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The consistency principle for set-valued solutions and a new direction for normative game theory[1]

Martin Dufwenberg[2], Henk Norde[3], Hans Reijnierse[4], Stef Tijs[5]

[2] Department of Economics, Stockholm University, SE-10691 Stockholm, Sweden (E-mail: md@ne.su.se)
[3] Department of Econometrics and CentER for Economic Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (E-mail: h.norde@kub.nl)
[4] Department of Econometrics and CentER for Economic Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (E-mail: j.h.reijnierse@kub.nl)
[5] Department of Econometrics and CentER for Economic Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (E-mail: s.h.tijs@kub.nl)

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Abstract. We extend the consistency principle for strategic games (Peleg and Tijs (1996)) to apply to solutions which assign to each game a collection of *product sets* of strategies. Such solutions turn out to satisfy desirable properties that solutions assigning to each game a collection of strategy *profiles* lack. Our findings lead us to propose a new direction for normative game theory.

Key words: Consistency, set-valued solution, normative game theory, self-enforcing recommendation

1 Introduction

A series of recent papers characterize solutions for strategic games using the axiom of "consistency", and some complementary axioms. This literature focuses on solutions that are *point-valued* in the sense that they assign to each game a collection of strategy *profiles*. In this paper we extend these ideas to apply to solutions that are *set-valued* in the sense that they assign a collection of *product sets* of strategies to each game. Our findings lead us to propose a new direction for normative game theory. The motivation of our study is as follows:

According to the classical view, game theory is a normative science with the aim to offer "self-enforcing recommendations" to rational players (see e.g. Kohlberg and Mertens (1986, footnote 3) or van Damme (1987, pp 1–3)). Most game theoretic solutions are point-valued. If the solution is a good one, each profile selected should have the property that, once recommended to the players, none of them should have an incentive to deviate.

However, there are several reasons why one might prefer to study set-valued solutions, where each player is recommended a *set* of strategies. First, as argued

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by Basu and Weibull (1991), there is no obvious reason why recommendations should take the form of a single strategy rather than a set of strategies. Second, if one does not consider mixed strategies as reasonable objects of choice (see e.g. Ariel Rubinstein's arguments in Osborne and Rubinstein (1994), Section 3.2.1) then in many games no equilibria exist while appropriate set-valued solutions might have no such problems. Third, some notions that arise in decision-theoretic approaches to analyzing games, like the product set of rationalizable strategies (Bernheim (1984), Pearce (1984)), fit quite nicely into the framework of set-valued solutions. Fourth, in many games some player will have no "strict" incentive to comply with a recommended profile because he has multiple optimal choices given that all others comply. If all such strategies are made part of the recommendation, this will come as a strategy set. Similar concerns presumably motivate Nash's (1951) notion of "strict solvability", and certainly motivate the work of Basu and Weibull (1991) and Hurkens (1995, see especially pp 13–14).[6]

Peleg and Tijs (1996) introduce the axiom of "consistency" for point-valued solutions and show that it is a useful axiom for characterizing and understanding these solutions.[7] Intuitively, a solution is consistent if any profile selected by this solution is also selected in any "reduced game", in which only a subset of the players is active as before while the remaining players make choices in accordance with the profile under consideration and then "leave the game". Given the classical view of game theory, consistency has a natural interpretation: If a subset of players commit to following an initially self-enforcing recommendation, then the recommendation is still self-enforcing for the remaining players. Arguably, this is a "desirable" feature of a recommendation.

A recent literature in non-cooperative game theory has emerged in which the consistency axiom is in focus.[8] This literature focuses exclusively on point-valued solutions. We show that the consistency principle and the notion of a reduced game can be readily extended to set-valued solutions. However, a "leaving player" of a reduced game is not necessarily restricted to make one particular choice, so such a game has as many players as its parent game. This is in contrast to the set-up of Peleg and Tijs (1996), where reductions always decrease the number of players. In order to allow for a comparison of results it is necessary to somewhat modify the traditional theory. A game, reduced with respect to some particular profile, is viewed as a game with the same number

[6] We note that in the early days of game theory set-valued solutions were in focus. In a two-person zero-sum game, the set of strategy profiles, in which each player uses a maxminimizer strategy, has a product structure, and this observation is central to von Neumann's (1928) claim that he can solve zero-sum games. Nash's (1951) various notions of solutions of (solvable) games are product sets of strategies (he never promotes equilibrium *points* as solutions!). In contemporary game theory set-valued solutions are in focus in Basu and Weibull (1991), Hurkens (1995, 1996), and also in Kohlberg and Mertens (1986). However, the "stable sets" of Kohlberg and Mertens need not have a product structure, and so fit less conveniently into the recommendation setting we have described.

[7] Confer also Aumann (1987, p 479) who anticipates these results and Salonen (1992) who conducts an analysis of the Nash equilibrium concept using an axiom closely related to consistency.

[8] See Peleg and Sudhölter (1994), Patrone, Pieri, Tijs, and Torre (1995), van Heumen, Peleg, Tijs, and Borm (1996), Norde, Potters, Reijnierse, and Vermeulen (1996), Peleg, Potters, and Tijs (1996), Ray (1996) and Güth (1998). We do not include here references to the large literature in cooperative game theory where a notion of consistency plays an important role. See Thomson (1996) for a general survey (which also covers point-valued solutions for non-cooperative games).

of players as the original game, but with a subset of players restricted to choose from singleton strategy sets containing only the strategy prescribed by the profile.

A central axiom, complementary to consistency, in Peleg and Tijs (1996) is that of "one-person rationality", which imposes a rationality requirement on decision making in games with only one player. This axiom has cutting power in the theory of Peleg and Tijs because reduced games have fewer players than parent games. With our new view of reductions this axiom no longer works however. We replace it by another axiom, "rationality", which imposes a rationality requirement on decision making in any game. Then, the essence of the analysis of Peleg and Tijs (1996) can be recaptured in the new framework we propose.

We then turn to set-valued solutions, focusing on the class of finite games. We generalize several axioms used in the traditional approach, present a few set-valued solutions, and investigate whether these satisfy the new axioms. We ask what set-valued solutions satisfy those axioms that generalize the axioms that can characterize the Nash equilibrium concept in the traditional approach. The answer is somewhat surprising: The collection of singleton sets, each involving a Nash equilibrium, is not uniquely implied. Other solutions too qualify, for example the collection of product strategy sets with the "best response property" in the sense of Pearce (1984), which turns out to be the largest solution satisfying consistency and rationality. We henceforth refer to this solution as BRP. This solution has the virtue of being non-empty for the class of finite games, something "the Nash singletons" solution does not achieve.

For the traditional approach, Norde, Potters, Reijnierse, and Vermeulen (1996) have shown that if one insists that a solution selects a non-empty collection of profiles for each game that possesses a Nash equilibrium, then one cannot move towards refinements of Nash equilibria without producing inconsistent solutions. As consistency is often viewed as a desirable property (see our remark above and also Aumann (1987, p 478–9) for a general appraisal), this result has been taken as a set-back for the theories of equilibrium refinements and equilibrium selection. For example, Eric van Damme and Robert Aumann express some concern in the interview Aumann (1996, pp 28–30) and Güth (1998) attempts to "avoid the impasse" by modifying the consistency requirement. We argue that, given that consistency is viewed as a desirable condition, the findings reported in the previous paragraph suggest a way out of this dilemma. Instead of refining the point-valued Nash equilibrium solution, one should focus on the set-valued solution BRP and try to refine that solution while retaining consistency and other properties deemed desirable.

In order to exemplify this line of research we use Basu and Weibull's (1991) notion of a set "closed under rational behavior" to isolate refinements of BRP which are set-valued analogues of the strict equilibrium solution (Harsanyi (1973)), and prove that the desirable properties satisfied by BRP still hold. As these refinements concern product sets of strategies rather than strategy profiles, they look quite different from standard refinements. They need not always imply Nash behavior, but in some cases they have considerably more cutting power than standard refinements. We illustrate this using an example due to Hurkens (1996).

Notation. Throughout this paper strict inclusion is denoted by $\subset$ and weak inclusion by $\subseteq$.

2 Point-valued solutions

The main aim of this section is to modify the axiom of "consistency" for point-valued solutions, introduced by Peleg and Tijs (1996), such that it may be viewed as a special case of the consistency principle for set-valued solutions, which will be defined in Section 3. Moreover we will rephrase some of the traditional results in this new setting.

Throughout this paper we focus on finite strategic games. Such a game is a tuple $G = \langle N, A, u\rangle$, where N is the finite player set, $A = \Pi_{i\in N} A_i$ is the product set of the finite strategy sets A_i $(i \in N)$, and $u = (u_i)_{i\in N}$ is the vector of payoff functions $u_i : A \to \mathbb{R}$ $(i \in N)$. If $|A_i| = 1$ then i is called a *dummy player* of the game G. Let Γ be the collection of all finite strategic games. A *point-valued solution* on Γ is a map ϕ which assigns to every game $G = \langle N, A, u\rangle \in \Gamma$ a collection of strategy profiles in A. An example of a point-valued solution is the solution NE which assigns to every game $G \in \Gamma$ the set of Nash equilibria of G:

$$\mathrm{NE}(G) = \{a : a \text{ is a Nash equilibrium of } G\}.$$

The central axiom in the traditional approach to characterization of point-valued solutions is that of consistency. The version of this axiom we use is based on the following notion of a reduced game.

For a finite game $G = \langle N, A, u\rangle$, for a coalition $S \subset N$, and for a strategy profile $a = (a_i)_{i\in N} \in A$ *the reduced game of* G *with respect to* S *and* a is the game $G^{S,\{a\}} = \langle N, \Pi_{i\in S}A_i \times \Pi_{i\in N\setminus S}\{a_i\}, \tilde{u}\rangle$, where $\tilde{u} = (\tilde{u}_i)_{i\in N}$ is the vector of restrictions of the payoff functions u_i $(i \in N)$ to $\Pi_{i\in S}A_i \times \Pi_{i\in N\setminus S}\{a_i\}$. Note that the reduced game $G^{S,\{a\}}$ belongs to Γ. $G^{S,\{a\}}$ has as many players as the game G, because the players in $N\setminus S$ are still present as dummy players, whereas in the traditional definition of the notion of reduced game these players leave the game. It is allowed that $S = \varnothing$, in which case the game $G^{S,\{a\}}$ has only dummy players.

Definition 2.1. A point-valued solution ϕ on Γ satisfies *consistency* (CONS) if for every $G = \langle N, A, u\rangle \in \Gamma$, $S \subset N$, $a \in \phi(G)$ we have $a \in \phi(G^{S,\{a\}})$.

A second common axiom in the characterizations in the traditional literature deals with optimization in one-person games. In Peleg and Tijs (1996) and Peleg, Potters, and Tijs (1996) the axiom of *one-person-rationality* (OPR) is used, requiring the selection of all maximizers in one-person games, whereas in Norde et al. (1996) the weaker axiom of *utility maximization* (UM) is used, which requires the selection of a subset of the set of all maximizers in one-person games. The axioms (OPR) and (UM) work well in these cases, because reduction of games involves a reduction of the number of players. However, in our present definition of the notion of reduced game, the number of players is not reduced and (OPR) or (UM) can not be used. As a substitute we propose the axiom of rationality. In the definition of this axiom below the set $\Delta(\Pi_{j\in N\setminus\{i\}} A_j)$ is the collection of probability distributions (beliefs) over $\Pi_{j\in N\setminus\{i\}} A_j$ and we write $u_i(a_i, \mu_{-i})$ for the expected utility for player i if he plays strategy a_i and the other players play a strategy profile according to the probability distribution $\mu_{-i} \in \Delta(\Pi_{j\in N\setminus\{i\}} A_j)$.

Definition 2.2. A point-valued solution ϕ on Γ satisfies *rationality* (RAT) if for every $G = \langle N, A, u \rangle \in \Gamma$, for every $b = (b_i)_{i \in N} \in \phi(G)$ and for every $i \in N$ there exists an $\mu_{-i} \in \Delta(\Pi_{j \in N \setminus \{i\}} A_j)$ such that $b_i \in \operatorname{argmax}_{a_i \in A_i} u_i(a_i, \mu_{-i})$.

The following proposition shows that point-valued solutions satisfying (CONS) and (RAT) are refinements of the Nash equilibrium concept (cf. Proposition 2.8 in Peleg and Tijs (1996)).

Proposition 2.1. *Let ϕ be a point-valued solution on Γ satisfying (CONS) and (RAT). Then $\phi(G) \subseteq NE(G)$ for every $G \in \Gamma$.*

Proof. Let $G = \langle N, A, u \rangle \in \Gamma$, $a = (a_i)_{i \in N} \in \phi(G)$, and $i \in N$. By (CONS) we have $a \in \phi(G^{\{i\}, \{a\}})$ and by (RAT) we get that a_i is a best response to $(a_j)_{j \neq i}$. Hence $a \in \mathrm{NE}(G)$. ■

Proposition 2.1 is still true if we replace the axiom of rationality by a weaker axiom, which requires that if a profile is selected in some game with one non-dummy player, then that player must choose a utility maximizing strategy.

The following axioms are important in the traditional approach:

Definition 2.3. A point-valued solution ϕ on Γ satisfies

i) *non-emptiness* (NEM) if for every $G \in \Gamma$ we have $\phi(G) \neq \emptyset$;
ii) *restricted non-emptiness* (r-NEM) if for every $G \in \Gamma$ with $\mathrm{NE}(G) \neq \emptyset$ we have $\phi(G) \neq \emptyset$.

In Norde et al. (1996) the Nash equilibrium concept on the class of mixed extensions of all finite games is characterized by utility maximization, consistency, and non-emptiness. For finite games this characterization was already given in Peleg, Potters, and Tijs (1996). Since the Nash equilibrium set may be empty in these games the axiom of non-emptiness had to be replaced by restricted non-emptiness. Both proofs in Norde et al. (1996) and Peleg, Potters, and Tijs (1996) use a construction which associates with every game G and every Nash equilibrium x of G an ancestor game H with a unique Nash equilibrium y such that G may be viewed as a reduced game of H. Since a point-valued solution, satisfying utility maximization, consistency, and (restricted) non-emptiness should select y in H, one infers, by consistency, that it allows x in G. In our present setting this argument breaks down because the ancestor game H has more players than G and reduced games do not have fewer players. However, we can overcome this problem by adding the dummy out property.

Definition 2.4. A point-valued solution ϕ on Γ satisfies the *dummy out property* (DOP) if for every $G = \langle N, A, u \rangle$ and for every $i \in N$ with $|A_i| = 1$ and $G' = \langle N \setminus \{i\}, \Pi_{j \in N \setminus \{i\}} A_j, (\tilde{u}_j)_{j \in N \setminus \{i\}} \rangle \in \Gamma$ we have $\phi(G) = A_i \times \phi(G')$. Here the payoff functions $\tilde{u}_j$ $(j \in N \setminus \{i\})$ are defined by $\tilde{u}_j(a_{-i}) = u_j(a_{-i}, a_i)$ where a_i is the unique element of A_i.

Proposition 2.2. *Let ϕ be a point-valued solution on Γ. Then ϕ satisfies (CONS), (RAT), (DOP), and (r-NEM) if and only if $\phi = NE$.*

Proof. One easily verifies that NE satisfies (CONS), (RAT), (DOP), and (r-NEM). In order to prove the only-if-part, suppose that ϕ satisfies (CONS), (RAT), (DOP), and (r-NEM). We have to show that $\phi(G) = \mathrm{NE}(G)$ for every $G \in \Gamma$. By Proposition 2.1 we get that $\phi(G) \subseteq \mathrm{NE}(G)$ for every $G \in \Gamma$. For the proof of the converse inclusion, let $G = \langle N, A, u \rangle \in \Gamma$ and $x \in \mathrm{NE}(G)$. The ancestor game $H = \langle N', B, v \rangle \in \Gamma$ is constructed in the same way as in the proof of Theorem 3 in Peleg, Potters, and Tijs (1996), i.e $N' = N \cup \{0\}$, $B_i = A_i$ for every $i \in N$, $B_0 = \{\alpha, \beta\}$, and the payoff function for player $i \in N$ is defined by

$$\begin{cases} v_i(\alpha, a) = u_i(a) & \\ v_i(\beta, a) = -1 & \text{if } a_i \neq x_i \\ v_i(\beta, a) = 1 & \text{if } a_i = x_i \end{cases}$$

for every $a \in A$ and the payoff function for player 0 is defined by

$$\begin{cases} v_0(\alpha, a) = 2 & \text{if } a = x \\ v_0(\alpha, a) = -1 & \text{if } a \neq x \\ v_0(\beta, a) = 0 & \end{cases}$$

for every $a \in A$. One easily verifies that (α, x) is the unique Nash equilibrium of H. Since $\phi(H) \subseteq \mathrm{NE}(H)$ we infer by (r-NEM) that $(\alpha, x) \in \phi(H)$. By (CONS) we get $(\alpha, x) \in \phi(H^{N,\{(\alpha,x)\}})$. Since player 0 is a dummy player in $H^{N,\{(\alpha,x)\}}$ we get, by (DOP), $x \in \phi(G)$, which finishes the proof. ■

In Peleg and Tijs (1996) the Nash equilibrium concept is characterized by one-person rationality, consistency, and converse consistency. This result could be "duplicated" in the style of Proposition 2.2 by adjusting the definition of (RAT) (such that it selects all maximizers in games with at most one non-dummy player) and by giving a definition of converse consistency, which takes into account the new notion of a reduced game. However, we will not focus on converse consistency in this paper.

3 Set-valued solutions

We now turn our attention to set-valued solutions and generalize the axioms, mentioned in Section 2, such that they apply to set-valued solutions. We then present some examples and results.

A *set-valued solution* on Γ is a map ψ which assigns to every game $G = \langle N, A, u \rangle \in \Gamma$ a collection $\psi(G)$ of product sets, which are non-empty subsets of A. With every point-valued solution ϕ we can associate the set-valued solution $\hat{\phi}$ which assigns to every $G \in \Gamma$ the collection $\hat{\phi}(G) = \{\{x\} : x \in \phi(G)\}$. In this way the set-valued solutions can be viewed as a generalization of the point-valued solutions.

In order to give the definition of the consistency axiom for set-valued solutions, we first have to define the notion of a reduced game with respect to some product set and some coalition.

For a $G = \langle N, A, u \rangle \in \Gamma$, for a coalition $S \subset N$, and for a product set $B = \Pi_{i \in N} B_i \subseteq A, B \neq \emptyset$ *the reduced game of* G *with respect to* S *and* B is the game $G^{S,B} = \langle N, \Pi_{i \in S} A_i \times \Pi_{i \in N \setminus S} B_i, \tilde{u} \rangle$, where $\tilde{u} = (\tilde{u}_i)_{i \in N}$ is the vector of restrictions of the payoff functions u_i $(i \in N)$ to $\Pi_{i \in S} A_i \times \Pi_{i \in N \setminus S} B_i$. Note that this game belongs to Γ, that it has $|N|$ players, regardless of whether any B_i is a singleton or not, and that, if B is a singleton set, this definition coincides with the definition of a reduced game in Section 2.

The definitions of consistency, rationality, non-emptiness, restricted non-emptiness, and the dummy out property for set-valued solutions are straightforward.

Definition 3.1. A set-valued solution ψ on Γ satisfies

(i) *consistency* (CONS) if for every $G \in \Gamma$, $S \subset N$, $B \in \psi(G)$ we have $B \in \psi(G^{S,B})$;
(ii) *rationality* (RAT) if for every $G = \langle N, A, u \rangle \in \Gamma$, $B \in \psi(G)$, $i \in N$, and $b_i \in B_i$ there exists a $\mu_{-i} \in \Delta(\Pi_{j \in N \setminus \{i\}} A_j)$ such that $b_i \in \operatorname{argmax}_{a_i \in A_i} u_i(a_i, \mu_{-i})$;
(iii) *non-emptiness* (NEM) if for every $G \in \Gamma$ we have $\psi(G) \neq \emptyset$;
(iv) *restricted non-emptiness* (r-NEM) if for every $G \in \Gamma$ with $\text{NE}(G) \neq \emptyset$ we have $\psi(G) \neq \emptyset$;
(v) the *dummy out property* (DOP) if for every $G = \langle N, A, u \rangle \in \Gamma$ and for every $i \in N$ with $|A_i| = 1$ and $G' = \langle N \setminus \{i\}, \Pi_{j \in N \setminus \{i\}} A_j, (\tilde{u}_j)_{j \in N \setminus \{i\}} \rangle \in \Gamma$ we have $\psi(G) = A_i \times \psi(G')$. Here, again, the payoff functions $\tilde{u}_j$ $(j \in N \setminus \{i\})$ are defined by $\tilde{u}_j(a_{-i}) = u_j(a_{-i}, a_i)$ where a_i is the unique element of A_i.

We now give several examples of set-valued solutions. The two first ones are included for illustrative purposes and the others turn out to be important for the results in this section.

Example 3.1. Examples of set-valued solutions are

(i) the solution EMP on Γ which assigns to every $G \in \Gamma$ the empty collection;
(ii) the solution ALL on Γ which assigns to every $G = \langle N, A, u \rangle \in \Gamma$ the collection of all non-empty product sets $B \subseteq A$;
(iii) the solution $\widehat{\text{NE}}$ on Γ, associated with the point-valued solution NE, which assigns to every $G \in \Gamma$ the collection of singleton sets that contain a Nash equilibrium;
(iv) the solution BRP on Γ, which assigns to every $G = \langle N, A, u \rangle \in \Gamma$ the collection of product sets, having the *best response property*, i.e. the collection of product sets $B = \Pi_{i \in N} B_i$ such that for every $i \in N$ and for every $b_i \in B_i$ there exists a $\mu_{-i} \in \Delta(\Pi_{j \in N \setminus \{i\}} B_j)$ with $b_i \in \operatorname{argmax}_{a_i \in A_i} u_i(a_i, \mu_{-i})$;
(v) the solution BRP^+ on Γ, which assigns to every $G \in \Gamma$ the collection of maximal product sets, having the best response property;
(vi) the solution BRP^- on Γ, which assigns to every $G \in \Gamma$ the collection of minimal product sets, having the best response property.

The first three examples are self-explanatory. BRP is a coarsening of $\widehat{\text{NE}}$. If x is a Nash equilibrium of a game G then $\{x\}$ has the best response property. However, elements of BRP(G) are not required to be singletons, so BRP(G) is a superset of $\widehat{\text{NE}}(G)$ for any game G. BRP^+ is a refinement of BRP. For every $G \in \Gamma$, $\text{BRP}^+(G)$ consists of all product sets B with the best response property, such that there is no product set $B' \supset B$ having this property. In fact, it follows from the work of Bernheim (1984) and Pearce (1984) that this last collection contains only one set, namely the product set of *rationalizable strategies*, where a strategy a_i of player i is rationalizable if there exists a $B = \Pi_{i\in N} B_i$ with the best response property such that $a_i \in B_i$.[9] For every $G \in \Gamma$, $\mathrm{BRP}^-(G)$ consists of all product sets B with the best response property, such that there is no product set $B' \subset B$ having this property.

In Proposition 2.1 we showed that every point-valued solution, satisfying (CONS) and (RAT), is a refinement of the Nash equilibrium concept. In the following proposition we show that set-valued solutions, satisfying (CONS) and (RAT), are refinements of BRP.

Proposition 3.1. *Let ψ be a set-valued solution on Γ satisfying (CONS) and (RAT). Then $\psi(G) \subseteq BRP(G)$ for every $G \in \Gamma$.*

Proof. Let $G = \langle N, A, u\rangle \in \Gamma$. If $|N| = 1$ then $\psi(G) \subseteq \mathrm{BRP}(G)$ follows by (RAT). Suppose now that G has at least two players. Let $B = \Pi_{i\in N} B_i \in \psi(G)$ and $i \in N$. By (CONS) we have $B \in \psi(G^{\{i\},B})$ and hence, by (RAT), we infer that every $b_i \in B_i$ is a best response (of all strategies in A_i) to some belief $\mu_{-i} \in \Delta(\Pi_{j\in N\setminus\{i\}} B_j)$. So, $B \in \mathrm{BRP}(G)$. ■

The following proposition shows which solutions in Example 3.1 satisfy (CONS) and (RAT).

Proposition 3.2. *The solutions EMP, $\hat{NE}$, BRP, and BRP^+ satisfy (CONS) and (RAT).*

Proof. One easily verifies that EMP satisfies (CONS) and (RAT). In order to prove that $\hat{\mathrm{NE}}$ and BRP satisfy (CONS) note that a (pure) Nash equilibrium a in a $G \in \Gamma$ remains a Nash equilibrium in the reduced game $G^{S,\{a\}}$ for every $S \subset N$ and every product set B with the best response property has still the best response property in the reduced game $G^{S,B}$ for every $S \subseteq N$. To prove that also BRP^+ satisfies (CONS) let $G \in \Gamma$, $S \subset N$, and $R = \Pi_{i\in N} R_i$ be the product set of rationalizable strategies in G. Denote furthermore by $R' = \Pi_{i\in N} R'_i$ the product set of rationalizable strategies in $G^{S,R}$. Since R has the best response property in G, it also has the best response property in $G^{S,R}$. Therefore $R_i \subseteq R'_i$ for every $i \in N$. In fact, by definition of $G^{S,R}$, $R_i = R'_i$ for every $i \in N\setminus S$. Since for every $i \in N\setminus S$ any $r_i \in R'_i (= R_i)$ is a best response to some belief $\rho_{-i} \in \Delta(\Pi_{j\in N\setminus\{i\}} R_j) \subseteq \Delta(\Pi_{j\in N\setminus\{i\}} R'_j)$ and for every $i \in S$ any $r_i \in R'_i$ is a best response to some belief $\rho_{-i} \in \Delta(\Pi_{j\in N\setminus\{i\}} R'_j)$ the set R' has the best response property in G. Therefore $R' \subseteq R$ and hence $R' = R$, which proves that R satisfies (CONS).

In order to prove that the solutions $\hat{\mathrm{NE}}$, BRP, and BRP^+ satisfy (RAT) it is sufficient to note that these solutions only select product sets with the best response property. ■

[9] We note that this definition allows for "correlated beliefs" which is common nowadays (see e.g. Osborne and Rubinstein (1994, Definition 55.1)) but was precluded in the original 1984 papers. See Bernheim (1986) for some related discussion.

An example of an inconsistent refinement of BRP is the solution BRP^-.

Example 3.2. Let $G = \langle N, A, u \rangle \in \Gamma$ be the bimatrix game with $N = \{1, 2\}$, $A_1 = \{a, b\}$, $A_2 = \{c, d, e\}$, and u given by

	c	d	e
a	1, 2	1, 1	2, 0
b	2, 0	1, 1	1, 2

One easily verifies that $B = \{a, b\} \times \{d\}$ is a minimal set having the best response property. However, with $S = \{1\}$, the reduced game of G with respect to S and B is the bimatrix game with payoff matrix

	d
a	1, 1
b	1, 1

which admits only $\{a\} \times \{d\}$ and $\{b\} \times \{d\}$ as minimal sets having the best response property. Therefore, the solution BRP^- on Γ does not satisfy (CONS).

For some finite games G the collection $\hat{\mathrm{NE}}(G)$ may be empty. Therefore $\hat{\mathrm{NE}}$ satisfies (r-NEM) but not (NEM). However, BRP^-, BRP, and BRP^+ all satisfy (NEM). In order to see this note that the mixed extension of any finite strategic game $G = \langle N, A, u \rangle$ possesses a Nash equilibrium $x = (x_i)_{i \in N}$ (Nash (1951)). Now let, for every $i \in N$, $B_i \subseteq A_i$ be the support of x_i. Then the product set $B = \Pi_{i \in N} B_i$ has the best response property. So every finite strategic game G admits a product set with the best response property and, a fortiori, a minimal set with the best response property. Since (NEM) is a stronger axiom than (r-NEM) we infer that the solutions BRP^-, BRP, and BRP^+ also satisfy (r-NEM). One easily verifies that all solutions in Example 3.1 satisfy (DOP).

If we consider the solutions mentioned in Example 3.1 on Γ then the following table summarizes the statements made above:

	EMP	ALL	$\hat{\mathrm{NE}}$	BRP	BRP^+	BRP^-
(CONS)	+	+	+	+	+	−
(RAT)	+	−	+	+	+	+
(NEM)	−	+	−	+	+	+
(r-NEM)	−	+	+	+	+	+
(DOP)	+	+	+	+	+	+

The axioms (CONS), (RAT), (r-NEM), and (DOP) are logically independent. To see this, note that the solution BRP^- satisfies (RAT), (r-NEM), and (DOP) but not (CONS), the solution ALL satisfies (CONS), (r-NEM), and (DOP) but not (RAT), and the solution EMP satisfies (CONS), (RAT), and (DOP) but not (r-NEM). The solution, which coincides with $\hat{\mathrm{NE}}$ for games with at most three players and which coincides with BRP for games with at least four players, satisfies (CONS), (RAT), and (r-NEM) but not (DOP).

In the case of point-valued solutions the Nash equilibrium concept NE is completely characterized on Γ by consistency, rationality, restricted non-emptiness, and dummy out property (Proposition 2.2). The table above shows that this is not the case for set-valued solutions. These axioms are not only satisfied by $\hat{\text{NE}}$ but also by BRP and BRP$^+$. Moreover, as seen above, the two latter solutions even have the virtue of being non-empty for *every* finite game.

4 Refining BRP

In this section we accept as a working hypothesis that the consistency criterion is "desirable". Then, as argued in the Introduction, the result that there is no proper refinement of the Nash equilibrium concept satisfying consistency, rationality, and non-emptiness is troubling for the theory of equilibrium refinements. In Section 3 it was shown that several set-valued solutions satisfy these properties, and the Propositions 3.1 and 3.2 together imply that BRP is the unique maximal such solution. In light of this result we suggest a new approach for normative game theory: Shift attention from the point-valued solution NE to the set-valued solution BRP and refine the latter while preserving consistency and other properties deemed desirable! In this section we suggest one way of following this line of research.

Say a product set of strategies is recommended to the players. One might argue that this recommendation is not really self-enforcing unless for every player i and every belief consistent with the other players confirming with the recommendation, no strategy outside i's recommended set is optimal for him to use. Basu and Weibull (1991), Hurkens (1995, pp 13–14), and also Nash (1951, pp 290–291) discuss related ideas. Here we make use of Basu and Weibull's (1991) notion of a set *closed under rational behavior* – a *curb* set. The definition of a curb set, as well as of two finer notions that turn out to be useful, are as follows:

Definition 4.1. Let $G = \langle N, A, u \rangle \in \Gamma$. A non-empty product set $B = \Pi_{i \in N} B_i \subseteq A$ is called

(i) *curb* if for every $i \in N$ and $a_i \in A_i$, which is a best response to some belief $\mu_{-i} \in \Delta(\Pi_{j \in N \setminus \{i\}} B_j)$, we have $a_i \in B_i$;
(ii) *tight curb* if B is curb and has the best response property;
(iii) *minimal curb* if B is curb and there is no product set $B' \subset B$ which is curb.

Basu and Weibull (1991) show that every finite game admits at least one minimal curb set and that the minimal curb sets and the minimal tight curb sets coincide. Therefore every finite game also possesses at least one tight curb set.

Since we are interested in refinements of BRP we will investigate whether the two set-valued solutions, which select the tight curb and minimal curb sets respectively, satisfy consistency and other properties. We hence define the following solutions on Γ:

$$\text{t-CURB}(G) = \{B \subseteq A : B \text{ is tight curb}\};$$

$$\text{min-CURB}(G) = \{B \subseteq A : B \text{ is minimal curb}\}$$

for every $G = \langle N, A, u \rangle \in \Gamma$. Every tight curb set or every minimal curb set which is singleton contains a strict equilibrium. Therefore t-CURB and min-CURB may be viewed as set-valued analogues of the point-valued solution assigning to every game the collection of strict equilibria. Of course there are finite games without strict equilibria. The following proposition shows that the solutions t-CURB and min-CURB satisfy (NEM) as well as (CONS), (RAT), and (DOP).

Proposition 4.1. *The solutions t-CURB and min-CURB satisfy (CONS), (RAT), (NEM), and (DOP).*

Proof. In order to prove that t-CURB satisfies (CONS) let $G = \langle N, A, u \rangle \in \Gamma$, $B = \Pi_{i \in N} B_i$ a tight curb set in G and $S \subset N$. Since BRP satisfies (CONS) and B has the best response property in G it also has the best response property in $G^{S,B}$. Since by changing from G to $G^{S,B}$ no best responses to beliefs in B are deleted B is also curb in $G^{S,B}$. Hence B is a tight curb set in $G^{S,B}$ which proves that t-CURB satisfies (CONS). For the proof of the consistency of min-CURB assume that $B = \Pi_{i \in N} B_i$ is a minimal curb set in $G = \langle N, A, u \rangle$ and $S \subset N$. Suppose there is $B' \subset B$ which is a curb set in $G^{S,B}$. One easily verifies in that case that B' is a curb set in G, which contradicts the minimality of B. Hence B is also a minimal curb set in $G^{S,B}$ which proves the consistency of min-CURB.

Since the solutions t-CURB and min-CURB only select sets with the best response property both solutions satisfy (RAT).

In Proposition 1 of Basu and Weibull (1991) the authors show that every finite game admits at least one minimal curb set and in Proposition 2 they show that the minimal curb sets and the minimal tight curb sets coincide. As a consequence we get that both solutions t-CURB and min-CURB satisfy (NEM).

One easily verifies that the solutions t-CURB and min-CURB satisfy (DOP). ■

Proposition 4.1 illustrates that the research program we have proposed is feasible. We believe the program promises to deliver solutions that have cutting power in certain applications. To argue this point, consider the following game which is a special case of a "Burning Money" example discussed by Hurkens (1996, Figure 2 with $c = 1$):

	e	f	g	h
a	9, 5	9, 5	0, 4	0, 4
b	4, 4	4, 4	6, 7	6, 7
c	8, 5	−1, 4	8, 5	−1, 4
d	3, 4	5, 7	3, 4	5, 7

In this game $\{a\} \times \{e, f\}$ is the unique minimal curb set (see Hurkens (1996, p 188) for a proof). Note that the strategies c and g are not involved, despite the fact that (c, g) is a proper equilibrium.[10]

This is not to say that minimal curb sets *always* promote sharper predictions than does standard equilibrium concepts. For example, in a "matching

[10] Note also that, as observed by Hurkens (1996), iterated elimination of weakly dominated strategies has no cutting power in this game. There are no weakly dominated strategies.

pennies" game the entire set of strategy profiles is the unique minimal curb set while no equilibrium exists. (If the game's mixed extension is considered the entire set of strategy profiles is again the unique minimal curb set, while the game has a unique equilibrium.)

5 Summary

The axiom of consistency for *point-valued* solutions of strategic games is introduced in Peleg and Tijs (1996). They show that any consistent point-valued solution which satisfies a rationality requirement must be a refinement of the Nash equilibrium concept. Norde et al. (1996) show that requiring also the solution to be non-empty in games that do possess equilibria leads to a *characterization* of the Nash equilibrium concept. Given that the consistency condition is regarded as "desirable", this result may be taken as troublesome for the theory of equilibrium refinements.

We argue that *set-valued* solutions are natural objects of study for the classical theory of games which is concerned with offering self-enforcing recommendations to rational players. We extend the axiom of consistency to apply to such solutions. In the new context the aforementioned problems disappear, although the Nash equilibrium concept no longer takes center stage. Any consistent set-valued solution satisfying a rationality requirement must be a refinement of BRP, the solution assigning to each game the collection of sets with the best response property.

BRP itself satisfies these properties and also many refinements do so. Based on this finding we propose to refine BRP instead of the Nash correspondence, while requiring that consistency and other properties deemed desirable are preserved. To exemplify this line of research, we use Basu and Weibull's (1991) notion of a curb set. This leads for example to the solution min-CURB, the refinement of BRP which selects all product sets which are minimal curb. This solution has considerable cutting power in certain games. We show that min-CURB satisfies consistency, a rationality requirement, and non-emptiness.

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