Well-posedness of a class of linear networks with ideal diodes

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Abstract

We consider electrical networks containing linear elements, independent voltage/current sources and ideal diodes. As a test of model validity, we have shown the well-posedness (in the sense of existence and uniquness of solutions) of such network models under a condition on the zero structure at infinity of the underlying linear system. It is also shown that this condition is implied by passivity. As an additional result, the set of initials states for which the corresponding solution trajectory is impulse-free is explicitly characterized.

1 Introduction

The appropriateness of a proposed mathematical model for a given physical system can be tested in various ways. A very basic test is the following: if the physical system that is being modeled is deterministic in the sense that it shows identical behavior under identical circumstances, then the mathematical model should have the same property. Model validity would be put into serious doubt if it would turn out that the equations of the mathematical model allow multiple solutions for some initial data. With any model formulation for a deterministic physical system it is therefore important to establish well-posedness of the model, i. e., existence and uniqueness of solutions for feasible initial conditions.

This paper considers the well-posedness of models for electrical networks with diodes. In the engineering literature, mathematical models that make use of the ideal diode characteristic are routinely used for such networks. Remarkably enough, it seems that the well-posedness of such models has not been rigorously established before. Although general results from the theory of ordinary differential equations may be used to establish well-posedness of network models containing elements with Lipschitzian characteristics (see for instance [13]) or in special cases even for non-Lipschitzian characteristics (see for instance [3, 8]), such results do not cover the ideal diode characteristic since it cannot be reformulated as a current or voltage-controlled resistor. Neither does it seem possible to derive general well-posedness results for network models with ideal diodes from the theory of differential equations with discontinuous righthand sides [4], which in network terminology is concerned with models involving ideal relay elements. The theory that we develop below will be based on the theory of *complementarity systems* that has been worked out in a series of recent papers [5–7, 10, 11]; see also [12].

It is easy to come up with examples of mathematical models involving ideal diode characteristics (which are equivalent to complementarity conditions) that are not well-posed; see for instance [10]. Therefore, some restrictions need to be imposed. In this paper we consider network models that contain only linear elements besides the ideal diodes. We will study this class of models in the more general setting of complementarity conditions coupled to linear dynamical systems with a special zero structure at infinity. Some might say that it is "intuitively clear" that such network models are well-posed; nevertheless, ideal diodes are only approximations to real diodes and so the fact that actual networks with diodes behave deterministically does not make it evident that the corresponding mathematical models with idealized elements have unique solutions. Rather, as argued above, one should consider well-posedness as a test of model validity.

The paper is organized as follows. In Section 2 we first of all develop a precise notion of solution for a network model with ideal diode elements. Some elements of the theory of distributions will be used in order to allow for possible jump solutions. Then in Section 3 we briefly discuss the linear complementarity problem (LCP) of mathematical programming that plays an important role in our development. The main results follow in Section 4. We present a general well-posedness result for linear passive lumped networks with ideal diodes. We also discuss the nature of solutions; in particular we show that a jump may only occur at the initial time instant and we characterize the set of initial conditions that give rise to jumps. The paper will be closed by conclusions in Section 5 and proofs in Section 6.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ nonnegative real numbers, $\mathbb{C}$ complex numbers, $\mathbb{R}(s)$ the field of real rational function in the variable s. All inequalities concerning vectors must be understood componentwise. For any integer m, $\overline{m}$ denotes the set $\{1, 2, \ldots, m\}$. For an index set $K \subseteq \overline{m}$, K^c and $|K|$ denote the set $\overline{m} \setminus K$ and the number of elemnets of K, respectively. For any $A \in \mathbb{R}^{nxm}$, $J \subseteq \overline{n}$, and $K \subseteq \overline{m}$, A_{JK} denotes the submatrix $\{A_{ij}\}_{j\in J, k\in K}$. If $J = \overline{n}$ ($K = \overline{m}$), we also write $A_{\bullet K}$ ($A_{J\bullet}$). $\dim(\mathcal{U})$ denotes the dimension of the linear space $\mathcal{U}$. The orthogonal space of $\mathcal{U} \subseteq \mathbb{R}^n$ is denoted by $\mathcal{U}^\perp := \{v \in \mathbb{R}^n \mid v^\top u = 0 \text{ for all } u \in \mathcal{U}\}$. Given a mapping $A : \mathcal{U} \to \mathcal{V}$, we denote the image of A by $\operatorname{im} A := \{v \in \mathcal{V} \mid v = Au \text{ for some } u \in \mathcal{U}\}$ and the kernel of A by $\ker A := \{u \in \mathcal{U} \mid Au = 0\}$. $A|_{\mathcal{W}}$ will denote the restriction of A to $\mathcal{W} \subseteq \mathcal{U}$. For any two real vectors v and w, we write $v \perp w$ if $v^\top w = 0$.

2 Linear Complementarity Systems

Throughout the paper, we consider linear networks with ideal diodes at each port. A standing assumption will be the following.

Assumption 2.1 The linear network admits a state space representation of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t)$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^p$, and A, B, C, D and E are matrices with appropriate sizes. Here each (u_k, y_k) pair belongs to the set $\{(-v_k, i_k), (i_k, -v_k)\}$ where v_k and i_k denote the voltage and the current of the diode coupled to kth port, and w represents the independent voltage and/or current sources contained in the network.

By taking into account the characteristics of the ideal diodes as shown in Figure 2, the overall system can be described as a *linear complementarity system* of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t) \tag{1a}$$
$$y(t) = Cx(t) + Du(t) \tag{1b}$$
$$0 \leq u(t) \perp y(t) \geq 0. \tag{1c}$$

We denote the above system by LCS(A, B, C, D, E). For the previous study on this class of systems, the reader is refered to [5–7,10,11]. From a *hybrid system* point of view, one can distinguish 2^m *modes* (or *circuit topologies* as it is sometimes called in circuit theory) depending on whether the diodes are conducting or blocking. Every index set $K \subseteq \overline{m}$ determines one of these modes by imposing the constraints $y_K = 0$ and $u_{K^c} = 0$. Associated to each mode K, there are a linear dynamics given by

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t)$$
$$y(t) = Cx(t) + Du(t)$$
$$y_K(t) = 0, \qquad u_{K^c}(t) = 0$$

and a set called *invariants* given by

$$y_{K^c}(t) \geq 0, \qquad u_K(t) \geq 0. \tag{3a}$$

Starting at a given mode, the system trajectories must obey the dynamics corresponding to this mode as long as they belong to the invariant set, i.e., satisfy the inequalities (3). Time instants at which the state variables tend to leave the invariant set are called *event times*. Whenever an event occurs, another mode will become active depending on the state variables and the values of the voltage/current sources at the event time. Before giving a precise definition of the solution concept, we illustrate the above features of the systems under consideration in the following example.

Example 2.2 Consider the linear RLC circuit (with $R = 1$ Ohm, $L = 1$ Henry and $C = 1$ Farad) coupled to two ideal diodes as shown in Figure 1. By choosing the voltage across the capacitor and the current through the inductor as the state variables and by taking into account the ideal diode characteristic depicted in Figure 2, the governing equations of the network can be given by

$$\tfrac{d}{dt}v_C = i_L - u_1 + u_2$$
$$\tfrac{d}{dt}i_L = -v_C - i_L - u_2$$
$$y_1 = -v_C, \quad y_2 = v_C + i_L + u_2$$
$$0 \leq u \perp y \geq 0$$

where u_k and y_k denote i_{D_k} and $-v_{D_k}$, respectively for $k = 1, 2$.

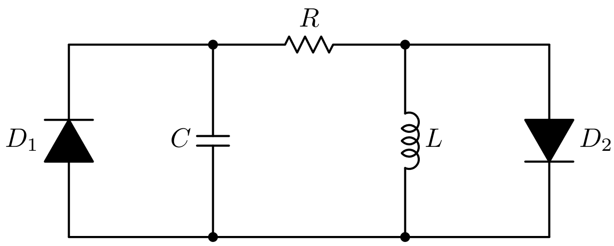

Figure 1: RLC circuit with ideal diodes

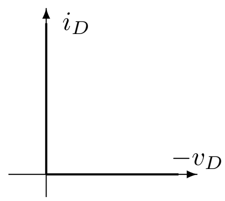

Figure 2: Ideal diode characteristic

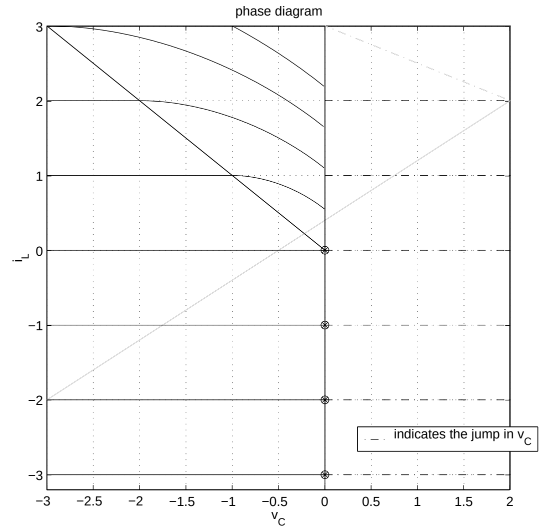

Figure 3: Phase diagram of the system given in Example 2.2

The phase diagram of the circuit is shown in Figure 3. We investigate the behaviour of the network for two initial conditions, namely for $(v_C(0), i_L(0)) = (-e, 1)$ and $(v_C(0), i_L(0)) = (1, 1)$.

- *Case 1:* $(v_C(0), i_L(0)) = (-e, 1)$
  The state trajectories can be computed for the initial state $(v_C(0), i_L(0)) = (-e, 1)$ as follows.

$$v_c(t) = \begin{cases} -e^{(t-1)} & \text{if } 0 \le t \le 1 \\ f(t) & \text{if } 1 \le t \le 1 + \frac{2\sqrt{3}}{9}\pi \\ 0 & \text{if } 1 + \frac{2\sqrt{3}}{9}\pi \le t \end{cases}$$

$$i_L(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ g(t) & \text{if } 1 \le t \le 1 + \frac{2\sqrt{3}}{9}\pi \\ e^{-(t+\frac{\pi}{6})} & \text{if } 1 + \frac{2\sqrt{3}}{9}\pi \le t \end{cases}$$

  where

$$\begin{aligned} f(t) &= -e^{-\frac{1}{2}(t-1)}[\cos(\tfrac{\sqrt{3}}{2}(t-1)) \\ &\quad - \tfrac{\sqrt{3}}{3}\sin(\tfrac{\sqrt{3}}{2}(t-1))] \\ g(t) &= e^{-\frac{1}{2}(t-1)}[\cos(\tfrac{\sqrt{3}}{2}(t-1)) \\ &\quad + \tfrac{\sqrt{3}}{3}\sin(\tfrac{\sqrt{3}}{2}(t-1))]. \end{aligned}$$

- *Case 2:* $(v_C(0), i_L(0)) = (1, 1)$
  The solution complying with the circuit can be given by

$$v_C(t) = 0,\ i_L(t) = e^{-t},\ i_{D_1}(t) = \delta + e^{-t},$$
$$v_{D_1}(t) = 0,\ i_{D_2}(t) = 0,\ v_{D_2}(t) = e^{-t}$$

  where δ denotes the Dirac impulse. The physical interpretation of the jump in the variable v_C is that there is an instantaneous discharge of the capacitor.

Example 2.2 indicates that the trajectories of the system are made of the concatenations of some trajectories produced by some linear systems as already expected due to the hybrid features of the system and also that the trajectories must incorporate Dirac impulses in order to capture the *inconsistent* initial states. Next, we recall the notion of *initial solution* which will serve as the 'atomic' element of the (global) solution concept. To do so, we need to introduce so-called *Bohl distributions*. A function $v : \mathbb{R}_+ \to \mathbb{R}$ is said to be a *Bohl function* if it has a rational Laplace transform. The set of all such functions is denoted by $\mathcal{B}$. In a similar fashion, a distribution v is said to be a *Bohl distribution* if it is of the form $v = v_{\text{imp}} + v_{\text{reg}}$ where the *impulsive* part $v_{\text{imp}} = v_0\delta$ for some $v_0 \in \mathbb{R}$ and the *regular* part v_{reg} belongs to $\mathcal{B}$. The set of all such distributions is denoted by $\mathcal{BD}$. $\mathcal{BD}$ can be viewed as the direct sum of the spaces $\delta\mathbb{R}$ and $\mathcal{B}$. We say that $v \in \mathcal{BD}$ is *initially nonnegative* if its Laplace transform $\hat{v}(s)$ satisfies $\hat{v}(\sigma) \ge 0$ for all sufficiently large $\sigma \in \mathbb{R}$. It is known (see [5]) that $v = v_0\delta + v_{\text{reg}}$ is initially nonnegative if and only if ($v_0 > 0$) or ($v_0 = 0$ and there exists $\epsilon > 0$ such that $v_{\text{reg}}(t) \ge 0$ for $t \in [0, \epsilon)$) .

Definition 2.3 The triple $(u, x, y) \in \mathcal{BD}^{m+n+m}$ is an *initial solution* of LCS(A, B, C, D, E) for the input $w \in \mathcal{B}$ and the initial state x_0 if there exists an index set $K \subseteq \overline{m}$ such that

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew + x_0\delta \\ y &= Cx + Du \\ y_K &= 0, \quad u_{K^c} = 0 \end{aligned}$$

hold in the distributional sense, and u and y are initially nonnegative.

To define the global solution concept, we need to introduce the space of *piecewise Bohl distributions* which are the solution candidates for linear complementarity systems. The notation $v|_\Omega$ denotes the restriction of the function v to the set Ω. A function $v : \mathbb{R}_+ \to \mathbb{R}$ is said to be a *piecewise Bohl function* if for each $t \in \mathbb{R}_+$ there exist an $\epsilon > 0$ and a $w \in \mathcal{B}$ such that $v|_{[t,t+\epsilon)} = w|_{[0,\epsilon)}$. The set of all such functions is denoted by $\mathcal{PB}$. The set $\mathcal{PBB}$ consists of all $\mathcal{PB}$-functions bounded on every compact set. In a similar fashion to Bohl distributions, a distribution v is said to be a *piecewise Bohl distribution* if it is of the form $v = v_{\mathrm{imp}} + v_{\mathrm{reg}}$ where the *impulsive* part $v_{\mathrm{imp}} = v_0\delta$ for some $v_0 \in \mathbb{R}$ and the *regular* part v_{reg} belongs to $\mathcal{PB}$. The set of all such distributions is denoted by $\mathcal{PBD}$.

Definition 2.4 The triple $(u, x, y) \in \mathcal{PBD}^{m+n+m}$ is a *(global) solution* of LCS(A, B, C, D, E) for the input $w \in \mathcal{PBB}^p$ and the initial state x_0 if the following conditions hold.

1. There exists an initial solution $(\bar{u}, \bar{x}, \bar{y})$ such that

$$(u_{\mathrm{imp}}, x_{\mathrm{imp}}, y_{\mathrm{imp}}) = (\bar{u}_{\mathrm{imp}}, \bar{x}_{\mathrm{imp}}, \bar{y}_{\mathrm{imp}}).$$

2. The equations

$$\dot{x} = Ax + Bu + Ew + x_0\delta$$
$$y = Cx + Du$$

hold in the distributional sense.

3. For all $t \in \mathbb{R}_+$,

$$0 \leq u_{\mathrm{reg}}(t) \perp y_{\mathrm{reg}}(t) \geq 0.$$

We say that the solution (u, x, y) is *impulse-free* whenever $u_{\mathrm{imp}} = 0$, i.e., there is no impulsive part.

The first item in the Definition 2.4 imposes a relation between the impulsive part and the rest of the solution. In the following example, we illustrate the necessity of such a connection.

Example 2.5 [1] Consider the simple circuit shown in Figure 4. By denoting the voltage across the capacitor and the diode by v_c and v_d, respectively and the current through the diode by i_d, one can obtain the circuit equations as follows:

$$\dot{v}_c = -i_d, \quad v_d = v_c$$
$$0 \geq v_d \perp i_d \geq 0.$$

They can be rewritten in the form of a linear complementarity system

$$\dot{x} = u, \quad y = x \tag{4a}$$
$$0 \leq u \perp y \geq 0 \tag{4b}$$

with the definitions $u = i_d$, $x = -v_c$, and $y = -v_d$. For the initial state $x_0 = -1$, the triple $(u, x, y) = (a\delta, a-1, a-1)$ with $a \geq 1$ satisfies the last two items of the Definition 2.4. However, $(a\delta, a-1, a-1)$ is only a solution for initial state $x_0 = -1$, if $a = 1$, since this is the only solution complying with the circuit from a physical point of view. Its interpretation is that there is an instantaneous discharge of the capacitor. Note that $(u, x, y) = (\delta, 0, 0)$ is indeed the unique initial solution and hence due to item 1 of Definition 2.4, the only allowed global solution.

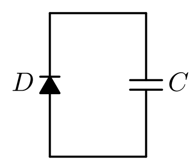

Figure 4: Circuit illustrating the need for Definition 2.4 item 1

3 Linear Complementarity Problem (LCP)

We briefly recall the linear complementarity problem of mathematical programming. For an extensive survey on the problem, the reader is referred to [2].

Problem 3.1 (LCP(q, M)) Given $q \in \mathbb{R}^m$ and $M \in \mathbb{R}^{m\mathrm{x}m}$, find $z \in \mathbb{R}^m$ such that

$$0 \leq z \perp q + Mz \geq 0 \tag{5}$$

We say that z *solves* LCP(q, M) if z satisfies (5). The set of all solutions of LCP(q, M) will be denoted by SOL(q, M). In general, SOL(q, M) may be empty set. K(M) denotes the set $\{q \mid \text{ SOL}(q, M) \neq \emptyset\}$. It is easy to see that $\mathbb{R}^m_+ \subseteq$ K(M) for all $M \in \mathbb{R}^{m\mathrm{x}m}$.

Next, we define some matrix clases used in the sequel.

Definition 3.2 A matrix $M \in \mathbb{R}^{m\mathrm{x}m}$ is called

- *nondegenerate* if all its principal matrices are nonsingular.
- a *P-matrix* if all its principal minors are positive.
- *positive (nonnegative) definite* if $x^\top Mx > 0$ (≥ 0) for all $0 \neq x \in \mathbb{R}^m$.

- *copositive* if $x^\top Mx \geq 0$ for all $x \geq 0$.
- *copositive-plus* if it is copositive and the following implication holds:

$$x^\top Mx = 0 \text{ and } x \geq 0 \Rightarrow (M + M^\top)x = 0.$$

For a given nonempty set $\mathcal{S}$, we say that the set $\{v \mid v^\top w \geq 0 \text{ for all } w \in \mathcal{S}\}$ is the *dual cone* of $\mathcal{S}$. It is denoted by $\mathcal{S}^*$. The next lemma states some of the standard results on the above defined matrix classes.

Lemma 3.3 *Let $M \in \mathbb{R}^{m\times m}$ be given. The following statements hold.*

1. *LCP(q, M) has a unique solution for all $q \in \mathbb{R}^m$ if and only if M is a P-matrix.*
2. *If M is copositive-plus then $K(M) = ($SOL$(0, M))^*$.*

The proofs of items 1 and 2 can be found in [2, Theorem 3.7.7 and Corollary 3.8.10].

4 Main Results

We will often use the following low index assumption.

Assumption 4.1 The matrix triple $(B, C, D) \subset \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$ is such that $D + \sigma^{-1}CB$ is a P-matrix for all sufficiently large $\sigma \in \mathbb{R}$ and for each index set $K \subseteq \overline{m}$

$$\operatorname{im} D_{KK} \oplus C_{K\bullet}B_{\bullet K}\,(\ker D_{KK}) = \mathbb{R}^{|K|}. \tag{6}$$

In the paper, we will show that for each $K \subseteq \overline{m}$ $s^{-1}(D_{KK} + s^{-1}C_{K\bullet}B_{\bullet K})^{-1}$ is proper under the above assumption. It can be also shown that if $D_{KK} + s^{-1}C_{K\bullet}B_{\bullet K}$ is invertible as a rational matrix and $s^{-1}(D_{KK} + s^{-1}C_{K\bullet}B_{\bullet K})^{-1}$ is proper then (6) holds.

The main result of the present paper is as follows.

Theorem 4.2 *Consider a linear network such that Assumption 2.1 holds. Let (A, B, C, D, E) be a state-space representation of the network as in* (1)*. Suppose that (B, C, D) satisfies Assumption 4.1. Then, for each $x_0 \in \mathbb{R}^n$ and for each $w \in \mathcal{PBB}^p$, there exists a unique global solution of LCS(A, B, C, D, E) for the input w and the initial state x_0. Moreover, the solution with initial state x_0 is impulse-free if and only if $Cx_0 \in K(D)$.*

As a consequence of the above theorem, we have the following result on linear passive networks with ideal diodes. Note that Assumption 4.1 is obviously weaker than passivity property since it does not depend on A.

Corollary 4.3 *Consider a linear network such that Assumption 2.1 holds. Let (A, B, C, D, E) be a state-space representation of the network as in* (1)*. Suppose that (A, B, C) is minimal, B is of full column rank, and (A, B, C, D) is passive (in the sense of [14]). Then, for each $x_0 \in \mathbb{R}^n$ and for each $w \in \mathcal{PBB}^p$, there exists a unique global solution of LCS(A, B, C, D, E) for the input w and the initial state x_0. Moreover, the solution with initial state x_0 is impulse-free if and only if $Cx_0 \in ($SOL$(0, D))^*$.*

Consider the network shown in Example 2.2. It can be easily seen that (A, B, C, D, E) given by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},\ B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},\ C = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

is a state-space representation as in Assumption 2.1. It can be easily verified that (A, B, C) is minimal and B is of full column rank. Obviously, the network is passive. Hence, Corollary 4.3 implies the existence and uniqueness of solutions of the network. Note that $\mathrm{SOL}(0, D) = \{z \mid z_1 \geq 0 \text{ and } z_2 = 0\}$. Then, $(\mathrm{SOL}(0, D))^* = \{z \mid z_1 \geq 0\}$. It follows that the solution with $(v_C(0), i_L(0))$ is impulse-free if and only if $v_C(0) \leq 0$ as already depicted in Figure 3.

5 Conclusions

In this paper we showed that a class of models for electrical networks with diodes passes the validity test of well-posedness. The used network models consist of linear passive elements (e.g. resistors, inductors, capacitors, etc.), independent voltage/current sources and ideal diodes. As a consequence, the model descriptions fall within the realm of linear complementarity systems with external inputs, which form a subclass of discontinuous dynamical systems with both discrete and continuous characteristics. Using this framework, we were able to prove the existence and uniqueness of solution trajectories under a condition on the zero structure of the underlying state space description. Since this condition is implied by passivity, the well-posedness of the network models was established. As an additional result we gave an explicit characterization of the consistent states for the network model, i.e., the initial states for which the corresponding solution trajectory is impulse-free.

This line of work is currently continued by investigating possible relaxations of the conditions used here. Moreover, we are interested in the numerical simulation of electrical networks by so-called time-stepping methods. For instance, in [1] we proved the consistency – in the sense of convergence of the approximations to the real transient solution – of a time-stepping method based on the backward Euler integration routine, when applied to linear passive electrical circuits with ideal diodes.

6 Proofs

Towards the proof of the main results, we begin with several technical lemmas on LCP. In the sequel, for a given nondegenerate matrix $M \in \mathbb{R}^{n\mathrm{x}n}$, $d(M)$ is defined as follows:

$$d(M) = 2^n(\max_{J\subseteq\overline{n}} \|M_{JJ}^{-1}\|).$$

Lemma 6.1 *Assume that $M \in \mathbb{R}^{n\mathrm{x}n}$ is a P-matrix. Let z^i be the unique solution of LCP(q^i, M) for $i = 1, 2$. Then, we have*

$$\|z^1 - z^2\| \leq d(M)\|q^1 - q^2\|.$$

Proof It is known that the mapping $q \mapsto z$ where z is the unique solution of LCP(q, M) is a piecewise function on some finite number of convex polyhedral cones $\mathcal{P}_J$ for $J \subseteq \overline{n}$ as given in [2, Proposition 1.4.6]. Besides, $q \in \mathcal{P}_J$ implies that the solution z of the LCP(q, M) is of the form $z_J = -M_{JJ}^{-1}q_J$ and $z_{J^c} = 0$. Since the line segment $[q^1, q^2]$ lies in only a finite number of these cones (at most 2^n), we get the desired inequality. ■

The following two lemmas will play quite an important role in the proof of the main results.

Lemma 6.2 *Let M, $N \in \mathbb{R}^{n\mathrm{x}n}$ be given. Suppose that*

$$\mathrm{im}\ N \oplus M(\ker N) = \mathbb{R}^n.$$

Then, the following statements hold.

1. *$N + s^{-1}M$ is invertible as a rational matrix and $s^{-1}(N + s^{-1}M)^{-1}$ is proper.*
2. *There exist matrices $P \in \mathbb{R}^{p\mathrm{x}n}$ and $Q \in \mathbb{R}^{(n-p)\mathrm{x}n}$ such that*

$$\begin{bmatrix} P \\ Q \end{bmatrix} \text{ and } \begin{bmatrix} PN \\ QM \end{bmatrix}$$

are both nonsingular and $QN = 0$.

Proof *1:* Let $Q \in \mathbb{R}^{n\mathrm{x}q}$ be such that $NQ = 0$ and $\ker Q = \{0\}$. Take any $P \in \mathbb{R}^{n\mathrm{x}(n-q)}$ such that $\begin{bmatrix} P & Q \end{bmatrix}$ is nonsingular. Then, $\mathrm{im}\ P \oplus \mathrm{im}\ Q = \mathbb{R}^n$. This implies that $\mathrm{im} NP = \mathrm{im} N$ because $N(\mathrm{im} Q) = \{0\}$. Since

$$\mathrm{im}\ N \oplus M(\ker N) = \mathbb{R}^n,$$

we get

$$\mathrm{im}\ NP \oplus \mathrm{im}\ MQ = \mathbb{R}^n.$$

Consequently, $\begin{bmatrix} NP & MQ \end{bmatrix}$ is nonsingular. On the other hand, we have

$$(N + s^{-1}M)\begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} NP + s^{-1}MP & s^{-1}MQ \end{bmatrix}$$
$$= \left(\begin{bmatrix} NP & MQ \end{bmatrix} + s^{-1}\begin{bmatrix} MP & 0 \end{bmatrix}\right)\begin{bmatrix} I & 0 \\ 0 & s^{-1}I \end{bmatrix}.$$

Since $\begin{bmatrix} NP & MQ \end{bmatrix}$ is nonsingular, the first term of the left hand side is biproper, i. e., it is proper, invertible as a rational matrix and its inverse is also proper. It follows that $(N + s^{-1}M)$ is invertible as a rational matrix and $s^{-1}(N + s^{-1}M)$ is proper.

2: Let $Q \in \mathbb{R}^{q\mathrm{x}n}$ be such that $\mathrm{im}\ Q^\top = \ker N^\top$ and $\ker Q^\top = \{0\}$. Clearly,

$$\ker Q = (\mathrm{im}\ Q^\top)^\perp = (\ker N^\top)^\perp = \mathrm{im}\ N.$$

Take any $P \in \mathbb{R}^{(n-q)\mathrm{x}n}$ such that $\begin{bmatrix} P \\ Q \end{bmatrix}$ is nonsingular. Suppose that

$$\begin{bmatrix} PN \\ QM \end{bmatrix} x = 0$$

for some $x \in \mathbb{R}^n$. This means that

$$PNx = 0 \tag{7}$$
$$QMx = 0. \tag{8}$$

Since $\ker Q = \mathrm{im}\ N$, we have

$$\begin{bmatrix} P \\ Q \end{bmatrix} Nx = 0.$$

from (7). This implies that, $Nx = 0$, i. e., $x \in \ker N$. Hence, $Mx \in M(\ker N)$. On the other hand, (8) yields $Mx \in \ker Q = \mathrm{im}\ N$. Therefore, we conclude from the hypothesis that $Mx = 0$. Since Nx is also zero, $(N + s^{-1}M)x = 0$ for all $s \in \mathbb{C}$. However, this can happen only if $x = 0$ due to item 1. Hence,

$$\begin{bmatrix} PN \\ QM \end{bmatrix}$$

is nonsingular.■

Lemma 6.3 *Let $G(s) = D + C(sI - A)^{-1}B \in \mathbb{R}^{m\mathrm{x}m}(s)$ be given. Assume that (B, C, D) satisfies Assumption 4.1. Then, there exists an $\alpha > 0$ such that $d(G(\sigma)) \leq \alpha\sigma$ for all sufficiently large σ.*

Proof Note that for each $J \subseteq \overline{m}$

$$G_{JJ}(s) = (D_{JJ} + s^{-1}C_{J\bullet}B_{\bullet J})$$
$$\times [I + s^{-2}(D_{JJ} + s^{-1}C_{J\bullet}B_{\bullet J})^{-1}CAB + \ldots].$$

Since $s^{-2}(D_{JJ} + s^{-1}C_{J\bullet}B_{\bullet J})^{-1}$ is strictly proper due to Lemma 6.2 item 1, the second term of the right hand side is biproper. Then, it follows that for some $\alpha_J > 0$ and $\alpha'_J > 0$

$$\|G_{JJ}^{-1}(\sigma)\| \leq \alpha_J \|(D_{JJ} + \sigma^{-1} C_{J\bullet} B_{\bullet J})^{-1}\| \leq \alpha'_J \sigma$$

for all sufficiently large σ. Therefore, $d(G(\sigma)) \leq \alpha\sigma$ for all sufficiently large σ where

$$\alpha = 2^n (\max_{J \subseteq \overline{m}} \alpha'_J). \blacksquare$$

Next, we recall the so-called *Rational Complementarity Problem* (see [5] for a detailed discussion).

Problem 6.4 (RCP($x_0, \hat{w}(s), A, B, C, D, E$)) Given $x_0 \in \mathbb{R}^n$, $\hat{w}(s)$, and (A, B, C, D, E) with $A \in \mathbb{R}^{n \mathrm{x} n}$, $B \in \mathbb{R}^{n \mathrm{x} m}$, $C \in \mathbb{R}^{m \mathrm{x} n}$, $D \in \mathbb{R}^{m \mathrm{x} m}$ and $E \in \mathbb{R}^{n \mathrm{x} p}$ find $\hat{u}(s) \in \mathbb{R}^m(s)$ such that

1. $\hat{u}(s) \perp \hat{y}(s)$ for all $s \in \mathbb{C}$.
2. $\hat{u}(\sigma) \geq 0$ and $\hat{y}(\sigma) \geq 0$ for all sufficiently large $\sigma \in \mathbb{R}$.

where

$$\begin{aligned}\hat{y}(s) = C(sI - A)^{-1}x_0 + C(sI - A)^{-1}E\hat{w}(s) \\ + [D + C(sI - A)^{-1}B]\hat{u}(s).\end{aligned}$$

For brevity of notation, we denote RCP($x_0, \hat{w}(s), A, B, C, D, E$) by RCP($x_0, \hat{w}(s)$). There is one-to-one correspondence between the proper solutions of RCP and initial solutions of LCS as described in the following lemma.

Lemma 6.5 *Consider a given matrix quintuple (A, B, C, D, E). The following statements hold.*

1. *Let $\hat{u}(s)$ be a proper solution of RCP($x_0, \hat{w}(s)$) for some x_0 and strictly proper $\hat{w}(s)$. Define $\hat{x}(s)$ and $\hat{y}(s)$ as follows*

$$\begin{aligned}\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s) \\ +(sI - A)^{-1}E\hat{w}(s), \\ \hat{y}(s) = C\hat{x}(s) + D\hat{u}(s).\end{aligned}$$

   Then, the inverse Laplace transform (u, x, y) of $(\hat{u}(s), \hat{x}(s), \hat{y}(s))$ is an initial solution of LCS(A, B, C, D, E) for the input w and the initial state x_0 where w is the inverse Laplace transform of $\hat{w}(s)$.

2. *Let (u, x, y) be an initial solution of LCS(A, B, C, D, E) for the input w and the initial state x_0 and let $\hat{u}(s)$ be the Laplace transform of u. Then, $\hat{u}(s)$ solves RCP($x_0, \hat{w}(s)$) where $\hat{w}(s)$ is the Laplace transform of w.*

Proof Evident from the proof of [6, Theorem 5.3]. $\blacksquare$

The following lemma will play the key role for the proof of Theorem 4.2.

Lemma 6.6 *Consider a given matrix quintuple (A, B, C, D, E). Suppose that (B, C, D) satisfies Assumption 4.1. Then the following statements hold.*

1. *RCP($x_0, \hat{w}(s)$) has a unique solution for all $x_0 \in \mathbb{R}^n$ and for all $\hat{w}(s) \in \mathbb{R}^p(s)$.*
2. *The unique solution of RCP($x_0, \hat{w}(s)$) is proper for all x_0 and for all strictly proper $\hat{w}(s)$. Moreover, it is strictly proper if and only if $Cx_0 \in$ K(D).*
3. *Let $\hat{u}(s)$ be the unique solution of RCP($x_0, \hat{w}(s)$) for some x_0 and strictly proper $\hat{w}(s)$. Then, $C(x_0 + Bu_0) \in K(D)$ where $u_0 = \lim_{s\to\infty} \hat{u}(s)$.*

Proof *1:* Since $D + \sigma^{-1}CB$ is a P-matrix for all sufficiently large σ, $D + C(\sigma I - A)^{-1}B$ is a P-matrix for all sufficiently large σ. Then, the statement follows from [5, Theorem 4.1] and Lemma 3.3 item 1.

2: Let $\hat{u}(s)$ be the unique solution of RCP($x_0, \hat{w}(s)$) for some x_0 and strictly proper $\hat{w}(s)$. Then, $\hat{u}(\sigma)$ solves LCP($q(\sigma), G(\sigma)$) for all sufficiently large σ where

$$\begin{aligned}q(s) = C(sI - A)^{-1}x_0 + C(sI - A)^{-1}E\hat{w}(s) \\ G(s) = D + C(sI - A)^{-1}B.\end{aligned}$$

Note that the unique solution of LCP($0, G(\sigma)$) is zero for all sufficiently large σ. Lemma 6.1, together with Lemma 6.3, yields for some $\alpha > 0$

$$\|\hat{u}(\sigma)\| \leq \alpha\sigma \|q(\sigma)\|$$

for all sufficiently large σ. Since $q(s)$ is strictly proper, this implies that for some $\beta > 0$

$$\|\hat{u}(\sigma)\| \leq \beta$$

for all sufficiently large σ. It follows that $\hat{u}(s)$ is proper. It remains to prove the second statement. For the 'only if' part, suppose that $\hat{u}(s)$ is stricly proper. Let the power series expansion around infinity of $\hat{u}(s)$ and $\hat{w}(s)$ be of the form

$$\hat{u}(s) = u_1 s^{-1} + u_2 s^{-2} + \ldots \tag{9a}$$

$$\hat{w}(s) = w_1 s^{-1} + w_2 s^{-2} + \ldots . \tag{9b}$$

Define

$$\begin{aligned}\hat{y}(s) = C(sI - A)^{-1}x_0 + C(sI - A)^{-1}E\hat{w}(s) \\ + [D + C(sI - A)^{-1}B]\hat{u}(s).\end{aligned}$$

By substituting (9) into the above equation, we get

$$\begin{aligned}\hat{y}(s) =&(Cx_0 + Du_1)s^{-1}\\ &+ (CAx_0 + CEw_1 + CBu_1)s^{-2} + \dots .\end{aligned}$$

It follows from the formulation of RCP($x_0, \hat{w}(s)$) that

$$\begin{aligned}u_1^\top (Cx_0 + Du_1) = 0\\ u_1 \geq 0 \text{ and } Cx_0 + Du_1 \geq 0.\end{aligned}$$

Consequently, LCP(Cx_0, D) is solvable. In other words, $Cx_0 \in$ K(D). To show the 'if' part, suppose that $Cx_0 \in$ K(D). Let $\bar{u}$ be a solution of LCP(Cx_0, D). It is clear that $\sigma^{-1}\bar{u}$ solves LCP($\sigma^{-1}Cx_0, D$) for all $\sigma > 0$. Then, it also solves LCP($\sigma^{-1}Cx_0 - \sigma^{-1}C(\sigma I - A)^{-1}B\bar{u}, G(\sigma)$). Lemma 6.1 together with Lemma 6.3 gives

$$\begin{aligned}\|\hat{u}(\sigma) - \sigma^{-1}\bar{u}\| \leq \alpha\sigma \| C[(\sigma I - A)^{-1} - \sigma^{-1} I]x_0\\ + C(\sigma I - A)^{-1}E\hat{w}(\sigma) + \sigma^{-1}C(\sigma I - A)^{-1}B\bar{u}\|\end{aligned} \tag{10}$$

for all sufficiently large σ. Note that for some $\beta > 0$ the last term of the righthand side is less than $\beta\sigma^{-2}$ for all sufficiently large σ. Therefore, (10) results in

$$\|\hat{u}(\sigma) - \sigma^{-1}\bar{u}\| \leq \alpha\beta\sigma^{-1}$$

for all sufficiently large σ. This implies that $\hat{u}(s)$ is strictly proper.

3: Let the power series expansion around infinity of $\hat{u}(s)$ and $\hat{w}(s)$ be of the form

$$\hat{u}(s) = u_0 + u_1 s^{-1} + u_2 s^{-2} + \dots \tag{11a}$$
$$\hat{w}(s) = w_1 s^{-1} + w_2 s^{-2} + \dots . \tag{11b}$$

Define

$$\begin{aligned}\hat{y}(s) = C(sI - A)^{-1}x_0 + C(sI - A)^{-1}E\hat{w}(s)\\ + [D + C(sI - A)^{-1}B]\hat{u}(s).\end{aligned}$$

By substituting (11) into the above equation, we get

$$\hat{y}(s) = Du_0 + (Cx_0 + CBu_0 + Du_1)s^{-1} + \dots .$$

It follows from the formulation of RCP($x_0, \hat{w}(s)$) that

$$(u_0 + u_1\sigma^{-1})^\top [Du_0 + (Cx_0 + CBu_0 + Du_1)\sigma^{-1}] = 0$$

for all σ, and

$$\begin{aligned}u_0 + u_1\sigma^{-1} \geq 0\\ Du_0 + (Cx_0 + CBu_0 + Du_1)\sigma^{-1} \geq 0.\end{aligned}$$

for all sufficiently large σ. Hence, we can conlude that $u_0 + u_1\sigma^{-1}$ solves LCP($Cx_0 + CBu_0, D$) for all sufficiently large σ. This means that $C(x_0 + Bu_0) \in$ K(D). ■

At this stage, we can state the following lemma which concerns the local existence of solutions. Later on, it will be used to show global existence.

Lemma 6.7 *Consider a given matrix quintuple* (A, B, C, D, E). *Suppose that* (B, C, D) *satisfies Assumption 4.1. For all* $w \in \mathcal{PBB}^p$ *and for all initial states* x_0 *with* $Cx_0 \in$ K(D), *there exist an* $\epsilon > 0$ *and a triple* $(u, x, y) \in \mathcal{B}^{m+n+m}$ *such that the equations*

$$\begin{aligned}\dot{x} &= Ax + Bu + Ew + x_0\delta\\ y &= Cx + Du\end{aligned}$$

hold in the distributional sense on $[0, \epsilon)$, *and for all* $t \in [0, \epsilon)$

$$0 \leq u(t) \perp y(t) \geq 0.$$

Moreover, $Cx(\epsilon) \in$K(D).

Proof Since $w \in \mathcal{PBB}^p$, there exist $\epsilon_1 > 0$ and $v \in \mathcal{B}^p$ such that $w|_{[0,\epsilon_1)} = v|_{[0,\epsilon_1)}$. Let $\hat{v}(s)$ be the Laplace transform of v. Lemma 6.6 items 1 and 2 implies that RCP($x_0, \hat{v}(s)$) has a unique strictly proper solution, say $\hat{u}(s)$. As a consequence of item 1 of Lemma 6.5, we know that there exists an initial solution (u, x, y) of LCS(A, B, C, D, E) for the input v and the initial state x_0. Note that $(u, x, y) \in \mathcal{B}^{m+n+m}$ since $\hat{u}(s)$ and $\hat{y}(s)$ are strictly proper. Then, there exists an $\epsilon_2 > 0$ such that $u(t)$ and $y(t)$ are nonnegative for all $t \in [0, \epsilon_2)$ since u and y are initially nonnegative. It is not difficult to see that $\epsilon = \min(\epsilon_1, \epsilon_2)$ and (u, x, y) satisfies the desired requirements. Note that $t \mapsto (u, x, y)(t + \rho)$ forms an initial solution of LCS(A, B, C, D, E) for the input $t \mapsto v(t+\rho)$ and the initial state $x(\rho)$ for all $\rho \in [0, \epsilon)$. Then, Lemma 6.6 item 2 implies that $Cx(\rho) \in$ K(D) for all $\rho \in [0, \epsilon)$. Since K(D) is closed and x is continuous (even Bohl), $Cx(\epsilon) \in$ K(D) as well. ■

As a final ingredient of the proof of Theorem 4.2, we need the following lemma.

Lemma 6.8 *Consider a given matrix quintuple* (A, B, C, D, E). *Suppose that* (B, C, D) *satisfies Assumption 4.1. For all* $K \subseteq \overline{m}$ *there exist matrices* F^K, G^K, H^K *and* J^K *such that if* $(u, x, y) \in \mathcal{BD}^{m+n+m}$ *satisfies*

$$\begin{aligned}\dot{x} &= Ax + Bu + Ew + x_0\delta\\ y &= Cx + Du\\ y_K &= 0\\ u_{K^c} &= 0\end{aligned}$$

in the distributional sense for some initial state x_0 *then*

$$\begin{aligned}\dot{x}_{\text{reg}} &= F^K x_{\text{reg}} + G^K w_{\text{reg}}\\ u_{\text{reg}} &= H^K x_{\text{reg}} + J^K w_{\text{reg}}\\ y_{\text{reg}} &= Cx_{\text{reg}} + Du_{\text{reg}}.\end{aligned}$$

The matrices F^K, G^K, H^K *and* J^K *only depend on* K *not on the particular choice of* x_0.

Proof Clearly, the regular part $(u_{\mathrm{reg}}, x_{\mathrm{reg}}, y_{\mathrm{reg}})$ satisfies

$$\dot{x}_{\mathrm{reg}} = Ax_{\mathrm{reg}} + Bu_{\mathrm{reg}} + Ew \tag{12a}$$
$$y_{\mathrm{reg}} = Cx_{\mathrm{reg}} + Du_{\mathrm{reg}} \tag{12b}$$
$$0 = C_{K\bullet}x_{\mathrm{reg}} + D_{KK}(u_{\mathrm{reg}})_K \tag{12c}$$
$$0 = (u_{\mathrm{reg}})_{K^c}. \tag{12d}$$

From Lemma 6.2 item 1, we know that there exist matrices P and Q such that

$$\begin{bmatrix} P \\ Q \end{bmatrix} \text{ and } \begin{bmatrix} PD_{KK} \\ QC_{K\bullet}B_{\bullet K} \end{bmatrix}$$

are both nonsingular and $QD_{KK} = 0$ since Assumption 4.1 holds for (B, C, D). By premultiplying (12c) by the first matrix in the above equation, we get

$$PC_{K\bullet}x_{\mathrm{reg}} + PD_{KK}(u_{\mathrm{reg}})_K = 0 \tag{13}$$
$$QC_{K\bullet}x_{\mathrm{reg}} = 0. \tag{14}$$

Differentiating (14) with respect to time, one gets

$$QC_{K\bullet}Ax_{\mathrm{reg}} + QC_{K\bullet}B_{\bullet K}(u_{\mathrm{reg}})_K + QC_{K\bullet}Ew = 0. \tag{15}$$

By combining (13) and (15), one can obtain

$$\begin{bmatrix} PD_{KK} \\ QC_{K\bullet}B_{\bullet K} \end{bmatrix}(u_{\mathrm{reg}})_K = -\begin{bmatrix} PC_{K\bullet} \\ QC_{K\bullet}A \end{bmatrix}x_{\mathrm{reg}} - \begin{bmatrix} 0 \\ QC_{K\bullet}E \end{bmatrix}w. \tag{16}$$

Since the first term of the lefthand side is nonsingular, the matrices H^K and J^K can be found by solving $(u_{\mathrm{reg}})_K$ from (16). F^K and G^K can be given as $F^K = A + BH^K$ and $G^K = E + BJ^K$. ■

After all these preparations, we can finally prove Theorem 4.2.

Proof of Theorem 4.2: We show first the existence of a solution for given input $w \in \mathcal{PBB}^p$ and initial state x_0. Since $w \in \mathcal{PBB}^p$, there exists $\mu > 0$ and $v \in \mathcal{B}^p$ such that $w|_{[0,\mu)} = v|_{[0,\mu)}$. Lemma 6.6 items 1 and 2 imply that RCP$(x_0, \hat{v}(s))$ has a proper solution. From Item 1 of Lemma 6.5, we can find an initial solution (u, x, y) of LCS(A, B, C, D, E) for the input v and the initial state x_0. Define

$$(\tilde{u}_{\mathrm{imp}}, \tilde{x}_{\mathrm{imp}}, \tilde{y}_{\mathrm{imp}}) := (u_{\mathrm{imp}}, x_{\mathrm{imp}}, y_{\mathrm{imp}}). \tag{17}$$

Properness of the solution of RCP$(x_0, \hat{v}(s))$ reveals that $u_{\mathrm{imp}} = u_0\delta$ where $u_0 = \lim_{s\to\infty}\hat{u}(s)$. Set $x_0^+ = x_0 + Bu_0$. It follows from Lemma 6.6 item 3 that $Cx_0^+ \in$ K(D). For the input w and initial state x_0^+, let $\epsilon_1 > 0$ and (u^1, x^1, y^1) be such that the conditions given in Lemma 6.7 hold. Note that $Cx^1(\epsilon_1) \in$ K(D). Now, for the input $w|_{[\sum_{l=1}^{k-1}\epsilon_l, \infty)}$ and initial state $x^{k-1}(\epsilon_{k-1})$ we can find $\epsilon_k > 0$ and (u^k, x^k, y^k) be such that the conditions given in Lemma 6.7 hold for $k = 2, 3, \ldots$ since $Cx^{k-1}(\epsilon_{k-1}) \in$ K(D). For $k = 1, 2, \ldots$, define

$$(\tilde{u}_{\mathrm{reg}}, \tilde{x}_{\mathrm{reg}}, \tilde{y}_{\mathrm{reg}})|_{[\sum_{l=0}^{k-1}\epsilon_l, \sum_{l=0}^{k}\epsilon_l)} = (u^k, x^k, y^k)|_{[0,\epsilon_k)}$$

with the convention $\epsilon_0 = 0$. By construction, $(\tilde{u}, \tilde{x}, \tilde{y})$ is a global solution candidate. The only possibility that obstructs it being a global solution can be that $\sum_{l=0}^{k}\epsilon_l = \tau$ and $\lim_{t\uparrow\tau}\tilde{x}_{\mathrm{reg}}(t)$ does not exist. Next, we will show that this is not the case. For brevity, we drop the subscript 'reg'. On an interval $(\rho, t) \subseteq [\epsilon_i, \epsilon_{i+1})$ for some i, $(\tilde{u}, \tilde{x}, \tilde{y})$ is governed by the dynamics $\dot{\tilde{x}} = F^K\tilde{x} + G^K w$ according to Lemma 6.8 for some K. Since $\tilde{x}$ and $t \mapsto e^{F^L t}G^L$ for $L \subseteq \overline{m}$ is continuous $[0, \tau)$ and $w \in \mathcal{PBB}^p$, they are all bounded on $[0, \tau)$, i.e., there exists an $M > 0$ such that $\|x(t)\| \leq M$ for all $t \in [0, \tau)$ and $\|e^{F^L t}G^L w(t)\| \leq M$ for all $t \in [0, \tau)$ and for all $L \subseteq \overline{m}$. Then, we have the following estimation

$$\begin{aligned}\|\tilde{x}(t) - \tilde{x}(\rho)\| &\leq \|e^{F^K(t-\rho)}\tilde{x}(\rho) - \tilde{x}(\rho)\| \\ &\quad + \|\int_\rho^t e^{F^K(t-s)}G^K w(s)ds\| \\ &\leq (1 + \alpha_K)M|t - \rho|\end{aligned} \tag{18}$$

since the function $t \mapsto \frac{e^{F^K t} - I}{t}$ is bounded, say by α_K. Hence, for $(\rho, t) \subseteq [0, \tau)$, we get from (18)

$$\|\tilde{x}(t) - \tilde{x}(\rho)\| \leq M[\max_{K\subseteq\overline{m}}(1 + \alpha_K)]|t - \rho|.$$

It follows that $\tilde{x}$ is Lipschitz continuous on $[0, \tau)$ and thus uniformly continuous. A standard result in mathematical analysis [9, Exercise 4.13] implies that $\lim_{t\uparrow\tau}\tilde{x}(t)$ exists. Therefore, $(\tilde{u}, \tilde{x}, \tilde{y})$ is a global solution of LCS(A, B, C, D, E) for the input w and the initial state x_0. The uniqness of follows from [5, Theorem 5.21].

This section will end with the proof of Corollary 4.3.

Proof of Corollary 4.3 We shall only show that Assumption 4.1 holds under the hypotheses of Corollary 4.3. The rest follows from Theorem 4.2 and Lemma 3.3 item 2. Since B is of full column rank, (A, B, C) is minimal and (A, B, C, D) is passive, it can be shown by using [1, Lemma 6.11 items 1 and 2] that D_{KK} is nonnegative definite and

$$w \neq 0,\ w^\top D_{KK}w = 0 \Rightarrow w^\top C_{K\bullet}B_{\bullet K}w > 0 \tag{19}$$

for any $K \subseteq \overline{m}$. Note that from the nonnegativity of D_{KK} we have the following statement

$$w^\top D_{KK} w = 0 \Leftrightarrow (D_{KK} + D_{KK}^\top) w = 0. \quad (20)$$

Suppose that

$$z \in \mathrm{im}\, D_{KK} \cap C_{K\bullet} B_{\bullet K}\, (\ker D_{KK}),$$

i. e., there exist v and w such that

$$z = D_{KK} v \quad (21)$$
$$z = C_{K\bullet} B_{\bullet K} w \quad (22)$$
$$D_{KK} w = 0. \quad (23)$$

Then, (23) implies that

$$w^\top D_{KK} w = 0. \quad (24)$$

It follows from (20), (23) and (24) that

$$D_{KK}^\top w = 0. \quad (25)$$

Note that we have

$$w^\top C_{K\bullet} B_{\bullet K} w = w^\top z = w^\top D_{KK} v = 0$$

from (21), (22) and (25). Consequently, (19) implies that $w = 0$. This means that $z = 0$ due to (22). Therefore,

$$\mathrm{im}\, D_{KK} \cap C_{K\bullet} B_{\bullet K}\, (\ker D_{KK}) = \{0\}. \quad (26)$$

It follows from (19) that $\ker C_{K\bullet} B_{\bullet K}|_{\ker D_{KK}} = \{0\}$. Hence, $\dim(C_{K\bullet} B_{\bullet K} (\ker D_{KK})) = \dim(\ker D_{KK})$. From, (26) and the fact that

$$\dim(\ker D_{KK}) + \dim(\mathrm{im}\, D_{KK}) = |K|,$$

we get

$$\mathrm{im}\, D_{KK} \oplus C_{K\bullet} B_{\bullet K}\, (\ker D_{KK}) = \mathbb{R}^{|K|}. \blacksquare$$

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