ECONOMIC RESEARCH REPORTS

A NOTE ON THE ESTIMATION OF NON-SYMMETRIC DYNAMIC FACTOR DEMAND MODELS

by

Dilip B. Madan

and

Ingmar R. Prucha

R.R. #87-25 July 1987

C. V. STARR CENTER
FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

A Note on the Estimation of Non-Symmetric Dynamic Factor Demand Models

Dilip B. Madan* and Ingmar R. Prucha**

May 1987

* Department of Economics, University of Maryland, College Park, MD 20742; and Department of Econometrics, University of Sydney, Sydney, N.S.W. 2006, Australia.

** Department of Economics, University of Maryland, College Park, MD 20742; National Bureau of Economic Research, Cambridge, MA 02138; and C.V. Starr Center for Applied Economics, New York University, New York, NY 10003.

Abstract

In a recent article Epstein and Yatchew (1985) introduced a simplified procedure for the estimation of symmetric dynamic factor demand models. This procedure hinges on a reparametrization of the model, results in closed form analytic expressions for the firm's factor demand, and can be carried out by standard econometric packages. The purpose of this note is to extend the procedure to the case of non-symmetric dynamic factor demand models.

1. Introduction[1]

In a recent article Epstein and Yatchew (1985) introduced a simplified procedure for the estimation of a class of linear dynamic factor demand systems. While the procedure is presented in terms of a dynamic factor demand model the procedure can also be applied towards the estimation of other linear rational expectations models which have a similar structure.

The Epstein and Yatchew procedure is similar to that suggested by Hansen and Sargent (1980, 1981) in that the solution to the firm's (stochastic closed loop) optimal control problem is obtained by solving the stable roots backwards and the unstable roots forwards, and it incorporates the transversality condition. The class of models considered by Epstein and Yatchew is less general than that considered by Hansen and Sargent in that adjustment costs only depend on first-order changes in the factor inputs.

Epstein and Yatchew take adjustment costs to be separable and assume that all new investment becomes immediately productive. This results in a symmetric second-order system of difference equations for the optimal factor inputs. The method suggested by Epstein and Yatchew hinges on a reparametrization of the model. Due to this reparametrization it is possible to obtain closed form analytic expressions for the firm's factor demand. As a consequence, this procedure can be carried out by standard econometric packages and avoids during estimation the need for the repeated solution of the firm's control problem by numerical methods.

Prucha and Nadiri (1986) adopt (and trivially modify) the Epstein and Yatchew procedure to the case where all new investment becomes productive with a one-period lag. Linear rational expectations models that correspond to a

[1]We would like to thank Harry Kelejian, Benedikt Pötscher and Robert Schwab for their helpful comments. However, we assume responsibility for any errors. This research was supported by the C.V. Starr Center's Focus Program for Capital Formation, Technological Change, Financial Structure and Tax Policy.

symmetric second-order difference equation system have recently also been discussed by Kollintzas (1985).

Many models of interest correspond to a set of non-symmetric difference equations. For example, symmetry is lost if we allow inputs to become productive at different points in time or if we allow for non-separability between the adjustment costs and the levels of the inputs.[2]

The purpose of this note is to extend the Epstein and Yatchew procedure to the non-symmetric case. This requires, firstly, that we demonstrate that both the backward- and the forward-looking portion of the optimal control solution can be expressed in terms of the accelerator matrix. In the symmetric case such a demonstration is readily achieved by "diagonalizing" the problem; compare, e.g., Lucas (1967) and Kollintzas (1985). In the non-symmetric case such a diagonalization is generally not possible, and hence, we introduce an alternative approach towards the construction of such an optimal control solution. Secondly, we extend the reparameterization method and derive closed form analytic expressions for the firm's factor demand.

2. A Generalization of a Closed Form Estimation Procedure

Consider a firm that produces output y_t from the kx1 input vector $x_t = [n_t', s_t']'$ where n_t is the k_1x1 vector of immediately productive inputs and s_t is the k_2x1 vector of inputs that only become productive with a one-period lag. The firm's production set is defined by the following production function:

$$(1) \qquad y_t = F(x_t, x_{t-1}, \Delta x_t) = a_0 + [n_t', s_{t-1}']a + \tfrac{1}{2}[n_t', s_{t-1}']A[n_t', s_{t-1}']' + \tfrac{1}{2}[\Delta n_t', \Delta s_t']B[\Delta n_t', \Delta s_t']' + [n_t', s_{t-1}']C[\Delta n_t', \Delta s_t']' \ ,$$

[2] We note that in empirical studies capital is often assumed to only become productive with a lag, while white collar workers are typically modeled as immediately productive; see, e.g., Berndt and Morrison (1981).

with

$$a = \begin{bmatrix} a_n \\ a_s \end{bmatrix}, \quad A = \begin{bmatrix} A_{nn} & A_{ns} \\ A_{sn} & A_{ss} \end{bmatrix}, \quad B = \begin{bmatrix} B_{nn} & B_{ns} \\ B_{sn} & B_{ss} \end{bmatrix}, \quad C = \begin{bmatrix} C_{nn} & C_{ns} \\ C_{sn} & C_{ss} \end{bmatrix},$$

and where $\Delta x_t = x_t - x_{t-1}$. We assume that the production function satisfies the usual properties. In particular, the matrix composed of A and B as diagonal blocks and C and C′ as off-diagonal blocks is assumed to be symmetric and negative definite. Adjustment costs in terms of forgone output are reflected by $\Delta x_t = [\Delta n_t', \Delta s_t']'$ as an argument in the production function.

The firm is assumed to choose its inputs according to a stochastic closed loop feed back control policy in order to maximize the expected present value of future profits.[3] More specifically, the firm sets the current input vector and chooses a contingency plan for setting its inputs in future periods according to the following optimization problem:

$$(2) \qquad \max E_t \textstyle\sum_{\tau=t}^{\infty} \gamma^{\tau-t} [F(x_\tau, x_{\tau-1}, \Delta x_\tau) - (Q_w w_\tau)' x_\tau - (Q_q q_\tau)'(x_\tau - (I-\delta) x_{\tau-1})]$$

for given initial stocks x_{t-1} and $\gamma = (1+r)^{-1}$. Here E_t denotes the expectations operator conditional on information available at time t. With w_t and q_t we denote m_1x1 and m_2x1 vectors of non-zero factor prices normalized by the output price where $m_i \leq k$ (i=1,2). The elements of w_t may be thought of as representing short-run costs such as wages, the elements of the q_t may be thought of as representing after tax acquisition prices. Since the dimension of w_t and q_t may be less than that of x_t we have introduced selector matrices Q_w and Q_q. Those matrices are of dimension k$\times m_1$ and k$\times m_2$, respectively, with rank$[Q_q, Q_w]$=k; they select the appropriate elements from the x_t vector. The

[3]Since B is negative definite all inputs are taken to be quasi-fixed. Variable factors can be readily incorporated by specifying the firm's technology in terms of the restricted profit function. We note that our discussion in terms of a profit maximization problem is only illustrative. The discussion also applies, with trivial modifications, to a cost minimization problem.

diagonal matrix of depreciation rates (some of which may be zero) is denoted by δ, the real discount rate is denoted by r.

The prices q_t and w_t are known at time t and are exogenous to the firm. As in Epstein and Yatchew (1985) we assume that they are determined by an autoregressive process:

$$(3) \quad [q_t', w_t']' = \nu + \Sigma_{i=1}^{p} \Theta_i [q_{t-i}', w_{t-i}']' + \xi_t \ ,$$

where the ξ_t's distributed i.i.d.. We assume that the mean exponential order of the price process is less than $(1+r)^{1/2}$.

The objective function in the above specified control problem is linear-quadratic. Certainty equivalence then implies that the optimal inputs in period t corresponding to the stochastic control problem (2) are identical to those obtained by solving the following nonstochastic control problem:

$$(4) \qquad \max \ \sum_{\tau=t}^{\infty} \gamma^{\tau-t} [F(x_\tau, x_{\tau-1}, \Delta x_\tau) - (E_t Q_w w_\tau)' x_\tau - (E_t Q_q q_\tau)'(x_\tau - (1-\delta) x_{\tau-1})]$$

for given initial stocks x_{t-1}. The input sequence optimizing (4) must satisfy the following set of deterministic Euler equation ($\tau = t, \ldots, \infty$):[4]

$$(5) \quad -\underline{B} x_{\tau+1} + \underline{G} x_\tau - (1+r)\underline{B}' x_{\tau-1} = -(1+r)\underline{a} + (1+r)\underline{a}_\tau \ ,$$

where

$$\underline{G} = \begin{bmatrix} (1+r)(A_{nn} + C_{nn} + C_{nn}') + (2+r)B_{nn} & (1+r)C_{ns} - C_{sn}' + (2+r)B_{ns} \\ (1+r)C_{ns}' - C_{sn} + (2+r)B_{sn} & A_{ss} - C_{ss} - C_{ss}' + (2+r)B_{ss} \end{bmatrix} ,$$

$$\underline{B} = \begin{bmatrix} B_{nn} + C_{nn}' & B_{ns} \\ B_{sn} + C_{ns}' - C_{sn} - A_{sn} & B_{ss} - C_{ss} \end{bmatrix} ,$$

$$\underline{a} = [a_n', a_s'/(1+r)]' \ , \qquad \underline{a}_\tau = E_t Q_w w_\tau + E_t Q_q q_\tau - [(I-\delta)/(1+r)] E_t Q_q q_{\tau+1}.$$

[4] Clearly, the optimal control solution can, e.g., also be found via iterations on the Ricatti equation. However the focus of this note is to obtain a closed form solution to (4) that can serve as a basis for a simplified estimation procedure.

Furthermore, this optimizing input sequence has to satisfy the transversality condition. We assume that $\underline{B}$ is nonsingular and that the characteristic roots of equation (5) are distinct. It is well known and simple to show that those roots come in pairs multiplying to (1+r). Equation (5) differs from the corresponding equation in, e.g., Epstein and Yatchew (1985), Kollintzas (1985) and Prucha and Nadiri (1986) in that the matrix $\underline{B}$ is (possibly) non-symmetric. We henceforth refer to situations with $\underline{B}=\underline{B}'$ and $\underline{G}=\underline{G}'$ as the symmetric case and to situations with $\underline{B}\neq\underline{B}'$ and $\underline{G}=\underline{G}'$ as the non-symmetric case.[5] Let Λ be the kxk diagonal matrix of stable roots of the homogeneous difference equation system corresponding to (5) and let V be the kxk matrix of solution vectors corresponding to those stable roots; further let $M = I - V\Lambda V^{-1}$. In the Appendix we prove the following theorem:

Theorem: *The optimal factor inputs at time t corresponding to control problem (3) and (because of the certainty equivalence principle) to control problem (2), are given by the following accelerator model:*

$$(6) \qquad x_t = M\bar{x}_t + (I-M)x_{t-1}, \qquad \bar{x}_t = \underline{A}^{-1}(I_t - \underline{a}),$$
$$I_t = D\sum_{\tau=t}^{\infty}(I+D)^{-(\tau-t+1)}\underline{a}_\tau, \qquad D = (1+r)(I-M')^{-1} - I,$$
$$\underline{A} = (I-M')^{-1}(rI+M')\underline{B}M/(1+r) = D\underline{B}M/(1+r)$$

The accelerator matrix M *satisfies:*

$$(7) \qquad -\underline{B}(I-M)^2 + \underline{G}(I-M) - (1+r)\underline{B}' = 0.$$

Furthermore $S = \underline{B}(I-M)$ *is symmetric.* ■

The structure of the above solution for x_t resembles closely that given

[5]Clearly, the present analysis of the non-symmetric case also covers the symmetric case since nowhere in the analysis do we impose $\underline{B}\neq\underline{B}'$.

in Epstein and Yatchew (1985) for the symmetric case.[6] However, there are subtle differences. Firstly, note that in the symmetric case $\underline{B}M = [\underline{B}M]'$; in the non-symmetric case $\underline{B}(I-M) = [\underline{B}(I-M)]'$ but in general $\underline{B}M \neq [\underline{B}M]'$. Secondly, note that in the symmetric case $\underline{A} = \underline{A}'$; however, in the non-symmetric case we find that generally $\underline{A} \neq \underline{A}' = \underline{B}M(M+rI)(I-M)^{-1}/(1+r)$.[7] Consequently, it is generally incorrect to use the latter formula for $\underline{A}$ in the non-symmetric case. The latter formula corresponds to formula (9) in Epstein and Yatchew (1985). Hence, while this formula is appropriate in the symmetric case only its transpose is appropriate in the non-symmetric case.

Given the structure of the above solution for x_t we can now extend the methodology introduced by Epstein and Yatchew to the non-symmetric case. A basic difficulty in estimating the dynamic factor demand model (6) stems from the fact that in general (7) cannot be solved explicitly for M in terms of the original model parameters. However, upon making use of the expressions for $\underline{B}$ and $\underline{G}$ given in (5), an inspection of (7) reveals that it is possible to solve this equation for A in terms of B, C and M, or alternatively, in terms of B, C and S:

$$(8)\quad A_{nn} = \{S_{nn} - (1+r)C_{nn} - (1+r)C'_{nn} - (2+r)B_{nn} + (1+r)[(B'_{nn}+C_{nn})S^{nn}(B_{nn}+C'_{nn}) + \bar{B}'_{sn}S^{sn}(B_{nn}+C'_{nn}) + (B'_{nn}+C_{nn})S^{ns}\bar{B}_{sn} + \bar{B}'_{sn}S^{ss}\bar{B}_{sn}]\}/(1+r) \quad ,$$

---

[6] In somewhat more detail, the case considered by Epstein and Yatchew corresponds to the following special case of the model considered here: $x_t=n_t=[\bar{n}'_t,\underline{n}'_t]'$, $a=a_n$, $A=A_{nn}$, $B=B_{nn}$ diagonal, $C=0$, $Q_q=[I_h,O_{k-hxh}]'$, $Q_w=[O_{kxk-h},I_{k-h}]'$ where k, h and k-h correspond to the dimensions of the vectors n_t, $\bar{n}_t$, and $\underline{n}_t$. Their analysis generalizes trivially to the general symmetric case.

[7] In the symmetric case define $\bar{A} = [\underline{G}-(2+r)\underline{B}]/(1+r)$. It is then easy to see that $V=W'P'$ where W and P are such that $\underline{B}^{-1}=W'W$ and $PW\bar{A}W'P'=\Phi$ with Φ diagonal and P orthogonal. Given the above expression for V, it is readily seen that $\underline{A}=\bar{A}$; hence $\underline{A}$ is symmetric. That in the non-symmetric case it is generally no longer true that $\underline{A}=\underline{A}'$ was checked in terms of a specific counter example. We mentioned above that in the symmetric case it is readily possible to "diagonalize" the difference equation system (5) . This fact is based on the observation that $V^{-1}\underline{B}^{-1}\bar{A}V=\Phi$.

$$A_{ss} = S_{ss} + C_{ss} + C'_{ss} - (2+r)B_{ss} + (1+r)[(B'_{ss}-C'_{ss})S^{ss}(B_{ss}-C_{ss}) + B'_{ns}S^{ns}(B_{ss}-C_{ss}) + (B'_{ss}-C'_{ss})S^{sn}B_{ns} + B'_{ns}S^{nn}B_{ns}] ,$$

$$A_{sn} = A'_{ns} = B_{sn} - \overline{B}_{sn} + C'_{ns} - C_{sn} ,$$

with

$$\overline{B}_{sn} = -[B'_{ns}S^{ns}+(B'_{ss}-C'_{ss})S^{ss}]^{-1}\{S_{sn} - (1+r)C'_{ns} + C_{sn} - (2+r)B_{sn} + (1+r)[B'_{ns}S^{nn}(B_{nn}+C'_{nn}) + (B'_{ss}-C'_{ss})S^{sn}(B_{nn}+C'_{nn})]\}/(1+r) ,$$

and where S_{ij} and S^{ij} denote the (i,j)-th block of, respectively, S and S^{-1} (i,j = n,s).

We expect that in most empirical applications the dimensions of the matrices in (8) will be small. Consequently, for typical applications, explicit expressions for the elements of A can be readily obtained from the above formulas. We note further that the above formulas simplify considerably if adjustment costs are taken to be separable, i.e. C=0, and the adjustment cost matrix B is diagonal.

In constructing I_t we assume that expectations on q_τ and w_τ are formed rationally from the autoregressive process (3). Consequently,

$$(9) \qquad I_t = \alpha + \Sigma_{i=0}^{p-1}\beta_i[q'_{t-i},w'_{t-i}]' .$$

Rather than express the vector α and the matrices β_i in terms of the vector ν and the matrices Θ_i, it is easier to express the latter in terms of the former. By, e.g., analogous argumentation as in Epstein and Yatchew (1985) it follows that:

$$(10) \qquad \nu = RD\alpha, \qquad R = \{\beta_0-D[(I-\delta)Q_q/(1+r),0]\}^{-1}$$

$$\Theta_1 = R\{(I+D)\beta_0-D[Q_q,Q_w]-\beta_1\}, \qquad \Theta_i = R\{(I+D)\beta_{i-1}-\beta_i\}, \quad i=2, \ , p,$$

where $\beta_p = 0$, $D = \underline{B}'S^{-1}/(1+r) - I$ and the null matrix in the expression for R is of dimension kxm_2. The above equations closely resemble analogous equations given in Epstein and Yatchew (1985).[8]

Based on (7) - (10) we can now reparametrize the production function (1), the system of factor demand equations (6), and the price process (3) in terms of a, B, C, S (or M), α, and $\beta_0,\ldots,\beta_{p-1}$.[9] Since the reparametrized model is described by closed form analytic expressions, it can be readily estimated by standard econometric packages such as TSP. Also, estimation of the model in its reparametrized form seems computationally advantageous: Firstly, it avoids repeated numerical solutions of the firm's optimization problem for different sets of trial parameter values; secondly, derivatives of the statistical objective function can be taken analytically rather than numerically.

Clearly, the estimation approach considered in this note is not restricted to the above specific dynamic factor demand model. The approach is, however, restricted to rational expectations models that result in a second order difference equation.[10]

---

[8] The model considered in Epstein and Yatchew corresponds to a model where $Q_q=[I,0]'$ and $Q_w=[0,I]'$; compare footnote 6. We note that completely analogous to Epstein and Yatchew it is readily possible to also incorporate deterministic time trends into the above analysis.

[9] For econometric estimation we need to add stochastic disturbance terms to (1) and (6). Following Epstein and Yatchew (1985), we may interprete those disturbance terms as measurement and random optimization errors. The latter may also be interpreted as random shocks to the technology that are observed by the firm but not by the researcher; compare Hansen and Sargent (1981).

[10] For example, the approach still applies if the production function (1) is generalized by adding the following adjustment cost term: $\frac{1}{2}\sum_i [\Delta n'_{t-i},\Delta s'_{t-i}]B_i[\Delta n'_{t-i},\Delta s'_{t-i}]'$.

Appendix: Proof of Theorem

It follows directly from the definition on Λ and V that:

$$-\underline{B}V\Lambda^2 + \underline{G}V\Lambda - (1+r)\underline{B}'V = 0. \quad (A.1)$$

Observe that by definition $M=I-V\Lambda V^{-1}$. Equation (7) then follows from (A.1) upon post-multiplication with V^{-1}. Next we demonstrate that S is symmetric. Define $\Omega = V'\Lambda V$, and let ω_{ij} and λ_i denote, respectively, the (i,j)-th element of Ω and the i-th diagonal element of Λ. Pre-multiplication of (A.1) with V' and post-multiplication with Λ^{-1} yields:

$$\Omega\Lambda + (1+r)\Omega'\Lambda^{-1} = V'\underline{G}V \quad (A.2)$$

Since $\underline{G}$ is symmetric it follows that also the matrix on the r.h.s. of the above equation is symmetric. The (i,j)-th and (j,i)-th elements of that matrix are given by, respectively, $\omega_{ij}\lambda_j+(1+r)\omega_{ji}/\lambda_j$ and $\omega_{ji}\lambda_i+(1+r)\omega_{ij}/\lambda_i$. Equating the two elements yields $\omega_{ij}\lambda_j = \omega_{ji}\lambda_i$. Hence $\Omega\Lambda$ and consequenetly $S = \underline{B}(I-M) = V'^{-1}\Omega\Lambda V^{-1}$ is symmetric.

It remains to be shown that (6) represents the optimal control solution. We define g_τ implicitly from the following equation ($\tau=t,\ldots,\infty$):

$$x_\tau = (I-M)x_{\tau-1} + g_\tau . \quad (A.3)$$

Clearly then $x_{\tau+1} = (I-M)^2x_{\tau-1} + (I-M)g_\tau + g_{\tau+1}$. Substitution of those expressions for x_τ and $x_{\tau+1}$ into (5) and making use of (7) yields:

$$\underline{B}g_{\tau+1} - (1+r)\underline{B}'(I-M)^{-1}g_\tau = -(1+r)h_\tau, \qquad h_\tau = -\underline{a} + \underline{a}_\tau . \quad (A.4)$$

We note that the roots of this first order difference equation are given by $(1+r)\Lambda^{-1}$ and are hence explosive. The backward solution for g_τ that satisfies the transversality condition for x_τ and hence for g_τ is easily seen to be given by

$$g_t = (1+r)\underline{B}^{-1}\sum_{\tau=t}^{\infty}[\underline{B}(I-M)\underline{B}'^{-1}/(1+r)]^{(\tau-t+1)}h_\tau . \quad (A.5)$$

Observe that $(I+D)^{-1} = \underline{B}(I-M)\underline{B}'^{-1}/(1+r) = (I-M')/(1+r)$. The solution given in (6) is then readily obtained upon substitution of (A.5) into (A.3). ■

References

Epstein, L.G. and A.J. Yatchew, 1985, The empirical determination of technology and expectations: A simplified procedure, Journal of Econometrics 27, 235-258.

Hansen, L.P. and T.J. Sargent, 1980, Formulating and estimating dynamic linear rational expectations models, Journal of Economic Dynamics and Control 2, 7-46.

Hansen, L.P. and T.J. Sargent, 1981, Linear rational expectations models for dynamically interrelated variables, in: R.E. Lucas, Jr. and T.J. Sargent, eds., Rational Expectations and Econometric Practice, Vol. I (University of Minnesota Press, Minneapolis, MN) 127-156.

Kollintzas, T., 1985, The symmetric linear rational expectations model, Econometrica 53, 963-976.

Lucas, R., 1967, Optimal investment policy and the flexible accelerator, International Economic Review 8, 78-85.

Morrison, C.J. and E.R. Berndt, 1981, Short-run labor productivity in a dynamic model, Journal of Econometrics 16, 339-365.

Prucha, I.R. and M.I. Nadiri, 1986, A comparison of alternative methods for the estimation of dynamic factor demand models under non-static expectations, Journal of Econometrics 33, 187-211.