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Existence of Nash Networks in One-Way Flow Models

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Abstract

This paper addresses the existence of Nash equilibria in one-way flow or directed network models in a number of different settings. In these models players form costly links with other players and obtain resources from them through the directed path connecting them. We find that heterogeneity in the costs of establishing links play a crucial role in the existence of Nash networks. We also provide conditions for the existence of Nash networks in models where costs and values of links are heterogeneous.

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1 Introduction

Galeotti (2006) characterized the Nash equilibria of one-way flow (or directed) networks under heterogeneity. He considers directed networks where agents must form costly links to obtain beneficial information from other agents. Heterogeneity in his formulation occurs in the value of information possessed by other players as well as in the cost of forming links. However, the question of existence of Nash equilibria for such models has not been resolved. Our paper complements the existing literature by addressing this issue.

The study of non-cooperative models of network formation was initiated by Bala and Goyal (2000). These authors examine both one-way flow and two-way flow (or undirected) networks. In the second type of networks, unlike in the former, a link between two players allows both players to get access to each other's resources regardless of who initiates the link. Bala and Goyal characterize and provide a constructive proof of existence of Nash equilibria for both directed and undirected networks under the assumption of homogeneous costs and benefits across players.

However, Bala and Goyal (2000) do not address the question of heterogeneity of costs and benefits of links. This is a critical shortcoming for two reasons. First, heterogeneity in costs and benefits is pervasive in social and economic networks. For instance, in the context of information networks, it is often the case that some individuals are better informed, which makes them more valuable contacts. Similarly, as individuals differ, it seems natural that forming links is cheaper for some individuals as compared to others. In other words, players can be distinct in terms of cultural, legal or geographical proximity, and it may be cheaper for a given player to form a link with a closer player.

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Second, the introduction of various heterogeneity conditions on costs and values provides a sensitivity check for the results obtained with homogeneous parameters. Thus, in this paper we ask if the introduction of different types of heterogeneity in the Bala and Goyal (2000) framework as analyzed by Galeotti (2006) alters existence results for Nash networks.

A few papers have explored heterogeneity in the context of Nash networks. Galeotti, Goyal and Kamphorst (2005) and Haller and Sarangi (2005) characterize Nash networks in two-way flow models. The existence of Nash networks in such models has been studied by Haller, Kamphorst and Sarangi (2005). Galeotti (2006) examines one-way flow models under value and cost heterogeneity while Billand and Bravard, 2005) take into account the role of congestion in Nash network models. Neither paper however addresses the issue of existence of such networks.[1]

In this paper we investigate the existence of Nash networks in the one-way flow model when costs and values of links are heterogeneous. We focus on one-way flow models with linear payoffs as described in Galeotti (2006). Moreover, we do not allow for decay and permit players to only use pure strategies.[2] We show that the Bala and Goyal results are not quite robust: there does not always exist a Nash network when heterogeneity in costs and values of links is introduced. More precisely, we find that, as in the two-way flow model, heterogeneity in costs of forming links plays a major role in

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[1]Necessary and sufficient conditions for the existence of pairwise stable networks – the other commonly used stability concept in this literature can be found in Jackson and Watts (2001).

[2]Following the networks literature we concentrate only on pure strategies since the idea of randomizing between different links is usually considered unrealistic. Further, all the network games considered here are finite and hence existence of mixed strategy equilibria is guaranteed.

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the non existence of Nash networks.[3] This is because cost heterogeneity provides link substitution possibilities as in Example 1. We then provide bounds on costs of forming links that guarantee the existence of Nash networks. We also show that if costs are homogeneous (and values are not), then Nash networks always exist.

The remainder of the paper is organized as follows. In Section 2 we set the basic one-way flow model. In Section 3 we present the results about the existence of Nash networks for this model. More precisely we study this problem under various heterogeneity conditions on costs and values. Section 4 concludes.

2 Model Setup

Let $N = \{1, \ldots, n\}$ be the set of players. The network relations among these players are formally represented by directed graphs whose nodes are the players. A network $\boldsymbol{g} = (N, E)$ is a pair of sets: the set N of players and the edges set $E(\boldsymbol{g}) \subset N \times N$ of directed links. A link initiated by player i to player j is denoted by $i\,j$. Each player i chooses a strategy $\boldsymbol{g}_i = (\boldsymbol{g}_{i1}, \ldots, \boldsymbol{g}_{ii-1}, \boldsymbol{g}_{ii+1}, \ldots, \boldsymbol{g}_{in})$, $\boldsymbol{g}_{ij} \in \{0, 1\}$ for all $j \in N \setminus \{i\}$, which describes the decision of establishing links. More precisely, $\boldsymbol{g}_{ij} = 1$ if and only if $i\,j \in E(\boldsymbol{g})$. The interpretation of $\boldsymbol{g}_{ij} = 1$ is that player i forms a link with player $j \neq i$, and the interpretation of $\boldsymbol{g}_{ij} = 0$ is that i does not form a link with player j. We assume in the following that every player is always trivially connected to herself, so $\boldsymbol{g}_{ii} = 1$ for all $i \in N$ and do not include it in $\boldsymbol{g}_i$. We only use pure strategies. Note that $\boldsymbol{g}_{ij} = 1$ does not necessarily imply that $\boldsymbol{g}_{ji} = 1$. Indeed it is possible i is linked to j, but j is not linked to i. Let $\mathcal{G} = \times_{i=1}^{n} \mathcal{G}_i$ be the set of all possible networks where $\mathcal{G}_i$ is the set

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[3]One-way flow models however are different from two-way flow models since cycles cannot be ruled out in the best response process. Thus the existence proofs are also different.

of all possible strategies of player $i \in N$. Finally, let $\mathcal{P}(\mathcal{G})$ be the power set of $\mathcal{G}$, that is the set of all subsets of $\mathcal{G}$.

We now provide some important graph theoretic definitions. For a directed graph, $\boldsymbol{g} \in \mathcal{G}$, a *path* $P(\boldsymbol{g})$ of length m in $\boldsymbol{g}$ from player j to i, $i \neq j$, is a finite sequence $i_0, i_1, \ldots, i_m$ of distinct players such that $i_0 = i$, $i_m = j$ and $\boldsymbol{g}_{i_k i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. If $i_0 = i_m$, then the path is a cycle. We denote the set of cycles in the network $\boldsymbol{g}$ by $\mathcal{C}(\boldsymbol{g})$. Let $C(\boldsymbol{g})$ be a typical member of $\mathcal{C}(\boldsymbol{g})$. In the empty network, $\dot{\boldsymbol{g}}$, there are no links between any agents.

To sum up, a link from a player j to a player i ($\boldsymbol{g}_{ij} = 1$) allows player i to get resources from player j but since we are in a one-way flow model, this link does not allow player j to obtain resources from i. Moreover, a player i may receive information from other players through a sequence of indirect links. To be precise, information flows from player j to player i, if i and j are linked by a path in $\boldsymbol{g}$ from j to i. Let

$$N_i(\boldsymbol{g}) = \{j \in N | \text{ there exists a path in } \boldsymbol{g} \text{ from } j \text{ to } i\},$$

be the set of players that player i can access in the network $\boldsymbol{g}$. By definition, we assume that $i \in N_i(\boldsymbol{g})$ for all $i \in N$ and for all $\boldsymbol{g} \in \mathcal{G}$. Let $n_i(\boldsymbol{g})$ be the cardinality of the set $N_i(\boldsymbol{g})$. Finally, we define $\eta : \mathcal{G} \to \mathbb{R}$, $\eta(\boldsymbol{g}) = \sum_{i \in N} n_i(\boldsymbol{g})$ as a function.

Information received from player j is worth V_{ij} to player i. Moreover, i incurs a cost c_{ij} when she initiates a direct link with j, i.e. when $\boldsymbol{g}_{ij} = 1$. We can now define the payoff function of player $i \in N$:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_{ij} - \sum_{j \in N} \boldsymbol{g}_{ij} c_{ij}. \tag{1}$$

We assume that $c_{ij} > 0$ and $V_{ij} > 0$ for all $i \in N$, $j \in N$, $i \neq j$. Also, we normalize $V_{ii} = 0$ for all $i \in N$. The next definition introduces the different notions of heterogeneity in our model.

Definition 1 *Values (or costs) are said heterogeneous by pairs of players if there exist $i, j, k \in N$ such that $V_{ij} \neq V_{ik}$ ($c_{ij} \neq c_{ik}$) and there exist $i', j', k' \in N$ such that $V_{j'i'} \neq V_{k'i'}$ ($c_{j'i'} \neq c_{k'i'}$). Values (or costs) are said heterogeneous by players if for all $i, j, k \in N$: $V_{ij} = V_{ik} = V_i$ ($c_{ij} = c_{ik} = c_i$) but there exist $i, i' \in N$ such that $V_i \neq V_{i'}$ ($c_i \neq c_{i'}$).*

We now provide some useful definitions for studying the existence of Nash networks. Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network obtained when all of player i's links are removed. The network $\boldsymbol{g}$ can be written as $\boldsymbol{g} = \boldsymbol{g}_{-i} \oplus \boldsymbol{g}_i$, where the operator $\oplus$ indicates that $\boldsymbol{g}$ is formed by the union of links in $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$. The strategy $\boldsymbol{g}_i$ is said to be a best response of player i to $\boldsymbol{g}_{-i}$ if:

$$\pi_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq \pi_i(\boldsymbol{g}_i' \oplus \boldsymbol{g}_{-i}), \text{for all } \boldsymbol{g}_i' \in \mathcal{G}_i.$$

The set of player i's best responses to $\boldsymbol{g}_{-i}$ is denoted by $\mathcal{BR}_i(\boldsymbol{g}_{-i})$. Furthermore, a network $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_i, \ldots, \boldsymbol{g}_n)$ is said to be a Nash network if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each $i \in N$.

Definition 2 *We say that two networks $\boldsymbol{g}$ and $\boldsymbol{g}'$ are adjacent if there is a unique player i such that $g_{ij} \neq g_{ij}'$ for at least one player $j \neq i$.*

An improving path is a sequence of adjacent networks that results when players form or sever links based on payoff improvement the new network offers over the current network. More precisely, each network in the sequence differs from the previous one by the links formed (or severed) by one unique player. Note that if a player changes her links, it must be that this player strictly benefits from such a change.

Definition 3 *Formally, an improving path from a network $\boldsymbol{g}$ to a network $\boldsymbol{g}'$ is a finite sequence of networks $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^\ell, \boldsymbol{g}^{\ell+1}, \ldots, \boldsymbol{g}^k$, with $\boldsymbol{g}^1 = \boldsymbol{g}$ and $\boldsymbol{g}^k = \boldsymbol{g}'$, such that the two following conditions are verified:*

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1. $\boldsymbol{g}^{\ell}$ and $\boldsymbol{g}^{\ell+1}$, are adjacent networks;

2. Let $i \in N$ be the player such that $g_{ij}^{\ell} \neq g_{ij}^{\ell+1}$ for at least one player $j \neq i$. Then, we have: $\boldsymbol{g}_{i}^{\ell+1} \in \mathcal{BR}_i(\boldsymbol{g}_{-i}^{\ell})$ and $\boldsymbol{g}_{i}^{\ell} \notin \mathcal{BR}_i(\boldsymbol{g}_{-i}^{\ell})$, that is $\boldsymbol{g}^{\ell+1}$ is a network where i plays a best response while $\boldsymbol{g}^{\ell}$ is a network where i does not play a best response.

Moreover, if $\boldsymbol{g}^1 = \boldsymbol{g}^k$, then the improving path is called an improving cycle.

It follows that a network $\boldsymbol{g}$ is a Nash network if and only if it has no improving path emanating from it.

3 Model with Heterogeneous Agents

Bala and Goyal (2000) outline a constructive proof of the existence of Nash networks in the one-way flow model when costs and values of links are homogeneous. We show that the introduction of heterogeneity in costs and values of links by pairs does not change the Nash networks existence result of Bala and Goyal when the number of players is $n = 3$. However, this result is no longer true if the number of players is $n > 3$.

Proposition 1 *Let the payoff be given by (1).*

1. *If values and costs of links are heterogeneous by pairs and $n = 3$, then a Nash network always exists.*

2. *If values and costs of links are heterogeneous by pairs and $n > 3$, then a Nash network does not always exist.*

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We prove only the first part of the proposition. Indeed, to prove the second part of the proposition, it is enough to give an example with $n > 3$ and where there does not exist any Nash network when values and costs of links are heterogeneous by pairs (Example 1). Note that in Example 1 we assume that values are the same for all players and costs are heterogeneous by pairs and we show that there does no exist any Nash network. However, by a continuity argument the non existence of Nash networks in such a context implies the non existence of Nash networks in contexts where values are heterogeneous by pairs.

Proof of Proposition 1 Let $N = \{1, 2, 3\}$. We begin with the empty network $\dot{g}$. Either $\dot{g}$ is a Nash network and we are done, or $\dot{g}$ is not a Nash network and there exists an improving path from $\dot{g}$ to an adjacent network g^1. That is, there exists a player, say without loss of generality player 1, such that $\dot{g}_1 \notin \mathcal{BR}_1(\dot{g}_{-1})$ and $g_1^1 \in \mathcal{BR}_1(\dot{g}_{-1})$. Since $1 \in N$ has no link in $\dot{g}$ and forms links in $g^1 = g_1^1 \oplus \dot{g}_{-1}$, we have $\eta(\dot{g}) < \eta(g^1)$. Now we repeat this step. Assume an improving path from a network g^1 to a network g^k where for each player $i \in N$, we have $N_i(g^{k-1}) \subseteq N_i(g^k)$. We show that if there exists an improving path from g^k to g^{k+1}, then for each player $i \in N$, $N_i(g^k) \subseteq N_i(g^{k+1})$. Let i be a player such that $g_i^{k+1} \in \mathcal{BR}_i(g_{-i}^k)$ and $g_i^k \notin \mathcal{BR}_i(g_{-i}^k)$. We show that if $j \in N_i(g^k)$, then $j \in N_i(g^{k+1})$. Indeed there are two possibilities for $j \in N_i(g^k)$.

1. Either $g_{ij}^k = 1$, that is i directly obtains the resources of player j. Then there are two possibilities.

   - If $V_{ij} - c_{ij} > 0$, then $j \in N_i(g^{k+1})$, otherwise i is not playing a best response in g^{k+1}.

   - If $V_{ij} - c_{ij} < 0$, then there is a network $g^{k'}$, $k' < k$, such that $\ell \in N_j(g^{k'})$ and

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$V_{ij} + V_{i\ell} - c_{ij} > \max\{0, V_{i\ell} - c_{i\ell}\}$, else $\boldsymbol{g}_{ij}^k = 0$. Since $N_j(\boldsymbol{g}^{k'}) \subseteq N_j(\boldsymbol{g}^k)$, for all $k' < k$ and for all $j \in N$, we have $\ell \in N_j(\boldsymbol{g}^k)$ and player i deletes her link with j only if $j \in N_\ell(\boldsymbol{g}^k)$ and $V_{ij} + V_{i\ell} - c_{ij} < V_{ij} + V_{i\ell} - c_{i\ell}$. In that case, i forms a link with ℓ and $j \in N_i(\boldsymbol{g}^{k+1})$.

2. Or $\boldsymbol{g}_{ij}^k = 0$, $\boldsymbol{g}_{i\ell}^k = 1$ and $\boldsymbol{g}_{\ell j}^k = 1$, that is i indirectly obtains the resources of player j. Then, we use the same argument as above to show that player i deletes her link with ℓ only if she has an incentive to form a link with j and $j \in N_i(\boldsymbol{g}^{k+1})$.

We now show that there does not exist a cycle in the improving path $\mathcal{Q} = \{\dot{\boldsymbol{g}}, \boldsymbol{g}^1, \ldots, \boldsymbol{g}^t, \ldots, \boldsymbol{g}^{t+\tau}, \ldots, \boldsymbol{g}^{t+\tau'}, \ldots\}$, with $\tau' > \tau > 0$. It suffices to show that if $\boldsymbol{g}_{ij}^t = 1$, $\boldsymbol{g}_{ij}^{t+\tau} = 0$, and $\boldsymbol{g}_{ij}^{t+\tau'} = 1$, then we have $N_i(\boldsymbol{g}^t) \subsetneq N_i(\boldsymbol{g}^{t+\tau'})$. Note that as $j \in N_i(\boldsymbol{g}^t)$ and $N_i(\boldsymbol{g}^t) \subseteq N_i(\boldsymbol{g}^{t+\tau})$, we have $j \in N_i(\boldsymbol{g}^{t+\tau})$. Also, as $\boldsymbol{g}_{ij}^{t+\tau} = 0$, we have $\boldsymbol{g}_{i\ell}^{t+\tau} = 1$ and $\ell \in N_i(\boldsymbol{g}^{t+\tau})$. Moreover, as $N_i(\boldsymbol{g}^{t+\tau}) \subseteq N_i(\boldsymbol{g}^{t+\tau'})$, we have $N_i(\boldsymbol{g}^{t+\tau'}) = \{j, \ell\}$.

Without loss of generality, let player i delete the link $i\,j$ for the first time, between t and $t+\tau$, in $\boldsymbol{g}^{t+\tau}$. Likewise, we assume that player i forms the link $i\,j$ for the first time, between $t + \tau$ and $t + \tau'$, in $\boldsymbol{g}^{t+\tau'}$.

We have two cases.

1. Suppose $\boldsymbol{g}_{i\ell}^t = 0$. To obtain a contradiction, assume that $\ell \in N_i(\boldsymbol{g}^t)$. It follows that $\boldsymbol{g}_{j\ell}^{t+\tau} = 1$ since player i does not form the link $i\,\ell$ between $\boldsymbol{g}^t$ and $\boldsymbol{g}^{t+\tau}$ if j preserves the link $j\,\ell$. Also j does not delete the link $j\,\ell$ between $\boldsymbol{g}^t$ and $\boldsymbol{g}^{t+\tau}$ if i does not form the link $i\,\ell$ (recall that in our process only one player changes her strategy in each period). Since player i chooses to delete the link $i\,j$ in $\boldsymbol{g}^{t+\tau}$, then she must form the link $i\,\ell$ and we must have $\boldsymbol{g}_{\ell j}^{t+\tau} = 1$, since $\ell \in N_i(\boldsymbol{g}^t) \subseteq N_i(\boldsymbol{g}^{t+\tau})$. Moreover, the substitution of the link $i\,j$ by the link $i\,\ell$ implies that $c_{ij} > c_{i\ell}$. By the same argument, player ℓ does not delete the link $\ell\,j$ between $\boldsymbol{g}^{t+\tau}$ and $\boldsymbol{g}^{t+\tau'}$.

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Therefore, if player i forms the link $i\,j$ in $\boldsymbol{g}^{t+\tau'}$ (and deletes the link $i\,\ell$), then we have $c_{ij} < c_{i\ell}$ and giving us the desired contradiction.

2. Suppose that $\boldsymbol{g}^t_{i\ell} = 1$. If player i deletes the link $i\,j$ in $\boldsymbol{g}^{t+\tau}$, then we obtain the situation in case 1 up to a permutation of players j and ℓ. Hence the proof follows.

$\square$

We have shown that if values and costs of links are heterogeneous by pairs and $n = 3$, a Nash network always exists. Note that this result is not true for the model with two-way flow of resources (see Haller, Kamphorst and Sarangi 2006 pg. 7). We now give an example with 4 players where there does not exist any Nash network.

Example 1 Let $N = \{1, 2, 3, 4\}$ be the set of players and $V_{ij} = V$ for all $i \in N$, $j \in N$. Moreover, we suppose that $c_{13} = V - V/16$ and $c_{12} = c_{14} = 4V$; $c_{21} = 2V - V/16$ and $c_{23} = c_{24} = 4V$; $c_{32} = 2V - V/8$, $c_{34} = 2V - V/6$ and $c_{31} = 4V$; $c_{41} = 3V - V/8$ and $c_{42} = c_{43} = 4V$.

1. In a best response, player 2 never forms a link with player 3 or player 4. Moreover, player 2 has an incentive to form a link with player 1 if the latter gets resources from player 3 or player 4.

2. In a best response, player 4 never forms links with player 3 or player 2.

3. Then the unique best response of player 1 to any network in which she does not observe player 3 is to add a link with player 3 (since player 2 and player 4 never form a link with player 3). Moreover, we note that player 1 never has an incentive to form a link with player 2 or player 4.

4. In a best response, player 3 never forms a link with player 1.

Now let us take those best replies for granted and consider best responses regarding the remaining links 2 1; 3 2; 3 4 and 4 1. If player 2 initiates the link 2 1 (see g^0 in figure 1), then player 3's best response is to initiate the link 3 2 (see g^1). In that case player 4 must initiate the link 4 1 (see g^2) and player 3 must replace the link 3 2 by the link 3 4 (see g^3). Then, player 4 must delete the link 4 1 (see g^4) and the player 3 must replace the link 3 4 by the link 3 2 (see g^1). Hence there do not exist any mutual best responses. Therefore, a Nash network does not exist.

Finally, by appropriately adjusting costs, it can be verified that this example holds even if we relax the assumption that $V_{ij} = V$ for all $i, j \in N$. *In particular, using a continuity argument it is possible to construct an example where values are heterogeneous by players, costs are heterogeneous by pairs, and a Nash network does not exist.*

Figure 1: Best responses process of Example 1
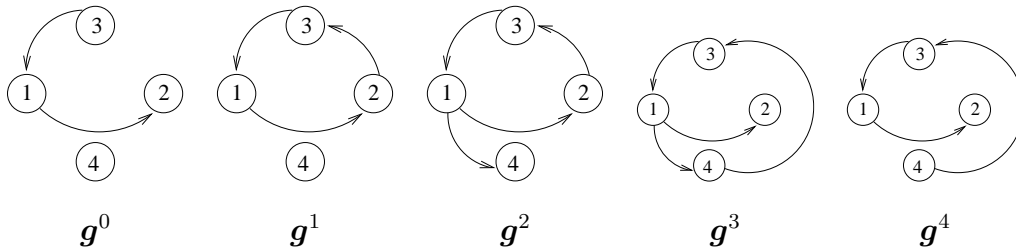
This example shows that unlike in two-way flow models existence results in one-way flow models with heterogeneity depend crucially on the number of players. Indeed, the proof of existence of Nash networks with three players is based on the following fact: after a given player i has played a best response, the set of players from whom she obtains resources always contains the set of players from whom she obtained resources before. Example 1 shows that this property does not hold anymore when $n > 3$. Indeed, in

this example, player 3's best response requires him not to obtain resources from player 2 in network $\boldsymbol{g}^4$. Given that heterogeneity often arises in reality, such a negative result suggests that one must be cautitious when using Nash networks. Later in the paper, we show that there exist bounds on costs which are sufficient for the existence of Nash equilibria when costs and values are heterogeneous by pairs (see Corollary 1).

3.1 Existence of Nash networks under heterogeneity of values by pairs

In this section, we present a proof of the existence of Nash networks in the one-way flow model where values are heterogeneous by pairs and costs are heterogeneous by players. Our proof is quite different from the proof of Haller, Kamphorst and Sarangi (2006) who address the Nash existence problem in the two-way flow model. Essentially, in the one-way flow model, unlike in the two-way flow model, we cannot rule out the existence of cycles in the best response process.[4] Our proof takes cycles into account by modifying the network obtained when a player plays a best response in such a way that no player has an incentive to remove any of her links.

The payoff function when values are heterogeneous by pairs and costs are heterogeneous by players is given by:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_{ij} - c_i \sum_{j \in N} \boldsymbol{g}_{ij}. \tag{2}$$

Let $\pi_i^j(\boldsymbol{g})$ be the marginal payoff of player i from player j in the network $\boldsymbol{g}$. If $\boldsymbol{g}_{ij} = 1$, then we have $\pi_i^j(\boldsymbol{g}) = \pi_i(\boldsymbol{g}) - \pi_i(\boldsymbol{g} \ominus i\ j)$. To take double counting into account

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[4]For an example showing such cycles, refer to the working paper version (Billand, Bravard and Sarangi, 2007, http:\\www.bus.lsu.edu\economics\papers\pap07_02.pdf).

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we define the following set. Let $\mathcal{K}(\boldsymbol{g}; i\ j) = N_i(\boldsymbol{g} \ominus i\ j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i\ j)$, where $\boldsymbol{g} \ominus i\ j$ denotes the network $\boldsymbol{g}$ without the link $i\ j$. We can rewrite $\pi_i^j(\boldsymbol{g})$ as follows:

$$\pi_i^j(\boldsymbol{g}) = \sum_{k \in N_i(\boldsymbol{g}_{-i} \oplus i\ j)} V_{ik} - \sum_{k \in \mathcal{K}(\boldsymbol{g}; i\ j)} V_{ik} - c_i. \tag{3}$$

We now define some classes of networks that are useful in the proof of the next proposition. We say that a network $\boldsymbol{g}$ is minimal if it is not possible to delete any link $i\ j \in E(\boldsymbol{g})$ formed by player i in $\boldsymbol{g}$ without altering the set $N_i(\boldsymbol{g})$. In other words, if a link of $\boldsymbol{g}$, say $i\ j$, is deleted, then $N_i(\boldsymbol{g} \ominus i\ j) \neq N_i(\boldsymbol{g})$. Let $\mathcal{G}^m$ be the set of minimal networks.

Let $\mathcal{G}^1 = \{\boldsymbol{g} \in \mathcal{G}^m | i \in N_j(\boldsymbol{g}), j \notin N_i(\boldsymbol{g}), k \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{ki} = 0\}$ be a subset of minimal networks. Let $\mathcal{G}^2 \subset \mathcal{G}^1$ be the set of networks which contain at most one cycle. If $\boldsymbol{g} \in \mathcal{G}^2$ and $\boldsymbol{g}$ contains a cycle, then we denote by $C(\boldsymbol{g})$ the cycle in the network $\boldsymbol{g}$. We denote by $N^{C(\boldsymbol{g})}$ the set of players who belong to the cycle $C(\boldsymbol{g})$, and $E^{C(\boldsymbol{g})} \subset N^{C(\boldsymbol{g})} \times N^{C(\boldsymbol{g})}$ the set of links which belong to the cycle $C(\boldsymbol{g})$. Let $\mathcal{G}^3 = \{\boldsymbol{g} \in \mathcal{G}^2 | i \in C(\boldsymbol{g}), j \notin C(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{ji} = 0\}$ be the set of networks where there does not exist a link between a player $i \in N^{C(\boldsymbol{g})}$ to a player $j \notin N^{C(\boldsymbol{g})}$. Next we present some properties of these minimal networks.

Lemma 1 *Suppose values of links are heterogeneous by pairs and costs of links are heterogeneous by players and $\boldsymbol{g} \in \mathcal{G}^3$.*

1. *For all $i, j \in N$, if $\boldsymbol{g}_{ji} = 1$, then there does not exist any player k such that $\boldsymbol{g}_{ki} = 1$.*

2. *For all $i, j \in N$, if $\boldsymbol{g}_{ij} = 1$, then $\mathcal{K}(\boldsymbol{g}; i\ j) = N_i(\boldsymbol{g} \ominus i\ j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i\ j)$ is an empty set.*

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Lemma 1 describes the properties of networks $\boldsymbol{g} \in \mathcal{G}^3$. The proof of this lemma is simple and can be found in Billand, Bravard and Sarangi (2007).[5] Lemma 1.1 says that, given any two agents j and k, only one of them will form a link to i. Essentially it would be better if they form the other link between them instead of going directly to i. Lemma 1.2 says that there is only one path between any two agents. Both properties follow from the minimality of $\boldsymbol{g}$ and the fact that $\mathcal{G}^3 \subset \mathcal{G}^2 \subset \mathcal{G}^1$.

Observe that if $\boldsymbol{g} \in \mathcal{G}^3$, then we can write $\pi_i^j(\boldsymbol{g})$ as follows:

$$\pi_i^j(\boldsymbol{g}) = \sum_{k \in N_i(\boldsymbol{g}_{-i} \oplus i \ j)} V_{ik} - c_i. \tag{4}$$

In the following lemma, which provides the best response properties of the networks $\boldsymbol{g} \in \mathcal{G}^3$, we let $\boldsymbol{g}_i' \in \mathcal{G}_i$ be a strategy of player i, with $\boldsymbol{g}_i' \neq \boldsymbol{g}_i$.

Lemma 2 *Suppose values of links are heterogeneous by pairs, costs of links are hetero-geneous by players and $\boldsymbol{g} \in \mathcal{G}^3$. Moreover, let the payoff function be given by (2).*

1. *Suppose players $i, j, k \in N$ are such that $j \notin N_i(\boldsymbol{g})$, $i \in N_j(\boldsymbol{g})$, $k \notin N_j(\boldsymbol{g})$. If $\boldsymbol{g}_{ki}' = 1$, then $\boldsymbol{g}_k' \notin \mathcal{BR}_k(\boldsymbol{g}_{-k})$.*

2. *Suppose $\boldsymbol{g}$ contains a cycle $C(\boldsymbol{g})$ and for all $i, j \in N^{C(\boldsymbol{g})}$, and for all $i \ j \in E^{C(\boldsymbol{g})}$, we have $\pi_i^j(\boldsymbol{g}) > 0$. If $\boldsymbol{g}_{ij}' = 0$, then $\boldsymbol{g}_i' \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$.*

3. *Suppose $i \in N$, $j \in N \setminus N^{C(\boldsymbol{g})}$ and $\boldsymbol{g}_{ij} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$. If $\boldsymbol{g}_{ij}' = 0$, then $\boldsymbol{g}_i' \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$.*

Again the proof is simple and can be found in Billand, Bravard and Sarangi (2007). Lemma 2.1 claims that if j observes i and i does not observe j, then k cannot be playing a best response if she establishes a link with i. This is because either k already observes

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[5]http:\\www.bus.lsu.edu\economics\papers\pap07_02.pdf.

i and hence it cannot be a best response to form a link with i, or she does not observe i, in which case she should form a link with j, since this will also allow her to obtain i's information. Lemma 2.2 states that if $i\ j \in E^{C(\boldsymbol{g})}$ and the marginal payoff from j is strictly positive, then deleting the link $i\ j$ is not a best response for player i. Lemma 2.3 applies the same argument to a player j who does not belong to the cycle $C(\boldsymbol{g})$.

Proposition 2 *Suppose values of links are heterogeneous by pairs and costs of links are heterogeneous by players. Moreover, let the payoff be given by (2). Then a Nash network always exists.*

The proof of Proposition 2 is long, and involves a number of lemmas. So we first provide a quick overview of the proof. It consists in constructing a sequence of networks, $\mathcal{Q} = (\boldsymbol{g}^0, \ldots, \boldsymbol{g}^{t-1}, \boldsymbol{g}^t, \ldots)$, beginning with the empty network. In each subsequent network, no player should have an incentive to decrease the amount of resources she obtains. Note that this sequence of networks is not an improving path. Indeed, we go from $\boldsymbol{g}^t$ to $\boldsymbol{g}^{t+1}$ in several operations. First, in $\boldsymbol{g}^t$ we let a player $i \in N$, who is not playing a best response in $\boldsymbol{g}^t$, play a best response (if such player does not exist, $\boldsymbol{g}^t$ is a Nash network) and we obtain a network called $\mathrm{br}_i(\boldsymbol{g}^t)$. Second, we modify the network $\mathrm{br}_i(\boldsymbol{g}^t)$ as follows: we construct a cycle using all players $j \in N$ who obtain resources from a player k who forms part of a cycle in $\mathrm{br}_i(\boldsymbol{g}^t)$, while preserving all links in $\mathrm{br}_i(\boldsymbol{g}^t)$ between a player $k \in N$ and a player j who is not part of a cycle in $\mathrm{br}_i(\boldsymbol{g}^t)$. We obtain a network called $h(\mathrm{br}_i(\boldsymbol{g}^t))$. Thirdly, we delete all links $i\ j$ which does not allow player i to obtain additional resources in $h(\mathrm{br}_i(\boldsymbol{g}^t))$. We obtain a network called $m(h(\mathrm{br}_i(\boldsymbol{g}^t))) = \overline{\boldsymbol{g}}_i^t$, and in the sequence $\mathcal{Q}$, we have $\boldsymbol{g}^{t+1} = \overline{\boldsymbol{g}}_i^t$.

When a player i receives an opportunity to revise her strategy, we go from a network $\boldsymbol{g}^t$ to a network $\boldsymbol{g}^{t+1}$, and we will show that $\eta(\boldsymbol{g}^{t-1}) < \eta(\boldsymbol{g}^t)$. Since the amount of resources

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that players can obtain in a network $\boldsymbol{g} \in \mathcal{Q}$ is finite, $\mathcal{Q}$ is finite and there exists a Nash network. More precisely, in the following we show that $\mathcal{G}^3$ contains all networks in the sequence $\mathcal{Q}$ (Lemma 5). Then, we use the condition which implies that no player has an incentive to delete a link in a network $\boldsymbol{g} \in \mathcal{G}^3$ (Lemma 2). Finally, we show that all networks $\boldsymbol{g}^t \in \mathcal{Q}$ satisfy this condition since the empty network satisfies this condition (Lemma 6).

We now introduce some additional definitions that are required to complete the proof. Let $\mathcal{MBR}_i(\boldsymbol{g}_{-i})$ be a modified version of the best response function of player $i \in N$. More precisely, $\boldsymbol{g}'_i \in \mathcal{MBR}_i(\boldsymbol{g}_{-i})$ if $\boldsymbol{g}'_i$ is a best response of player i against $\boldsymbol{g}_{-i}$ and if player i does not form any links that yield zero marginal payoffs. Let $\mathrm{br}_i : \mathcal{G} \to \mathcal{G}$, $\boldsymbol{g} \mapsto \mathrm{br}_i(\boldsymbol{g})$ be a function. The network $\mathrm{br}_i(\boldsymbol{g}) = (\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i})$ is a network where $\boldsymbol{g}'_i \in \mathcal{MBR}_i(\boldsymbol{g}_{-i})$, and all other players $j \neq i$ have the same links as in the network $\boldsymbol{g}$ (since $\mathrm{br}_i(\boldsymbol{g})$ is a network, $\mathrm{br}_i(\boldsymbol{g})_{ij} \in \{0, 1\}$ indicates if player i forms a link with player j). In other words, in $\mathrm{br}_i(\boldsymbol{g})$, we have $\mathrm{br}_i(\boldsymbol{g})_{ij} = 1 \Rightarrow \pi_i^j(\mathrm{br}_i(\boldsymbol{g})) > 0$ and $\mathrm{br}_i(\boldsymbol{g})_{ij} = 0 \Rightarrow \pi_i^j(\mathrm{br}_i(\boldsymbol{g})) \leq 0$.

Let $\mathcal{H} : \mathcal{G} \to \mathcal{P}(\mathcal{G})$ be a correspondence. A network $h(\boldsymbol{g}) \in \mathcal{H}(\boldsymbol{g})$ is a network associated with $\boldsymbol{g}$ such that $h(\boldsymbol{g})$ contains at most one cycle, $C(h(\boldsymbol{g}))$. Since $h(\boldsymbol{g})$ is a network, $h(\boldsymbol{g})_{ij} \in \{0, 1\}$ indicates if player i forms a link with player j in $h(\boldsymbol{g})$. We now state the rules for constructing the network $h(\boldsymbol{g})$. If k is such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{C(\boldsymbol{g})}$, then $k \in N^{C(h(\boldsymbol{g}))}$. If $k \notin N^{C(h(\boldsymbol{g}))}$, then for all $\ell \in N$, we have $g_{\ell k} = h(\boldsymbol{g})_{\ell k}$. This is different from the networks in $\mathcal{G}^2$ since there is no minimality restriction here. This operation creates one cycle leaving unchanged the strategies of those players who do not belong to the cycle.

Let $\mathcal{M} : \mathcal{G} \to \mathcal{P}(\mathcal{G})$, $\boldsymbol{g} \mapsto \mathcal{M}(\boldsymbol{g})$ be a correspondence. Let $m(\boldsymbol{g})$ be a typical element

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of $\mathcal{M}(\boldsymbol{g})$. Since $m(\boldsymbol{g})$ is a network, $m(\boldsymbol{g})_{ij} \in \{0,1\}$ indicates if player i forms a link with player j in $m(\boldsymbol{g})$. Next, we provide the rules for constructing the network $m(\boldsymbol{g})$ from a network $\boldsymbol{g} \in \mathcal{G}$. A network $m(\boldsymbol{g})$ is such that, for all $i, j \in N$, $N_i(\boldsymbol{g}) = N_i(m(\boldsymbol{g}))$ and if $m(\boldsymbol{g})_{ij} = 1$, then $j \notin N_i(m(\boldsymbol{g}) \ominus i\ j)$ and $\boldsymbol{g}_{ij} = 1$. It is obvious that $m(\boldsymbol{g})$ is a minimal network. Obviously, we have $\eta(\boldsymbol{g}) = \eta(m(\boldsymbol{g}))$. In the following, without loss of generality, we can select any element of $\mathcal{M}(\boldsymbol{g})$.

Observe that for all $\boldsymbol{g} \in \mathcal{G}$ and for all $k \in N$, we have, by construction, for all $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\boldsymbol{g})$, $N_k(\boldsymbol{g}) \subseteq N_k(\boldsymbol{g}')$. Finally, we define

$$\overline{\boldsymbol{g}}^i \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_i(\boldsymbol{g}), \tag{5}$$

as a network obtained from $\boldsymbol{g}$ after performing the three operations defined above. Note that the superscript in $\overline{\boldsymbol{g}}^i$ refers to the fact that in this network player i is playing a best response. Since $\overline{\boldsymbol{g}}^i$ is a network, $\overline{\boldsymbol{g}}^i_{ij} \in \{0,1\}$ indicates if player i forms a link with player j in $\overline{\boldsymbol{g}}^i$, and $\overline{\boldsymbol{g}}^i_{-i}$ denote the network obtained when all of player i's links are removed in $\overline{\boldsymbol{g}}^i$. In the next two lemmas we describe properties of the networks $\overline{\boldsymbol{g}}^i$ and $\mathrm{br}_i(\boldsymbol{g})$.

Lemma 3 *Let the payoff function be given by (2) and let $\overline{\boldsymbol{g}}^i$ be defined as in equation (5). Suppose $\boldsymbol{g} \in \mathcal{G}^3$ and for all $k \in N$, $j \in N$, $\boldsymbol{g}_{kj} = 1 \Rightarrow \pi_k^j(\boldsymbol{g}) > 0$.*

1. *If $k \in N_j(\boldsymbol{g})$, then $k \in N_j(\mathrm{br}_i(\boldsymbol{g}))$.*

2. *If $k \in N_j(\boldsymbol{g})$, then $k \in N_j(\overline{\boldsymbol{g}}^i)$.*

3. *If $\boldsymbol{g}_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$, then $\eta(\boldsymbol{g}) < \eta(\overline{\boldsymbol{g}}^i)$.*

The proof of this lemma can be found in the appendix.

We denote $\boldsymbol{g} \setminus \mathcal{MBR}_i(\boldsymbol{g}_{-i})$ by $\boldsymbol{gm}$. Then $\boldsymbol{gm} \oplus i\ j$ is the network obtained from $\mathrm{br}_i(\boldsymbol{g})$ when player i forms no link except the link $i\ j$.

Lemma 4 *Let the payoff function be given by (2) and let $\overline{\boldsymbol{g}}^i$ be defined as in equation (5). Suppose $\boldsymbol{g} \in \mathcal{G}^3$.*

1. *If $\overline{\boldsymbol{g}}_{ij}^i = \mathrm{br}_i(\boldsymbol{g})_{ij} = 1$, then, for all $j \in N \setminus \{i\}$, $N_j(\boldsymbol{gm} \oplus i\ j) \subseteq N_j(\overline{\boldsymbol{g}}_{-i}^i \oplus i\ j)$.*

2. *Suppose for all $i \in N$, $j \in N$, $\boldsymbol{g}_{ij} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$. If $\overline{\boldsymbol{g}}_{k\ell}^i = \boldsymbol{g}_{k\ell} = 1$, then $N_\ell(\boldsymbol{g}_{-k} \oplus k\ \ell) \subseteq N_\ell(\overline{\boldsymbol{g}}_{-k}^i \oplus k\ \ell)$.*

The proof of this lemma can be found in the appendix.

Lemma 5 *Let the payoff function be given by (2) and let $\overline{\boldsymbol{g}}^i$ be defined as in equation (5). If $\boldsymbol{g} \in \mathcal{G}^3$, then $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$.*

Proof We must show that $\overline{\boldsymbol{g}}^i$ has the following four properties: it is a minimal network, it contains at most one cycle, there does not exist a link from $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$ to $k \in N^{C(\overline{\boldsymbol{g}}^i)}$ and if $\ell \in N_j(\overline{\boldsymbol{g}}^i), j \notin N_\ell(\overline{\boldsymbol{g}}^i), k \notin N_j(\overline{\boldsymbol{g}}^i)$ then $\overline{\boldsymbol{g}}_{k\ell}^i = 0$. The first property follows from the correspondence $\mathcal{M}$ and the next two from the correspondence $\mathcal{H}$. We just need to verify that the last property holds.

First, we show that in $\mathrm{br}_i(\boldsymbol{g})$, we have $\ell \in N_j(\mathrm{br}_i(\boldsymbol{g})), j \notin N_\ell(\mathrm{br}_i(\boldsymbol{g})), i \notin N_j(\mathrm{br}_i(\boldsymbol{g}))$ $\Rightarrow \mathrm{br}_i(\boldsymbol{g})_{i\ell} = 0$. We know that in $\boldsymbol{g}$ we have $\ell \in N_j(\boldsymbol{g}), j \notin N_\ell(\boldsymbol{g}), i \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{i\ell} = 0$ since $\boldsymbol{g} \in \mathcal{G}^3$. By definition, we have $\mathrm{br}_i(\boldsymbol{g})_k = \boldsymbol{g}_k$, for all $k \in N \setminus \{i\}$. Hence, if we show that player $i \notin N_j(\mathrm{br}_i(\boldsymbol{g}))$ has not formed a link $i\ \ell$ with a player ℓ such that $\ell \in N_j(\mathrm{br}_i(\boldsymbol{g}))$ and $j \notin N_\ell(\mathrm{br}_i(\boldsymbol{g}))$ in $\mathrm{br}_i(\boldsymbol{g})$, then we will have shown the conclusion for $\mathrm{br}_i(\boldsymbol{g})$. But, by Lemma 2.1, we know that if i has formed a link with player ℓ, then i is not playing a best response which is a contradiction.

Second, by construction, if $\boldsymbol{g}$ is such that $\ell \in N_j(\boldsymbol{g}), j \notin N_\ell(\boldsymbol{g}), k \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{k\ \ell} = 0$, then $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\boldsymbol{g})$ is such that $\ell \in N_j(\boldsymbol{g}'), j \notin N_\ell(\boldsymbol{g}'), k \notin N_j(\boldsymbol{g}') \Rightarrow \boldsymbol{g}'_{k\ell} = 0$. The conclusion follows. $\qquad\square$

Lemma 6 *Let the payoff function be given by (2) and let $\overline{\boldsymbol{g}}^i$ be defined as in equation (5).*

1. *If $\boldsymbol{g} \in \mathcal{G}^3$, then $\overline{\boldsymbol{g}}^i_{ij} = 1 \Rightarrow \pi^j_i(\overline{\boldsymbol{g}}^i) > 0$.*

2. *If for all $i,j \in N$, $\boldsymbol{g}_{ij} = 1 \Rightarrow \pi^j_i(\boldsymbol{g}) > 0$, then for all $i \in N \setminus \{k\}$, $j \in N$, $\overline{\boldsymbol{g}}^k_{ij} = 1 \Rightarrow \pi^j_i(\overline{\boldsymbol{g}}^k) > 0$.*

Proof We prove both parts of the lemma successively.

1. (a) First, we show that this property is true if $\overline{\boldsymbol{g}}^i_{ij} = 1$ and $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$. If $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, then by construction $\mathrm{br}_i(\boldsymbol{g})_{ij} = 1$ and so $\pi^j_i(\mathrm{br}_i(\boldsymbol{g})) > 0$. Using Lemma 4.1, Lemma 5, and the marginal payoff function defined in equation (4) we have:

$$
\pi^j_i(\overline{\boldsymbol{g}}^i) \;=\; \sum_{k \in N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i\ j)} V_{ik} - c_i
$$

$$
\geq \;\sum_{k \in N_j(\boldsymbol{gm} \oplus i\ j)} V_{ik} - \sum_{k \in \mathcal{K}(\mathrm{br}_i(\boldsymbol{g}); i\ j)} V_{ik} - c_i
$$

$$
=\; \pi^j_i(\mathrm{br}_i(\boldsymbol{g})) > 0
$$

(b) Second, we show that this property is true if $\overline{\boldsymbol{g}}^i_{ij} = 1$ and $j \in N^{C(\overline{\boldsymbol{g}}^i)}$. By construction if $\overline{\boldsymbol{g}}^i_{ij} = 1$ and $j \in N^{C(\overline{\boldsymbol{g}}^i)}$, then $i \in N^{C(\overline{\boldsymbol{g}}^i)}$. If $i \in N^{C(\overline{\boldsymbol{g}}^i)}$, then by construction of $\overline{\boldsymbol{g}}^i$, there is at least one player $\ell \in N^{C(\overline{\boldsymbol{g}}^i)}$, such that $\pi^\ell_i(\mathrm{br}_i(\boldsymbol{g})) > 0$. So for all players $\ell' \in N^{C(\overline{\boldsymbol{g}}^i)}$, there exists a network $(\overline{\boldsymbol{g}}^i)' \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_i(\boldsymbol{g})$ where player i forms a link with player ℓ', and by construction $\pi^j_i(\overline{\boldsymbol{g}}^i) = \pi^{\ell'}_i((\overline{\boldsymbol{g}}^i)')$. We know by Lemma 4.1, that $N_j(\boldsymbol{gm} \oplus i\ j) \subseteq N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i\ j)$. Finally, by Lemma 5, we know that $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$. Hence

using the marginal payoff function defined in equation (4) we have:

$$\pi_i^j(\overline{\boldsymbol{g}}^i) \;=\; \sum_{k\in N_j(\overline{\boldsymbol{g}}^i_{-i}\oplus i\;\; j)} V_{ik} - c_i = \sum_{k\in N_\ell((\overline{\boldsymbol{g}}^i_{-i})'\oplus i\;\; \ell)} V_{ik} - c_i$$

$$\geq\; \sum_{k\in N_\ell(\boldsymbol{gm}\oplus i\;\; \ell)} V_{ik} - \sum_{k\in \mathcal{K}(\boldsymbol{gm}\oplus i\;\; \ell; i\;\; \ell)} V_{ik} - c_i$$

$$=\; \pi_i^\ell(\mathrm{br}_i(\boldsymbol{g})) > 0.$$

2. First, we show that for all $i \in N \setminus \{k\}$, and for all $j \notin N^{C(\overline{\boldsymbol{g}}^k)}$, if $\boldsymbol{g}_{ij} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$, then $\overline{\boldsymbol{g}}^k_{ij} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^k) > 0$. Indeed, if player $i \in N \setminus \{k\}$ has initiated a link with player $j \notin N^{C(\overline{\boldsymbol{g}}^k)}$ in $\overline{\boldsymbol{g}}^k$, then, by construction of $\overline{\boldsymbol{g}}^k$, player i has formed a link with player j in $\boldsymbol{g}$, so $\pi_i^j(\boldsymbol{g}) > 0$. We know from Lemma 4.2, that for all $j \in N$, we have $N_j(\boldsymbol{g}_{-i} \oplus i\;\; j) \subseteq N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i\;\; j)$. Moreover, by Lemma 5, $\overline{\boldsymbol{g}}^k \in \mathcal{G}^3$. So using the marginal payoff function defined in equation (4) we have:

$$\pi_i^j(\overline{\boldsymbol{g}}^k) \;=\; \sum_{\ell\in N_j(\overline{\boldsymbol{g}}^k_{-i}\oplus i\;\; j)} V_{i\ell} - c_i$$

$$\geq\; \sum_{\ell\in N_j(\boldsymbol{g}_{-i}\oplus i\;\; j)} V_{i\ell} - c_i$$

$$=\; \pi_i^j(\boldsymbol{g}) > 0.$$

Next, we show that for all $i \in N \setminus \{k\}$, and for all $j \in N^{C(\overline{\boldsymbol{g}}^k)}$, if $\boldsymbol{g}_{ij} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$, then $\overline{\boldsymbol{g}}^k_{ij} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^k) > 0$. Since $\overline{\boldsymbol{g}}^k \in \mathcal{G}^3$ and there exists a link from player j to player i, we have $i \in N^{C(\overline{\boldsymbol{g}}^k)}$. If $i \in N^{C(\overline{\boldsymbol{g}}^k)}$, then there are two possibilities: either $k \in N_i(\mathrm{br}_k(\boldsymbol{g}))$ or $i \in N^{C(\boldsymbol{g})}$. We deal with these two possibilities successively.

(a) If $k \in N_i(\mathrm{br}_k(\boldsymbol{g}))$, then there exists in $\mathrm{br}_k(\boldsymbol{g})$ a link $i\;\; \ell$ such that $\mathrm{br}_k(\boldsymbol{g})_{i\ell} = \boldsymbol{g}_{i\ell} = 1$ and $k \in N_\ell(\mathrm{br}_k(\boldsymbol{g}))$. Since, $\boldsymbol{g}_{i\ell} = 1$, we have $\pi_i^\ell(\boldsymbol{g}) > 0$. Furthermore,

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by construction, player $\ell \in N^{C(\overline{g}^k)}$, since $k \in N_\ell(\mathrm{br}_k(g))$. We note that for all players $h' \in N^{C(\overline{g}^k)}$, there exists a network $(\overline{g}^k)' \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_k(g)$ where player i forms a link with player h', and by construction $\pi_i^j(\overline{g}^k) = \pi_i^{h'}((\overline{g}^k)')$. We know from Lemma 4.2 that for all $j \in N$, we have $N_j(g_{-i} \oplus i\ j) \subseteq N_j(\overline{g}_{-i}^k \oplus i\ j)$. Finally, we know by Lemma 5 that $\overline{g}^i \in \mathcal{G}^3$. Hence, using the marginal payoff function defined by equation (4), we obtain:

$$\pi_i^j(\overline{g}^k) \quad = \quad \sum_{\ell' \in N_j(\overline{g}_{-i}^k \oplus i\ j)} V_{i\ell'} - c_i = \sum_{\ell' \in N_\ell((\overline{g}_{-i}^k)' \oplus i\ \ell)} V_{i\ell'} - c_i$$

$$\geq \quad \sum_{\ell' \in N_\ell(g_{-i} \oplus i\ \ell)} V_{i\ell'} - c_i = \pi_i^\ell(g) > 0.$$

(b) If $i \in N^{C(g)}$, then we have $\pi_i^\ell(g) > 0$ for $i\ \ell \in E^{C(g)}$. We assume, without loss of generality, that player i forms in $C(\overline{g}^i)$ a link with a player j such that $\pi_i^j(\mathrm{br}_i(g)) > 0$. By construction of $\overline{g}^k$ we have $N^{C(g)} \subseteq N^{C(\overline{g}^k)}$ and by Lemma 4.2, we have $N_j(g_{-i} \oplus i\ j) \subseteq N_j(\overline{g}_{-i}^k \oplus i\ j)$ for all $j \in N$. Note that for all players $h' \in N^{C(\overline{g}^k)}$, there exists a network $(\overline{g}^k)' \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_k(g)$ where player i forms a link with player h'. Also by construction $\pi_i^j(\overline{g}^k) = \pi_i^{h'}((\overline{g}^k)')$. We know by Lemma 5 that $\overline{g}^i \in \mathcal{G}^3$. Again, using the marginal payoff function defined by equation (4), we obtain:

$$\pi_i^j(\overline{g}^k) \quad = \quad \sum_{\ell' \in N_j(\overline{g}_{-i}^k \oplus i\ j)} V_{i\ell'} - c_i = \sum_{\ell' \in N_\ell((\overline{g}_{-i}^k)' \oplus i\ \ell)} V_{i\ell'} - c_i$$

$$\geq \quad \sum_{\ell' \in N_\ell(g_{-i} \oplus i\ \ell)} V_{i\ell'} - c_i = \pi_i^\ell(g) > 0.$$

$\square$

Proof of Proposition 2 We start with the empty network $\dot{g} = g^0$. It is easy to check that $g^0 \in \mathcal{G}^3$. Either g^0 is a Nash network, and we are done, or there exists a player, say i, who does not play a best response in g^0. In that case, we construct the network

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$\boldsymbol{g}^1 \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_i(\boldsymbol{g}^0)$. We know from Lemma 3.3 that $\eta(\boldsymbol{g}^0) < \eta(\boldsymbol{g}^1)$. From Lemma 5, $\boldsymbol{g}^1 \in \mathcal{G}^3$ and from Lemma 6.1 and 6.2, we know that for all players $j \in N$ and $\ell \in N$, $\boldsymbol{g}^1_{j\ell} = 1 \Rightarrow \pi^\ell_j(\boldsymbol{g}^1) > 0$. Either $\boldsymbol{g}^1$ is a Nash network, and we are done, or there exists a player, say j, who does not play a best response in $\boldsymbol{g}^1$. In that case, we construct the network $\boldsymbol{g}^2 \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_j(\boldsymbol{g}^1)$. We know from Lemma 3.3 that $\eta(\boldsymbol{g}^1) < \eta(\boldsymbol{g}^2)$. Again from Lemma 5, $\boldsymbol{g}^2 \in \mathcal{G}^3$ and from Lemma 6.1 and 6.2, we know that for all players $j \in N$ and $\ell \in N$, $\boldsymbol{g}^2_{j\ell} = 1 \Rightarrow \pi^\ell_j(\boldsymbol{g}^2) > 0$. It follows that we can construct a sequence of networks $\{\boldsymbol{g}^0, \boldsymbol{g}^1 \ldots, \boldsymbol{g}^t, \ldots\}$ such that in $\boldsymbol{g}^{t-1}$, there exists a player, say k, who does not play a best response, and $\boldsymbol{g}^t \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_k(\boldsymbol{g}^{t-1})$, $\eta(\boldsymbol{g}^{t-1}) < \eta(\boldsymbol{g}^t)$, $\boldsymbol{g}^t \in \mathcal{G}^3$ and for all $j \in N$, $\boldsymbol{g}^t_{j\ell} = 1 \Rightarrow \pi^\ell_j(\boldsymbol{g}^t) > 0$. This sequence is finite since $\eta(\boldsymbol{g}) \le n^2$, for all $\boldsymbol{g} \in \mathcal{G}$.

$\square$

Proposition 2 establishes that if values of links are heterogeneous by pairs of players and costs of links are heterogeneous by players, then a Nash network always exists. Note that although this result is similar to the result of Haller et al. for two-way flow models, the proof is quite different. Indeed in Haller et al. (2007) it is sufficient to reduce the networks associated to the best response process to minimal networks for convergence to a Nash network. This procedure cannot be used in our formulation because of the existence of cycles in the best response process.

3.2 Existence of Nash networks under heterogeneity of costs by pairs

We now study one-way flow models when values of links are heterogeneous by players and costs of links are heterogeneous by pairs of players. costs of We can write the payoff

function for this case as follows:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_i - \sum_{j \in N} g_{ij} c_{ij}. \tag{6}$$

Let $\pi_i^j(\boldsymbol{g})$ denote the marginal payoff of player i from player j in the network $\boldsymbol{g}$. If $\boldsymbol{g}_{ij} = 1$, then $\pi_i^j(\boldsymbol{g}) = \pi_i(\boldsymbol{g}) - \pi_i(\boldsymbol{g} \ominus i\ j)$. We can rewrite $\pi_i^j(\boldsymbol{g})$ as follows:

$$\pi_i^j(\boldsymbol{g}) = \sum_{k \in N_i(\boldsymbol{g}_{-i} \oplus i\ j)} V_i - \sum_{k \in \mathcal{K}(\boldsymbol{g}; i\ j)} V_i - c_{ij}. \tag{7}$$

To prove the following proposition, we need an additional definition. First, note that we cannot use our previous re-composition of the best response network. More precisely, the definition of $\mathcal{H}$ is not appropriate in the case of heterogeneous cost. Indeed, in the previous section, we could place players in the cycle without restrictions because there is no difference for player i to form a link with either player j or player k since all links costs are the same. However, this is not true in the case of heterogeneous costs. So, let $\mathcal{H}_i : \mathcal{G} \to \mathcal{G}$ be a correspondence where $h_i(\boldsymbol{g}) \in \mathcal{H}_i(\boldsymbol{g})$ satisfies the following conditions.

- If $\boldsymbol{g}$ contains at most one cycle and there does not exist a link from a player $j \notin C(\boldsymbol{g})$ to a player $k \in C(\boldsymbol{g})$, then $\boldsymbol{g} = h_i(\boldsymbol{g})$.

- If player i has formed a link with no player $j \in N^{\mathcal{C}(\boldsymbol{g})}$ or with at least two players $j \in N^{\mathcal{C}(\boldsymbol{g})}$ in $\boldsymbol{g}$, then

    1. if k is such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{\mathcal{C}(\boldsymbol{g})}$, then $k \in N^{C(h_i(\boldsymbol{g}))}$;

    2. if $k \notin N^{C(h_i(\boldsymbol{g}))}$, then for all $\ell \in N$, we have $g_{\ell k} = h_i(\boldsymbol{g})_{\ell k}$.

- If player i has formed a link with one and only one player $j \in N^{\mathcal{C}(\boldsymbol{g})}$ in $\boldsymbol{g}$, then:

    1. if k is such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{\mathcal{C}(\boldsymbol{g})}$, then $k \in N^{C(h_i(\boldsymbol{g}))}$;

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2. if $k \notin N^{C(h_i(\boldsymbol{g}))}$, then for all $\ell \in N$, we have $g_{\ell k} = h_i(\boldsymbol{g})_{\ell k}$;

3. player i and player j belong to $N^{C(h_i(\boldsymbol{g}))}$ and the link $i\,j$ belongs to $E(h_i(\boldsymbol{g}))$.

We now define $\hat{\boldsymbol{g}}^i$ as follows: $\hat{\boldsymbol{g}}^i \in \mathcal{M} \circ \mathcal{H}_i \circ br_i(\boldsymbol{g})$. We can now state the next proposition which says that Nash networks always exist when the costs of link formation are not very different from each other relative to the value of information that the player can obtain. If on the other hand the range of cost heterogeneity is large, then non existence cannot be ruled out.

Proposition 3 *Consider a game where values of links are heterogeneous by players and costs of links are heterogeneous by pairs. Moreover, let the payoff function be given by (6). There always exists a Nash network if for all $i, j, j' \in N$: $|c_{ij} - c_{ij'}| < V_i$.*

Proof The proof of this proposition is rather long. It is similar to the proof of the proposition 2 with $\hat{\boldsymbol{g}}^i$ playing the same role as $\overline{\boldsymbol{g}}^i$ and hence is omitted. $\qquad\square$

We turn now to the case where values and costs are heterogeneous by pairs. We give a sufficient conditions which allow to guarantee the existence of Nash networks.

Corollary 1 *Suppose a game where values and costs of links are heterogeneous by pairs. Moreover, let the payoff function be given by (1). If for all $i, j, j' \in N$: $|c_{ij} - c_{ij'}| < \min_{k \in N}\{V_{ik}\}$, then there always exists a Nash network.*

Recall that we have already shown that when values and costs are heterogeneous by pairs and $n > 3$ a Nash network does not always exist. The above corollary is in the nature of a silver lining since it provides a sufficient condition under which Nash networks will always exist. The importance of these results stems from the fact that they identify conditions under which Nash networks always exist under heterogeneity.

4 Discussion

The existing literature on one-way flow models shows that over some parameters ranges, Nash networks with specific properties exist. This amounts to providing sufficient conditions for the existence of Nash networks. However, these conditions often do not cover the entire parameters space and are unable to answer if Nash networks always exist. Our paper fills this void in the literature.

To sum up our results we find that as in two-way flow models, cost heterogeneity plays a key role in the non-existence of Nash networks in pure strategies. Indeed, if values are heterogeneous, but costs are not, then Nash networks always exist. The reason for this is that cost heterogeneity offers agents the possibility of substituting one link for another. This can lead to cyclical behavior, i.e., a sequence of link switches that never converges. We also find that when the costs are not too different from each other (relative to values) Nash networks will always exist. There are however some differences with two-way flow models. In one way-flow models, unlike in two-way flow models, it is not possible to rule out the existence of cycles in the best response process. This completely changes the nature of proofs. Furthermore, we see that when there are too few players ($n < 4$), then heterogeneity in values or costs cannot affect the existence of Nash networks in one-way flow models. This is also due to the possibility of cycles in one-way flow models. In other words, we need a large enough set of players along with heterogeneity to get non existence of Nash equilibria. This interaction between the player set and heterogeneity does not arise in two-way flow models.

Finally, our different results raise two questions for future research. The first is: Can the introduction of a decay assumption change the different results? Billand, Bravard

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and Sarangi (2006)[6] show that there does not always exist a Nash network even in a framework with homogeneous costs, heterogeneous values (by pairs) under decay for two-way flow models. This issue would be interesting in the context of directed networks. Next, how sensitive are the results of the paper to the assumption of linearity in values and costs? The issue of characterization and existence of Nash equilibria for networks using more general payoffs under heterogeneity are interesting questions requiring further work.

5 Appendix

Proof of Lemma 3 We successively prove each part of the Lemma.

1. Observe that for all $k \neq i$, and for all $j \in N$, we have $\boldsymbol{g}_{kj} = \mathrm{br}_i(\boldsymbol{g})_{kj}$. Hence, if $N_j(\boldsymbol{g}) \not\subseteq N_j(\mathrm{br}_i(\boldsymbol{g}))$, then there exists a player k such that $k \in N_i(\boldsymbol{g})$ and $k \notin N_i(\mathrm{br}_i(\boldsymbol{g}))$. Since $\boldsymbol{g} \in \mathcal{G}^3$, we know from Lemma 2.2 and 2.3, that player i will not be playing a best response if she deletes one of her links. Hence, if $k \in N_i(\boldsymbol{g})$, then $k \in N_i(\mathrm{br}_i(\boldsymbol{g}))$, and we obtain the desired conclusion.

2. We know from the first part of the lemma that $N_j(\boldsymbol{g}) \subseteq N_j(\mathrm{br}_i(\boldsymbol{g}))$. Also we know that $N_j(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_j(\boldsymbol{g}')$, for all $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\mathrm{br}_i(\boldsymbol{g}))$. The result follows.

3. From the second part of the lemma, we know that $N_j(\boldsymbol{g}) \subseteq N_j(\overline{\boldsymbol{g}}^i)$ for all $j \neq i$. We now show that if $\boldsymbol{g}_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$, then $N_i(\boldsymbol{g}) \subset N_i(\overline{\boldsymbol{g}}^i)$. By Lemma 2.2 and 2.3, we know that player i cannot be playing a best response if she deletes links. Hence, if she is playing a best response, it must be that $N_i(\boldsymbol{g}) \subset N_i(\mathrm{br}_i(\boldsymbol{g}))$. Since,

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[6]Billand, Bravard and Sarangi, Heterogeneity in Nash networks, *Working Paper* (2006).

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we know that, for all $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\mathrm{br}_i(\boldsymbol{g}))$, $N_i(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_i(\boldsymbol{g}')$, we conclude that $N_i(\boldsymbol{g}) \subset N_i(\overline{\boldsymbol{g}}^i)$. Therefore, $\eta(\boldsymbol{g}) < \eta(\overline{\boldsymbol{g}}^i)$.

$\square$

Proof of Lemma 4 We only prove the first part of this lemma. The second part can be proved using similar arguments. If $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, then $N_j(\overline{\boldsymbol{g}}_{-i}^i) = N_j(\overline{\boldsymbol{g}}^i)$. Indeed, since $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$, $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, and $\overline{\boldsymbol{g}}_{ij}^i = 1$, player j does not obtain any resources from player i. Moreover, we have by construction, $N_j(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_j(\overline{\boldsymbol{g}}^i)$. It follows that $N_j(\boldsymbol{gm} \oplus i\ j) \subseteq N_j(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_j(\overline{\boldsymbol{g}}^i) = N_j(\overline{\boldsymbol{g}}_{-i}^i) \subseteq N_j(\overline{\boldsymbol{g}}_{-i}^i \oplus i\ j)$.

Assume that $j \in N^{C(\overline{\boldsymbol{g}}^i)}$, $\overline{\boldsymbol{g}}_{ij}^i = \mathrm{br}_i(\boldsymbol{g})_{ij} = 1$ and there exists a player ℓ such that $\ell \in N_j(\boldsymbol{gm} \oplus i\ j)$ and $\ell \notin N_j(\overline{\boldsymbol{g}}_{-i}^i \oplus i\ j)$. So in $\mathrm{br}_i(\boldsymbol{g})$, player i obtains resources from player ℓ through a path containing j, and in $\overline{\boldsymbol{g}}^i$ player i obtains resources from player ℓ through a path that does not contain j, since for all $k \in N$, $N_k(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_k(\overline{\boldsymbol{g}}^i)$. Hence, there is a player j' where $j' \in N_i(\overline{\boldsymbol{g}}^i)$, $j' \notin N^{C(\overline{\boldsymbol{g}}^i)}$ and $j' \in N_j(\overline{\boldsymbol{g}}^i)$ who has formed a link with player ℓ between $\mathrm{br}_i(\boldsymbol{g})$ and $\overline{\boldsymbol{g}}^i$. This is not possible by construction.

$\square$

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