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Discrete Mathematics 189 (1998) 233–257

DISCRETE MATHEMATICS

Small regular graphs with four eigenvalues

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Received 15 January 1997; accepted 11 August 1997

Abstract

For most feasible spectra of connected regular graphs with four distinct eigenvalues and at most 30 vertices we find all such graphs, using both theoretic and computer results. © 1998 Elsevier Science B.V. All rights reserved

AMS classification: primary 05C30; secondary 05E99

Keywords: Graphs; Eigenvalues

1. Introduction

Graphs with a few eigenvalues in general have nice combinatorial properties and a rich structure. A well-investigated family of such graphs comprises the strongly regular graphs (the regular graphs with three eigenvalues) and these too have a nice combinatorial characterization. Two combinatorial generalizations of these strongly regular graphs are the distance-regular graphs, and going a step further, the association schemes (cf. [1]). The first stage of investigation after strongly regular graphs would be to consider three-class association schemes (cf. [5]). In such a scheme all graphs are regular with at most four eigenvalues, but of course not all graphs with four eigenvalues are in a three-class association scheme (indeed most are not). Even so, as mentioned earlier, such graphs have some interesting combinatorial properties and were studied previously by the first author [4]. Using the results from [4] we generated a list of feasible spectra for regular graphs with four eigenvalues and at most 30 vertices (for the definition of feasible, see Section 2). Using both theoretic and computer results we were able to find *all* graphs when a graph did exist, or show that none exists, for 214 of the 244 feasible spectra thus found. To be precise, we know that there does not exist a graph for 68 of these feasible spectra. In 15 of the remaining 30 cases whose classification

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PII S0012-365X(98)00085-5

is not yet finished, we have found some graphs after an incomplete computer search, while the other 15 cases are completely open. Of these, the smallest unsolved case is one on 28 vertices.

Sections 2–4 of this paper contain results from [4] that are relevant to our search, while Section 5 is used to prove the non-existence of graphs for a considerable number of the feasible spectra. Section 6 contains an explanation of the methods used in the computer search and the appendix includes the lists of feasible spectra together with the number of graphs for each spectrum. Moreover, all new *transitive* graphs found by computer are also there. In the case of a spectrum for which new graphs were discovered, none of which was transitive, we also include as a representative one with the largest automorphism group.

2. Feasible spectra

If G is a connected regular graph on v vertices with four distinct eigenvalues, then (cf. [4])

(i) G has four integral eigenvalues, or

(ii) G has two integral eigenvalues, and two eigenvalues of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b \in \mathbb{Z}$, $b > 0$, with the same multiplicity, or

(iii) G has one integral eigenvalue, its degree k, and the other three have the same multiplicity $m = \frac{1}{3}(v-1)$, and $k = m$ or $k = 2m$.

In addition, if G has v vertices and spectrum $\{[\lambda_0 = k]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}\}$, then the following three equations uniquely determine the multiplicities m_1, m_2 and m_3 from v and the eigenvalues $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ (cf. [3,4]):

$$1 + m_1 + m_2 + m_3 = v,$$

$$\lambda_0 + m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 = 0,$$

$$\lambda_0^2 + m_1\lambda_1^2 + m_2\lambda_2^2 + m_3\lambda_3^2 = vk.$$

The second equation follows from the trace of A (the adjacency matrix of G), and the third from the trace of A^2. Using these conditions we are able to generate all possible spectra for regular graphs with four eigenvalues and at most 30 vertices. Different algorithms were used in each of the three cases above and they in turn were checked for the following further conditions.

Since a regular graph with four eigenvalues is walk-regular (cf. [4]), it follows that the number of triangles through a given vertex x is independent of x, and equals

$$\Delta = \frac{1}{2}A_{xx}^3 = \frac{\mathrm{Trace}(A^3)}{2v} = \frac{1}{2v}\sum_{i=0}^{3} m_i\lambda_i^3.$$

This expression gives a further feasibility condition for the spectrum of G since Δ should be a non-negative integer. In general, it follows that

$$\theta_r = \frac{1}{v}\sum_{i=0}^{3} m_i\lambda_i^r$$

is a non-negative integer. Since the number of closed walks of odd length r is even, θ_r should be even if r is odd. For even r we can sharpen the condition since the number of non-trivial closed walks (those containing a cycle) is even. When $r=4$ the number of trivial closed walks through a given vertex (i.e. passing through one or two other vertices only) equals $2k^2-k$, and this means that

$$\Xi = \frac{\theta_4 - 2k^2 + k}{2}$$

is a non-negative integer, and it equals the number of quadrangles through a vertex. Here we allow the quadrangles to have diagonals. When $r=6$, the number of non-trivial closed walks through a vertex equals $\theta_6 - k(5k^2-6k+2)$, which should be even.

The complement of a connected regular graph with four eigenvalues is also such a graph unless it is disconnected. By generating only those spectra for which $k \geqslant v-1-k$, we ensured that any putative graph would be connected, but in the appendix we printed the complementary spectrum, unless it implied disconnectivity.

In the algorithm to generate spectra with four integral eigenvalues we checked that θ_r was an even non-negative integer for $r=3,5,7,9$ and 11 and that it was a non-negative integer in the cases $r=8,10$ and 12. In addition we tested to see that both $\theta_4 - 2k^2 + k$ and $\theta_6 - k(5k^2-6k+2)$ were even non-negative integers, and that the complementary spectrum gave rise to numbers of triangles and quadrangles through a vertex that were also non-negative integers. For technical reasons we checked different conditions in the case of two integral eigenvalues, namely the conditions on θ_r for $r=3,\dots,6$ and the complementary θ_r for $r=3,\dots,8$. Finally, in the remaining case of one integral eigenvalue it was not necessary to implement so many conditions. Here we checked only the conditions on θ_3 and θ_4. When a putative spectrum satisfies all of the above conditions, it is termed *feasible.*

3. Special spectral properties

3.1. A useful idea

Let G be a connected k-regular graph on v vertices with four eigenvalues k, λ_1, λ_2 and λ_3. If G has a simple eigenvalue, besides k, say λ_3, then G admits a regular partition into halves with degrees $(\frac{1}{2}(k+\lambda_3), \frac{1}{2}(k-\lambda_3))$, that is, we can partition the vertices into two parts of equal size such that every vertex has $\frac{1}{2}(k+\lambda_3)$ neighbours in its own part and $\frac{1}{2}(k-\lambda_3)$ neighbours in the other part (cf. [4]). A consequence of this is that

$k - \lambda_3$ is even, a condition which was proved by Godsil and McKay [6]. This condition eliminates the existence of a graph with spectrum $\{[14]^1, [2]^9, [-1]^{19}, [-13]^1\}$. Moreover, v is a divisor of

$$(k-\lambda_1)(k-\lambda_2)+(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$$

and

$$(k-\lambda_1)(k-\lambda_2)-(\lambda_3-\lambda_1)(\lambda_3-\lambda_2),$$

from which we derive that there is no graph with spectrum $\{[8]^1, [2]^7, [-2]^9, [-4]^1\}$.

It is possible to prove all this, using an idea on which most of our results, both theoretic and computer, are built. Let A be the adjacency matrix of G. The matrix $C = C(\lambda_1, \lambda_2)$ defined by

$$C(\lambda_1,\lambda_2)=(A-\lambda_1 I)(A-\lambda_2 I)-\frac{(k-\lambda_1)(k-\lambda_2)}{v}J$$

is a symmetric matrix with row sums zero and one non-zero eigenvalue $(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$ with multiplicity m_3 (the multiplicity of λ_3 as an eigenvalue of G). Now C or $-C$ is a positive semi-definite matrix of rank m_3, and C has constant diagonal $k+\lambda_1\lambda_2-(k-\lambda_1)(k-\lambda_2)/v$. Of course, as A^2 is a matrix with non-negative integral entries and A is a $(0,1)$-matrix, the other entries of C are very restricted. Especially when m_3 is small we get strong restrictions on the structure of G. This enables us to show uniqueness of the graph in Proposition 1, and prove the non-existence of graphs in a substantial number of cases in Section 5. It also proved to be a powerful tool in our computer search, as is explained in more detail in Section 6.

3.2. Disconnected complement

If G has v vertices, degree k, and an eigenvalue $k - v$, then the complement of G is disconnected, and then it must be the disjoint union of strongly regular graphs with the same spectrum and hence with the same parameters.

3.3. Graphs with least eigenvalue at least -2

If G has least eigenvalue at least -2, then it must be C_7, or the line graph of some graph H, where H is a strongly regular graph, or H is the incidence graph of a symmetric design, or H is a complete bipartite graph, or G is one of the graphs found by Bussemaker, Cvetković and Seidel [2]:

BCS_9: one graph on 12 vertices with spectrum $\{[4]^1, [2]^3, [0]^3, [-2]^5\}$,
BCS_{70}: one graph on 18 vertices with spectrum $\{[7]^1, [4]^2, [1]^5, [-2]^{10}\}$,
BCS_{153}–BCS_{160}: eight graphs on 24 vertices with spectrum $\{[10]^1, [4]^4, [2]^3, [-2]^{16}\}$,
BCS_{179}: one graph on 18 vertices with spectrum $\{[10]^1, [4]^2, [1]^4, [-2]^{11}\}$,
BCS_{183}: one graph on 24 vertices with spectrum $\{[14]^1, [4]^4, [2]^2, [-2]^{17}\}$.

Note that the complement of a connected regular graph with least eigenvalue -2, is a graph with second largest eigenvalue 1.

4. Constructions and small examples

4.1. *Distance-regular graphs and association schemes*

In [7] (almost) all graphs with the spectrum of a distance-regular graph with at most 30 vertices are found. Most of these graphs have four distinct eigenvalues.

A substantial number of examples of distance-regular graphs with four eigenvalues are given by the incidence graphs of symmetric designs. In fact, any bipartite graph with four eigenvalues must be the incidence graph of a symmetric design, and hence is distance-regular (cf. [3, p. 166]).

Some other graphs can be obtained by merging classes in distance-regular graphs (cf. [1]) or association schemes. By G_i we denote the distance i graph of G. For example, the distance 3 together with the distance 5 relation in the Dodecahedron gives a graph with four eigenvalues. Also the distance 4 relation in the Coxeter graph gives such a graph.

In the tables of the appendix we have added, for every spectrum, the number of graphs with that spectrum or complementary spectrum that are a relation in some three-class association scheme. These numbers are obtained from [5].

4.1.1. *Pseudocyclic association schemes*

A three-class association scheme is said to be pseudocyclic if there are three eigenvalues with the same multiplicity. If the number of vertices q is a prime power and $q \equiv 1 \pmod 3$, then the cyclotomic scheme, which has the third power cyclotomic classes of $GF(q)$ as classes, is an example. For $q>4$, the corresponding graph Cycl(q) has four distinct eigenvalues and is obtained by making two elements of $GF(q)$ adjacent if their difference is a cube. The smallest example is the 7-cycle C_7. It is determined by its spectrum, as are Cycl(13) and Cycl(19), which we can prove by hand. On 28 vertices two schemes are known. Mathon [9] found one, and Hollmann [8] proved that there are precisely two.

4.2. *Product constructions*

If G is a graph with adjacency matrix A, then we denote by $G \otimes J_n$ the graph with adjacency matrix $A \otimes J_n$ ($\otimes$ denotes the Kronecker product), and by $G \circledast J_n$ the graph with adjacency matrix $(A+I)\otimes J_n - I$. Note that $\overline{G \otimes J_n} = \overline{G} \circledast J_n$, where $\overline{G}$ is the complement of G. The first construction adds an eigenvalue 0 to the spectrum, while the second construction adds an eigenvalue -1.

Thus, if we have a strongly regular graph or a connected regular graph with four distinct eigenvalues of which one is 0 or -1, then this construction produces a bigger graph with four distinct eigenvalues.

For any n, $C_5 \otimes J_n$ and $C_5 \circledast J_n$ are uniquely determined by their spectra. Furthermore, if $\mathrm{IG}(l, l-1, l-2)$ denotes the incidence graph of the unique (trivial) 2-$(l, l-1, l-2)$ design, then for each l and n, the graph $\mathrm{IG}(l, l-1, l-2) \circledast J_n$ is uniquely determined by its spectrum (cf. [4]).

If A is the adjacency matrix of a graph G, then the graph with adjacency matrix

$$\begin{pmatrix} A & I \\ I & J-I-A \end{pmatrix}$$

is called the twisted double td$G$ of G. Now let $v = 4\mu + 1$ and $k = 2\mu$. Then G is a graph with spectrum $\{[k+1]^1, [k-1]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{v+4}]^{2k}, [-\frac{1}{2} - \frac{1}{2}\sqrt{v+4}]^{2k}\}$ if and only if G is the twisted double of a conference graph on v vertices (cf. [4]). Since the conference graphs $L_2(3)$ on 9 vertices and $P(13)$ on 13 vertices are unique, also their twisted doubles are uniquely determined by their spectra.

Let G and G' be graphs with adjacency matrices A and A', and eigenvalues λ_i, $i = 1, 2, \ldots, v$, and λ_i', $i = 1, 2, \ldots, v'$, respectively. Then the graph with adjacency matrix $A \otimes I_{v'} + I_v \otimes A'$ has eigenvalues $\lambda_i + \lambda_j'$, $i = 1, 2, \ldots, v$, $j = 1, 2, \ldots, v'$. This graph, which is sometimes called the sum [3] or the Cartesian product of G and G', will be denoted by $G \oplus G'$. An example with four distinct eigenvalues is $G \oplus K_m$, where G is the complete bipartite graph $K_{m,m}$ or the lattice graph $L_2(m)$. Here we present our first new result.

Proposition 1. *The graph $K_{3,3} \oplus K_3$ is uniquely determined by its spectrum.*

Proof. Let G be a graph with spectrum $\{[5]^1, [2]^6, [-1]^9, [-4]^2\}$ and adjacency matrix A. Then G is a 5-regular graph on 18 vertices with one triangle through each vertex. The matrix $C = C(2, -1) = A^2 - A - 2I - J$, as defined in Section 3.1 is a positive semi-definite integral matrix of rank two with diagonal 2. Thus, C is the Gram matrix of a set of vectors in $\mathbb{R}^2$ of length $\sqrt{2}$ such that their inner products are $\pm 2, \pm 1$ or 0.

Note that if two vertices are adjacent and the vectors representing these vertices have inner product -1, then they are in a triangle. This implies that any vertex is adjacent to precisely two vertices such that their inner product is -1, and that the inner product between those two vertices is also -1. If two vertices have inner product -2 then they are adjacent, and if they have inner product 1 or 2 then they are not adjacent.

Without loss of generality we assume that there is a vertex represented by vector $\sqrt{2}(1, 0)^{\mathrm{T}}$. This vertex must be in a triangle with vertices represented by vectors $\sqrt{2}(-\frac{1}{2}, \frac{1}{2}\sqrt{3})^{\mathrm{T}}$ and $\sqrt{2}(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})^{\mathrm{T}}$. Furthermore, it is adjacent to three vertices represented by $\sqrt{2}(-1, 0)^{\mathrm{T}}$. In turn, such a vertex is in a triangle with vertices represented by vectors $\sqrt{2}(\frac{1}{2}, \frac{1}{2}\sqrt{3})^{\mathrm{T}}$ and $\sqrt{2}(\frac{1}{2}, -\frac{1}{2}\sqrt{3})^{\mathrm{T}}$, and is adjacent to three vertices represented by $\sqrt{2}(1, 0)^{\mathrm{T}}$.

In this way we find 18 vertices: each of the 6 mentioned vectors represents 3 vertices. Now, up to isomorphism, all adjacencies follow from the inner products and the fact that every vertex is in one triangle. The graph we obtain is $K_{3,3} \oplus K_3$. $\square$

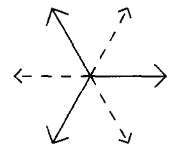

Fig. 1. Vectors representing the vertices of $K_{3,3} \oplus K_3$.

4.3. Examples from strongly regular graphs

If G is a non-bipartite strongly regular graph on v vertices, with spectrum $\{[k]^1, [r]^f, [s]^g\}$, and C is a coclique of size c meeting the Hoffman bound, i.e. $c = -vs/(k-s)$, then the induced subgraph $G \backslash C$ on the vertices not in C is a regular, connected graph with four eigenvalues if $c < g$. By looking at the complement of the graph, a similar construction works for cliques instead of cocliques. An example is obtained by removing a 3-clique (a line) in the point graph of the generalized quadrangle GQ(2,2), and the resulting graph has spectrum $\{[5]^1, [1]^6, [-1]^2, [-3]^3\}$.

If G contains a spread, that is, a partition of the vertices into Hoffman cliques, then by removing the edges of this spread, we also obtain a regular graph with four eigenvalues. For example, the complement of the triangular graph $T(n)$ contains (many) spreads for every even n.

Also some subconstituents (i.e. induced subgraphs on the set of (non-)neighbours of a given vertex) of strongly regular graphs have four distinct eigenvalues, for example the second subconstituent $\mathrm{GQ}(3,3)_2(x)$ of the point graph of a generalized quadrangle GQ(3,3) (cf. [4]).

4.4. Covers

In [4] n-covers of $C_3 \otimes J_n$, $C_3 \circledast J_n = K_{3n}$, $C_5 \circledast J_n$, $C_6 \circledast J_n$ and Cube $\circledast J_n$, having four distinct eigenvalues are constructed. The 2-cover of $C_3 \otimes J_2$ is isomorphic to the line graph of the Cube, the 2-cover of $C_3 \circledast J_2$ is isomorphic to the Icosahedron, and the 3-cover of $C_3 \otimes J_3$ is cospectral but not isomorphic to $H(3,3)$. The graphs in Fig. 2 are the three remaining covers with at most 30 vertices. They were shown, using the computer, to be uniquely determined by their spectra.

4.5. Switching

Let G be a graph, and suppose we have a partition of the vertices into two parts. Switching G with respect to this partition is the operation of interchanging the edges and non-edges between the two parts (thus two vertices in different parts are adjacent in the new graph if and only if they are not adjacent in the original graph G) and leaving all other edges the same (thus two vertices in the same part are adjacent in the new graph if and only if they were also adjacent in G). If the partition, say with parts

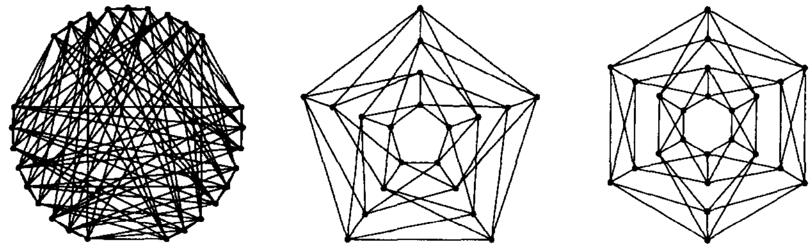

Fig. 2. A 3-cover of $C_3 \circledast J_3$ and 2-covers of $C_5 \circledast J_2$ and $C_6 \circledast J_2$.

V_1 and V_2, is regular, that is, if every vertex in V_i is adjacent to a constant number of vertices in the same part, and to a constant number in the other part, for $i=1,2$ then switching with respect to this partition changes at most two of the eigenvalues of G.

If a graph G admits a regular partition of the vertices into two parts, such that every vertex is adjacent to half of the vertices in the other part, then switching with respect to this partition gives a graph with the same spectrum as G, but which is possibly different from G. This way we computed several cospectral regular graphs with four eigenvalues, as is explained in more detail in Section 6.

Switching was introduced by Seidel (cf. [11]), and it has proved to be a powerful tool to construct cospectral graphs. Switching in general however changes the spectrum. For example, by switching with respect to some special regular partitions in strongly regular graphs it is possible to construct regular graphs with four eigenvalues (cf. [4]).

Here we also use switching to construct regular graphs with four eigenvalues from other ones, with a different spectrum.

Proposition 2. *Let G be a regular graph on v vertices with spectrum $\{[k]^1,[\lambda_1]^{m_1},[\lambda_2]^{m_2},[\lambda_3]^{m_3}\}$ where $\lambda_i=k-\frac{1}{2}v$ for some i, say $i=3$. Suppose G admits a partition of the vertices into two parts of equal size such that every vertex is adjacent to $\frac{1}{2}(k-\lambda_j)$ vertices in the other part, for some $j\neq i$, say $j=1$. Switching with respect to this partition gives a regular graph with spectrum $\{[\lambda_1+\frac{1}{2}v]^1,[\lambda_1]^{m_1-1},[\lambda_2]^{m_2},[\lambda_3=k-\frac{1}{2}v]^{m_3+1}\}$.*

For example, consider the incidence graph $\mathrm{IG}(2n,2n-1,2n-2)$ of the trivial design on $2n$ points, i.e. we have a graph on vertices p_i and b_i, $i=1,\ldots,2n$, where p_i is adjacent to b_j if and only if $i\neq j$, and with no other edges. Take for one part of the partition the set of vertices $\{p_i,b_{i+n}\,|\,i=1,\ldots,n\}$. Switching with respect to this partition gives the complement of the disjoint union of two cocktail party graphs $\mathrm{CP}(n)$, which has spectrum $\{[2n+1]^1,[1]^{2n-2},[-1]^{2n},[-2n+1]^1\}$, while the spectrum of the incidence graph was $\{[2n-1]^1,[1]^{2n-1},[-1]^{2n-1},[-2n+1]^1\}$.

5. Non-existence results

Let G be a k-regular graph on v vertices with Δ triangles and Ξ quadrangles through every vertex. Fix a vertex x, and let σ_i be the number of vertices y adjacent to x, such that $A^2_{xy}=i$, and let τ_i be the number of vertices y not adjacent to x, such that $A^2_{xy}=i$. Then counting arguments show that

$$\sum_i \sigma_i = k, \qquad \sum_i i\sigma_i = 2\Delta, \qquad \sum_i \tau_i = v-k-1,$$
$$\sum_i i\tau_i = k(k-1)-2\Delta \quad \text{and} \quad \sum_i \binom{i}{2}(\sigma_i+\tau_i)=\Xi.$$

We shall call this system of equations the (σ,τ)-system.

In the following we examine several feasible spectra and prove the non-existence of a graph in each case. In each of the proofs of the following propositions we assume the existence of a graph G with the given spectrum and A will denote its adjacency matrix.

Proposition 3. *There are no graphs with spectrum* $\{[7]^1,[2]^{15},[-2]^5,[-3]^9\}$, $\{[6]^1,[2]^9,[1]^9,[-3]^{11}\}$, $\{[7]^1,[2]^{12},[1]^5,[-3]^{12}\}$, $\{[6]^1,[1+\sqrt{10}]^2,[-1]^{10},[1-\sqrt{10}]^2\}$, $\{[7]^1,[1+2\sqrt{3}]^2,[-1]^{11},[1-2\sqrt{3}]^2\}$ *or* $\{[8]^1,[-1+\sqrt{6}]^7,[1]^6,[-1-\sqrt{6}]^7\}$.

Proof. A graph with the first spectrum would be 7-regular on 30 vertices with $\Delta=3$ triangles and $\Xi=12$ quadrangles through every vertex. Using the idea of Section 3.1, let $C=C(2,-3)=A^2+A-6I-\frac{5}{3}J$, then $-C$ is a positive semi-definite matrix with diagonal $\frac{2}{3}$. It follows that C can only have entries $-\frac{2}{3}$ and $\frac{1}{3}$, and so if x and y are adjacent then $A^2_{xy}=0$ or 1, and if x and y are not adjacent then $A^2_{xy}=1$ or 2. But now the (σ,τ)-system does not have a solution, so we have a contradiction. The other cases go similarly. □

Proposition 4. *There are no graphs with spectrum* $\{[8]^1,[2+3\sqrt{2}]^3,[-1]^{20},[2-3\sqrt{2}]^3\}$ *or* $\{[9]^1,[7]^3,[-1]^{24},[-3]^2\}$.

Proof. The first spectrum would give an 8-regular graph on 27 vertices with $\Delta=22$ triangles and $\Xi=102$ quadrangles through every vertex. The matrix C as defined in Section 3.1 by $C=C(2+3\sqrt{2},-1)=A^2-(1+3\sqrt{2})A-(2+3\sqrt{2})I-(2-\sqrt{2})J$, is a positive semi-definite matrix with diagonal $4-2\sqrt{2}$. It follows that if x and y are adjacent then $A^2_{xy}=5,6$ or 7, and if x and y are not adjacent then $A^2_{xy}=0$ or 1. Now the (σ,τ)-system has one solution $\sigma_7=2$, $\sigma_6=0$, $\sigma_5=6$, $\tau_1=12$, $\tau_0=6$. But then $G=H \circledast J_3$, for some graph H. It follows that H must have spectrum $\{[2]^1,[\sqrt{2}]^3,[-1]^2,[-\sqrt{2}]^3\}$, but since such a graph does not exist, we have a contradiction. Similarly, a graph with the second spectrum must be of the form $H \circledast J_2$, where H has spectrum $\{[4]^1,[3]^3,[-1]^9,[-2]^2\}$, which is impossible by the results of Section 3.3. □

The next proposition uses the fact that if the number of triangles through an edge is constant, then the number of quadrangles through an edge is also constant (cf. [4]). If ξ is the (constant) number of quadrangles through an edge, and if Ξ is the number of quadrangles through a vertex, then $\xi = 2\Xi/k$.

Proposition 5. *There are no graphs with spectrum* $\{[8]^1, [-1+\sqrt{21}]^4, [0]^{21}, [-1-\sqrt{21}]^4\}$ *or* $\{[4]^1, [-\frac{1}{2}+\frac{1}{2}\sqrt{21}]^4, [0]^6, [-\frac{1}{2}-\frac{1}{2}\sqrt{21}]^4\}$.

Proof. Note that if H is a graph with the second spectrum, then $H \otimes J_2$ is a graph with the first spectrum. Thus it suffices to show that there is no graph with the first spectrum. Suppose G is such a graph, then G is 8-regular on 30 vertices without triangles, such that every vertex is in $\Xi = 84$ quadrangles and every edge is in $\xi = 21$ quadrangles.

Suppose first of all that G has diameter 2. Suppose x and z are two non-adjacent vertices such that $A^2_{xz} = 1$ and let y be their common neighbour. Now the 21 quadrangles through $\{x, y\}$ and the 21 quadrangles through $\{z, y\}$ are distinct, and since there are 42 edges between $G(y) \backslash \{x, z\}$ and $G_2(y)$, all these edges contain a neighbour of x or z. Then it follows that the number of vertices at distance 2 from y is 14, and so G has diameter 3, which is a contradiction. Thus for any two non-adjacent vertices x and z we must have $A^2_{xz} \geqslant 2$. But then the (σ, τ)-system has no non-negative integral solution. Thus G has diameter 3.

Take a vertex x and let y be a vertex at distance 3 from x. Let A be partitioned into two parts, where one part contains y and the neighbours of x. Then

$$A = \begin{pmatrix} O_{9,9} & N \\ N^T & B \end{pmatrix}.$$

Since $\mathrm{rank}(A) = 9$, it follows that $\mathrm{rank}(N) \leqslant 4$. Now write

$$N = \begin{pmatrix} \underline{1}_8 & N_1 \\ 0 & N_2 \end{pmatrix} \quad \text{and} \quad N' = \begin{pmatrix} \underline{0}_8 & N_1 \\ 1 & N_2 \end{pmatrix}.$$

Since the all-one vector is in the column space of N (N has constant row sums 8), $\mathrm{rank}(N') \leqslant \mathrm{rank}(N)$, so $\mathrm{rank}(N_1) \leqslant 3$. Moreover, N_1 has constant row sums 7, and so it follows that N_1 is of the form

$$N_1 = \begin{pmatrix} J_{m_1,7-t_1-t_2} & J_{m_1,t_1} & O_{m_1,t_1} & J_{m_1,t_2} & O_{m_1,t_2} & O_{m_1,13-t_1-t_2} \\ J_{m_2,7-t_1-t_2} & J_{m_2,t_1} & O_{m_2,t_1} & O_{m_2,t_2} & J_{m_2,t_2} & O_{m_2,13-t_1-t_2} \\ J_{m_3,7-t_1-t_2} & O_{m_3,t_1} & J_{m_3,t_1} & J_{m_3,t_2} & O_{m_3,t_2} & O_{m_3,13-t_1-t_2} \\ J_{m_4,7-t_1-t_2} & O_{m_4,t_1} & J_{m_4,t_1} & O_{m_4,t_2} & J_{m_4,t_2} & O_{m_4,13-t_1-t_2} \end{pmatrix},$$

with $m_1 + m_2 + m_3 + m_4 = 8$, and $t_1, t_2 \neq 0$, or that N_1 has at most 3 distinct rows. Suppose we are in the first case. If we count the number of quadrangles through x and a vertex z which corresponds to one of the first m_1 rows, then we see that

$$\xi = 7(m_1 - 1) + (7 - t_2)m_2 + (7 - t_1)m_3 + (7 - t_1 - t_2)m_4.$$

If we count the number of quadrangles through x and a vertex corresponding to one of the m_2 rows of the second block, then

$$\xi = 7(m_2 - 1) + (7 - t_2)m_1 + (7 - t_1)m_4 + (7 - t_1 - t_2)m_3.$$

From this it follows that $m_1+m_3 = m_2+m_4 = 4$, and $t_1+t_2 = 7$. Similarly it follows that $m_1+m_2 = m_3+m_4 = 4$, and so that $m_1 = m_4$ and $m_2 = m_3$. This implies that $G_3(x)$ has 7 vertices and that every vertex in $G_2(x)$ has 4 neighbours in $G(x)$. From the Hoffman polynomial it follows that if y is a vertex at distance 3 from x, then $A^3_{xy} = 16$, so in turn every index in $G_3(x)$ has 4 neighbours in $G_2(x)$. But then the induced subgraph on $G_3(x)$ is 4-regular on 7 vertices, and this is not possible without triangles.

Thus, we are in the second case. Suppose N_1 has 4 identical rows. By counting the number of quadrangles through x and a vertex corresponding to one of these 4 rows it follows that the other 4 rows are disjoint from the first 4. Further counting gives that the other 4 rows must also be the same, and again we have that $G_3(x)$ has 7 vertices and that every vertex in $G_2(x)$ has 4 neighbours in $G(x)$, which leads to a contradiction. It follows that we have one row occurring twice and two rows occurring three times. By counting quadrangles through x and a vertex corresponding to one of the rows occurring twice, we see that

$$\xi = 7 + 3t_1 + 3t_2,$$

for some t_1, t_2, and so 14 should be divisible by 3, which is a contradiction. □

Next we shall prove the non-existence of some graphs, assuming that they have an eigenvalue with multiplicity 2.

Proposition 6. *There are no graphs with spectrum* $\{[7]^1, [3]^6, [-1]^{15}, [-5]^2\}$, $\{[10]^1, [2]^3, [0]^{18}, [-8]^2\}$, $\{[10]^1, [4]^2, [0]^{18}, [-6]^3\}$ *or* $\{[9]^1, [4]^6, [-1]^{21}, [-6]^2\}$.

Proof. A graph with the first spectrum is 7-regular on 24 vertices with 5 triangles through each vertex. Let $C = C(3, -1) = A^2 - 2A - 3I - \frac{4}{3}J$, then C is a positive semi-definite matrix of rank two with row sums zero and diagonal $\frac{8}{3}$. Thus, C can only have entries $-\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{5}{3}$ and $\frac{8}{3}$.

Now suppose that $C_{xy} = -\frac{1}{3}$ for some vertices x and y. Let z be another arbitrary vertex. Since C has rank two it follows that the principal submatrix of C on vertices x, y and z has zero determinant, and so either $C_{xz} = \frac{8}{3}$ and $C_{yz} = -\frac{1}{3}$ or $C_{xz} = -\frac{1}{3}$ and $C_{yz} = \frac{8}{3}$. But then x and y cannot both have row sums zero, and it follows that C has no entries $-\frac{1}{3}$. Similarly it follows that C cannot have entries $\frac{5}{3}, -\frac{7}{3}$ and $\frac{2}{3}$. Thus C can only have entries $\frac{8}{3}$ and $-\frac{4}{3}$.

Now fix x. For all vertices y adjacent to x, we must have $A^2_{xy} = 2$ or 6. But x has 7 neighbours, giving that x is in at least 7 triangles, which is a contradiction. The other cases go similarly. □

Proposition 7. *There is no graph with spectrum* $\{[12]^1, [3]^2, [0]^{22}, [-9]^2\}$.

Proof. Here we would have a 12-regular graph on 27 vertices with $\Delta = 6$ triangles and $\Xi = 492$ quadrangles through every vertex. The matrix $C(0, -9)$ is positive semi-definite of rank two with row sums zero and diagonal $\frac{8}{3}$. Thus, C can only have entries $-\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{5}{3}$ and $\frac{8}{3}$.

Now suppose that $C_{xy} = -\frac{1}{3}$ for some vertices x and y. Let z be another arbitrary vertex. Since C has rank two it follows that the principal submatrix of C on vertices x, y and z has zero determinant, and so either $C_{xz} = \frac{8}{3}$ and $C_{yz} = -\frac{1}{3}$ or $C_{xz} = -\frac{1}{3}$ and $C_{yz} = \frac{8}{3}$. But then x and y cannot both have row sums zero. Thus, C has no entries $-\frac{1}{3}$. This implies that if x and y are adjacent then $A^2_{xy} \neq 0$, and since there are only 6 triangles through every vertex, it follows that $A^2_{xy} = 1$ ($\sigma_1 = 12$), and so $C_{xy} = \frac{2}{3}$. Again, let z be another vertex, then it follows that $C_{xz} = \frac{2}{3}$, $\frac{8}{3}$ or $-\frac{7}{3}$. Now it follows that if x and z are not adjacent, then $A^2_{xz} = 7, 10$ or 12. But then the (σ, τ)-system has no integral solution, giving a contradiction. □

Proposition 8. *There are no graphs with spectrum* $\{[9]^1, [3]^8, [-1]^{19}, [-7]^2\}$ *or* $\{[10]^1, [5]^2, [0]^{18}, [-5]^4\}$.

Proof. A graph with the first spectrum would be 9-regular on 30 vertices with $\Delta = 4$ triangles and $\Xi = 124$ quadrangles through every vertex. Take $C(3, -1)$, which is a positive semi-definite integral matrix of rank two with diagonal 4. Thus, C can only have entries $-4, -3, \ldots, 3$ and 4. Note that since there are 4 triangles through a vertex, it follows that if x and y are adjacent then $A^2_{xy} \leqslant 4$.

Now suppose that $C_{xy} = 0$ for some vertices x and y. Let z be another arbitrary vertex. Since C has rank two it follows that the principal submatrix of C on vertices x, y and z has zero determinant, and so $C_{xz} = 0$ or ± 4. This implies that if x and z are adjacent then $A^2_{xz} = 0$ or 4, and if x and z are not adjacent then $A^2_{xz} = 2$ or 6. But then the (σ, τ)-system has no solution, so C has no entries 0. Similarly, we can show that C has no entries ± 1 and ± 3. Thus, C only has entries ± 2 and ± 4. This implies that if x and y are adjacent then $A^2_{xy} = 0$ or 2, and if x and y are not adjacent, then $A^2_{xy} = 0, 4$ or 6. The (σ, τ)-system now has one solution $\sigma_0 = 5$, $\sigma_2 = 4$, $\tau_0 = 6$, $\tau_4 = 10$, $\tau_6 = 4$. Now it is not hard to show that a graph with these parameters does not exist. The other spectrum is even easier, since here none of the (σ, τ)-systems has a solution. □

Proposition 9. *There is no graph with spectrum* $\{[13]^1, [3 + 2\sqrt{10}]^2, [-1]^{25}, [3 - 2\sqrt{10}]^2\}$.

Proof. Such a graph is 13-regular on 30 vertices with $\Delta = 62$ triangles and $\Xi = 570$ quadrangles through every vertex. Take the matrix $C(3 + 2\sqrt{10}, -1)$, so $C = A^2 - (2 + 2\sqrt{10})A - (3 + 2\sqrt{10})I - \frac{7}{15}(10 - 2\sqrt{10})J$, which is a positive semi-definite matrix of rank two with diagonal $\frac{8}{15}(10 - 2\sqrt{10})$. From this it follows that if $A_{xy} = 1$ then $A^2_{xy} = 9, 10, 11$ or 12, and if $A_{xy} = 0$ then $A^2_{xy} = 0, 1, 2$ or 3. For a non-negative integral solution of the (σ,τ)-system we have $\sigma_9 \geqslant 6$ and $\sigma_{12} \leqslant 2$. Now fix a vertex x, and let y and z be two vertices with $A^2_{xy} = A^2_{xz} = 9$, then $C_{xy} = C_{xz} = \frac{8}{15}(10 - 2\sqrt{10}) - 3$. Since the principal submatrix on the vertices x, y and z has zero determinant, it follows that $C_{yz} = \frac{8}{15}(10 - 2\sqrt{10})$, so $A^2_{yz} = 12$. For fixed y we have at least 5 choices for z left ($\sigma_9 \geqslant 6$), so for y we have $\sigma_{12} \geqslant 5$, which is a contradiction. $\square$

We finish by giving a case where we use the same technique as in the uniqueness proof of the graph $K_{3,3} \oplus K_3$.

Proposition 10. *There is no graph with spectrum* $\{[6]^1, [3]^5, [-1]^{13}, [-4]^2\}$.

Proof. Here we have a 6-regular graph on 21 vertices with $\Delta = 5$ triangles and $\Xi = 20$ quadrangles through every vertex. Here we take the matrix $C(3,-1)$, then C is a positive semi-definite matrix of rank two with row sums zero and diagonal 2. Thus C is the Gram matrix of a set of vectors in $\mathbb{R}^2$ of length $\sqrt{2}$ with mutual inner products ± 2, ± 1 or 0. Note that not both 0 and ± 1 can occur as inner product, since then also inner products that are not allowed occur.

Suppose that inner product 0 occurs. Without loss of generality we assume that there is a vertex represented by vector $\sqrt{2}(1,0)^{\mathrm{T}}$. The only vectors that can occur now are $\pm\sqrt{2}(1,0)^{\mathrm{T}}$ and $\pm\sqrt{2}(0,1)^{\mathrm{T}}$. Since C has row sums zero, it follows that the number of vertices represented by $\sqrt{2}(1,0)^{\mathrm{T}}$ equals the number of vertices represented by $-\sqrt{2}(1,0)^{\mathrm{T}}$, and the number of vertices represented by $\sqrt{2}(0,1)^{\mathrm{T}}$ equals the number of vertices represented by $-\sqrt{2}(0,1)^{\mathrm{T}}$. But the number of vertices is odd, which is a contradiction.

It follows that if x and y are adjacent then $A^2_{xy} = 1, 2, 4$ or 5 and if x and y are not adjacent then $A^2_{xy} = 0, 2$ or 3. Now we have the (σ,τ)-system

$$\sigma_1 + \sigma_2 + \sigma_4 + \sigma_5 = k = 6,$$

$$\sigma_1 + 2\sigma_2 + 4\sigma_4 + 5\sigma_5 = 2\Delta = 10,$$

$$\tau_0 + \tau_2 + \tau_3 = v - k - 1 = 14,$$

$$2\tau_2 + 3\tau_3 = k(k-1) - 2\Delta = 20,$$

$$\sigma_2 + 6\sigma_4 + 10\sigma_5 + \tau_2 + 3\tau_3 = \Xi = 20,$$

which has three solutions:

(i) $\sigma_5 = 1,\ \sigma_4 = 0,\ \sigma_2 = 0,\ \sigma_1 = 5,\ \tau_3 = 0,\ \tau_2 = 10,\ \tau_0 = 4$.
(ii) $\sigma_5 = 0,\ \sigma_4 = 1,\ \sigma_2 = 1,\ \sigma_1 = 4,\ \tau_3 = 2,\ \tau_2 = 7,\ \tau_0 = 5$.
(iii) $\sigma_5 = 0,\ \sigma_4 = 0,\ \sigma_2 = 4,\ \sigma_1 = 2,\ \tau_3 = 4,\ \tau_2 = 4,\ \tau_0 = 6$.

By looking at out vector representation we see that if there is a vertex for which we are in case (ii), then there are vertices (those represented by vectors opposite to the vector representing our original vertex) for which the (σ,τ)-system does not hold. Similarly, if there is a vertex for which we are in case (iii), then there must be vertices for which we are in case (i).

Thus we may assume that there is a vertex x for which we are in case (i). Let y be the vertex adjacent to x with $A^2_{xy} = 5$, then the other neighbours of x and y are the same, say $1,2,3,4$ and 5. Now $A^2_{xi} = 1$ for $i = 1,\dots,5$, so $C_{xi} = -2$, and i and j are not adjacent, for all $i,j = 1,\dots,5$. From the principal submatrix of C on vertices x,i and j it follows that $C_{ij} = 2$, but then $A^2_{ij} = 3$, so besides x and y, i and j only have one common neighbour. This implies that we can identify the 10 vertices z not adjacent to x such that $A^2_{xz} = 2$ with the pairs $\{i,j\}$, $i,j = 1,\dots,5$, $i \neq j$, in such a way that i and j are adjacent to $\{i,j\}$. From the principal submatrix of C on vertices $x, i, \{j,k\}$, with $i \neq j$, k, it follows that $C_{i\{j,k\}} = -1$, and so $A^2_{i\{j,k\}} = 0$. This implies that the subgraph on the pairs $\{i,j\}$, $i,j = 1,\dots,5$ is empty, so that all 10 pairs must be adjacent to the remaining four vertices, which is a contradiction. Thus, we may conclude that there is no graph with the given spectrum. □

6. Computer results

When the existence or full classification of a graph with a given set of four eigenvalues could not be determined without the use of a computer, we used basically the same methods as [7]. In some situations, however, it was still not possible to classify the graphs completely on account of the vast amount of CPU time required. Indeed, in some cases we were unable to discover whether or not a graph existed at all.

As before, we used two programs to determine the graphs, one for the case when all the eigenvalues were integral and another when only two of the eigenvalues were integers. Both methods, however, had a common element, which we now briefly describe. Let A be the adjacency matrix of a graph G on v vertices. In our (backtracking) search for zero-one (symmetric) matrices A with four eigenvalues we have to ensure, as far as possible, that we avoid a path of the search tree that would yield an isomorphic copy of a graph that had already been discovered. This we did by demanding that the matrix A be in *standard form*. Thus, A is the greatest adjacency matrix amongst the adjacency matrices of all graphs isomorphic to G. Here the ordering involved is the lexicographical one on the binary integer obtained by concatenating the rows of the upper triangular part of A. A simple observation is that if r rows of A have been determined in the form

$$\begin{pmatrix} A_r & N_r \\ N_r^{\mathrm{T}} & O \end{pmatrix},$$

where A_r is a principal sub-matrix of A of order r, then this matrix itself must be in standard form. Testing to see that this was the case was very efficient, at least for small values of r.

To use the eigenvalues of the given graph in our search, observe that, as outlined in Section 3.1, we can use the eigenvalue k and two of the other three eigenvalues to determine constants a, b, c and d such that the matrix B defined by

$$B = aA^2 + bA + cI + dJ$$

is positive semi-definite and has rank equal to the multiplicity, ρ say, of the remaining eigenvalue. This can be done for each of the three possible choices of two eigenvalues. The matrix B then will have two eigenvalues, the non-zero one of which we denote by θ. It is then clear that every principal sub-matrix of B must have rank at most ρ and have eigenvalues, all of which lie between 0 and θ. Thus, for all $r \leqslant v$ the matrix B_r defined by

$$B_r = a(A_r^2 + N_r N_r^{\mathrm{T}}) + bA_r + cI + dJ$$

must satisfy these two conditions.

The method used for testing the rank of B_r depended on whether the constants a, b, c and d above could all be chosen as integers. This is certainly the case when all the eigenvalues of G are integers, but not always so when only two of the eigenvalues are integral. For details as to how the methods differed the reader is referred to [7]. In the cases when ρ was small (no more than 5) testing the rank condition was reasonably efficient, and when used in conjunction with the above bounds on the eigenvalues of B_r, it generally enabled the computer search to be completed. However, other determining factors in the completion of the computer searches were the number of triangles and upper bounds to λ and μ (the number of common neighbours of two adjacent and non-adjacent vertices, respectively). The smaller these numbers were, the more likely it was that the classification was feasible, but this was not always so. Overall, a complete classification was achieved in all but 30 cases, 18 of which were graphs with four integral eigenvalues, and 11 of which were graphs with two integral eigenvalues. In 13 of these 30 cases we used other computer methods to obtain a partial classification, and there remain 15 sets of four eigenvalues for which the existence of a graph (on at most 30 vertices) is still in doubt. We briefly describe the methods used in this situation.

First we mention that an incomplete search in case 162 found 1487 graphs. In fact these are all graphs having a (Hoffman-)clique of size six, which can be partitioned into three pairs of vertices such that for each pair there are eight vertices outside the clique that are adjacent to both its vertices and to no other vertices of the clique. The induced subgraph on the vertices outside the clique is then a graph of case 91, and each and every one of the 28 graphs of this case actually appears in this way.

One method of establishing the existence of some graphs with four integral eigenvalues is to take a strongly regular graph and remove a clique or a co-clique of appropriate size, as in Section 4.3. This we did by utilising the strongly regular graphs with parameters (36, 15, 6, 6), (36, 21, 12, 12) and (35, 18, 9, 9), and their complements, that were found in [13], where a partial classification of regular two-graphs on 36 vertices was made. This enabled the quoted lower bounds to be obtained for numbers 110, 112, 157, 164 and 170. In the case of number 157 the initial lower bound found was later increased by *switching* (see below). It seems rather surprising that of the 32 548 graphs found in [13] with parameters (36, 14, 4, 6) only two possess a co-clique of size 8. These two give rise to the lower bound for number 110.

It might be of interest at this stage to point out that the 11 graphs that were found in case 130 *all* come from graphs co-spectral with GQ(3,3) by the removal of a 10-coclique. This was established by examining the 27 strongly regular graphs with parameters (40,12,2,4) that were obtained in [12].

In Proposition 2 there were described arithmetical conditions on the eigenvalues of a graph (with four integral eigenvalues) which, if satisfied, might lead to the possibility of constructing from the graph further graphs with four eigenvalues, but with (possibly) different spectra. We have examined all the graphs with four integral eigenvalues (on at most 30 vertices) and have noted the ones that satisfy this condition. Those that might be switchable are (44,49,55), (48,54), (70,78,83), (71,76,93), (72,91), (75,86), (110,111,112), (137,168), (138,148), (144,159,161), (147,155), (149,163), (157,170) and (158,166). Of course, we can only use the idea of Proposition 2 to possible advantage if we know of the existence of at least one graph belonging to the above pairs or triples. Some we had already worked out by virtue of our exhaustive search, and others yielded no information, as in the case (149,163). Nevertheless, we were able to use the triples (71,76,93), (144,159,161) and (110,111,112) to advantage to produce new graphs. In the first of these, the four graphs that were found by exhaustive search in the case of number 71 were switchable into five and sixteen new ones corresponding to numbers 76 and 93, respectively. We had been unable to produce any graphs at all in these two cases using our exhaustive search. However, the transitive graphs of McKay and Royle [10] contained among them five graphs all of which were among the ones found by switching (two in case 76 and three in case 93). In the case of the triple (110,111,112) the 8472 graphs from number 112 could be switched into 10 350 graphs with spectrum that of number 111 (all new) and the two graphs of number 110 (which had already been found). Finally, the three graphs of number 144 were switchable into 50 graphs corresponding to number 159 of the triple (144,159,161). No information was obtained about number 161. Although we had already found graphs in cases 157 and 170, we discovered that by switching the 24 931 graphs of number 170 we could increase the number of graphs of number 157 from the 66 986 found from $L_3(6)\backslash$6-coclique, to 68 876.

Overall there were so many graphs found that it is not possible to list them all in this paper. However, in the cases where there were graphs discovered by the computer that were not already known, we list one graph as a representative of these new graphs. Further, all new *transitive* graphs are also listed.

In Appendix A.1, where there is a table of feasible spectra for regular graphs with four integral eigenvalues, the numbers for which a computer was needed to establish the existence of at least one graph, are 37, 69, 70, 71, 76, 78, 83, 93, 100, 110, 111, 130, 144, 157, 159, 164, and 170. In all of cases except 70, 100, 110 and 111, there are transitive graphs. Indeed, in some instances there are several such graphs. For example, number 78 has four transitive graphs, number 83 and 93 have three, and numbers 71, 76, 157 and 170 have two each. New graphs were also discovered with spectra numbered 39, 44, 48, 49, 54, 72, 91, 104, 112, 138, 148, and 154. In each of these cases, at least one graph was already known. The corresponding graphs are listed at the end of

Appendix A.1. The graphs corresponding to numbers 55, 97, 99, 125 and 129 are already contained in [7] and as a consequence are not included here.

For spectra containing only two integral eigenvalues, there was precisely one number in the listing of Appendix A.2, namely 23, for which the computer established the existence of a graph when none was already known. In this case there is a unique graph, and it too is transitive. New graphs were also found for spectra numbered 25 and 28. Numbers 45 and 54 were considered in [7], where the quoted lower bounds were obtained. However, none of the graphs coming from number 45 was listed there, and so, for the sake of completeness, we include here those transitive graphs found. The graphs in all these cases are given in Appendix A.2.

Corresponding to each graph listed there are two lines of data. The first contains the *hexadecimal* form of the graph. This is obtained by expressing the binary integer, derived by concatenating the rows of the upper triangular part of the adjacency matrix, as a hexadecimal integer. The next line contains the order of the automorphism group of the graph, together with its orbits, unless the graph is transitive.

The reader who wishes to have copies of any, or indeed all, of the graphs found in the above investigation, may obtain them by accessing the second author's home page on the Internet at http://gauss.maths.gla.ac.uk/~ted/.

Appendix A

In this appendix we list all feasible spectra for connected regular graphs with four eigenvalues and at most 30 vertices. If both the spectrum and its complementary spectrum correspond to connected graphs then only the one with least degree is mentioned. # denotes the number of graphs. In between brackets the number of such graphs or their complements that are a relation in a three-class association scheme is denoted (if positive). The references refer to the sections or the literature.

A.1. Four integral eigenvalues

No.	v	Spectrum	Δ	Ξ	#	Notes	References
1	6	$\{[\ 2]^1, [1]^2, [-1]^2, [\ -2]^1\}$	0	0	1 (1)	C_6	4.1
2	8	$\{[\ 5]^1, [1]^2, [-1]^4, [\ -3]^1\}$	6	22	1 (1)	$\bar{G}=2C_4$	3.2
3	8	$\{[\ 3]^1, [1]^3, [-1]^3, [\ -3]^1\}$	0	3	1 (1)	Cube	4.1
4	10	$\{[\ 4]^1, [1]^4, [-1]^4, [\ -4]^1\}$	0	12	1 (1)	IG(5, 4, 3)	4.1
5	12	$\{[\ 9]^1, [1]^3, [-1]^6, [\ -3]^2\}$	28	204	1 (1)	$\bar{G}=3C_4$	3.2
6	12	$\{[\ 8]^1, [2]^2, [-1]^8, [\ -4]^1\}$	19	123	1 (1)	$\bar{G}=2K_{3,3}$	3.2
7	12	$\{[\ 4]^1, [2]^3, [\ 0]^3, [\ -2]^5\}$	2	2	2	L(Cube), BCS$_9$	3.3
8	12	$\{[\ 7]^1, [1]^4, [-1]^6, [\ -5]^1\}$	9	81	1 (1)	$\bar{G}=2$CP(3)	3.2
9	12	$\{[\ 5]^1, [1]^5, [-1]^5, [\ -5]^1\}$	0	30	1 (1)	$IG(6,5,4)$	4.1
10	12	$\{[\ 5]^1, [1]^3, [\ 0]^6, [\ -4]^2\}$	0	25	0	$\lambda_1=1$	3.3
11	12	$\{[\ 5]^1, [1]^6, [-1]^2, [\ -3]^3\}$	2	14	1	GQ(2,2)\3-clique $\overline{L(\mathrm{CP}(3))}$	3.3, 4.3

No.	v	Spectrum	Δ	Ξ	#	Notes	References
12	12	$\{[5]^1, [3]^2, [-1]^8, [-3]^1\}$	6	14	1	$C_6 \circledast J_2$	4.2
13	12	$\{[5]^1, [2]^2, [1]^3, [-2]^6\}$	4	9	1 (1)	$L(K_{3,4})$	3.3
14	14	$\{[6]^1, [1]^6, [-1]^6, [-6]^1\}$	0	60	1 (1)	IG(7, 6, 5)	4.1
15	15	$\{[4]^1, [2]^5, [-1]^4, [-2]^5\}$	2	0	1 (1)	L(Petersen)	3.3, 4.1
16	15	$\{[4]^1, [3]^3, [-1]^9, [-2]^2\}$	4	4	0	$\lambda_3 = -2$	3.3
17	15	$\{[6]^1, [1]^6, [0]^5, [-4]^3\}$	1	36	0	$\lambda_1 = 1$	3.3
18	15	$\{[6]^1, [3]^2, [1]^4, [-2]^8\}$	7	20	1 (1)	$L(K_{3,5})$	3.3
19	16	$\{[13]^1, [1]^4, [-1]^8, [-3]^3\}$	66	738	1 (1)	$\bar{G} = 4C_4$	3.2
20	16	$\{[11]^1, [3]^2, [-1]^{12}, [-5]^1\}$	39	367	1 (1)	$\bar{G} = 2K_{4,4}$	3.2
21	16	$\{[6]^1, [4]^2, [0]^6, [-2]^7\}$	9	27	0	$\lambda_3 = -2$	3.3
22	16	$\{[9]^1, [1]^6, [-1]^8, [-7]^1\}$	12	204	1 (1)	$\bar{G} = 2\mathrm{CP}(4)$	3.2
23	16	$\{[7]^1, [1]^7, [-1]^7, [-7]^1\}$	0	105	1 (1)	IG(8, 7, 6)	4.1
24	16	$\{[7]^1, [1]^8, [-1]^5, [-5]^2\}$	3	69	0	$\lambda_1 = 1$	3.3
25	16	$\{[7]^1, [3]^3, [-1]^{11}, [-5]^1\}$	9	57	1	Cube$\circledast J_2$	4.2
26	18	$\{[14]^1, [2]^3, [-1]^{12}, [-4]^2\}$	73	894	1 (1)	$\bar{G} = 3k_{3,3}$	3.2
27	18	$\{[13]^1, [1]^6, [-1]^9, [-5]^2\}$	54	666	1 (1)	$\bar{G} = 3CP(3)$	3.2
28	18	$\{[13]^1, [1]^8, [-2]^8, [-5]^1\}$	56	652	1 (1)	$\bar{G} = 2L_2(3)$	3.2
29	18	$\{[5]^1, [2]^6, [-1]^9, [-4]^2\}$	1	1c2	1	$K_{3,3} \oplus K_3$	4.2
30	18	$\{[5]^1, [2]^7, [-1]^1, [-2]^9\}$	3	2	0	$\lambda_3 = -2$	3.3
31	18	$\{[11]^1, [2]^4, [-1]^{12}, [-7]^1\}$	28	360	1 (1)	$\bar{G} = 2K_{3,3,3}$	3.2
32	18	$\{[6]^1, [3]^4, [0]^4, [-2]^9\}$	7	16	1	$L(L_2(3))$	3.3
33	18	$\{[7]^1, [1]^{11}, [-2]^4, [-5]^2\}$	2	58	1	$\overline{\mathrm{BCS}_{179}}$	3.3
34	18	$\{[7]^1, [4]^2, [1]^5, [-2]^{10}\}$	11	40	2 (1)	$L(K_{3,6})$, BCS_{70}	3.3
35	18	$\{[8]^1, [1]^8, [-1]^8, [-8]^1\}$	0	168	1 (1)	IG(9, 8, 7)	4.1
36	18	$\{[8]^1, [2]^7, [-2]^9, [-4]^1\}$	12	68	0		3.1
37	18	$\{[8]^1, [2]^6, [-1]^8, [-4]^3\}$	10	78	2		6
38	18	$\{[8]^1, [5]^2, [-1]^{14}, [-4]^1\}$	19	96	1	$C_6 \circledast J_3$	4.2
39	18	$\{[8]^1, [2]^4, [0]^9, [-4]^4\}$	8	84	3 (1)	$L_2(3) \otimes J_2$	4.2, 6
40	18	$\{[8]^1, [4]^3, [-1]^8, [-2]^6\}$	18	78	0	$\lambda_3 = -2$	3.3
41	20	$\{[17]^1, [1]^5, [-1]^{10}, [-3]^4\}$	120	1816	1 (1)	$\bar{G} = 5C_4$	3.2
42	20	$\{[16]^1, [1]^8, [-2]^{10}, [-4]^1\}$	99	1401	1 (1)	$\bar{G} = 2$Petersen	3.2
43	20	$\{[14]^1, [4]^2, [-1]^{16}, [-6]^1\}$	66	817	1 (1)	$\bar{G} = 2K_{5,5}$	3.2
44	20	$\{[6]^1, [2]^5, [0]^{10}, [-4]^4\}$	0	27	2 (1)	Petersen$\otimes J_2$	4.2, 6
45	20	$\{[6]^1, [3]^4, [1]^4, [-2]^{11}\}$	6	12	1	L(IG(5, 4, 3))	3.3
46	20	$\{[13]^1, [1]^{10}, [-2]^8, [-7]^1\}$	45	615	1 (1)	$\bar{G} = \overline{2\mathrm{Petersen}}$	3.2
47	20	$\{[7]^1, [2]^4, [0]^{12}, [-5]^3\}$	0	63	0		6
48	20	$\{[7]^1, [2]^8, [-1]^5, [-3]^6\}$	6	30	9	SR(26, 10, 3, 4)\6-cocl. Dodecahedron$_{3,5}$	4.1, 4.3, 6
49	20	$\{[7]^1, [3]^5, [-1]^{10}, [-3]^4\}$	9	33	4 (1)	Petersen$\circledast J_2$	4.2, 6
50	20	$\{[7]^1, [3]^3, [2]^4, [-2]^{12}\}$	9	27	1 (1)	$L(K_{4,5})$	3.3
51	20	$\{[11]^1, [1]^8, [-1]^{10}, [-9]^1\}$	15	415	1 (1)	$\bar{G} = 2\mathrm{CP}(5)$	3.2
52	20	$\{[9]^1, [1]^9, [-1]^9, [-9]^1\}$	0	252	1 (1)	IG(10, 9, 8)	4.1
53	20	$\{[9]^1, [3]^4, [-1]^{14}, [-7]^1\}$	12	156	1	IG(5, 4, 3))$\circledast J_2$	4.2
54	20	$\{[9]^1, [2]^8, [-1]^4, [-3]^7\}$	15	105	26	$L_3(5)$\5-coclique	4.3, 6
55	20	$\{[9]^1, [3]^5, [-1]^9, [-3]^5\}$	18	108	9 (1)	$J(6, 3)$	4.1, [7]
56	21	$\{[6]^1, [3]^5, [-1]^{13}, [-4]^2\}$	5	20	0		5
57	21	$\{[8]^1, [5]^2, [1]^6, [-2]^{12}\}$	16	72	1 (1)	$L(K_{3,7})$	3.3
58	22	$\{[10]^1, [1]^{10}, [-1]^{10}, [-10]^1\}$	0	360	1 (1)	IG(11, 10, 9)	4.1
59	24	$\{[21]^1, [1]^6, [-1]^{12}, [-3]^5\}$	190	3630	1 (1)	$\bar{G} = 6C_4$	3.2
60	24	$\{[20]^1, [2]^4, [-1]^{16}, [-4]^3\}$	163	2961	1 (1)	$\bar{G} = 4K_{3,3}$	3.2

No.	v	Spectrum	Δ	Ξ	#	Notes	References
61	24	$\{[19]^1, [3]^3, [-1]^{18}, [-5]^2\}$	139	2395	1 (1)	$\bar{G}=3K_{4,4}$	3.2
62	24	$\{[19]^1, [1]^8, [-1]^{12}, [-5]^3\}$	135	2403	1 (1)	$\bar{G}=4\mathrm{CP}(3)$	3.2
63	24	$\{[5]^1, [3]^6, [-1]^{14}, [-3]^3\}$	4	6	1	2-cover $C_6 \circledast J_2$	4.4, 6
64	24	$\{[5]^1, [2]^8, [0]^8, [-3]^7\}$	0	5	0		6
65	24	$\{[17]^1, [5]^2, [-1]^{20}, [-7]^1\}$	100	1536	1 (1)	$\bar{G}=2K_{6,6}$	3.2
66	24	$\{[6]^1, [4]^4, [0]^8, [-2]^{11}\}$	8	19	0	$\lambda_3=-2$	3.3
67	24	$\{[17]^1, [1]^9, [-1]^{12}, [-7]^2\}$	88	1560	1 (1)	$\bar{G}=3\mathrm{CP}(4)$	3.2
68	24	$\{[7]^1, [3]^6, [-1]^{15}, [-5]^2\}$	5	41	0		6
69	24	$\{[7]^1, [3]^3, [1]^{11}, [-3]^9\}$	4	25	5		6
70	24	$\{[8]^1, [2]^{11}, [-2]^9, [-4]^3\}$	7	48	5		6
71	24	$\{[8]^1, [2]^8, [0]^9, [-4]^6\}$	4	60	4		6
72	24	$\{[8]^1, [4]^3, [0]^{15}, [-4]^5\}$	8	68	5	$L(\mathrm{Cube})\otimes J_2$, $\mathrm{BCS}_9\otimes J_2$	6
73	24	$\{[15]^1, [3]^4, [-1]^{18}, [-9]^1\}$	57	981	1 (1)	$\bar{G}=2K_{4,4,4}$	3.2
74	24	$\{[8]^1, [4]^3, [2]^5, [-2]^{15}\}$	13	48	1 (1)	$L(K_{4,6})$	3.3
75	24	$\{[9]^1, [1]^{17}, [-3]^2, [-5]^4\}$	4	116	1	GQ(2,4)\3-cl, $\overline{\mathrm{BCS}_{183}}$	3.3, 4.3
76	24	$\{[9]^1, [3]^7, [-1]^9, [-3]^7\}$	15	84	$\geqslant 5$		6
77	24	$\{[14]^1, [2]^6, [-1]^{16}, [-10]^1\}$	37	822	1 (1)	$\bar{G}=2K_{3,3,3,3}$	3.2
78	24	$\{[9]^1, [3]^4, [1]^9, [-3]^{10}\}$	12	84	87		6
79	24	$\{[9]^1, [6]^2, [1]^7, [-2]^{14}\}$	22	119	1 (1)	$L(K_{3,8})$	3.3
80	24	$\{[10]^1, [2]^3, [0]^{18}, [-8]^2\}$	0	285	0		5
81	24	$\{[10]^1, [1]^{16}, [-2]^4, [-6]^3\}$	7	196	0	$\lambda_1=1$	3.3
82	24	$\{[10]^1, [4]^2, [0]^{18}, [-6]^3\}$	10	205	0		5
83	24	$\{[10]^1, [2]^{11}, [-2]^8, [-4]^4\}$	16	141	183		6
84	24	$\{[10]^1, [4]^5, [0]^3, [-2]^{15}\}$	25	145	0	$\lambda_3=-2$	3.3
85	24	$\{[13]^1, [1]^{10}, [-1]^{12}, [-11]^1\}$	18	738	1 (1)	$\bar{G}=2\mathrm{CP}(6)$	3.2
86	24	$\{[10]^1, [4]^4, [2]^3, [-2]^{16}\}$	24	141	9	$L(\mathrm{CP}(4))$, $\mathrm{BCS}_{153-160}$	3.3
87	24	$\{[11]^1, [1]^{11}, [-1]^{11}, [-11]^1\}$	0	495	1 (1)	IG(12,11,10)	4.1
88	24	$\{[11]^1, [3]^5, [-1]^{17}, [-9]^1\}$	15	335	1	IG(6,5,4) $\circledast J_2$	4.2
89	24	$\{[11]^1, [5]^3, [-1]^{19}, [-7]^1\}$	28	279	1	Cube$\circledast J_3$	4.2
90	24	$\{[11]^1, [1]^{16}, [-1]^2, [-5]^5\}$	15	255	0	$\lambda_1=1$	3.3
91	24	$\{[11]^1, [3]^6, [-1]^{14}, [-5]^3\}$	23	239	28	(GQ(2,2)\3-clique)$\circledast J_2$	4.2, 6
92	24	$\{[11]^1, [7]^2, [-1]^{20}, [-5]^1\}$	39	303	1	$C_6 \circledast J_4$	4.2
93	24	$\{[11]^1, [3]^7, [-1]^8, [-3]^8\}$	27	215	$\geqslant 16$		6
94	25	$\{[10]^1, [5]^2, [0]^{18}, [-5]^4\}$	15	180	0		5
95	26	$\{[12]^1, [1]^{12}, [-1]^{12}, [-12]^1\}$	0	660	1 (1)	IG(13,12,11)	4.1
96	27	$\{[22]^1, [1]^{12}, [-2]^{12}, [-5]^2\}$	191	3892	1 (1)	$\bar{G}=3L_2(3)$	3.2
97	27	$\{[6]^1, [3]^6, [0]^{12}, [-3]^8\}$	3	12	4 (1)	$H(3,3)$, 3-cover $C_3\otimes J_3$	4.1, 4.4, [7]
98	27	$\{[20]^1, [2]^6, [-1]^{18}, [-7]^2\}$	136	2664	1 (1)	$\bar{G}=3K_{3,3,3}$	3.2
99	27	$\{[8]^1, [2]^{12}, [-1]^8, [-4]^6\}$	4	48	13 (3)	GQ(2,4)\ spread (2×) $H(3,3)_3$, GQ$(3,3)_2(x)$	4.1, 4.3, [7]
100	27	$\{[8]^1, [5]^4, [-1]^{20}, [-4]^2\}$	16	72	1		6
101	27	$\{[10]^1, [4]^6, [1]^2, [-2]^{18}\}$	23	124	1	$L(K_{3,3,3})$	3.3
102	27	$\{[10]^1, [7]^2, [1]^8, [-2]^{16}\}$	29	184	1 (1)	$L(K_{3,9})$	3.3
103	27	$\{[12]^1, [3]^2, [0]^{22}, [-9]^2\}$	6	492	0		5
104	27	$\{[12]^1, [3]^4, [0]^{18}, [-6]^4\}$	18	348	5 (1)	$L_2(3)\otimes J_3$	4.2, 6
105	27	$\{[12]^1, [3]^8, [0]^6, [-3]^{12}\}$	30	276	$\geqslant 1$ (1)	$H(3,3)_2$	4.1
106	28	$\{[25]^1, [1]^7, [-1]^{14}, [-3]^6\}$	276	6372	1 (1)	$\bar{G}=7C_4$	3.2
107	28	$\{[6]^1, [5]^4, [-1]^{20}, [-2]^3\}$	12	36	0	$\lambda_3=-2$	3.3
108	28	$\{[20]^1, [6]^2, [-1]^{24}, [-8]^1\}$	141	2587	1 (1)	$\bar{G}=2K_{7,7}$	3.2
109	28	$\{[9]^1, [5]^3, [2]^6, [-2]^{18}\}$	18	81	1 (1)	$L(K_{4,7})$	3.3
110	28	$\{[10]^1, [2]^{14}, [-2]^7, [-4]^6\}$	12	117	$\geqslant 2$	SR(36,14,4,6)\8-cocl.	4.3, 6

No.	v	Spectrum	Δ	Ξ	#	Notes	References
111	28	$\{[11]^1, [3]^7, [\ 1]^7, [\ -3]^{13}\}$	21	175	$\geqslant$10350		6
112	28	$\{[12]^1, [2]^{14}, [-2]^6, [\ -4]^7\}$	24	270	$\geqslant$8472 (56)	$\overline{T(8)}\backslash$spread SR(35,16,6,8)\7-cocl.	4.3, 6
113	28	$\{[15]^1, [1]^{12}, [-1]^{14}, [-13]^1\}$	21	1197	1 (1)	$\bar{G}=2\text{CP}\ (7)$	3.2
114	28	$\{[13]^1, [1]^{13}, [-1]^{13}, [-13]^1\}$	0	858	1 (1)	IG(14,13,12)	4.1
115	28	$\{[13]^1, [3]^6, [-1]^{20}, [-11]^1\}$	18	618	1	IG(7, 6, 5)$\circledast J_2$	4.2
116	28	$\{[13]^1, [5]^5, [-1]^6, [\ -2]^{16}\}$	48	408	0	$\lambda_3=-2$	3.3
117	30	$\{[26]^1, [2]^5, [-1]^{20}, [\ -4]^4\}$	289	6972	1 (1)	$\bar{G}=5K_{3,3}$	3.2
118	30	$\{[26]^1, [1]^{12}, [-2]^{15}, [\ -4]^2\}$	289	6966	1 (1)	$\bar{G}=3$Petersen	3.2
119	30	$\{[25]^1, [1]^{10}, [-1]^{15}, [\ -5]^4\}$	252	5940	1 (1)	$\bar{G}=5\text{CP}(3)$	3.2
120	30	$\{[24]^1, [4]^3, [-1]^{24}, [\ -6]^2\}$	226	5022	1 (1)	$\bar{G}=-3K_{5,5}$	3.2
121	30	$\{[\ 6]^1, [2]^9, [\ 1]^9, [\ -3]^{11}\}$	0	6	0		5
122	30	$\{[23]^1, [2]^{10}, [-2]^{18}, [\ -7]^1\}$	196	4194	1 (1)	$\bar{G}=2\text{GQ}(2,2)$	3.2
123	30	$\{[\ 6]^1, [3]^8, [\ 1]^4, [\ -2]^{17}\}$	5	4	0	$\lambda_3=-2$	3.3
124	30	$\{[23]^1, [1]^{15}, [-2]^{12}, [\ -7]^2\}$	190	4230	1 (1)	$\bar{G}=3\overline{\text{Petersen}}$	3.2
125	30	$\{[\ 7]^1, [2]^{14}, [-2]^{14}, [\ -7]^1\}$	0	42	4 (4)	IG(15,7,3)	4.1, [7]
126	30	$\{[\ 7]^1, [2]^{15}, [-2]^5, [\ -3]^9\}$	3	12	0		5
127	30	$\{[\ 7]^1, [4]^5, [\ 0]^{15}, [\ -3]^9\}$	7	28	0		6
128	30	$\{[\ 7]^1, [\ 2]^{12}, [\ 1]^5, [-3]^{12}\}$	2	14	0		5
129	30	$\{[\ 8]^1, [\ 2]^{14}, [-2]^{14}, [-8]^1\}$	0	84	4 (4)	IG(15, 8, 4)	4.1, [7]
130	30	$\{[\ 8]^1, [\ 2]^{15}, [-2]^9, [-4]^5\}$	4	36	11	GQ(3, 3)\10-coclique	4.3, 6
131	30	$\{[\ 8]^1, [\ 3]^9, [-1]^{15}, [-4]^5\}$	7	42	0		6
132	30	$\{[\ 8]^1, [\ 4]^7, [-1]^8, [-2]^{14}\}$	14	42	0	$\lambda_3=-2$	3.3
133	30	$\{[21]^1, [\ 1]^{12}, [-1]^{15}, [-9]^2\}$	130	3030	1 (1)	$\overline{G}=3\text{CP}(5)$	3.2
134	30	$\{[\ 8]^1, [\ 4]^5, [\ 2]^5, [-2]^{19}\}$	12	36	1	$L(\text{IG}(6,5,4))$	3.3
135	30	$\{[21]^1, [\ 1]^{18}, [-3]^{10}, [-9]^1\}$	138	2934	1 (1)	$\overline{G}=\overline{2\text{GQ}(2,2)}$	3.2
136	30	$\{[\ 9]^1, [\ 3]^8, [-1]^{19}, [-7]^2\}$	4	124	0		5
137	30	$\{[\ 9]^1, [\ 4]^6, [-1]^{21}, [-6]^2\}$	11	102	0		5
138	30	$\{[\ 9]^1, [\ 3]^5, [\ 0]^{20}, [-6]^4\}$	0	126	2 (1)	Petersen$\otimes J_3$	4.2, 6
139	30	$\{[\ 9]^1, [\ 4]^4, [\ 0]^{20}, [-5]^5\}$	6	102	0		6
140	30	$\{[\ 9]^1, [\ 3]^{10}, [-1]^9, [-3]^{10}\}$	12	60	$\geqslant$ 17		6
141	30	$\{[\ 9]^1, [\ 5]^5, [-1]^{19}, [-3]^5\}$	20	92	1	L(Petersen)$\circledast J_2$	4.2, 6
142	30	$\{[\ 9]^1, [\ 7]^3, [-1]^{24}, [-3]^2\}$	28	156	0		5
143	30	$\{[\ 9]^1, [\ 4]^4, [\ 3]^5, [-2]^{20}\}$	16	62	1 (1)	$L(K_{5,6})$	3.3
144	30	$\{[10]^1, [\ 2]^{15}, [-2]^{10}, [-5]^4\}$	9	120	3		6
145	30	$\{[19]^1, [\ 4]^4, [-1]^{24}, [-11]^1\}$	96	2082	1 (1)	$\overline{G}=2K_{5,5,5}$	3.2
146	30	$\{[10]^1, [\ 5]^4, [\ 2]^5, [-2]^{20}\}$	23	120	1	$L(\overline{\text{Petersen}})$	3.3
147	30	$\{[11]^1, [\ 2]^{16}, [-3]^9, [-4]^4\}$	16	162	0		6
148	30	$\{[11]^1, [\ 5]^5, [-1]^{20}, [-4]^4\}$	28	198	8 (1)	Petersen$\circledast J_3$	4.2, 6
149	30	$\{[11]^1, [\ 2]^{10}, [\ 1]^9, [-4]^{10}\}$	13	174	?		
150	30	$\{[11]^1, [\ 6]^4, [-1]^{20}, [-3]^5\}$	34	222	0		6
151	30	$\{[11]^1, [\ 5]^5, [\ 1]^4, [-2]^{20}\}$	30	186	0	$\lambda_3=-2$	3.3
152	30	$\{[11]^1, [\ 8]^2, [\ 1]^9, [-2]^{18}\}$	37	270	1 (1)	$L(K_{3,10})$	3.3
153	30	$\{[12]^1, [\ 2]^6, [\ 0]^{20}, [-8]^3\}$	4	414	0		6
154	30	$\{[12]^1, [\ 2]^9, [\ 0]^{15}, [-6]^5\}$	12	318	2 (1)	GQ(2, 2)$\otimes J_2$	4.2, 6
155	30	$\{[12]^1, [\ 2]^{16}, [-3]^8, [-4]^5\}$	22	244	?		
156	30	$\{[12]^1, [\ 2]^{14}, [\ 0]^5, [-4]^{10}\}$	20	254	?		
157	30	$\{[12]^1, [\ 3]^{10}, [\ 0]^5, [-3]^{14}\}$	27	240	$\geqslant$ 68876	$L_3(6)$\6-coclique	4.3, 6
158	30	$\{[17]^1, [\ 2]^8, [-1]^{20}, [-13]^1\}$	46	1590	1 (1)	$\overline{G}=2K_{3,3,3,3,3}$	3.2
159	30	$\{[12]^1, [\ 4]^5, [\ 1]^{10}, [-3]^{14}\}$	28	248	$\geqslant$ 50		6
160	30	$\{[13]^1, [\ 1]^{20}, [-1]^5, [-7]^4\}$	14	474	0	$\lambda_1=1$	3.3

No.	v	Spectrum	Δ	Ξ	#	Notes	References
161	30	$\{[13]^1, [\ 2]^{15}, [-2]^9, [-5]^5\}$	27	372	?		
162	30	$\{[13]^1, [\ 3]^9, [-1]^{15}, [-5]^5\}$	30	378	$\geqslant$ 1487 (1)	GQ(2,2) $\circledast J_2$	4.2, 6
163	30	$\{[13]^1, [\ 3]^{11}, [-2]^8, [-3]^{10}\}$	36	344	?		
164	30	$\{[13]^1, [\ 3]^9, [\ 1]^5, [-3]^{15}\}$	34	346	$\geqslant$ 82	$L_3(6)\backslash$6-clique	4.3, 6
165	30	$\{[14]^1, [\ 1]^{14}, [-1]^{14}, [-14]^1\}$	0	1092	1 (1)	IG(15, 14, 13)	4.1
166	30	$\{[14]^1, [\ 2]^9, [-1]^{19}, [-13]^1\}$	10	930	0		3.1
167	30	$\{[14]^1, [\ 5]^4, [-1]^{24}, [-10]^1\}$	37	660	1	IG(5, 4, 3) $\circledast J_3$	4.2
168	30	$\{[14]^1, [\ 4]^6, [-1]^{20}, [-6]^3\}$	41	542	?		
169	30	$\{[14]^1, [\ 9]^2, [-1]^{26}, [-6]^1\}$	66	692	1	$C_6 \circledast J_5$	4.2
170	30	$\{[14]^1, [\ 2]^{15}, [-1]^4, [-4]^{10}\}$	37	498	$\geqslant$ 24931	SR(35, 16, 6, 8)$\backslash$5-cl.	4.3, 6

No.	v	Hexadecimal form/orbits	**|Aut|**
37	18	FF0078E00FC024F09784F84CCB339878F3D4A90	**432**
39	18	FF0078E0093C24F096696670DCCE7818C3DFE00 (1 2 7 8 9 10 11 12) (3 4 5 6 15 16 17 18) (13 14)	**512**
44	20	FC0000F8007C0043C0878111C88E00C0C049A4DB00063C28 (1 8 9 12 13 16 19 20) (2 3 4 5 6 7 10 11 14 15 17 18)	**3072**
48	20	FE00187802238112611 0B228C1A528B49A9984D154D52C80 (1 5 6 9 10 11 12 13 15 17) (2 3 4 7 8 14 16 18 19 20)	**20**
49	20	FE001F80043C003C0207801E100F00F998664CCCD52AA548 (1 2 3 4) (5 6 7 8 9 10 11 12) (13 14 15 16 17 18 19 20)	**1024**
54	20	FF801E1E04C0F00CCD7513AA299C66C0B653B5BA469CB748	**40**
69	24	FE000187800223802207004 01C0924C2495124B000E48C92A89D070AA58C2124604A0	**1152**
70	24	FF0001E3800003F00030F001998019FCC03A30C21E1E01CC303C1542A892610A52592 (1 2 7 8 9 10 11 12) (3 4 5 6 13 14 15 16 17 18 19 20 21 22 23 24)	**384**
71	24	FF000103F000007E21C61006394968824AAA91A2D8827088D254416039892224 5044F FF000183E00400F800007E318C0318C72C1583C0F2C043A4B0CC121CDA04493294C10	**1152** **144**
72	24	FF0001E3800093C0093C009660259807030700CE07801800300F1E078033331E78CDE (1 2 7 8 9 10 11 12) (3 4 5 6 19 20 21 22) (13 14 15 16 17 18 23 24)	**4096**
76	24	FF8001E1E004C8E0034983140E12827210B4296A09C11CA5A170B6A662B0A38D6256D FF8001E1E00680F0108CC10C4634047408BD21430A6383B0BA172A0E1B34C1F58E931	**144** **48**
78	24	FF8001E1E004CCC0033C0201F80061F0CCC0C33E18781E42A8156528B599662D4A58C FF8001E1E004CCC0033C0201F80061F0CCC0C33E15582AC30C0F2528B599662D4A58C FF8001C1F0018C7014A0E18C039424C911988E10E4236F12552AAA91256D8C199D119 FF8001E1E00488F0044CC24456088B8949A68662CC34C8A1A5255492C553A3292223F	**1152** **1152** **1152** **48**
83	24	FFC001E0F800C63C0A5238A52698C6632692C2759B194B04F045CCD2CBE83CCCA8859 FFC001E0F804843E04231E425B084DB36232E1CF913D29079248C8E71352E4A44F28F FFC001E0F804C03E00631D635216AA46919B26A091E8B3A85CC37518F23CBA3DA8908	**192** **384** **48**
91	24	FFE001FF800703F0303F02001F8007FCF30CF30819E067FC3FC3E661993333E79F321 FFE001F878043C3C003FC2666606799CC3C333C5554AD5296956952B4A5555861CCFF	**196608** **384**
93	24	FFE001F87807223C31133211158222FF0C30FC3F961E264CA8C57D86992A15E67FB73 FFE001F87806333812AA622AA30662BF02B0F2BC9648E1E0F093764AD99336AEEF9A9 FFE001F878063B30122CE1227C36469443757035 05F8556314F1AA5AABDC8BB4AFF13	**384** **192** **48**

No.	v	Hexadecimal form/orbits	|Aut|
100	27	FF00003F80001F800010F00003C00081E0001E00403C000F0380718071001C007E1C6 1C40701FF1F11C7C40E (1 2 3 22 23 24 25 26 27) (4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21)	**3981312**
104	27	FFF0003F07C000C63F018C7E0631F8286E3A86E3D0DC70F8007C0E7C70F8FFE3F0038 038380E07E7EFFF0000 (1 10 11 18) (2 9 12 13 14 15 16 17) (3 4 5 6 7 8 22 23 24 25 26 27) (19 20 21)	**995328**
110	28	FFC0001E0F8000843F00423CC04245708495A268592645670527C0B184C305168C2AA 4C719330C21F07A868AA498800 (1 2 11 16) (3 4 5 6 7 8 9 10 12 13 14 15 19 20 25 26) (17 18 21 22 23 24 27 28)	**32**
111	28	FFE0001F878006223E021121E011223F106C94449C228A3325A4AA954E457C8248E4A C63A0E112EB1A1C9091FB67C84 (1 2 19 20 24 25 27 28) (3 4 5 6 7 8 17 18 21 22 23 26) (9 10 11 13 14 15) (12 16)	**96**
112	28	FFF0001E03F804C183E030631F10A5B1284DB20B4E4B623A4B54670E7B226E94A8F24 52A923C71239C4D4B92913CA3C	**1344**
130	30	FF0000040FC0000007E00861070008607124124804924AA030A04A2A050229108382B 40085284692068528105C12982306882232210C0	**1440**
138	30	FF80000007F80000FF00003FC000181F80181F80303F00A083E5041F5041F00000000 E00E00118E231C8C739C39C00001C0E0EE380890 (1 11 17 18 23 24 28 29) (2 3 4 5 6 7 8 9 10 14 15 16 20 21 22 25 26 27) (12 13 19 30)	**17915904**
144	30	FFC0000703F0001801F8018061E00C0C8E3300E801872C415270A14DA160BAC0AB306 6031E4B60C6C26424C38842C0AAA0E2E450C0040	**60**
148	30	FFE00007FE00007FC0000C0FC00207E00007E001803F0200FC0007E06001FC003F000 FF1C711C7038E31C78E38E3D26499B24D6593498 (1 2 3 4 5 6) (7 8 9 10 11 12 13 14 15 16 17 18) (19 20 23 24 26 27 28 29) (21 22 25 30)	**5971968**
154	30	FFF00007E0F8000210FF00843FC0211E1E211E1E2103FC840FF0F8078F8781FF007F3 33E666001E03C0F006661998330CD99980C36600 (1 2 9 10 11 12 13 14 15 16 17 18) (3 4 5 6 7 8 19 20 21 22 23 24 25 26 27 28 29 30)	**73728**
157	30	FFF00007C0FC00661878006199C00C331DAAA092AAA1275052554152C264E0331C8CD 8899C5AA26AA4E4AA1955AA4D49188D70E3188FE	**720**
		FFF00007C0FC0064107E01C390605A262997011C4A69C0C0CCAB2B03124DC461952A9 0C229F295A8967D8141BE228E28AEE5B3901B638	**60**
159	30	FFF00007F0F0007E01E00C4447821113A0088B91C2083E428271280BCA61E36F02ACA E36C82DD02C9E0C3AB8219C439911059F5E7D61E	**60**
162	30	FFF80007FF8000700FF00C03FC020001FE0001FFC3C3C30F0F08078780787FF00FFC0 3E1FE01FE13330CCC999999C330CF9998A9552A4 (1 2 3 4) (5 6) (7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22) (23 24 25 26 27 28 29 30)	**65536**
164	30	FFF80007E07E00730C3C0C30C0F206066600199EEBA0AADB6812E0D55CDA223299D91 D08AE2362C59AADD0552E8E14D359A74A1D97D38	**240**
170	30	FFFC0007E03F8073040FC230E38C544D3603AAAF0C2CF93B0CDA8D96AD6C97DC589CD 188B6A8CC7EAE23A430789DFD45A8E3138CD53C8	**60**
		FFFC0007E03F8063860F881C631C010E7FC909ED494AB49295BE4D87526AA936A3F8C C66AD49D39F3272554D1D3519CF2724C9912E51E	**720**

A.2. Two integral eigenvalues

No.	v	Spectrum	Δ	Ξ	#	Notes	References
1	10	$\{[\ 7]^1,[-3]^1,[\ 0.618]^4,[-1.618]^4\}$	15	80	1 (1)	$\bar{G}=2C_5$	3.2
2	10	$\{[\ 4]^1,[\ 0]^5,[\ 1.236]^2,[-3.236]^2\}$	0	10	1 (1)	$C_5\otimes J_2$	4.2
3	12	$\{[\ 5]^1,[-1]^5,[\ 2.236]^3,[-2.236]^3\}$	5	10	1 (1)	Icosahedron	4.1, [6]
4	12	$\{[\ 5]^1,[\ 1]^3,[\ 0.732]^4,[-2.732]^4\}$	2	13	0	$\lambda_1=1$	3.3
5	14	$\{[\ 3]^1,[-3]^1,[\ 1.414]^6,[-1.414]^6\}$	0	0	1 (1)	IG(7,3,1)	4.1
6	14	$\{[\ 4]^1,[-4]^1,[\ 1.414]^6,[-1.414]^6\}$	0	6	1 (1)	IG(7,4,2)	4.1
7	14	$\{[\ 6]^1,[\ 0]^7,[\ 1.646]^3,[-3.646]^3\}$	3	33	0		6
8	15	$\{[12]^1,[-3]^2,[\ 0.618]^6,[-1.618]^6\}$	55	560	1 (1)	$\bar{G}=3C_5$	3.2
9	15	$\{[\ 4]^1,[\ 0]^6,[\ 1.791]^4,[-2.791]^4\}$	0	4	0		5
10	15	$\{[\ 6]^1,[-1]^{10},[\ 4.162]^2,[-2.162]^2\}$	11	32	0		5
11	15	$\{[\ 6]^1,[-1]^6,[\ 2.449]^4,[-2.449]^4\}$	7	20	0		6
12	15	$\{[\ 6]^1,[\ 0]^{10},[\ 1.854]^2,[-4.854]^2\}$	0	48	1 (1)	$C_5\otimes J_3$	4.2
13	16	$\{[\ 7]^1,[-1]^{11},[\ 4.464]^2,[-2.464]^2\}$	15	57	0		5
14	18	$\{[\ 5]^1,[\ 3]^1,[\ 1.303]^8,[-2.303]^8\}$	2	4	1	td$L_2(3)$	4.2
15	20	$\{[17]^1,[-3]^3,[\ 0.618]^8,[-1.618]^8\}$	120	1815	1 (1)	$\bar{G}=4C_5$	3.2
16	20	$\{[\ 5]^1,[-1]^5,[\ 2.236]^7,[-2.236]^7\}$	3	2	1	2-cover $C_5 \circledast J_2$	4.4, 6
17	20	$\{[\ 7]^1,[-1]^{15},[\ 5.873]^2,[-1.873]^2\}$	18	75	0	$\lambda_3 > -2$	3.3
18	20	$\{[\ 8]^1,[-2]^9,[\ 3.236]^5,[-1.236]^5\}$	15	60	0	$\lambda_3=-2$	3.3
19	20	$\{[\ 8]^1,[\ 0]^{15},[\ 2.472]^2,[-6.472]^2\}$	0	132	1 (1)	$C_5\otimes J_4$	4.2
20	20	$\{[\ 8]^1,[\ 0]^{11},[\ 2.317]^4,[-4.317]^4\}$	6	80	0		6
21	21	$\{[\ 4]^1,[-2]^8,[\ 2.414]^6,[-0.414]^6\}$	2	0	1 (1)	L(IG(7,3,1))	3.3, 4.1
22	21	$\{[\ 6]^1,[-1]^6,[\ 2.449]^7,[-2.449]^7\}$	5	10	0		6
23	21	$\{[\ 6]^1,[\ 0]^8,[\ 2.193]^6,[-3.193]^6\}$	2	16	1		6
24	21	$\{[\ 8]^1,[-1]^{14},[\ 4.742]^3,[-2.742]^3\}$	18	78	0		6
25	21	$\{[\ 8]^1,[-1]^8,[\ 2.828]^6,[-2.828]^6\}$	12	56	6 (1)	L(IG(7,3,1))$_3$	4.1, 6
26	21	$\{[\ 8]^1,[\ 1]^{12},[-0.209]^4,[-4.791]^4\}$	2	88	0	$\lambda_1=1$	3.3
27	21	$\{[\ 8]^1,[\ 1]^6,[\ 1.449]^7,[-3.449]^7\}$	6	62	0		5
28	21	$\{[\ 8]^1,[\ 2]^8,[-0.586]^6,[-3.414]^6\}$	8	60	28 (1)	L(IG(7,3,1))$_2$	4.1, 6
29	22	$\{[\ 5]^1,[-5]^1,[\ 1.732]^{10},[-1.732]^{10}\}$	0	10	1 (1)	IG(11,5,2)	4.1
30	22	$\{[\ 5]^1,[\ 0]^{11},[\ 2.372]^5,[-3.372]^5\}$	0	10	0		6
31	22	$\{[\ 6]^1,[-6]^1,[\ 1.732]^{10},[-1.732]^{10}\}$	0	30	1 (1)	IG(11,6,3)	4.1
32	22	$\{[10]^1,[\ 0]^{11},[\ 2.317]^5,[-4.317]^5\}$	15	175	0		6
33	24	$\{[\ 7]^1,[-1]^{15},[\ 4.464]^4,[-2.464]^4\}$	13	41	0		6
34	24	$\{[\ 7]^1,[-1]^7,[\ 2.646]^8,[-2.646]^8\}$	7	21	10 (1)	Klein	4.1, [7]
35	24	$\{[\ 8]^1,[\ 0]^{15},[\ 2.873]^4,[-4.873]^4\}$	3	78	0		6
36	24	$\{[\ 9]^1,[\ 1]^{15},[-0.551]^4,[-5.449]^4\}$	2	134	0	$\lambda_1=1$	3.3
37	24	$\{[\ 9]^1,[\ 1]^7,[\ 1.646]^8,[-3.646]^8\}$	8	91	1 (1)	Klein$_{1,3}$	4.1, 6
38	24	$\{[11]^1,[-1]^{17},[\ 5.472]^3,[-3.472]^3\}$	35	255	1	Icosahedron $\circledast J_2$	4.2, 6
39	25	$\{[22]^1,[-3]^4,[\ 0.618]^{10},[-1.618]^{10}\}$	210	4220	1 (1)	$\bar{G}=5C_5$	3.2
40	25	$\{[10]^1,[\ 0]^{20},[\ 3.090]^2,[-8.090]^2\}$	0	280	1 (1)	$C_5\otimes J_5$	4.2
41	26	$\{[\ 4]^1,[-4]^1,[\ 1.732]^{12},[-1.732]^{12}\}$	0	0	1 (1)	IG(13,4,1)	4.1

No.	v	Spectrum	Δ	Ξ	#	Notes	References
42	26	$\{[19]^1,[-7]^1,[\ 1.303]^{12},[-2.303]^{12}\}$	123	2208	1 (1)	$\bar{G}=2P(13)$	3.2
43	26	$\{[\ 7]^1,[\ 5]^1,[\ 1.562]^{12},[-2.562]^{12}\}$	6	24	1	td$P(13)$	4.2
44	26	$\{[\ 9]^1,[-9]^1,[\ 1.732]^{12},[-1.732]^{12}\}$	0	180	1 (1)	IG(13,9,6)	4.1
45	26	$\{[12]^1,[\ 0]^{13},[\ 2.606]^6,[-4.606]^6\}$	24	318	$\geqslant 85$ (1)	$P(13)\otimes J_2$	4.2, 6, [7]
46	27	$\{[\ 8]^1,[-1]^{20},[\ 6.243]^3,[-2.243]^3\}$	22	102	0		5
47	27	$\{[\ 8]^1,[-1]^{14},[\ 3.854]^6,[-2.854]^6\}$	13	48	1	3-cover $C_3 \circledast J_3$	4.4, 6
48	28	$\{[\ 6]^1,[-2]^{15},[\ 3.414]^6,[\ 0.586]^6\}$	6	9	1	L(IG(7,4,2)), Coxeter$_4$	3.3, 4.1
49	28	$\{[\ 7]^1,[-1]^7,[\ 2.646]^{10},[-2.646]^{10}\}$	6	15	0		6
50	28	$\{[\ 9]^1,[-1]^{21},[\ 6.583]^3,[-2.583]^3\}$	27	144	0		6
51	28	$\{[\ 9]^1,[\ 0]^{21},[\ 3.623]^3,[-6.623]^3\}$	0	153	0		6
52	28	$\{[12]^1,[\ 0]^{21},[\ 3.292]^3,[-7.292]^3\}$	12	390	0		6
53	28	$\{[12]^1,[\ 0]^{15},[\ 2.873]^6,[-4.873]^6\}$	21	300	?		
54	28	$\{[13]^1,[-1]^{13},[\ 3.606]^7,[-3.606]^7\}$	39	390	$\geqslant 515$ (1)	Taylor	4.1, [7]
55	30	$\{[27]^1,[-3]^5,[\ 0.618]^{12},[-1.618]^{12}\}$	325	8150	1 (1)	$\bar{G}=6C_5$	3.2
56	30	$\{[\ 7]^1,[-3]^9,[\ 2.732]^{10},[-0.732]^{10}\}$	5	16	?		
57	30	$\{[\ 8]^1,[\ 0]^{21},[\ 3.583]^4,[-5.583]^4\}$	0	84	0		5
58	30	$\{[\ 9]^1,[\ 3]^5,[\ 1.236]^{12},[-3.236]^{12}\}$	8	62	?		
59	30	$\{[10]^1,[\ 0]^{19},[\ 3.359]^5,[-5.359]^5\}$	7	151	0		6
60	30	$\{[11]^1,[-1]^{11},[\ 3.317]^9,[-3.317]^9\}$	22	165	?		
61	30	$\{[11]^1,[\ 1]^{19},[\ 0.162]^5,[-6.162]^5\}$	3	249	0	$\lambda_1=1$	3.3
62	30	$\{[11]^1,[\ 5]^5,[-0.382]^{12},[-2.618]^{12}\}$	29	190	?		
63	30	$\{[12]^1,[\ 0]^{25},[\ 3.708]^2,[-9.708]^2\}$	0	510	1 (1)	$C_5\otimes J_6$	4.2
64	30	$\{[13]^1,[-2]^{19},[\ 5.372]^5,[-0.372]^5\}$	47	388	0	$\lambda_3=-2$	3.3
65	30	$\{[13]^1,[-1]^{25},[\ 9.325]^2,[-3.325]^2\}$	62	570	0		5
66	30	$\{[13]^1,[-1]^{21},[\ 5.899]^4,[-3.899]^4\}$	46	410	?		
67	30	$\{[13]^1,[\ 3]^9,[-0.268]^{10},[-3.732]^{10}\}$	32	358	?		
68	30	$\{[14]^1,[-2]^{21},[\ 5.791]^4,[-1.209]^4\}$	56	532	0	$\lambda_3=-2$	3.3
69	30	$\{[14]^1,[\ 0]^{15},[\ 2.873]^7,[-4.873]^7\}$	35	525	?		
70	30	$\{[14]^1,[\ 2]^{11},[\ 0.449]^9,[-4.449]^9\}$	34	513	?		

No.	v	Hexadecimal form/orbits	**|Aut|**
23	21	FC0008780001E04C880066180B130A8051224912B06104B4590A4	**42**
25	21	FF000F1C01C0E0120E0849824AE4332233C1CB52390BAA691ED18	**21**
		FF000E1E0066602A8C15430C51489303902E2A542CD47B3B65950	**336**
28	21	FF000E1E0060780A23819A4A27154B358934382E30E88CD2263A8	**42**
		FF000C1F00843C211991915498B62570C0F4446A4A324B524C93C	**336**
45	26	FFF0007E0F806390F049A23909C1E281FB867A014DF8B98CD630FAB F64B6EA25304E6C5D2BC52ED908	**78**
		FFF000780FE00E1C3C3870F1326673266607F00730F3987B87E70FC730F 9860003CCF367E78C31FE00	**638976**

A.3. One integral eigenvalue

No.	v	Spectrum	Δ	Ξ	#	Notes	References
1	7	$\{[2]^1, [1.247]^2, [-0.445]^2, [-1.802]^2\}$	0	0	1 (1)	C_7	3.3, 4.1.1
2	13	$\{[4]^1, [1.377]^4, [\ 0.274]^4, [-2.651]^4\}$	0	4	1 (1)	Cycl(13)	4.1.1
3	19	$\{[6]^1, [2.507]^6, [-1.222]^6, [-2.285]^6\}$	6	12	1 (1)	Cycl(19)	4.1.1
4	28	$\{[9]^1, [2.604]^9, [-0.110]^9, [-3.494]^9\}$	9	72	$\geqslant 2$ (2)	Mathon, Hollmann	4.1.1

Acknowledgements

We wish to thank Chris Godsil for checking the transitive graphs computed by McKay and Royle (cf. [10]) for the ones with four eigenvalues. At an earlier stage of our search he indeed found some which we didn't know of. Later we found these graphs too using switching.

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