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D. Jibetean · E. de Klerk

Global optimization of rational functions: a semidefinite programming approach

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Abstract. We consider the problem of global minimization of rational functions on $\mathbb{R}^n$ (unconstrained case), and on an open, connected, semi-algebraic subset of $\mathbb{R}^n$, or the (partial) closure of such a set (constrained case). We show that in the univariate case ($n = 1$), these problems have exact reformulations as semidefinite programming (SDP) problems, by using reformulations introduced in the PhD thesis of Jibetean [6]. This extends the analogous results by Nesterov [13] for global minimization of univariate polynomials.

For the bivariate case ($n = 2$), we obtain a fully polynomial time approximation scheme (FPTAS) for the unconstrained problem, if an a priori lower bound on the infimum is known, by using results by De Klerk and Pasechnik [1].

For the NP-hard multivariate case, we discuss semidefinite programming-based relaxations for obtaining lower bounds on the infimum, by using results by Parrilo [15], and Lasserre [12].

1. Introduction

In this paper we study semidefinite programming relaxations of the problem of minimizing a rational objective function over some feasible set. Formally, we consider

$$p^* := \inf_{x \in S, q(x) \neq 0} \frac{p(x)}{q(x)}, \tag{1}$$

where $p(x)$, $q(x)$ are relatively prime polynomials (no common factors) with real coefficients and $S \subseteq \mathbb{R}^n$ is an open connected set or the (partial) closure of such a set.

Rational functions play an important role in engineering design, since Padé approximation of data using rational functions is usually an attractive alternative to polynomial approximation. Another type of application is in H_2 model reduction; see Jibetean and Hanzon [7].

Note that we do not assume that the infimum is attained (or is finite).

We will further restrict the feasible set S to the two special cases where:

– $S = \mathbb{R}^n$ (unconstrained minimization of rational functions);

D. Jibetean: Eindhoven University of Technology, Dept. Mathematics and Computer Science, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. e-mail: djibetean@win.tue.nl

E. de Klerk: Tilburg University, Department of Econometrics and Operations Research, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. e-mail: E.deklerk@uvt.nl

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– S is a semi-algebraic set, i.e. defined by finitely many polynomial inequalities (polynomially constrained minimization of rational functions). In this case we will also assume that S is the closure of some open bounded set.

In these cases, problem (1) is already an NP-hard problem, with the exception of a few special cases (like $n = 1$).

1.1. Possible solution approaches

Techniques from real algebraic geometry The first order optimality conditions of problem (1) can be written as a system of polynomial equations, which can in turn be solved using techniques from real algebraic geometry. A modern review of techniques for solving polynomial equations is the book by Sturmfels [23]. The difficulty is that the solution of the first order optimality conditions provides no information if the infimum is not attained in problem (1). In the case of a polynomial objective function, it is possible to use symbolic perturbation of the objective function in order to ensure that the infimum of the perturbed problem is attained, and then to take the limit as the perturbation parameter goes to zero (see e.g. Hanzon and Jibetean [3]). We do not know of similar techniques in the literature for rational objective functions. Moreover, the abovementioned techniques may involve linear algebra with prohibitively large matrices, even for relatively small values of n and the degrees of p and q; see Parrilo and Sturmfels [16].

Global optimization techniques Several global optimization codes are available for problems like (1), but Lipschitz continuity is usually required in order to guarantee global convergence, which does not hold in general for rational functions. Moreover, some problem instances involving 10 variables and as many constraints already pose problems for state-of-the-art solvers.

Convex relaxation Convex relaxation aims to give a tight lower bound on p^*. A popular modern technique is to use semidefinite programming (SDP) to obtain such relaxations. Kojima and Tunçel [8] have formulated a hierarchy of semi-infinite SDP relaxations that yield the convex hull of a quite general class of nonconvex sets, but in the authors' own words this method is 'mainly of theoretical interest'. Discrete (finite) variants of this method (see Kojima and Tunçel [9]), have been implemented by Takeda et. al [24], but the computational results are somewhat disappointing. One should mention, though, that the general methodology by Kojima and Tunçel in [8] apply to more general nonconvex sets than semi-algebraic ones.

Nesterov [13] has shown that the case $n = 1$ of problem (1) can be reformulated exactly as an SDP if $q(x) \equiv 1$. In another seminal work, Lasserre [12] has derived a hierarchy of SDP relaxations such that the optimal values converge asymptotically to p^*, if $q(x) \equiv 1$ and S is a compact semi-algebraic set that meets some technical condition. These relaxations seem to be more promising from a computational point of view than those in [8], and have now been implemented in the software Gloptipoly [4]. This software is quite useful in solving small scale optimization problems involving polynomials to global optimality (see [4]).

The aim of this paper is to generalize the above mentioned results by Nesterov and Lasserre to include rational objective functions.

Jibetean [5] considered a particular SDP relaxation of problem (1) in the unconstrained case ($S = \mathbb{R}^n$). We will also extend this approach to a hierarchy of SDP relaxations that converge to the infimum under suitable assumptions, by using a methodology due to Parrilo [15].

1.2. Outline of this paper

We first show in Section 2 that if $p^* > -\infty$, then q cannot change sign on S. As a consequence, one can assume without loss of generality that $q(x) \geq 0$ for all $x \in S$. Under this assumption one has

$$p^* = \sup\{\alpha \quad : \quad p(x) - \alpha q(x) \geq 0, \quad \forall x \in S\}.$$

This reformulation involves the nonnegativity condition of the polynomial $p(x)-\alpha q(x)$. (We view this as a polynomial in the variables x with an unknown parameter α.) In Section 3 we therefore discuss how a sufficient condition for nonnegativity, namely the sums of squares condition, can be written as a system of linear matrix inequalities (LMI's). This leads us to SDP relaxations of problem (1) in Sections 4 and 5. In Section 4 we treat the unconstrained case $S = \mathbb{R}^n$ and treat the special univariate ($n = 1$) and bivariate ($n = 2$) cases separately. In Section 5 we treat the constrained case where S is a semi-algebraic set. Once again, the univariate case is treated separately.

1.3. Notation

We will use the following (more-or-less standard) notation throughout the paper:

- $\mathbb{R}[x_1, \ldots, x_n]$: polynomials defined on $\mathbb{R}^n$ with real coefficients;
- For $f \in \mathbb{R}[x_1, \ldots, x_n]$, we write $f(x) = \sum_\beta a_\beta x^\beta$, where $\beta := [\beta_1, \ldots, \beta_n]$ is a nonnegative integer vector, and $x^\beta := x_1^{\beta_1} \ldots x_n^{\beta_n}$; also $|\beta| := \sum_{i=1}^n \beta_i$;
- $\mathcal{P}_{n,d}$: elements of $\mathbb{R}[x_1, \ldots, x_n]$ of (total) degree at most d that are nonnegative on $\mathbb{R}^n$;
- $\Sigma^2_{n,d} = \left\{r \in \mathcal{P}_{n,d} \ : \ r = \sum_i r_i^2 \text{ for some } r_i \in \mathbb{R}[x_1, \ldots, x_n]\ \forall i\right\}$; We will refer to $\Sigma^2_{n,d}$ as the 'sum of squares (s.o.s.) cone of degree at most d'; $\Sigma^2_{n,\infty}$ will refer to the union $\cup_{d \in \mathbb{N}} \Sigma^2_{n,d}$.

2. Problem reformulation

We start by giving a reformulation of problem (1) that only involves polynomials (in stead of rational functions). The proof — taken from the PhD thesis of Jibetean [6] — is included for the sake of completeness.

Theorem 1. *Let $a(x), b(x)$ be relatively prime polynomials and* **B** *an open ball in* $\mathbb{R}^n$. *One has $a(x)b(x) \geq 0,\ \forall x \in \mathbf{B}$, if and only if one of the two following statements holds:*

- $a(x) \geq 0, \quad b(x) \geq 0 \ \forall x \in \mathbf{B}$,
- $a(x) \leq 0, \quad b(x) \leq 0 \ \forall x \in \mathbf{B}$.

Proof. Assume that a changes sign on **B**, therefore there must exist an irreducible factor of a, denoted a_1, which changes sign on **B**.

We follow the proof of Lemma 6.14 of [10]. We want to prove that $f = a_1$ divides $g = b$. We know that f changes sign in **B**, that is there exist two points $\tilde{x},\ \hat{x} \in \mathbf{B}$ such that $f(\tilde{x}) > 0$ and $f(\hat{x}) < 0$. Let us make a suitable change of coordinates such that $f(y, z_1) < 0 < f(y, z_2)$ where $y \in \mathbb{R}^{n-1},\ z_1, z_2 \in \mathbb{R}$. This can be achieved by considering a system of coordinates for which one axis passes through $\hat{x}$ and $\tilde{x}$. After the change of coordinates, **B** becomes the ball $\tilde{\mathbf{B}}$. Let $G = \mathbb{R}[x_1, \ldots, x_{n-1}]$ and F the quotient ring of G. View f and g as polynomials in x_n in the ring $G[x_n] \subset F[x_n]$. Suppose that f does not divide g in $G[x_n](= \mathbb{R}[x_1, \ldots, x_n])$. We know that f remains irreducible in $F[x_n]$ and f does not divide g also in $F[x_n]$. Since $F[x_n]$ is a principal ideal domain, there exist $\rho, \gamma \in F[x_n]$ such that $f\rho + g\gamma = 1$. Write $\rho = \rho_0/h$ and $\gamma = \gamma_0/h$, where $\rho_0, \gamma_0 \in G[x_n]$ and $0 \neq h \in G$. Then $f\rho_0 + g\gamma_0 = h$. Choose a neighborhood V of y in $\mathbb{R}^{n-1}$ such that $V \times \{z_1\},\ V \times \{z_2\} \subset \tilde{\mathbf{B}}$ and $f(V, z_1) < 0 < f(V, z_2)$. For any $v \in V$, $f(v, z_1) < 0 < f(v, z_2)$ implies that $f(v, b_v) = 0$ for some b_v between z_1 and z_2. Actually, since $f(x)g(x) \geq 0$ we have $g(V, z_1) \leq 0 \leq g(V, z_2)$ and there exists a b_v where both $f(v, b_v) = 0$ and $g(v, b_v) = 0$. Therefore $f\rho_0 + g\gamma_0 = h$ implies that $h(v) = 0,\ \forall v \in V$ and so $h(x_1, \ldots, x_{n-1})$ vanishes on a non-empty open set in $\mathbb{R}^{n-1}$. This forces $h \equiv 0$, a contradiction. Hence $a_1 = f$ divides $b = g$, but this contradicts the hypothesis that a and b are relatively prime. Hence, a cannot change sign on **B**. □

Remark 1. In Theorem 1 the condition $a(x)b(x) \geq 0,\ \forall x \in \mathbf{B}$ is equivalent to, and therefore can be replaced by, $a(x)/b(x) \geq 0,\ \forall x \in \mathbf{B}$, with $b(x) \neq 0$.

Corollary 1. *Let $p(x)/q(x)$ be a rational function with $p(x),\ q(x)$ relatively prime polynomials. If $q(x)$ changes sign on* **B** *then $p^* := \inf_{x \in \mathbf{B}} p(x)/q(x) = -\infty$.*

Proof. Assume, by way of contradiction, that $p^* > -\infty$. Then there exists an $\alpha \leq p^*$ ($\alpha \in \mathbb{R}$). For every $x \in \mathbf{B}$, with $q(x) \neq 0$, we have

$$\frac{p(x)}{q(x)} \geq \alpha \iff \frac{p(x) - \alpha q(x)}{q(x)} \geq 0 .$$

Applying Theorem 1, we deduce that both $p(x) - \alpha q(x)$ and $q(x)$ do not change sign on **B**, which contradicts the hypothesis. □

Notice that the converse does not hold in general, as is shown by the example $\inf_{|x| \leq 1} \frac{-1}{x^2} = -\infty$.

The following corollary is another easy consequence of the last theorem.

Corollary 2. *Corollary 1 remains valid if the open ball* **B** *is replaced by any open connected set, or the (partial) closure of such a set.*

Proof. Let S be an open connected set or the (partial) closure of an open set, and let $p(x)/q(x)$ be a rational function with $p(x),\ q(x)$ relatively prime polynomials. If q changes sign on S, then there exists an open ball $\mathbf{B} \subset S$ such that q changes sign on $\mathbf{B}$. By Corollary 1 one now has $\inf_{x\in\mathbf{B}} p(x)/q(x) = -\infty$, which implies $\inf_{x\in S} p(x)/q(x) = -\infty$. □

We arrive at the following reformulation of problem (1).

Theorem 2. *Assume that the set S in problem (1) is an open connected subset of* $\mathbb{R}^n$*, or the (partial) closure of such a set.*

1. *If* q *changes sign on* S*, then* $p^* = -\infty$.
2. *If* q *is nonnegative on* S*, one has*

$$p^* = \sup\{\alpha \ : \quad p(x) - \alpha q(x) \geq 0, \ \ \forall x \in S\}. \tag{2}$$

□

We can therefore obtain p^* in two steps:

1. Decide if q changes sign on S; If $S = \mathbb{R}^n$ one can use techniques from [3] or [16] to find the global minimum of q, and if S is a compact semi-algebraic set then techniques from [11] or [23] may be used;
   [1a] if q changes sign on S, then $p^* = -\infty$, STOP;
   [1b] if q does not change sign but is nonpositive on S, replace q by $-q$ and p by $-p$; go to step 2.
2. Now solve (2) to obtain p^*.

In the rest of the paper we will therefore assume without loss of generality that q is nonnegative on S, and will focus on SDP-based procedures for solving (2) to obtain p^*.

The next example casts some light on the assumptions in Theorem 2.

Example 1.

$$\begin{aligned} p^* &= \inf_{x\in S} \frac{p(x)}{q(x)} := \inf_{x\in S} \frac{x_1 - x_2 + x_3 + 1}{x_1 + x_2 + x_3 + 1} \\ S &:= \{x \in \mathbb{R}^3 \ : \quad x_2^2 + x_3^2 = 0\}. \end{aligned}$$

Here the numerator and denominator in the objective function are relatively prime polynomials. However, when restricted to the feasible set

$$S := \{(x_1, 0, 0) \mid x_1 \in \mathbb{R}\},$$

which is a 'thin' connected set, the rational objective function becomes $(x_1 + 1)/(x_1 + 1) = 1,\ \forall x_1 \in \mathbb{R}$. Thus, $p^* = 1$. On the other hand, q changes sign on S. This shows that the first part of Theorem 2 no longer holds if one drops the requirement that S must be an open set or the (partial) closure of such a set.

Moreover, one has

$$\begin{aligned} &\sup\{\alpha \ : \quad p(x) - \alpha q(x) \geq 0 \ \ \forall x \in S\} \\ &= \sup\{\alpha \ : \ x_1 + 1 - \alpha(x_1 + 1) \geq 0 \ \ \forall x_1 \in \mathbb{R}\} \\ &= 1 = p^*. \end{aligned}$$

In other words, the reformulation (2) is valid for this example, even though it does not meet the conditions of Theorem 2. □

The reformulation in Theorem 2 (see (2)) involves the nonnegativity condition

$$p(x) - \alpha q(x) \in \mathcal{P}_{n,d}$$

where $d = \max\{\deg(p), \deg(q)\}$. This brings us to the theory of nonnegative polynomials and their representations.

3. Nonnegativity vs. sums of squares

3.1. Nonnegativity on $\mathbb{R}^n$

Not all nonnegative polynomials can be written as sums of squares of other polynomials. Formally, one only has

$$\Sigma^2_{n,d} = \mathcal{P}_{n,d}$$

in the following three cases:

- $n = 1$, i.e. nonnegative univariate polynomials may be written as sums of squares;
- $d = 2$, i.e. nonnegative quadratic polynomials are sums of squares;
- $n = 2$ and $d = 4$, i.e. nonnegative bivariate polynomials of degree at most 4 are sums of squares.

Note that $\mathcal{P}_{n,d} = \emptyset$ if d is odd. For $n = 2$ and $d = 6$ one already has $\Sigma^2_{n,d} \neq \mathcal{P}_{n,d}$. For an excellent review of these historical results which date back to Hilbert's 17th problem, see Reznick [21].

S.o.s. representable polynomials are of interest from a computational point of view, since they can be represented via LMI's. Formally, one can model the constraint $f \in \Sigma^2_{n,d}$ via LMI's as follows.

Theorem 3. *One has* $f \in \Sigma^2_{n,2d}$ *if and only if*

$$f(x) = \tilde{x}^T_{n,d} M \tilde{x}_{n,d}, \tag{3}$$

where $\tilde{x}_{n,d} = [1, x_1, x_2, \ldots, x_1^2, x_1x_2, \ldots, x_n^d]^T$ *is the canonical basis for the real n-variate polynomials of degree at most d, and M is a positive semidefinite matrix of size* $\binom{n+d}{d} \times \binom{n+d}{d}$. □

Equating the corresponding coefficients on the left and right hand side of equation (3) yields the following reformulation of the theorem.

Corollary 3. *One has* $f := \sum_\beta a_\beta x^\beta \in \Sigma^2_{n,2d}$ *if and only if*

$$a_\beta = \sum_{i+j=\beta} M_{ij}$$

where M is a positive semidefinite matrix of size $\binom{n+d}{d} \times \binom{n+d}{d}$ *with rows and columns indexed by all nonnegative integer vectors* β *satisfying* $\sum_{i=1}^n \beta_i \leq d$. □

The bivariate case For the cone of nonnegative bivariate polynomials, De Klerk and Pasechnik [1] have used an old lemma by Hilbert to show that

$$f \in \mathcal{P}_{2,2d} \Leftrightarrow \exists g \in \Sigma_{2,s} \text{ such that } fg \in \Sigma^2_{2,2d+s},$$

where $s = \lfloor \frac{3}{2}d^2 \rfloor$.

Thus, the authors show that for a given $f \in \mathbb{R}[x_1, x_2]$ of degree $2d$, one can answer the question 'is $f \in \mathcal{P}_{n,2d}$?' by deciding if the corresponding system of LMI's has a non-zero solution. Formally, the result is as follows.

Theorem 4 (De Klerk–Pasechnik [1]). *Given $f(x) := \sum_\beta a_\beta x^\beta \in \mathbb{R}[x_1, x_2]$ of degree $2d$, one has $f \in \mathcal{P}_{n,2d}$ if and only if the following system of LMI's has a non-zero solution:*

$$\sum_{i+j+k=\beta} a_i M^{(1)}_{jk} = \sum_{i+j=\beta} M^{(2)}_{ij} \quad \forall \beta \in \mathbb{Z}^2_+ \text{ such that } |\beta| \le 2d + 3d^2,$$

where $M^{(1)} \succeq 0$ of size $(s_1 \times s_1)$ and $M^{(2)} \succeq 0$ of size $(s_2 \times s_2)$,

$$s_1 := \binom{2 + \lfloor \frac{3}{2}d^2 \rfloor}{\lfloor \frac{3}{2}d^2 \rfloor}, \quad s_2 := \binom{2 + \lfloor 2d + \frac{3}{2}d^2 \rfloor}{\lfloor 2d + \frac{3}{2}d^2 \rfloor}.$$

The solution of this system of LMI's yields the decomposition $fg = h$ with $g \in \Sigma^2_{2,\frac{3}{2}d^2}$ and $h \in \Sigma^2_{2,2d+\frac{3}{2}d^2}$, by setting

$$g(x) := \tilde{x}^T_{2,s_1} M^{(1)} \tilde{x}_{2,s_1}, \quad h(x) := \tilde{x}^T_{2,s_2} M^{(2)} \tilde{x}_{2,s_2}. \tag{4}$$

□

3.2. Nonnegativity on a semi-algebraic set

We first state two classical theorems that characterize nonnegative univariate polynomials on a line segment or a half-line. See Powers and Reznick [17] and the references therein for more background on these results.

Theorem 5 (M. Fekete). *Let $n = 1$ and $S = [a, b]$ for some $a < b$. Any $f \in \mathbb{R}[x]$ of degree d such that $f(x) \ge 0$ for all $x \in S$ can be decomposed as*

$$f \in \Sigma^2_{1,2d} + (x - a)(b - x)\Sigma^2_{1,2d-2}. \quad □$$

Theorem 6 (Pólya-Szegö). *If S is a half line $S = [a, \infty)$ for some $a \in \mathbb{R}$, then any $f \in \mathbb{R}[x]$ of degree d such that $f(x) \ge 0$ for all $x \in S$ can be decomposed as*

$$f = \Sigma^2_{1,d} + (x - a)\Sigma^2_{1,d-1}. \quad □$$

We now consider the multivariate case. Assume that $S \subset \mathbb{R}^n$ is a semi-algebraic set defined by

$$S = \left\{x \in \mathbb{R}^n \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, k)\right\}, \tag{5}$$

where the $p_i \in \mathbb{R}[x_1, \ldots, x_n]$ are given polynomials.

Assumption 1 *S is compact and there exists a*

$$\bar{p} \in \Sigma^2_{n,\infty} + p_1 \Sigma^2_{n,\infty} + \cdots + p_k \Sigma^2_{n,\infty}$$

such that $\{x \ : \ \bar{p}(x) \geq 0\}$ is compact.

Theorem 7 (Putinar [19]). *Let S be a semi-algebraic set of the form (5) for which Assumption 1 holds. If a given $p_0 \in \mathbb{R}[x_1, \ldots, x_n]$ satisfies $p_0(x) > 0$ for all $x \in S$, then*

$$p_0 \in \Sigma^2_{n,\infty} + p_1 \Sigma^2_{n,\infty} + \cdots + p_k \Sigma^2_{n,\infty}. \qquad \square$$

4. Unconstrained optimization of rational functions: an SDP approach

In this section we treat the unconstrained problem

$$p^* := \inf_{x \in \mathbf{R}^n,\ q(x) \neq 0} \frac{p(x)}{q(x)} \quad \text{with} \quad p(x),\ q(x) \in \mathbb{R}[x_1, \ldots, x_n] \ \text{relatively prime.} \tag{6}$$

4.1. The univariate case

The univariate case ($n = 1$) of problem (6) can be solved in polynomial time, by applying techniques from real algebraic geometry (see e.g. Parrilo and Sturmfels [16]) to the reformulation in Theorem 2. Our aim in this section is to show that the univariate case also has an *exact* SDP reformulation, which generalizes the analogous result for global minimization of univariate polynomials by Nesterov [13].

If p and q are univariate polynomials then the condition

$$p(x) - \alpha q(x) \geq 0 \ \ \forall x \in \mathbb{R}$$

is equivalent to

$$p(x) - \alpha q(x) \in \Sigma^2_{1,2d},$$

where $2d = \max\{\deg(p), \deg(q)\}$. Applying Theorem 2, and using $\Sigma^2_{1,d} = \mathcal{P}_{1,2d}$, we obtain the following *exact* SDP formulation of problem (6) in the univariate case.

Theorem 8. *Consider problem (6) with $n = 1$. For any $\alpha \in \mathbb{R}$, we denote*

$$p(x) - \alpha q(x) := \sum_{\beta} a_\beta(\alpha) x^\beta,$$

where the coefficients $a_\beta(\alpha)$ depend affinely on α. One now has

$$p^* = \sup \alpha$$

subject to

$$a_\beta(\alpha) = \sum_{i+j=\beta} M_{ij}$$

where M is a positive semidefinite matrix of size $(d+1) \times (d+1)$. □

Theorem 8 generalizes the result by Nesterov [13] for global minimization of univariate polynomials. The theorem actually follows from the remarks in §4.3 of Nesterov [13], if we use the fact that we may assume without loss of generality that q is nonnegative on $\mathbb{R}$.

Example 2. Consider the problem of finding p^*, where

$$p^* = \inf_{x \in \mathbb{R}} \frac{p(x)}{q(x)} := \frac{x^2 - 2x}{(x+1)^2}.$$

Here $p^* = -1/3$ which is attained at $x = \frac{1}{2}$.

The equivalent SDP problem is: $\sup \alpha$ such that

$$(1-\alpha)x^2 - 2(1+\alpha)x - \alpha = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}, \tag{7}$$

for some $M \succeq 0$.

From (7) we have:

$$M_{00} = -\alpha, \quad M_{01} = M_{10} = -(1+\alpha), \quad M_{11} = 1 - \alpha.$$

We therefore get the SDP problem

$$p^* = \min_{x \in \mathbb{R}} \frac{p(x)}{q(x)} = \max_{\alpha, M} \alpha$$

such that

$$M = \begin{bmatrix} -\alpha & -(1+\alpha) \\ -(1+\alpha) & 1-\alpha \end{bmatrix} \succeq 0.$$

Note that the optimal value is $p^* = -1/3$.

The dual SDP problem is

$$\min -2x_{12} + x_{22}$$

such that

$$x_{11} + 2x_{12} + x_{22} = 1, \ \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \succeq 0.$$

Note that the optimal solution here is the rank one matrix

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} = \frac{4}{9}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} = \frac{4}{9}\begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} \text{ if } x = \tfrac{1}{2},$$

from which we may extract the optimal solution $x = \frac{1}{2}$ where the infimum is attained.

□

4.2. *The bivariate case*

We treat the bivariate case ($n = 2$) of problem (6) separately as well. This problem can again be solved in polynomial time, by applying techniques from real algebraic geometry (see e.g. Parrilo and Sturmfels [16]) to the reformulation in Theorem 2. (In fact, this observation remains true for any fixed number of variables, i.e. if $n = O(1)$.)

We do not know if the bivariate problem allows an exact SDP reformulation, but will show that the weaker decision problem 'Given $\alpha \in \mathbb{R}$, is $p^* \le \alpha$?' does allow an exact SDP reformulation. One can therefore use SDP in conjunction with bisection to estimate p^*, if an a priori lower bound on p^* is known.

If p and q are bivariate polynomials and $2d = \max\{\deg p, \deg q\}$, then the condition

$$p(x) - \alpha q(x) \ge 0 \ \ \forall x \in \mathbb{R}^n$$

is equivalent to

$$(p(x) - \alpha q(x))r(x) \in \Sigma^2_{2,2d+\frac{3}{2}d^2}$$

for some $r \in \Sigma^2_{2,\frac{3}{2}d^2}$, by Theorem 4.

We can therefore solve the decision problem: 'given $\alpha \in \mathbb{R}$, is $\alpha \le p^*$?', by solving a system of LMI's.

Example 3. Consider the problem

$$p^* =: \inf_{x_1,x_2} \frac{x_1^6 + x_2^2 + x_2^4 - 3x_1^2x_2^2}{x_1^2 - 2x_1x_2 + x_2^2} := \frac{p(x)}{q(x)}.$$

Note that $p^* \le 0$ (look at $x_1 = 1, x_2 = -1$).

We can prove that '$\alpha := 0 \le p^*$' by considering the bivariate polynomial

$$p(x) - 0q(x) = p(x) = x_1^6 + x_2^2 + x_2^4 - 3x_1^2x_2^2.$$

One can now use Theorem 4 to show using SDP that this polynomial is nonnegative on $\mathbb{R}^2$. The SDP approach (using equation (4)) yields the decomposition

$$(p(x) - 0q(x))(1 + x_1^2 + x_2^2) = \left(x_1x_2 - x_2x_1^3\right)^2 + \left(x_2^2x_1 - x_1^3\right)^2 + \left(x_2^2 - x_1^4\right)^2 + \\ + \left(\frac{1}{2}x_2^3 - \frac{1}{2}x_2\right)^2 + \sqrt{3}^2\left(\frac{1}{2}x_2^3 + \frac{1}{2}x_2 - x_2x_1^2\right)^2 \\ \in \Sigma^2_{2,8}.$$

We conclude that $p^* = 0$. □

4.3. *The multivariate case*

We consider the problem

$$\inf_{x\in\mathbb{R}^n,\ q(x)\neq 0} \frac{p(x)}{q(x)}.$$

This is an NP-hard problem in general. If we assume that the infimum is attained in the ball

$$S := \{x \in \mathbb{R}^n \ : \ \|x\| \leq R\},$$

for some known parameter R, then we can treat this problem as the constrained problem

$$\inf_{x\in S,\ q(x)\neq 0} \frac{p(x)}{q(x)}.$$

and subsequently use the techniques that will be described in Section 5.2. Note that the set S meets Assumption 1. Of course, the parameter R will not in general be known a priori.

An alternative approach was investigated by Jibetean in [5], where the author considered the SDP-based lower bound obtained by computing

$$\sup\left\{\alpha \ : \ p(x) - \alpha q(x) \in \Sigma^2_{n,d}\right\},$$

where $d = \max\{\deg(p), \deg(q)\}$. One can extend this approach by considering a hierarchy of SDP based lower bounds

$$\bar{p}^{(r)} := \sup\left\{\alpha \ : \ (p(x) - \alpha q(x))\left(1 + \sum_{i-1}^{n} x_i^2\right)^r \in \Sigma^2_{n,d+2r}\right\}, \tag{8}$$

for $r = 0, 1, 2, \ldots$. Note that the relaxation by Jibetean [5] is obtained when $r = 0$. These types of relaxations were first studied in the context of global optimization of polynomials by Parrilo [14, 15]. Under the assumption that the homogeneous form associated with the polynomial $p - p^*q$ is positive definite on $\mathbb{R}^n$, it follows from a theorem by Reznick [20] that $\lim_{r\to\infty} \bar{p}^{(r)} = p^*$. This assumption is difficult to check in practice. If the assumption does not hold, we still obtain a hierarchy of lower bounds

$$\bar{p}^{(r)} \leq \bar{p}^{(r+1)} \leq p^* \text{ for } r = 0, 1, 2, \ldots,$$

but it may happen that the sequence $\{\bar{p}^{(r)}\}$ does not converge to p^*.

Example 4. We consider the problem in Example 3 again. Note that in this case one has $\bar{p}^{(1)} = p^* \equiv 0$, where $\bar{p}^{(1)}$ is defined in (8). □

5. Constrained optimization of rational functions: an SDP approach

In this section we consider the constrained problem

$$p^* := \inf_{x \in S,\; q(x) \neq 0} \frac{p(x)}{q(x)},$$

where $S \subset \mathbb{R}^n$ is a connected semi-algebraic set that satisfies certain additional assumptions.

Before we treat the general multivariate case, we again look at the polynomially solvable univariate case and show that — similar to the unconstrained case — it has an *exact* SDP reformulation. This generalizes the analogous result for global minimization of univariate polynomials on line segments and half-lines by Nesterov [13].

5.1. The univariate case

Consider

$$p^* := \inf_{x \in S,\; q(x) \neq 0} \frac{p(x)}{q(x)},$$

where S is an interval $S = [a, b]$, and $d = \max\{\deg p, \deg q\}$.

Assuming w.l.o.g. that $q(x) \geq 0$ for all $x \in S$, and applying Theorems 2, 5 and 3 in turn yields

$$\begin{aligned} p^* &= \sup\{\alpha \;:\; p(x) - \alpha q(x) \geq 0 \;\forall\, x \in S\} \\ &= \sup\left\{\alpha \;:\; p(x) - \alpha q(x) = \Sigma^2_{1,2d} + (x-a)(b-x)\Sigma^2_{1,2d-2}\right\} \\ &= \sup\left\{\alpha \;:\; p(x) - \alpha q(x) = \tilde{x}^T_{1,d} M_1 \tilde{x}_{1,d} + (x-a)(b-x)\tilde{x}^T_{1,d-1} M_2 \tilde{x}_{1,d-1}\right\}, \end{aligned}$$

where $\tilde{x}_{1,d} = [1, x, x^2, \ldots, x^d]^T$ as before, and M_1 and M_2 are positive semidefinite matrices.

Similary to the unconstrained case, we can denote

$$p(x) - \alpha q(x) := \sum_{\beta} a_\beta(\alpha) x^\beta,$$

to obtain the exact SDP reformulation:

$$p^* = \sup_{\alpha, M_1, M_2} \alpha$$

subject to

$$a_\beta(\alpha) = \sum_{i+j=\beta} (M_1)_{ij} - ab \sum_{i+j=\beta} (M_2)_{ij} + (a+b) \sum_{i+j=\beta-1} (M_2)_{ij} - \sum_{i+j=\beta-2} (M_2)_{ij}$$

where M_1, M_2 are positive semidefinite matrix variables of size $(d+1)\times(d+1)$ and $d \times d$ respectively.

Univariate optimization over a half-line $[a, \infty)$ can be reformulated as an SDP problem in the same way, by using Theorem 6.

5.2. The multivariate case

We now consider the problem

$$p^* := \inf_{x\in S,\ q(x)\neq 0} \frac{p(x)}{q(x)}, \tag{9}$$

where S is the semi-algebraic set

$$S := \left\{x \in \mathbb{R}^n \ : \ g_i(x) \geq 0,\ i = 1, \ldots, m\right\}. \tag{10}$$

This problem is again NP-hard, and we are interested in obtaining lower bounds on p^* in polynomial time using SDP.

In addition to Assumption 1 we make the following assumption about S:

Assumption 2 *S is the closure of some open connected set.*

By Theorem 2 we know that — under these assumptions — one has

$$p^* = \sup\{\alpha \ : \ p(x) - \alpha q(x) \geq 0 \ \ \forall x \in S\}.$$

We show in the next lemma that the inequality can be replaced by strict inequality under the following assumption.

Assumption 3 *The polynomials p and q have no common real roots in S.*

Lemma 1. *Under Assumption 3 and the assumptions of Theorem 2, one has*

$$p^* = \sup\{\alpha \ : \ p(x) - \alpha q(x) > 0 \ \ \forall x \in S\}.$$

Proof. Assume $\alpha < p(x)/q(x)$ for all $x \in S$ such that $q(x) \neq 0$. We know that q must be nonnegative on S in this case. In other words

$$q(x) \neq 0 \Leftrightarrow q(x) > 0 \text{ if } x \in S.$$

We therefore have that

$$p(x) - \alpha q(x) > 0 \text{ for all } x \in S \text{ with } q(x) \neq 0.$$

Now we use the assumption that p and q have no common real roots: since $p(x) - \alpha q(x)$ is nonnegative on S, $q(x) = 0$ implies $p(x) > 0$. We therefore have that

$$\alpha < p(x)/q(x) \text{ for all } x \in S \text{ with } q(x) \neq 0 \Leftrightarrow p(x) - \alpha q(x) > 0 \text{ for all } x \in S.$$

The required result follows. □

Remark 2. Assumption 3 may be checked in practice by determining whether the polynomial $p^2 + q^2$ is strictly positive on S. As before, these conditions may be checked using techniques from [12] or from real algebraic geometry.

By the theorem of Putinar (Theorem 7), the condition

$$p(x) - \alpha q(x) > 0 \ \ \forall x \in S$$

implies

$$p(x) - \alpha q(x) \in \Sigma^2_{n,\infty} + \sum_{j=1}^{m} g_j(x)\Sigma^2_{n,\infty}.$$

Following Lasserre [12], we define a hierarchy of SDP relaxations

$$p^{(r)} = \sup\left\{\alpha \ : \ p(x) - \alpha q(x) \in \Sigma^2_{n,2r} + \sum_{j=1}^{m} g_j(x)\Sigma^2_{n,2r}\right\}, \tag{11}$$

for $r = 1, 2, \ldots$. Note that the computation of $p^{(r)}$ involves solving an SDP problem of size polynomial in m, n and in the degrees of p and q for any *fixed* r.

By Theorem 7, if $p^* > -\infty$ one will have

$$\lim_{r\to\infty} p^{(r)} = p^*,$$

as well as $p^{(r)} \le p^{(r+1)} \le p^*$ for $r = 1, 2, \ldots$.

We can summarize these results as the following theorem.

Theorem 9. *Consider problem (9), where S is a compact semi-algebraic set of the form (10) that meets Assumptions 1, 2 and 3. If $p^* = -\infty$, then one has*

$$p^{(r)} = -\infty \textit{ for all } r = 1, 2, \ldots,$$

where $p^{(r)}$ is defined in (11). If $p^* > -\infty$, one has

$$p^{(r)} \le p^{(r+1)} \le p^* \textit{ for all } r = 1, 2, \ldots,$$

as well as $\lim_{r\to\infty} p^{(r)} = p^*$. □

Example 5. Consider the constrained optimization problem

$$p^* := \inf \frac{{x_3}^2 + {x_2}^2\,(x_1 - 1)^2 - x_2 + 5}{{x_3}^2\,(x_1 - 4)^3\,(x_2 - 5) + (x_1 - 1)^2}$$

$$\begin{aligned} \text{s.t. } & x_1^4 + x_2^4 + x_3^4 \le 100 \\ & 3x_1 + 2x_2 - x_3 \ge -3 \\ & x_2 - x_1^2 - x_3^3 \le 1. \end{aligned}$$

It is straightforward to verify that the feasible set S satisfies all the hypothesis of Theorem 9.

We used the program SOSTools [18] to compute the lower bounds $p^{(r)} \leq p^*$ in (11) for $r = 1, 2, 3$, to obtain

$$p^{(1)} = 4.76 \times 10^{-7}, \; p^{(2)} = p^{(3)} = 3.707 \times 10^{-3}.$$

By using the optimization solver CONOPT [2] we obtained the KKT point

$$x_1 = -1.674, \; x_2 = 0.247, \; x_3 = -1.526,$$

with objective value 3.707×10^{-3}. This shows that — for this example — one has $p^{(r)} = p^*$ for $r \geq 2$. It also illustrates the usefulness of the approach for *proving* global optimality of a given solution. □

Remark 3. Note that in the univariate case $n = 1$ we obtain an *exact* reformulation of problem (9) without the assumption of compactness.

One can at least avoid the second part of Assumption 1 in Putinar's theorem, by replacing the theorem of Putinar by Schmüdgen's Positivstellensatz [22]. Schmüdgen's theorem states that the condition

$$p(x) - \alpha q(x) > 0 \;\; \forall x \in S$$

implies

$$p(x) - \alpha q(x) \in \Sigma^2_{n,\infty} + \left(\sum_{I \subseteq \{1,\ldots,m\}} \prod_{i \in I} g_i(x) \right) \Sigma^2_{n,\infty}.$$

Here we only assume that S is non-empty, compact, and semi-algebraic of the form (10).

Thus we can define lower bounds for p^* in a similar way as we did using Putinar's theorem. The disadvantage is that the representation of positive polynomials via Schmüdgen's Positivstellensatz is clearly more complicated than when using Putinar's theorem.

6. Conclusions and discussion

In this paper we have extended the results by Nesterov [13], Lasserre [12], and De Klerk and Pasechnik [1] for global optimimization of polynomial functions to include rational objective functions. In particular, we have shown that global minimization of univariate rational functions over a connected subset of $\mathbb{R}$ has a reformulation as a semidefinite program. In the unconstrained bivariate case we have shown how to use bisection to obtain a arbitrarily good approximation of the optimal value, thus extending the scope of the results by De Klerk and Pasechnik [1]. For the multivariate case, we have derived various semidefinite programming based lower bounds on the infimum, by extending the methodologies of Lasserre [12], Jibetean [5], and Parrilo [15].

All these extensions relied on a reformulation of the nonnegativity of rational functions in terms of nonnegativity of suitable polynomials, as introduced in the PhD thesis of Jibetean [6].

Since the ideas of Lasserre [12] have been implemented in the software GloptiPoly [4] by Henrion and Lasserre, we hope that our work will lead to an extension of this software to include rational objective functions in the near future. An important issue here is how to extract solutions for the original problem (1) from a solution of the SDP relaxation. In particular, one should investigate whether the extraction procedure used in the GloptiPoly software can be extended to the more general problem.

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