paradox can occur with only two propositions:

**Example 6 (basic paradox, with abstention and with ties for fewest: 2 propositions).** Suppose there are 15 voters who cast the following numbers of votes for the $3^2 = 9$ different combinations:

YY: 0 YN: 3 NY: 3 YA: 3 AY: 3 NA: 1 AN: 1 NN: 1 AA: 0.

There is a paradox, because YY wins under office voting by 6 votes to 5 but, as a combination, it ties for the fewest votes (i.e., 0) with AA. The notion that AA receives 0 votes—or any other number—is somewhat misleading, however, because it is often impossible to ascertain the numbers who "chose" AA if these voters did not go to the polls (later we shall count, as abstainers, voters who cast ballots but abstain on the propositions being considered).

It is worth noting in this example that YY wins in large part because 6 voters (the YA and AY voters) abstained on one office and supported Y for the other. An A, however, is not necessarily to be interpreted as a vote against YY. A's are qualitatively different from other votes, in part because they are never a component of a winning combination.

**Example 7 (complete-reversal paradox, with abstention and without ties for fewest: 3 propositions).** Suppose there are 52 voters who cast the following numbers of votes for the 27 ($3^3$) combinations:

| | | | | | | |
|---|---|---|---|---|---|---|
| YYY: 0 | YYN: 4 | YNY: 4 | NYY: 4 | YYA: 4 | YAY: 4 | AYY: 4 |
| YNN: 1 | NYN: 1 | NNY: 1 | YAA: 1 | AYA: 1 | AAY: 1 | |
| NAA: 1 | NAN: 1 | NNA: 1 | NYA: 1 | ANY: 1 | YAN: 1 | AAA: 5 |
| ANN: 1 | ANA: 1 | AAN: 1 | AYN: 1 | NAY: 1 | YNA: 1 | NNN: 5. |

There is a complete-reversal paradox, because YYY wins under proposition voting by 20 votes to 16 on each proposition but, as a combination, it has the fewest (i.e., 0) votes. On the other hand, the other two "pure" combinations, AAA and NNN (the latter might be considered the opposite of YYY[4]), have the most votes (i.e., 5).

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[4]"Opposite" is ambiguous, of course, when there are more than two options. In the case of the three options postulated in Example 7, for instance, the opposite of YYY might be not only NNN but also the 7 other combinations that do not include any Ys. It turns out

The foregoing examples illustrate a range of discrepancies between aggregating votes by proposition and aggregating them by combination. We next analyze the general conditions that give rise to these discrepancies, focusing on the basic paradox in the two-option, three-proposition case and the "coherence" of voter support.

### 3. The Coherence of Support for Winning Combinations

Having demonstrated the existence of a multiple-election paradox and some variants of it, we turn in this section to the analysis of conditions that give rise to it. In particular, we distinguish between voting directly for a combination and voting indirectly for it by supporting some of its parts.

This distinction is illustrated by Example 2. The three voters who vote for YYN, YNY, and NYY give Y a 2-to-1 margin of victory for each proposition, resulting in the choice of YYY by proposition aggregation. But this indirect support by the three voters for YYY is indistinguishable under proposition aggregation from the direct support that one hypothetical voter, voting for YYY, would give to this combination.

In effect, this one voter would contribute three times as much support to YYY as does any of the three voters who "tilts" toward YYY by agreeing with it on two of the three offices. Not only is the support of this one voter more potent, but we also consider it more "coherent" because there is no question that if YYY prevails, the YYY voter supported it.

To make these ideas more precise, we define a quantitative measure Q of the support for some combination UVW comprising three propositions. Q possesses two properties:

1. It is the sum of the coherent (C) and incoherent (I) contributions of voters:

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that not even a basic paradox can be constructed when each of the 8 non-Y combinations must have more votes than each of the 19 Y combinations (with, say, 0 votes each)—and one of the latter combinations must also be the proposition-aggregation winner. Necessary and sufficient conditions for the paradox in the three-proposition case, but without abstention, are given in section 3, but these conditions can be generalized to non-binary elections in which, for example, A is a third option.

$$Q(UVW) = C(UVW) + I(UVW). \quad (1)$$

where the C and I components will be defined shortly.

2. The winning combination according to proposition aggregation is that which maximizes Q (to be proved in Theorem 1).

To construct the C and I terms, let n(UVW) denote the number of votes cast for combination UVW. We define four differences between "opposites," using our original binary distinction between Y votes and N votes on three propositions:

$n_0 = n(YYY) - n(NNN)$
$n_1 = n(YYN) - n(NNY)$
$n_2 = n(YNY) - n(NYN)$
$n_3 = n(NYY) - n(YNN)$.

These differences are set up to favor YYY, with the positive term in each difference agreeing with YYY in more than half the propositions and the negative term agreeing with YYY in fewer than half the propositions.

Given these differences, we define

$C(YYY) = 3n_0$ and $I(YYY) = n_1 + n_2 + n_3$,

based on the intuition, in the example just discussed, that a direct vote has three times the effect of indirect votes that tilt in favor of YYY.[5] Substituting into (1),

$Q(YYY) = 3n_0 + n_1 + n_2 + n_3$.

Q values for combinations other than YYY are similarly defined, but they require the insertion of some minus signs to compensate for the arbitrary choices of signs in the definitions of $n_0, \ldots, n_3$. For example,

$Q(NNY) = -3n_1 - n_0 + n_2 + n_3$,

because negative values of $n_1$ and $n_0$ indicate agreement with NNY in more than half the propositions.

[5]As a measure of the tilt toward YYY, I(YYY) is analogous to the "spin," or cyclic component, of the total vote (Zwicker, 1991).

**Theorem 1.** *The winning combination according to proposition aggregation maximizes Q.*

**Proof.** Define the difference (d) for proposition 1 as

$d_1(Y > N)$ = no. of voters voting Y - no. of voters voting N

on this proposition. Note that $d_1(Y > N) = - d_1(N > Y)$. Similarly, define $d_2$ and $d_3$ to be the differences for propositions 2 and 3.

Given any combination, such as NNY, define the following sum (S):

$S(NNY) = d_1(N > Y) + d_2(N > Y) + d_3(Y > N)$.

Note that the N or Y for each proposition in NNY matches the N or Y that is assumed greater in each $d_i$ term on the right-hand side of the equation. It is apparent that NNY will win the election if and only if each of the $d_i$'s is positive (we ignore here the possibility of ties and how they might be broken to determine a winner).

Assume NNY is the winning combination according to proposition aggregation. The S for any nonwinning combination sums the same three numbers as for YNN, but with one or more sign changes. Necessarily, at least one of the numbers for the nonwinning combinations is negative. Not only is the winning combination the only one for which each of the three $d_i$'s is positive, but this combination is also the one that maximizes S.

To complete the proof, it remains only to show that S(UVW) = Q(UVW) for any combination UVW. We do this for YYY and leave the other combinations for the reader to check:

$$d_1(Y > N) = n_0 + n_1 + n_2 - n_3;$$
$$d_2(Y > N) = n_0 + n_1 - n_2 + n_3;$$
$$d_3(Y > N) = n_0 - n_1 + n_2 + n_3.$$

When we sum the three $d_i$'s given above, we get

$$S(YYY) = 3n_0 + n_1 + n_2 + n_3 = Q(YYY),$$

as desired. Q.E.D.

Theorem 1 shows that the winning combination according to proposition aggregation is the one with the largest Q value. The fact that this value has both a C and an I component enables us to judge the extent to which a victorious combination owes its triumph to ***coherent***, or direct, support rather than to incoherent, or indirect ("tilt"), support.

The paradox of multiple elections, as illustrated in all the examples in section 3 except Example 5 (the two-office paradox), describes the extreme case wherein *all* the support for each winning combination is incoherent—no voter votes for this combination. To be sure, the paradox may occur when the winning combination receives some, but fewer, votes than any other combination.

It is worth noting that the Q value bears some resemblance to the Borda count. Imagine that a person voting for the combination UVW actually awards some points to each of the eight combinations, with the rule being that +1 point is awarded for each office on which UVW agrees with the combination in question, and -1 point for each office on which UVW disagrees. For example, a vote for YYY awards +1+1+1 = 3 points to YYY itself, whereas it awards -1+1+1 = 1 to NYY.

There are four possible levels of agreement and disagreement. Specifically, a vote for YYY awards

+3 points to YYY;
+1 point to YYN, YNY, NYY;
-1 point to YNN, NYN, NNY;
-3 points to NNN.

It is easy to show that if we add the total number of points awarded by all voters to a particular combination (e.g., YYY), the resulting sum equals Q(YYY).[6] Hence, the winning combination according to proposition aggregation—that is, the combination that maximizes Q—is the one with the highest Borda score (as we have interpreted it here).

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[6]The Borda score for YYY can be seen as identical to Q(YYY) and S(YYY), with points grouped differently. Generalizations of this scoring system to elections with any number of offices, and with abstention allowed, are given in Brams, Kilgour, and Zwicker (1996).

Thus, the voting system defined by the preceding system of awarding points is fully equivalent to the system of proposition aggregation currently in use. This correspondence shows how our present system of voting on propositions presumes an underlying cardinal evaluation of combinations of propositions. Thus, a ballot cast for YYY gives, effectively, a ranking of the eight combinations—but only an *incomplete* ranking, truncated into four levels of agreement and disagreement separated by equal intervals of 2 points, as we just showed. Not only does a YYY ballot contribute more to, say, YYN than YNN, but it does so by the same amount that other ballots that agree in two versus one office do to other combinations.

Of course, if the standard version of the Borda count were applied directly to the combinations, a voter could give a complete ranking of all eight combinations. We consider this possibility later.

Next we consider under what conditions Q(YYY) is maximal and therefore the combination YYY is winning. Then we discuss the more stringent conditions that render this winning combination paradoxical.

**Theorem 2.** *A necessary and sufficient condition for YYY to be winning is that* $Q(YYY) > 2n_0 + 2[\max\{n_1, n_2, n_3\}]$. *(Other combinations are governed by similar inequalities.)*

**Proof.** What we need to show is that YYY maximizes the Q value, and is therefore the winning combination, if and only if the above inequality is satisfied. To do this, note that YYY is winning precisely when each term of

$$d_1(Y > N) + d_2(Y > N) + d_3(Y > N)$$

is positive, or

$$n_0 + n_1 + n_2 > n_3;$$
$$n_0 + n_1 + n_3 > n_2; \text{ and} \quad (2)$$
$$n_0 + n_2 + n_3 > n_1.$$

We begin by showing that the three inequalities given by (2) are all necessary. Assume that $n_3 \geq \max\{n_1, n_2\}$, which we shall refer to as case 1. Then the second and third inequalities are true if the first is true: interchanging $n_3$ and $n_2$, and thereby obtaining the second inequality from the first, preserves the truth of the inequality, as does interchanging $n_3$ and $n_1$. But the first inequality must be true for all three to be satisfied, so all three inequalities are true if and only if the first is true.

Now the first inequality is equivalent to

$$n_0 + n_1 + n_2 + n_3 > 2n_3,$$

which is equivalent to

$$3n_0 + n_1 + n_2 + n_3 > 2n_0 + 2n_3.$$

This is the same as

$$Q(YYY) > 2n_0 + 2n_3,$$

which implies

$$Q(YYY) > 2n_0 + 2[\max\{n_1, n_2, n_3\}]. \quad (3)$$

Inequality (3) is the condition of Theorem 2. Because it is symmetric in $n_1$, $n_2$, and $n_3$, it is similarly implied by either of the other two cases, corresponding to the possibility that $n_2$ or $n_1$ is largest, or tied for largest, among $n_1$, $n_2$, and $n_3$.

To show sufficiency, note that if inequality (3) holds, regardless of which case prevails, the steps of the earlier proof are reversible, as they are with the interchange of $n_2$ and $n_3$, or $n_1$ and $n_3$. Thereby we can establish that the three inequalities of (2) all hold, ensuring that YYY is the winning combination. Q.E.D.

To obtain further insight into the conditions of the paradox, observe that Theorem 2 says that YYY is the winning combination if and only if

$$n_0 + n_1 + n_2 + n_3 > 2[\max\{n_1, n_2, n_3\}]. \quad (4)$$

Inequality (4) says that the total margin by which voters favor combinations with more Ys than Ns over their opposites must be greater than twice the maximum of the margins corresponding to indirect support.

If YYY receives the fewest votes—or ties for the fewest—then $n_0 \leq 0$ in inequality (4), which makes this inequality more difficult to satisfy than were $n_0 > 0$. Let us momentarily drop the $n_0$ term from (4) (e.g., assume $n_0 = 0$ because YYY and NNN receive equal numbers of votes). Then inequality (4) becomes

$$n_1 + n_2 + n_3 > 2[\max\{n_1, n_2, n_3\}]. \quad (5)$$

Roughly speaking, inequality (5) is satisfied when $n_1$, $n_2$, and $n_3$ are all positive (i.e., all the tilt terms favor YYY over NNN) and, in addition, $n_1$, $n_2$, and $n_3$ are close to each other in size (i.e., the tilt is evenly spread).

When we go back to inequality (4), how does the presence of $n_0$ affect this observation? The more direct support that YYY receives over its opposite, NNN (i.e., the larger $n_0$ is), then the less uniform the tilt must be in order for YYY to prevail.

On the other hand, if YYY receives no votes—making a win paradoxical—the tilt must be spread fairly evenly for YYY to win. At the same time, the larger the vote for NNN (i.e., the more negative $n_0$ is), the more evenly spread as well as larger the tilt must be to produce the paradox.

These considerations suggest two conditions sufficient to guarantee that the paradox does *not* occur for YYY. First, if $n_0$ is negative with absolute value either equal to or greater than the largest of $n_1$, $n_2$, or $n_3$, then inequality (4) cannot be satisfied and, hence, there can be no paradox. Second, the paradox is also precluded if even one of the tilt terms, $n_1$, $n_2$, or $n_3$, is less than or equal to 0; this happens when one or more of the combinations—YYN, YNY, or NYY (i.e., those that tilt towards YYY)—receive no (net) votes.

In summary, there must be more-or-less-equal positive differences between the mixed combinations that favor Y (YYN, YNY, and NYY) and their opposites (NNY, NYN, and YNN) for the paradox to occur. These positive differences overwhelm the greater direct support that NNN enjoys over YYY, enabling YYY to win even though it receives fewer votes than any other combination.

We turn next to three empirical cases. The first involves voting on multiple propositions, the second has a divided-government interpretation, and the third raises questions about the coherence of legislative choices. A genuine multiple-election paradox occurred in the first case. Although there was no full-fledged paradox in either of the latter two cases, which involved voting for different offices in an election and different bills in Congress, they illustrate situations in which there was a discrepancy between the two kinds of aggregation we have discussed in a referendum.

## 4. Empirical Cases

### Case 1: Voting on Propositions

On November 7, 1990, California voters were confronted with a dizzying array of choices on the general election ballot: 21 state, county, and municipal races, several local initiatives and referendums, and 28 statewide propositions. We analyze here only voting results on the 28 propositions, which concerned such issues as alcohol and drugs, child care, education, the environment, health care, law enforcement, transportation, and limitations on terms of office.

The data are images from actual ballots cast by approximately 1.8 million voters in Los Angeles county (Dubin and Gerber, 1992). Voters could vote yes (Y), no (N), or abstain (A)—abstention being the residual category of voting neither Y nor N—with a proposition passing if the number of its Ys exceeded the numbers of its Ns, and failing otherwise. In Los Angeles county, 11 of the 28 propositions passed, but several of these were defeated statewide, and some of the defeated propositions in Los Angeles county passed statewide.

For the purposes of this analysis, we consider only the results for Los Angeles county and ask how many voters voted for the winning combination, NNNYNNYNYNNNNNNYNYYYNYYNNYNY, on propositions 124 - 151. The answer is that nobody did, so there was a multiple-election paradox.

Because there are $2^{28} \cong 268.4$ million possible Y-N combinations, however, this is no great surprise. With fewer than 2 million voters, more than 99% of the combinations *must* have received 0 votes, even if each of the voters voted for a different combination.

In fact, however, this was not the case. "All abstain" received the most votes (1.75%), and "all no" was a close second (1.72%). Ranking fifth (0.29%) among the combinations were the recommended votes of the *Los Angeles Times*, and ranking ninth was "all yes" (0.20%).[7] Thus, the effect of the *Times* recommendations, at least for the complete list of propositions, was marginal. Nonetheless, it was greater than what Mueller (1969) found in the 1964 California general election, in which absolutely nobody in his sample of approximately 1,300 voters backed either all, or all except one, of the *Times* recommendations on 19 propositions.

Although a paradox occurred in voting on all 28 propositions, it was not a complete-reversal paradox, because the opposite of the winning combination did not garner the most votes (it, too, received zero votes). Moreover, because voters evidently considered abstain (A)—in addition to Y and N—as a voting option, it seems proper to use the $3^{28} \cong 22.9$ trillion combinations, which include A as well as Y and N as choices, in asking whether anybody voted for the winning combination. While A was "selected" by between 7.1% and 16.3% of voters on each of the propositions, its choice over Y or N could never elect A but could influence whether Y or N won.

What a voter's choice of A on any proposition did preclude was that voter's voting for the winning combination, thereby decreasing the already small likelihood that the winning combination received any votes. One could, of course, count A as a vote for both Y and N, thereby increasing the number of combinations that a voter supports; alternatively, one could give each voter one vote, splitting it among all combinations with which he or she agrees on each proposition by choosing Y, N, or A. In fact, we investigated these two different ways of aggregating votes to determine a winner and found that they would have given different results for three related propositions—all

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[7]The total for all abstain, all no, and all yes is only 3.7%. Thus, the vast majority of voters (96.7%) were not "pure" types but discriminated among propositions by choosing mixed combinations that included both Ys and Ns.

environmental bond issues—that were on the 1990 ballot (Brams, Kilgour, and Zwicker, 1996).

In the case of these three propositions, we also checked for a possible multiple-election paradox. The winning combination according to proposition aggregation was YNY, but it was supported by fewer than 6% of the voters, placing it fifth out of the eight possible combinations. While not a full-fledged paradox, the poor showing of YNY illustrates how an unpopular compromise may defeat more popular "pure" combinations (YYY was supported by 26% of the voters, NNN by 25%). The winning combination in this case was not only incoherent in the mathematical sense used earlier but also in a more substantive sense: it was pro-environment on two bond issues, anti-environment on the third, rendering policy choices by the voters somewhat of a hodgepodge that, as we pointed out, had little direct support. Furthermore, two other mixed combinations received more support.

**Case 2: Voting in Federal Elections**

The multiple-election paradox may occur not only in voting on multiple propositions but also in voting for multiple offices in an election. For example, in a presidential election year, a voter might vote for the Republican candidate for president, the Democratic candidate for Senate, and the Republican candidate for House—that is, the combination RDR. Just as votes are aggregated by proposition in a referendum, votes can be aggregated by office in an election (which we call *office aggregation*), so in theory RDR could receive, as a combination, the fewest votes.

Now insofar as the federal government is conceived as a single entity, normative arguments can be made that the most popular combination *should* win. But most voters, it seems, do not think in terms of electing a combination, at least not at a conscious level (Fiorina, 1992, pp. 65). Nevertheless, in voting for their favorite candidates for each office, they may worry about the consequences of divided (or unified) government—and possibly act on this concern in choosing a combination.

In so doing, voters seem to apply different criteria in selecting presidents and legislators. For example, voters until 1994 tended to favor Democrats for the benefits, protections, and services they provided at the district and state level, but Republicans for the discipline and responsibility, especially on economic matters, that they exercised at the national level (Zuppan, 1991; Jacobson, 1992, pp. 71-75). Thereby they hedged their bets, especially if they were "sophisticated," and opted for "balance" in the government (Alesina and Rosenthal, 1995). In the view of some (e.g., Conlan, 1991; Mayhew, 1991; Fiorina, 1992), divided government, which has been the norm from 1968 until 1996 (it occurred in 22 of the 28 years), did not impede the passage of major legislation.

To most voters, there is nothing incoherent in voting for a combination like RDR. Moreover, if this combination wins according to office aggregation, there is nothing paradoxical about the fact that the voters, collectively, chose divided government.

It is instructive to contrast two cases. In 1976, a Democratic president, Senate, and House were elected, giving DDD. In the absence of reliable combination voting data for the three offices (either from actual ballots cast by individual voters for the three offices or from voter surveys), we treated the 435 congressional districts *as if* they were voters, which raises difficulties we shall discuss shortly. We then classified the subset of districts with senatorial races (about 2/3) according to the eight combinations, depending on which party (D or R) won each of the three offices in a district.

We caution that, unlike the hypothetical examples for propositions given in section 2, we consider the winners in the Senate and House to be the party that wins a majority of seats in each house, not the party with the greatest number of Senate or House votes nationwide. Also, because many voters base their choices less on party than on the individual candidates running, interpreting the combination that wins as indicating a preference for either divided or unified government is somewhat questionable.

Bearing these caveats in mind, the results for 1976 are that the DDD combination was the most popular, being the choice of 40.8% of the 316 districts with senatorial races, whereas the next-most-popular combination, RRR, won in only 20.6% of the districts (see Table 1). In short, unified government garnered more than 3/5 of the vote that year,

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Table 1 about here

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at least as indicated by the congressional district results with senatorial races; and the most popular of these combinations, DDD, concurred with the office-aggregation winner.

By contrast, RRD was the winning combination in 1980, but it was only the fourth-most-popular voting combination (again, by congressional districts with senatorial races, of which there were 315), as shown in Table 1. As in 1976, the straight-ticket voting combinations won in the most districts (28.6% for RRR and 22.2% for DDD). Although RRD was not the least popular combination, its fourth-place finish with 14.3% seems at least semi-paradoxical.

To be sure, the contrast between 1976 and 1980, with unified government coinciding with the winning combination in 1976 and divided government coinciding with the fourth-place combination in 1980, could be happenstance. Unfortunately, combination-voting data for the three federal offices seem to have been collected only for 1976 and 1980 (Gottron, 1983), so we cannot test for the paradox in other presidential election years.[8]

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[8]Inexplicably, voting returns reported in Congressional Quarterly's *Congressional Districts in the 1990s: A Portrait of America* (CQ Press, 1993) for the 1988 presidential election are for the 1992 congressional districts, based on the the 1990 census, so they do not accurately reflect the results of the 1988 election. While Congressional Quarterly's annual *Politics in America* and National Journal's annual *Almanac of American Politics* give presidential-election returns by congressional district and state, senatorial returns for each congressional district are not available (except in 1976 and 1980). Although senatorial returns are broken down by county in Congressional Quarterly's *America Votes* series, congressional districts often divide counties, requiring that one use precinct-level data to determine senatorial results by district. While such data for approximately 190,000 precincts have been collected for the period 1984-90 by a now-defunct group called "Fairness for the 90s," an officially nonpartisan and nonprofit organization, most of the data are not currently in a form amenable to computer analysis (King, 1996).

We have, however, analyzed combination-voting data for the two-office elections of president/senator and president/representative for the five presidential elections between 1976 and 1992. In such elections, it will be recalled from Example 5 in section 2, the winning combination by office aggregation can rank as low as third out of four. Treating the 50 states (actually, only the 33 or 34 states that had senatorial contests in each year) as if they were voters in the president/senate comparisons, and the 435 House districts as if they were voters in the president/representative comparisons, we found that in only two of the ten comparisons—the president/representative comparisons in 1980 and 1988—did the winning combination by office aggregation (i.e., RD) come in even as low as second (RR in each of these years won according to combination aggregation).[9]

The absence of even a mildly paradoxical third-place finish of the combination winner (the two-office paradox) may well be attributable to aggregating voters by district and state and treating these large units as if they were individual voters. It seems likely that this aggregation wipes out numerous mixed combinations—one of which might win according to office aggregation with relatively few votes—that one would pick up from individual ballots.[10] Furthermore, the fact that the second-place finishes occur only in the president/representative comparisons and not in the president/senate comparisons is *prima facie* evidence that more aggregative units (i.e., states rather than congressional districts) have this wipe-out effect, decreasing the probability of observing a paradox.

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[9]In the president/senator comparison in 1988, there was a tie for first, according to combination aggregation, between RD and RR (RD was the office-aggregation winner that year). Although the Republican presidential candidate (George Bush) prevailed in both combinations, this fact does not ensure that such a candidate, who may win in a majority of states, would win a majority of either popular or electoral votes should these states be mostly small.

[10]In particular, a district that roughly reflects the nation might appear to vote for the national winning combination, even though that vote in fact represents a paradoxical combination that individual ballot data would have revealed.

**Case 3: Voting on Bills in Congress**

There was a two-office paradox in the case of what are generally acknowledged to be the two most important votes to come before the House of Representatives during the first two years of Bill Clinton's administration—that on the budget on August 5, 1993, and that on NAFTA on November 17, 1993. On these two bills, NY got 36%, YN got 32%, YY got 18%, and NN got 14%; the winning combination was YY. The explanation for why only 18% of the House—all Democrats—supported President Clinton on *both* bills lies in the fact that the party split was very different on the two bills: 84% of Democrats and no Republicans voted Y on the budget bill, whereas 40% of Democrats and 75% of Republicans voted Y on NAFTA bill.

It is appropriate to ask whether there is anything problematic about the winning combination of YY receiving so few votes, given that it can be viewed as a compromise between the two more popular alternatives, NY and YN. Indeed, 86% of House members saw their preferred position enacted on *at least one* of the two bills. By this measure of satisfaction, YY is better than YN (64% satisfied), NY (68% satisfied), and NN (82% satisfied).

We shall return to a consideration of the normative implications of the paradox in the concluding section. But first take up the connection between the multiple-election paradox and the most venerable of all aggregation paradoxes—the paradox of voting.

## 5. The Paradox of Voting

To illustrate the linkage between these two paradoxes, we present two examples from voting in Congress. Because the paradox of voting assumes that voters have certain preferences over a set of alternatives, the previous analysis, based solely on a numerical comparison of winning combinations under two different aggregation methods, must be extended. The preferences we assume enable us to create an isomorphism that renders the multiple-election paradox a natural generalization of the paradox of voting.

Our first congressional example concerns the Wilmot Proviso, which prohibited slavery in land acquired from Mexico in the Mexican war. On August 8, 1846, there were several votes in the House of Representatives for attaching this proviso to a $2 million appropriation to facilitate President James K. Polk's negotiation of a territorial settlement with Mexico. The three possible outcomes were:

*a*. Appropriation without the proviso;
*b*. Appropriation with the proviso;
*c*. No action.

Riker (1982, pp. 223-227) reconstructs the preferences of eight different groups of House members for these outcomes, where *xyz* indicates a group prefers *x* to *y*, *y* to *z*, and *x* to *z* (the groups are assumed to have transitive preferences). The preferences of these groups, which comprise a total of 172 House members, are shown in Table 2.

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Table 2 about here

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To simplify the subsequent analysis, assume that the 8 border Democrats split 4 - 4 for each of their two possible preference scales shown in Table 2, and the 3 border Whigs split 1 1/2 - 1 1/2 for their two possible preference scales (not actually possible, of course, but the subsequent results do not depend on how we split the votes of either group). Then it is easy to show that majorities are cyclical: *b* beats *a* (as happened) by 93 to 79 votes, *a* beats *c* by 129 1/2 to 42 1/2 votes, and *c* beats *b* by 107 to 65 votes. Consequently, there is no *Condorcet outcome* that defeats each of the other outcomes in pairwise comparisons, which makes the social choice an artifact of the order of voting.

To establish an isomorphism between the paradox of voting and the multiple-election paradox, assume that the eight votes actually taken on the proviso in the House on August 8, 1846, can be reduced to three hypothetical pairwise contests between (1) *a* and *b*, (2) *b* and *c*, and (3) *c* and *a*. Assume further that, given its preferences, each group can answer yes (Y) or no (N) about whether it prefers the first member of each pair to the second. (We assume, as before, that the border Democrats and border Whigs split 50-50 on their preferences for second and third choices.)

Answers to these three questions give what we call an *answer sequence*. For example, an answer sequence of YYN indicates that the group prefers *a* to *b*, *b* to *c*, but not *c* to *a*, so its preference scale is *abc*. (In the remainder of this section, we assume for simplicity that preferences are strict.) Likewise, we can associate five other mixed answer sequences of Ys and Ns with the preference scales shown below each:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Preference: | *abc* | *cab* | *bca* | *acb* | *bac* | *cba* | ? | ? |
| Sequence: | YYN | YNY | NYY | YNN | NYN | NNY | YYY | NNN |

The question marks indicate intransitive preferences. Thus, for a group to answer Y to all three questions indicates a preference cycle *abca*; to answer N to these questions reverses the direction of the cycle, giving *cbac*. Although we assume that groups of like-minded House members have, like individuals, transitive preferences, we shall return to this matter later.

In voting on the Wilmot Proviso, observe that the winner by combination aggregation is *acb* (YNN) with 66 votes, comprising 4 border Democrats, 46 Southern Democrats, and 16 Southern and border Whigs:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Sequence: | YYN | YNY | NYY | YNN | NYN | NNY | YYY | NNN |
| Votes: | 11 | 2 | 1 1/2 | 66 | 52 1/2 | 39 | 0 | 0 |

By contrast, the winner by what we will call *bill aggregation*, which is analogous to proposition aggregation and office aggregation, in pairwise contests (1), (2), and (3) is NNN. Specifically, N beats Y in contest (1) by 93 to 79 votes, in contest (2) by 107 to 65 votes, and in contest (3) by 129 1/2 to 42 1/2 votes. Thus, we have a basic multiple-election paradox: the winner by bill aggregation (NNN) ties for the fewest votes (with the other intransitive sequence, YYY).[11]

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[11]It is perhaps more accurate to call this the "multiple-vote paradox," because the multiple votes in Congress are not really multiple elections, as in the case of voting on propositions in a referendum or for different offices in an election. For simplicity, we will stick with "multiple-election paradox," but it is worth noting two distinct features of

This coincidence of the paradox of voting and the multiple-election paradox is no accident. If there is a paradox of voting, the outcome is cyclical majorities, which in our isomorphism translates into either YYY or NNN. But since no group with transitive preferences has these sequences, they must, according to combination aggregation, receive 0 votes. Consequently, the winning combination according to bill aggregation (either YYY or NNN) when there is a paradox of voting must tie for the fewest votes (with the other intransitive sequence). Thus we have

**Theorem 3.** *If the preferences of individual voters (or like-minded groups) are transitive with respect to pairwise contests among three or more alternatives, then a paradox of voting, based on the pairwise contests, implies a multiple-election paradox.*

Whether the reverse implication holds turns on the number of alternatives being ranked. For three alternatives, it turns out that none of the six mixed combinations can win, according to bill aggregation, and also receive 0 votes when YYY and NNN do, too. To show that there is no such example for three pairwise contests, associate the following numbers of voters with the six mixed-answer sequences:

| Sequence: | YYN | YNY | NYY | YNN | NYN | NNY |
|---|---|---|---|---|---|---|
| Number: | 0 | $v$ | $w$ | $x$ | $y$ | $z$ |

Without loss of generality, we have assumed that YYN receives the fewest votes, and that this number of votes is 0. The other numbers are all nonnegative. Now in order for Y to win the first and second offices by bill aggregation, we need

$$v + x > w + y + z$$
$$w + y > v + x + z.$$

Adding these inequalities gives $0 > 2z$, which is impossible since $z \geq 0$. This contradiction shows that YYN cannot win by bill aggregation and receive the fewest votes.

voting in Congress: (1) the non-election quality of votes on bills and amendments and (2) the sequential nature of voting, which gives voters information about the results of previous votes that simultaneous voting on multiple propositions, or for multiple offices, does not provide.

On the other hand, if there are four alternatives we have

**Example 8 (basic paradox, with ties for fewest: 4 outcomes and 6 pairwise contests).** Suppose there are 3 voters, whose preferences among the alternatives *a*, *b*, *c*, and *d* are as follows:

*bacd*: 1 *cabd*: 1 *dabc*: 1.

Then their votes on the six questions of whether their first alternative is preferred to their second for the six possible pairwise contests—*a* and *b*, *b* and *c*, *c* and *d*, *a* and *c*, *a* and *d*, and *b* and *d*—will be

NYYYYY: 1 YNYNYY: 1 YYNYNN: 1.

Now YYYYYY is the winning combination according to bill aggregation, corresponding to the transitive ordering *abcd* for which none of the voters voted. Thus, we have an example of a multiple-election paradox that does not arise, given our particular enumeration of pairwise contests, from a paradox of voting. Note that the Condorcet alternative, *a*, is not ranked first by any of the voters.

More generally, this example, together with our earlier argument that a transitive combination cannot win according to bill aggregation when there are only three pairwise contests, yield the following strengthening of Theorem 3:

**Theorem 4.** *Assume there are three or more alternatives over which voters have transitive preferences. Then every paradox of voting corresponds to a multiple-election paradox. The reverse correspondence holds for three alternatives but fails for more than three.*

Theorem 4 shows that, given our isomorphism, the multiple-election paradox is a generalization of the paradox of voting, because whenever the latter occurs so does the former, but not vice versa if there are more than three alternatives.

We caution that Theorem 4 should not be construed as an empirical law in situations in which voters may, for a variety of reasons, not express transitive preferences and therefore not meet the condition of Theorem 4. For example, it may not be clear at the outset that they will vote in a particular sequence in three pairwise contests, so the question of being consistent is not a primary consideration.

Even if it is, voters may decide to vote YYY or NNN if such ostensibly inconsistent behavior on the part of enough voters leads to a contradiction, which in turn triggers a default option that these voters prefer. For example, assume that an NNN sequence indicates that a voter votes "no" on three pairwise contests between three levels or types of regulation; if none wins, the status quo of no regulation prevails, which the voter prefers. Then the apparent contradiction of preferring none of the three levels—when matched in pairs against each other—is really no contradiction, given a preference for the default option.

That some voters are, at least on the surface, inconsistent in this sense is observable in actual legislative contests. Blydenburgh (1971) studied the voting behavior of members of the House on Representatives in voting on three provisions of the Revenue Act of 1932: the first to delete a sales tax, the second to add an income tax, and the third to add an excise tax.

Let *a* be the status quo (SQ) without a sales tax, *b* be the SQ with an income tax, and *c* be the SQ with an excise tax. Based on his reconstruction of voter preferences, Blydenburgh (1971) argued that there was a paradox of voting *abca*, so majorities would answer YYY in each of the three pairwise contests.

In fact, however, there were no such contests, because the voting was sequential under the amendment procedure. The first contest was *a* versus SQ; when *a* passed, the second contest was *a* versus *a* plus *b* (i.e., SQ with both a sales and income tax); when the latter failed, the third contest was *a* versus *a* plus *c* (i.e., SQ and both a sales and excise tax), which passed. Thus, the winner by bill aggregation in *these* three pairwise contests was YNY. This combination was chosen by 38 voters, ranking fifth of the eight combinations according to combination aggregation.[12]

The fact that there was no multiple-election paradox shows there is obviously some slippage between our theoretical results and their empirical reality. We take the fifth-place finish of the winning combination, nonetheless, as partial confirmation of a discrepancy between—if not a paradoxical aspect of—the two different ways of aggregating votes.

The significance of this discrepancy is underscored by the linkage of the multiple-election paradox to the paradox of voting. The paradox of voting has produced an enormous literature since the pioneering work of Black (1958) and Arrow (1963), which first appeared in the late 1940s and early 1950s, that extended and generalized the original paradox discovered by Condorcet in the late 18th century (see Black, 1958; McLean and Urken, 1995). The multiple-election paradox, we believe, casts the paradox of voting in a new light that illuminates, especially, its implications for making coherent social choices using different aggregation procedures.

### 6. Normative Questions and Democratic Political Theory

Let us return to the case of voting on propositions, with which we introduced the analysis. Given that the winner under proposition aggregation can receive the fewest votes under combination aggregation—and even that the two methods of aggregation can produce diametrically opposed social choices (when there is a complete-reversal paradox)—it is legitimate to ask which choice, if either, is the proper one. In defining "proper," one might apply such social-choice criteria as the election of Condorcet outcomes (if they exist), the selection of Pareto-efficient outcomes, the existence of

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[12]The winning combination was YYY with 85 votes. Because the pairwise contests were not among $a$, $b$, and $c$ but partially overlapping sets of alternatives (see text), it is not inconsistent for individual voters to have a preference order associated with YYY in this case.

incentives to vote sincerely, and so on.[13] We shall not pursue this line of inquiry here, however, but instead ask an explicitly normative question: Is a conflict between the proposition and combination winners necessarily bad?

In addressing this question, we first consider whether this conflict comes as any great surprise. If there is one thing that social choice theory has taught us over the last several decades, it is that strange things may happen when we try to aggregate individual choices into some meaningful whole. Thus, the whole may lose important properties that the parts had, such as transitivity of preferences when there is a paradox of voting.

Whether the intransitivity of social preferences caused by the paradox of voting is a serious social problem has been much debated in the literature (Riker, 1982, and Miller, 1983, give representative views). The multiple-election paradox shows up a different aspect of this problem by drawing our attention to the discrepancy between aggregating votes by proposition and by combination. From a theoretical viewpoint, what is interesting about the multiple-election paradox is that it is a more general phenomenon than the paradox of voting—at least under our isomorphism—but we have not analyzed in detail those situations that give the multiple-election paradox, and not the paradox of voting, to see precisely where the differences between the two paradoxes lie.

From a practical viewpoint, we are led to ask whether, given the multiple-election paradox, it would be advisable for voters to vote directly for combinations rather than for

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[13]Benoit and Kornhauser (1994) focus on the inefficiency of assemblies elected by office aggregation, given that voters have separable preferences over all possible combinations of candidates for the assembly. (An inefficient assembly is one in which the candidates elected by office aggregation are worse for all voters than some other assembly—possibly one elected by combination aggregation—and so might receive zero votes when pitted against it.) A crucial difference between our model and theirs is that there is no restriction on the number of propositions that can pass in a referendum, or bills in a legislature, in our model, whereas in their model the number of representatives to be elected to the legislature is predetermined. Although the multiple-election paradox is based purely on numerical comparisons, it may be explicitly linked to preference-based models like that of Benoit and Kornhauser, as we illustrated in the case of the paradox of voting through the answer-sequence isomorphism. See also Lacy and Niou (1994) and Brams, Kilgour, and Zwicker (1996), who analyze referenda in which voters have nonseparable preferences.

individual propositions, offices, and bills. We have our doubts in the case of different offices, in part because it is not clear how combination aggregation would work in the election of bodies like the Senate or House. In the case of the president, one could prescribe that if the winning combination includes, say, D for president, the Democrat would be elected. But if the winning combination turns out to be DDD, as occurred in 1992, what does it mean to elect a Democratic Senate and a Democratic House, and in what proportions in what states?[14]

In voting in other arenas, such as on a referendum or in a legislature, we believe the choices that legislators and voters can now make substantially restrict their ability to express their preferences. Thus, legislators cannot express support for different packages of amendments, such as the amendments sequentially voted on in the 1932 Revenue Act (section 5). If they vote YYY, for example, this contributes nothing to NNN, even though the latter package might be their second choice. Likewise, there is no way under the present system that legislators can support exactly the six mixed combinations.

A possible solution to this problem is to use approval voting (Brams and Fishburn, 1983), whereby voters could, in the present instance, vote for as many combinations as they wish. Thus, a proponent of all the amendments or of none—assuming he or she regards these as the only acceptable packages—could indeed vote for both YYY and NNN, just as a proponent of some but not all of the amendments could vote for from one to six of the mixed combinations.

But approval voting for combinations is not the only way of expanding voter choices. Other means for producing more coherent social choices, in light of the paradox,

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[14]Fiorina (1992, p. 120) argues that the eight combinations the voter can choose for the three federal offices in the United States are more numerous than voters have in many multiparty systems; furthermore, unlike in multiparty systems, voters can "vote directly for the coalition they most prefer." As the multiple-election paradox dramatically demonstrates, however, this *expression* of preference for a combination means little, because the combination with the least support can actually win, vitiating the vaunted "popular will" from being expressed.

include allowing voters to rank the combinations under a system like the Borda count. This would enable voters to make more fine-grained choices than does the crude variation of the Borda count, discussed in section 3, that corresponds to the present system.

To be sure, if there are more than eight or so combinations to rank, the voter's task could become burdensome. How to package combinations (e.g., of different propositions on a referendum, different amendments to a bill) so as not to swamp the voter with inordinately many choices—some perhaps inconsistent—is a practical problem that will not be easy to solve.[15]

We raise these questions about packaging and voting systems not so much to provide answers but rather to show how the mulitple-election paradox invests them with important social-choice consequences. Their ramifications, especially for reform, need to be analyzed in concrete empirical settings. As we indicated in the case of the 1990 California referendum (section 4), different ways of casting votes and counting abstentions would almost surely have led to different outcomes on a set of propositions related to the environment (Brams, Kilgour, and Zwicker, 1996).

The multiple-election paradox tends to create the greatest problem when the issues being voted on in a referendum are linked, but the voter has to make simultaneous choices. (This was not the case for House members in the 1993 votes on NAFTA and the budget, in which the votes were not simultaneous and also reflected very different party alignments.) As a case in point, suppose 1/3 of the electorate favors proposition U = do *a* alone, 1/3 V = do *a* and *b*, and 1/3 W = do *a*, *b*, and *c*. (These measures might represent different levels of environmental cleanup.) Now if a voter votes only for his or her first choice, then 1/3 of the electorate will vote YNN, 1/3 YNY, and 1/3 NNY, yielding NNN under proposition aggregation—nothing passes. But the choice that best reflects the will

[15]"Yes-no voting" (Fishburn and Brams, 1993), whereby a voter can indicate multiple packages of propositions he or she supports, would render practicable voting on large numbers of propositions.

of the electorate is V, a moderate level of cleanup, which completely satisfies 1/3 of the voter and partially satisfies the remaining 2/3. It seems likely that V would have won had there been approval voting for the eight combinations.

At a minimum, a heightened awareness of the multiple-election paradox alerts us to unintended and often deleterious consequences that may attend the tallying of votes by proposition aggregation. The paradox does not just highlight problems of aggregation and packaging, however, but strikes at the core of social choice—both what it means and how to uncover it. In our view, the paradox shows there may be a clash between two different meanings of social choice, leaving unsettled the best way to uncover what this elusive quantity is.

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**Table 1**

**Combination Returns for 1976 and 1980 Presidential Elections, by Congressional Districts, in States with Senatorial Contests**

| *Combination* | *1976* | *1980* |
|---|---|---|
| 1. DDD | 40.8%* | 22.2% |
| 2. DDR | 5.7 | 2.5 |
| 3. DRD | 2.6 | 2.2 |
| 4. DRR | 1.9 | 0.6 |
| 5. RDD | 11.7 | 16.8 |
| 6. RDR | 8.2 | 12.7 |
| 7. RRD | 8.5 | 14.3* |
| 8. RRR | 20.6 | 28.6 |
| *Total* | 100.0 (*n* = 316) | 100.0 (*n* = 315) |

*Winner by office aggregation

**Table 2**

**Preferences of Groups of Members of the House of Representatives in Voting on the Wilmot Proviso (1846)**

| *Group* | *No. of Members* | *Preferences* |
|---|---|---|
| Northern Administration Democrats | 7 | abc |
| Northern Free Soil Democrats | 51 | bac |
| Border Democrats | 8 | abc or acb |
| Southern Democrats | 46 | acb |
| Northern prowar Whigs | 2 | cab |
| Northern antiwar Whigs | 39 | cba |
| Border Whigs | 3 | bac or bca |
| Southern and border Whigs | 16 | acb |
| *Total* | 172 | |

*Source:* Riker (1982, p. 227)