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On (v, k, λ) Graphs and Designs with Trivial Automorphism Group

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1. INTRODUCTION AND SUMMARY OF RESULTS

A (v, k, λ) graph is a strongly regular graph with $\lambda = \mu$. In terms of the vertex-vertex $(1, 0)$-adjacency matrix A of size v this is expressed by

$$A^2 = (k - \lambda) I + \lambda J, \qquad AJ = kJ, \qquad A^t = A, \qquad \operatorname{diag} A = 0.$$

(I is the identity matrix; J is the all-one matrix.) The same equations also define the point-block incidence matrix of a symmetric (v, k, λ) design, having a polarity and no absolute points, cf. [9]. The (v, k, λ) design may have automorphisms ($PAQ = A$) which the (v, k, λ) graph does not possess ($PAP^t \neq A$). Moreover, nonisomorphic (v, k, λ) graphs may correspond to isomorphic (v, k, λ) designs. To illustrate this: for $(v, k, \lambda) = (16, 6, 2)$ there are two nonisomorphic graphs which correspond to the same design (having larger automorphism group). In [8] it has been proved that the second phenomenon can only happen if the automorphism groups of the

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graphs have even order. Therefore nonisomorphic (v, k, λ) graphs with a trivial automorphism group correspond to nonisomorphic (v, k, λ) designs. For instance, the 15,417 graphs with trivial automorphism group among the 16,448 nonisomorphic (36, 15, 6) graphs, cf. [2], correspond to 15,417 nonisomorphic (36, 15, 6) designs. The question arises whether these non-isomorphic (v, k, λ) designs also have a trivial automorphism group. The first cases where this question makes sense are (35, 18, 9), (36, 15, 6), (36, 21, 12). In the present paper we show that in these cases the answer to our question is always affirmative, with just one exception. The same holds for (40, 27, 18), without exceptions. However, we do not expect the same to hold for larger (v, k, λ).

The main theorem 4 in Section 2 gives strong conditions for the internal structure of (v, k, λ) graphs having trivial automorphism group which correspond to (v, k, λ) design admitting nontrivial automorphisms. These conditions follow from $A = PAP$ for a permutation matrix P of prime order $p > 2$. The orbits of P on the vertices of the graph form m cocliques of size p and $v - mp$ fixed points, and induce a partition of A into $m \times m$ anticyclic blocks of size $p \times p$, bordered by blocks with constant columns (rows). In the special cases of Sections 3–5, Theorem 4 forces a number of possibilities for p and m, which are handled case by case.

The most interesting cases are $v = 35$, $p = 3$, $m = 10$ in Section 3, $v = 36$, $p = 3$, $m = 12$ and 10 in Section 4, and $v = 36$, $p = 3$, $m = 12$ in Section 5. In each of these cases there is a partition of the adjacency matrix A into blocks having constant row sums. In each case we are able to determine the quotient matrix B, whose entries are the row sums of the blocks. In order to construct the larger matrix A from the quotient B we used a computer search, which confirmed our earlier complicated and tedious mathematical reasonings. As a result we did find a (36, 15, 6) graph with trivial automorphism group whose corresponding (36, 15, 6) design has 3 automorphisms. Its matrix is reproduced in Section 4.

In the Sections 3–5 it is shown that this example is the only one for the cases (35, 18, 9), (36, 15, 6), (36, 21, 12), and that all other such graphs with a trivial automorphism group correspond to designs with a trivial automorphism group. Thus, from [2] we obtain 1576 designs (35, 18, 9), 15,444 designs (36, 15, 6) (15,416 from (36, 15, 6) graphs and 28 from (36, 21, 12) graphs), all having a trivial automorphism group. Only for $v = 36$ one such (v, k, λ) design was explicitly known before, cf. [1]. We remark that (40, 27, 18) graphs are easily dealt with by use of the results obtained in Sections 2 and 4. The answer to our question is affirmative in this case. We know of one (40, 27, 18) graph with trivial automorphism group, namely the third one of Appendix B in [7].

As a general reference we use [4]. We recall the meaning of the term "switching a graph on v vertices with respect to a subset of p vertices." Let

A denote the $(1, 0)$-adjacency matrix of the graph, then the switched graph has the adjacency matrix

$$A' = A + K_{p+q} \pmod 2, \qquad K_{p+q} = P^t \begin{bmatrix} 0_p & J \\ J & 0_q \end{bmatrix} P,$$

for $p + q = v$ and some permutation matrix P of size $v \times v$. It is well known [2, 3] that the (v, k, λ) graphs on 35 and 36 vertices are related by switching in the following sense. The 35-graph is obtained from a 36-graph by isolating any vertex x by switching (with respect to the vertices adjacent to x). Conversely, given any 35-graph and an isolated vertex, one can obtain a (36, 15, 16) graph by switching with respect to an induced subgraph of size 15 and valency 6, and a (36, 21, 12) graph by switching with respect to an induced subgraph of size 21 and valency 12. Consequently, for a (36, 15, 6) graph and a (36, 21, 12) graph one may be obtained from the other by switching with respect to a suitable subgraph.

2. The General Case

In this section necessary conditions are obtained for the occurrence of the phenomenon described in the introduction.

THEOREM 1. *Let A be the adjacency matrix of a (v, k, λ) graph with trivial automorphism group. If $QAR = A$ for permutation matrices Q and R, then $Q = R$ and Q has odd order.*

Proof. Put $P = RQ$, then

$$RAR^t = RQARR^t = PA, \qquad Q^tAQ = Q^tQARQ = AP.$$

Since A is symmetric, so are PA and AP, hence

$$AP = P^tA, \qquad PA = AP^t, \qquad PAP = P^tAP^t = A.$$

If the order of P were even, $2m$ say, then

$$A = P^mP^mA = P^mA(P^m)^t,$$

hence $P^m = I$, a contradiction. Hence P has odd order, $2k + 1$ say. It follows that

$$A = P^{k+1}AP^{k+1} = P^k(PA)(P^k)^t = P^kRAR^t(P^k)^t,$$

hence $P^kR = I$, $R = P^{k+1}$. Similarly $Q = P^{k+1}$, and we have proved $Q = R$. Since $Q^2 = P$, it is seen that Q has odd order $2k + 1$ as well. ∎

LEMMA 2. *Let A be the adjacency matrix of a (v, k, λ) graph G. Let $P \neq I$ be a permutation matrix of odd order such that $PAP = A$. Then*

(i) *the orbits of P on G are cocliques,*

(ii) *the number of orbits is at most $v-2k+2\lambda-\alpha+1$, where α is the size of the largest orbit.*

Proof. $PAP=A$ implies

$$P^m A = \begin{cases} P^{(1/2)m} A (P^{(1/2)m})^t & \text{if } m \text{ is even,} \\ P^{(1/2)(n+m)} A (P^{(1/2)(n+m)})^t & \text{if } m \text{ is odd,} \end{cases}$$

where n is the order of P. Hence $P^m A$ has zero diagonal for all m. All entries of A, which are moved to the diagonal by P^m, are zero. This proves (i).

Let x and y be two vertices in the largest orbit. Consider the set S of the vertices which are adjacent to x and y, or not adjacent to x and y (including x, y). Then $|S| = v-2k+2\lambda$. Within any one orbit the number of vertices adjacent to x equals the number of vertices adjacent to y. Since the orbit has odd size, it must contain at least one vertex from S. Now (ii) follows, since the orbit of x and y has α elements from S. ∎

We need the following coclique bound, which is due to Delsarte [5] for strongly regular graphs, to Hoffman for regular graphs, and to Haemers [7] for nonregular graphs.

RESULT 3. *A coclique in a (v, k, λ) graph has at most $(k\sqrt{k-\lambda} - k+\lambda)/\lambda$ vertices. If equality holds, then each vertex outside the coclique is adjacent to precisely $\sqrt{k-\lambda}$ vertices of the coclique.*

The next theorem is the major tool in the forthcoming sections. We recall the following definition. A square matrix M of size n is *anti-cyclic* whenever

$$M_{i,j+1 (\mathrm{mod}\, n)} = M_{i+1 (\mathrm{mod}\, n), j} \qquad \text{for} \quad i, j = 1, ..., n.$$

Notice that an anti-cyclic matrix is symmetric and has constant row and column sums.

THEOREM 4. *Let A be the adjacency matrix of the (v, k, λ) graph G. Suppose that as a graph G has trivial automorphism group, while as a design G has a non-trivial automorphism. Then there exist a prime $p>2$, an integer $m \leqslant v/p$, and a partitioning*

$$A = \begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m} \\ \vdots & \vdots & & \vdots \\ A_{m,0} & A_{m,1} & \cdots & A_{m,m} \end{bmatrix},$$

such that

(i) $p\lambda \leqslant k\sqrt{k-\lambda}-k+\lambda$;

(ii) $(p-1)(m-1) \geqslant 2(k-\lambda)$;

(iii) $A_{i,i}=0$, *for* $i=1, ..., m$;

(iv) $A_{i,j}=A_{j,i}$ *is anti-cyclic of size* $p \times p$, *for* $i, j=1, ..., m$;

(v) $A_{i,0}=A^t_{0,i}$ *consists of* p *identical rows, for* $i=1, ..., m$.

Proof. Theorem 1 implies $A=QAQ$ for a permutation matrix $Q \neq I$ of odd order n, say. Let $p>2$ be a prime dividing n. Then $P=Q^{n/p}$ has the order p and $PAP=A$. Hence the orbits of P on G have size 1 or p. Let m denote the number of orbits of size p. We partition A according to the set of the $v-mp$ fixed points and the m orbits of size p. Then Lemma 2 implies (iii) and (ii), and Result 3 yields (i). We may write

$$P=\operatorname{diag}(1, ..., 1, P_1, ..., P_1),$$

where the diagonal blocks P_1 are cyclic permutation matrices of size p. Furthermore, $PAP=A$ implies

$$P_1 A_{ij} P_1 = A_{ij}, \qquad P_1 A_{i0} = A_{i0},$$

which proves (iv) and (v). ∎

We will need the following result about eigenvalues, cf. [6, Theorem 0.12; 7, Theorem 1.2.3].

RESULT 5. *Let* A *be a symmetric* $v \times v$ *matrix which is partitioned into blocks* A_{ij} *with square* A_{ii}, $i, j=1, ..., n$. *Let* B *be the* $n \times n$ *quotient matrix whose entry* b_{ij} *is the average row sum of* A_{ij}. *If each block has constant row and column sums, then any eigenvalue of* B *is also an eigenvalue of* A.

3. THE CASE (35, 18, 9)

Throughout this section, let G denote any (35, 18, 9) graph with a trivial automorphism group:

$$A^2=9I+9J, \qquad AJ=18J, \qquad \operatorname{spec} A=(18^1, 3^{14}, (-3)^{20}).$$

We shall prove that G, considered as a design, has a trivial automorphism group as well. Supposing the contrary, we apply Theorem 4, and its notation: $mp \leqslant 35$, $p \leqslant 5$, $(p-1)(m-1) \geqslant 18$. This leaves just four possibilities for p and m: $(p, m) = (5, 7)$, $(5, 6)$, $(3, 11)$, $(3, 10)$. We shall show that each of these cases leads to a contradiction.

3.1. *Case* $p = 5$, $m = 7$. There are no matrices $A_{0,i}$. Since the coclique bound of Result 3 is tight, all $A_{i,j}$ must have row and column sum 3, for $i \neq j$. Up to cyclic shifts there are just two such anti-cyclic matrices, namely

$$C_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

The matrix C_1 and its shifts, and C_2 and its shifts satisfy

$$C_1 C_1^t = \begin{bmatrix} 3 & 2 & 1 & 1 & 2 \\ 2 & 3 & 2 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 2 & 3 \end{bmatrix}, \qquad C_2 C_2^t = \begin{bmatrix} 3 & 1 & 2 & 2 & 1 \\ 1 & 3 & 1 & 2 & 2 \\ 2 & 1 & 3 & 1 & 2 \\ 2 & 2 & 1 & 3 & 1 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix}.$$

From $\sum_{j=1}^{r} A_{ij} A_{ij}^t = 9I + 9J$ it follows that each blockrow of A contains 3 matrices of type C_1 and 3 matrices of type C_2. Hence the total number of matrices of type C_1 is $7 \times 3 = 21$. However, the symmetry of A requires this number to be even, which yields a contradiction.

3.2. *Case* $p = 5$, $m = 6$. Again the coclique bound is tight, hence each column of $A_{i,0}$ contains exactly 3 ones, $i = 1, ..., 6$. Since $A_{i,0}$ has 5 rows we have a contradiction to (v) of Theorem 4.

3.3. *Case* $p = 3$, $m = 11$. Let r denote the row sum of $A_{i,0}$, then

$$A_{i,0} A_{i,0}^t = rJ, \qquad \sum_{j=0}^{11} A_{ij} A_{ij}^t = 9I + 9J.$$

Let a_l, $l = 0, 1, 2, 3$, denote the number of matrices A_{ij}, $j = 1, ..., 11$, in blockrow i having row sum l. Then the matrix equation imply $a_1 + 2a_2 + 3a_3 + r = 18$, $a_2 + 3a_3 + r = 9$, hence $a_1 + a_2 = 9$. In total there are $11 \times 9 = 99$ matrices A_{ij} with row sum 1 or 2. Since the symmetry of A requires this number to be even, we have a contradiction.

3.4. *Case* $p=3$, $m=10$. Unlike the previous cases the present case does admit matrices A satisfying the necessary conditions (i)–(v) of Theorem 4. This is seen as follows. From (v) of Theorem 4, and $A^2=9I+9J$ applied to block row 0, we obtain $A_{0,0}^2 \equiv 0 \pmod 3$ whence $A_{0,0}=0$, a coclique of size 5 which meets the coclique bound. Hence each $A_{i,0}$ has three all-one columns and two all-zero columns, each of size 3.

We partition A following $35=5\times 1+10\times 3$, the 5 fixed points and the 10 orbits of length 3. The quotient matrix B of the row sums is not symmetric, yet B has the eigenvalues 3, -3, 18 of A (by result 5). We write

$$A=\begin{bmatrix} 0 & A_0 \\ A_0^t & A_1 \end{bmatrix}, \qquad B=\begin{bmatrix} 0 & 3B_0 \\ B_0^t & B_1 \end{bmatrix},$$

with A_1 of size 30 and B_1 of size 10. Then $B_0B_0^t=3I+3J$, $B_0J=6J$, $B_0^tJ=3J$, $B_1J=15J$; hence B_0 is the incidence matrix of the design $2-(5,3,3)$: all triples from a 5-set. These data imply rank $(B^2-9I)=1$ and, in fact,

$$B^2-9I=9\begin{bmatrix} J & 3J \\ J & 3J \end{bmatrix}, \qquad B_1B_0=9J, \qquad B_1^2+3B_0^tB_0=9I+27J.$$

Hence B_1 has off-diagonal entries 1 and 2 (from $\operatorname{diag}(B_1^2)=27I$, $B_1J=15J$), and is the (1, 2)-adjacency matrix of the Petersen graph (from $B_1^2\equiv 0$, $B_1J\equiv 0 \pmod 3$, and replacing 2 by -1 in B_1).

Once B is known, we may take $A_0=B_0\otimes J_{1\times 3}$ and construct A_1 from B_1 by replacing the entry (x, y) by K, L, or M if x and y are adjacent, by $J-K$, $J-L$, or $J-M$ if x and y are nonadjacent, and by 0 if $x=y$ in the Petersen graph. Here

$$K=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad L=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

are the only anti-cyclic matrices of size 3 and row sums 1. Both by mathematical reasoning and by use of a computer it is seen that there are two solutions which yield the adjacency matrix A of a (35, 18, 9) graph. The first solution is obtained by substituting only K and $J-K$. However, the resulting graph has a large automorphism group. Any automorphism of the Petersen graph induces an automorphism; another one (which fixes the blocks) is obtained by interchanging the first and the third vertex of every block of size 3×3.

By use of a computer it has been verified that up to isomorphism there is precisely one further such realization for the matrix A, namely with the following A_1 (write $\bar{K}=J-K$, $\bar{L}=J-L$).

$$
\left[\begin{array}{cccccccccc}
0 & K & K & K & \bar{K} & \bar{K} & \bar{K} & \bar{K} & \bar{K} & \bar{K} \\
 & 0 & \bar{L} & \bar{L} & K & \bar{K} & \bar{L} & K & \bar{L} & \bar{K} \\
 & & 0 & \bar{L} & \bar{L} & K & \bar{K} & \bar{K} & K & \bar{L} \\
 & & & 0 & \bar{K} & \bar{L} & K & \bar{L} & \bar{K} & K \\
 & & & & 0 & \bar{L} & \bar{L} & \bar{K} & K & L \\
 & & & & & 0 & \bar{L} & L & \bar{K} & K \\
 & & & & & & 0 & K & L & \bar{K} \\
 & & & & & & & 0 & \bar{L} & \bar{L} \\
 & & & & & & & & 0 & \bar{L} \\
 & & & & & & & & & 0
\end{array}\right]
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \end{array}
$$

The block permutation $(bcd)(efg)(hij)$ is an automorphism of A_1 and induces an automorphism of the corresponding (35, 18, 9) graph. This shows that the case $p=3$, $m=10$ cannot exist. Thus we have proved the claim at the beginning of this paragraph that the (35, 18, 9) designs G have a trivial automorphism group as well.

Remark. For future reference we notice that the full automorphism group of the second realization for A_1 is the following. The automorphisms map blocks to blocks, and the corresponding automorphisms of the Petersen graph are precisely the 6 symmetries of the planar representation given by Fig. 1. The vertices $a, b, \ldots, j$ of Fig. 1 correspond to the block-rows of matrix A_1 above.

4. The Case (36, 15, 6)

Throughout this section, let G denote any (36, 15, 6) graph with a trivial automorphism group:

$$A^2 = 9I + 6J, \quad AJ = 15J, \qquad \operatorname{spec} A = (15^1, 3^{15}, (-3)^{20}).$$

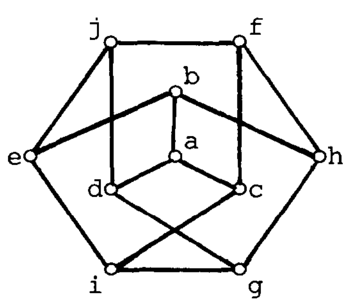


Figure 1

We shall prove that G, considered as a design, has a trivial automorphism group as well, up to exactly one exception. Supposing the contrary, we apply Theorem 4: $mp \leqslant 36$, $p \leqslant 5$, $(p-1)(m-1) \geqslant 18$. This leaves five possibilities for p and m: $(p, m) = (5, 7)$, $(5, 6)$, $(3, 11)$, $(3, 12)$, $(3, 10)$.

4.1. *Cases* $(5, 7)$, $(5, 6)$, $(3, 11)$. There is at least one fixed point. We isolate this point by switching and observe that the switching goes blockwise, that is, for any block either all or no vertices belong to the switching set. On the remaining 35 vertices we obtain a $(35, 18, 9)$ graph with the structure $(p, m) = (5, 7)$, $(5, 6)$, $(3, 11)$ of Section 3. In that section we proved that no matrices with this structure exist which satisfy the necessary conditions of Theorem 4. Hence the present cases cannot exist.

4.2. *Case* $p = 3$, $m = 12$. There are no matrices $A_{0,i}$, and the matrix A is partitioned into 12×12 blocks of size 3×3. Since these blocks have constant row sums $\in \{3, 2, 1, 0\}$, the symmetric quotient matrix B satisfies

$$\operatorname{spec} B = (15^1, 3^3, (-3)^8), \qquad B^2 = 9I + 18J, \qquad BJ = 15J.$$

The format of B is determined by consideration of

$$C := 6I + 2B - 3J, \qquad \operatorname{spec} C = (12^3, 0^9), \qquad CJ = 0, \qquad C^2 = 12C.$$

Each row of C has three entries ± 3 and nine entries ± 1 (from $\operatorname{diag} C^2 = 36I$), and C is the Gram matrix of 12 vectors in $\mathbb{R}^3$ having $(x, y) \in \{-3, -1, 1, 3\}$. Hence these vectors span four lines at $\cos \varphi = \frac{1}{3}$, the diameters of a cube in $\mathbb{R}^3$. Since the vectors are balanced (sum up to zero), they consist of the 4 vertices of a regular tetrahedron, each 3 times, or each 2 times and their opposites once. Hence B has the following format.

$$B = \begin{bmatrix} S & T & T & T \\ T & S & T & T \\ T & T & S & T \\ T & T & T & S \end{bmatrix},$$

$$S = 3J - 3I, \qquad T = J,$$

or

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

For the moment we proceed with the first case for B. The question now is how to fill the entries of B, of size 12×12, by blocks of size 3×3 so as to obtain the adjacency matrix A of G, of size 36×36. This amounts to the distribution of the blocks

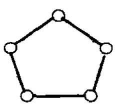
$$K=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\qquad L=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\qquad M=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

over the entries 1 of B (the diagonal entries 0 become 0, the entries 3 become J). We have solved this problem both without and with the use of a computer. The result is that A is isomorphic to one of the two following matrices:

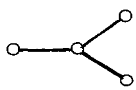
$$\begin{bmatrix}
0&J&J&K&K&K&K&K&K&K&K&K\\
&0&J&K&K&K&L&L&L&M&M&M\\
&&0&K&K&K&M&M&M&L&L&L\\
&&&0&J&J&K&L&M&K&L&M\\
&&&&0&J&L&M&K&M&K&L\\
&&&&&0&M&K&L&L&M&K\\
&&&&&&0&J&J&K&M&L\\
&&&&&&&0&J&M&L&K\\
&&&&&&&&0&L&K&M\\
&&&&&&&&&0&J&J\\
&&&&&&&&&&0&J\\
&&&&&&&&&&&0
\end{bmatrix}
\qquad
\begin{bmatrix}
0&J&J&K&K&K&K&K&K&K&K&K\\
&0&J&K&K&K&L&L&L&M&M&M\\
&&0&K&K&K&M&M&M&L&L&L\\
&&&0&J&J&K&L&M&K&L&M\\
&&&&0&J&L&M&K&M&K&L\\
&&&&&0&M&K&L&L&M&K\\
&&&&&&0&J&J&L&K&M\\
&&&&&&&0&J&K&M&L\\
&&&&&&&&0&M&L&K\\
&&&&&&&&&0&J&J\\
&&&&&&&&&&0&J\\
&&&&&&&&&&&0
\end{bmatrix}
\begin{array}{c}
\mathrm{a}\\ \mathrm{b}\\ \mathrm{c}\\ \mathrm{d}\\ \mathrm{e}\\ \mathrm{f}\\ \mathrm{g}\\ \mathrm{h}\\ \mathrm{i}\\ \mathrm{j}\\ \mathrm{k}\\ \mathrm{l}
\end{array}$$

Both matrices are unchanged under the block-permutation $(bc)(ef)$ $(gj)(hk)(il)$, hence the present case cannot occur. In fact, by computer we found the orders 1296 and 648, respectively, for the full automorphism group. The automorphisms map blocks to blocks. Therefore also the second case of B cannot lead to a matrix A with a trivial automorphism group. Indeed, the matrix A of the first case can be switched into the matrix A of the second case. There are 81 choices for a switching set (for each vertex of the tetrahedron one of the three vectors is switched into its opposite). Hence for each switching set there are at least $\frac{648}{81}=8$ commuting automorphisms, and after switching these remain automorphisms.

4.3. *Case* $p=3$, $m=10$. By switching we isolate a fixed vertex. Since the switching goes block-wise (from (v), Theorem 4), we obtain a (35, 18, 9) graph of the type $p=3$, $m=10$, with 5 fixed vertices. Conversely, any realization of the present case of (36, 15, 6) graphs can be obtained by blockwise switching of such a (35, 18, 9) graph. In Section 3 it was observed that there exist two such (35, 18, 9) graphs, both consisting of a 5-coclique G_0 and a 30-subgraph G_1. We have seen that the first graph possesses at least one automorphism which fixes the blocks of G_1, hence which remains an automorphism for the (36, 15, 6) graph. So this case can be discarded. The other graph possesses only 6 automorphisms, and the question is whether these can disappear after switching. In order to answer this question we distinguish the possible switching sets, which all are induced subgraphs on 15 vertices, regular with valency 6. These switching sets have to correspond to the following subgraphs of the underlying Petersen graph:



a pentagon (15 vertices from G_1);



a star (12 vertices from G_1, 3 from G_0);

a path (12 vertices from G_1, 3 from G_0).

Now consider Fig. 1. Any pentagon in Fig. 1 has at least one symmetry, which remains an automorphism of G after switching. The star *bfgh*, and the paths *abce*, *abch*, *abfh*, *abgh*, *befh*, *begh*, *bfhj*, *bghi* are the relevant subgraphs which do not possess any symmetry of Fig. 1. We found by computer that the corresponding switching sets lead to only three non-isomorphic (36, 15, 6) graphs. One of these has 4 automorphisms (corresponding to the star *bfgh*), one has 2 automorphisms (corresponding to path *abce* or *abch*), and one has a trivial automorphism group (corresponding to any of the 6 remaining paths). The last (36, 15, 6) graph is the exception to the phenomenon under consideration. This graph is reproduced hereafter, with

$$r_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\bar{r}_i = J - r_i$, $\bar{K} = J - K$, $\bar{L} = J - L$,

$$\begin{bmatrix}
0 & J & r_2 & r_1 & r_3 & r_3 & 0 & r_2 & \bar{r}_2 & \bar{r}_3 & r_1 & \bar{r}_1 \\
 & 0 & r_1 & r_1 & r_2 & r_3 & 0 & r_3 & \bar{r}_2 & \bar{r}_1 & r_2 & \bar{r}_3 \\
 & & 0 & \bar{K} & K & K & \bar{K} & \bar{K} & K & K & K & \bar{K} \\
 & & & 0 & L & L & \bar{K} & K & \bar{L} & K & \bar{L} & K \\
 & & & & 0 & \bar{L} & \bar{L} & K & K & K & \bar{K} & \bar{L} \\
 & & & & & 0 & \bar{K} & \bar{L} & \bar{K} & L & K & K \\
 & & & & & & 0 & \bar{L} & L & K & \bar{K} & L \\
 & & & & & & & 0 & L & \bar{L} & K & K \\
 & & & & & & & & 0 & K & L & K \\
 & & & & & & & & & 0 & \bar{L} & L \\
 & & & & & & & & & & 0 & L \\
 & & & & & & & & & & & 0
\end{bmatrix}$$

This (36, 15, 6) graph has a trivial automorphism group. However, considered as a (36, 15, 16) design it has automorphism group $\{I, P, P^2\}$, where

$$P = \operatorname{diag}(P_1, \ldots, P_{12}); \qquad P_1 = P_2 = I;$$

$$P_3 = \cdots = P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \qquad PAP = A.$$

5. The Case (36, 21, 12)

In this section, let G denote any (36, 21, 12) graph with a trivial automorphism group:

$$A^2 = 9I + 12J, \qquad AJ = 21J, \qquad \operatorname{spec} A = (21^1, 3^{14}, (-3)^{21}).$$

We shall prove that G, considered as a design, has a trivial automorphism group as well. Suppose the contrary; we obtain from Theorem 4 three posibilities for p and m: $(p, m) = (3, 12)$, $(3, 11)$, $(3, 10)$.

5.1. *Case* (3, 12). There is a partition of A into blocks of size 3×3, with constant row sums $\in \{3, 2, 1, 0\}$. The symmetric quotient matrix B satisfies

$$\operatorname{spec} B = (21^1, 3^2, (-3)^9), \qquad B^2 = 9I + 36J, \qquad BJ = 21J.$$

B is determined via $C := 6I + 2B - 4J$, with

$$\operatorname{spec} C = (12^2, 0^{10}), \qquad C^2 = 12C, \qquad CJ = 0.$$

The rows of C have entries 6×0 and $6 \times (\pm 2)$ (from $\operatorname{diag} C^2 = 24I$) and C is the Gram matrix of the 4 vectors of length $\sqrt{2}$ of an orthogonal cross in $\mathbb{R}^2$, each $3\times$, hence

$$B = \begin{bmatrix} 3J-3I & J & 2J & 2J \\ J & 3J-3I & 2J & 2J \\ 2J & 2J & 3J-3I & J \\ 2J & 2J & J & 3J-3I \end{bmatrix}.$$

$$\hat{B} := \begin{bmatrix} 3J-3I & J & J & J \\ J & 3J-3I & J & J \\ J & J & 3J-3I & J \\ J & J & J & 3J-3I \end{bmatrix}$$

is the quotient matrix of the adjacency matrix $\tilde{A}$ of the $(36, 15, 6)$ graph which is obtained from the $(36, 21, 12)$ graph A by switching with respect to the first 18 vertices. This switching goes blockwise. Up to the pairing of the four diagonal blocks $\tilde{A}$ must be one of the matrices obtained in Section 4.2. We have seen that $\tilde{A}$ has at least 648 automorphisms. So, for each pairing of the diagonal blocks, at least $648/3 = 216$ automorphisms fix the pairing and therefore must have been automorphisms before switching. Hence there are no graphs satisfying 5.1.

5.2. *Case* (3, 11). Isolating one of the fixed vertices we obtain a $(35, 18, 9)$ graph with the structure $p = 3$, $m = 11$. In Section 3.3 we have shown that there are no such graphs.

5.3. *Case* (3, 10). Consider the subgraph on the set of the fixed vertices, which is represented by $A_{0,0}$. It easily follows that this graph must be the complete bipartite graph $K_{3,3}$ or a coclique of size 6. Since the coclique bound of result 3 yields $4\frac{1}{2}$, $A_{0,0}$ must represent $K_{3,3}$. From the parameters it follows that any two nonadjacent vertices x and y of $A_{0,0}$ are nonadjacent to four other vertices. One belongs to $A_{0,0}$ and the other three must be the vertices of one orbit. But then this orbit and x and y form a coclique of size 5, which is impossible. Therefore, the present case, hence the whole case $(36, 21, 12)$ does not occur.

Note added in proof. Meanwhile the second and fourth authors have proved that there is no other (v, k, λ) graph with $k - \lambda = g$ that satisfies the phenomenon of the present paper. We thank Walter Harnan for his remarks.

REFERENCES

1. J. W. BROWN AND E. T. PARKER, A symmetric BIBD with trivial automorphism group, *J. Combin. Theory Ser. A* **40** (1985), 1–8.
2. F. C. BUSSEMAKER, R. A. MATHON, AND J. J. SEIDEL, Tables of two-graphs, 70-112 *in* "Combinatorics and Graph Theory" (S. B. Rao, Ed.), pp. 70–112, Lecture Notes in Math. Vol 885, Springer-Verlag, New York/Berlin, 1981; (extended version) "Tables of Two-graphs," TH-Report 79-WSK-05, Technical University Eindhoven, 1979.
3. F. C. BUSSEMAKER AND J. J. SEIDEL, Symmetric Hadamard matrices of order 36, *in* "Internat. Conf. Combin. Math." (A. Gewirtz and L. V. Quintas, Ed.), pp. 66–79, New York Acad. Sci., New York, 1970; (extended version) "Symmetric Hadamard Matrices of Order 36," TH-Report 70-WSK-02, Technical University, Eindhoven, 1970.
4. P. J. CAMERON AND J. H. VAN LINT, "Graphs, Codes and Designs," London Math. Soc. Lecture Notes Series Vol. 43, Cambridge Univ. Press, Cambridge/New York, 1980.
5. P. DELSARTE, An algebraic approach to the association schemes of coding theory, *Philips Res. Rep. Suppl.* **10** (1973).
6. D. M. CVETKOVIĆ, M. DOOB, AND H. SACHS, Spectra of graphs, VEB Deutscher Verlag der Wiss, 1980.
7. W. H. HAEMERS, "Eigenvalue Techniques in Design and Graph Theory," Math. Centre Tract Vol. 121, Math Centrum, Amsterdam, 1980.
8. W. H. HAEMERS, Dual Seidel switching, *in* "Papers Dedicated to J. J. Seidel" (P. J. de Doelder, J. de Graaf, and J. H. van Lint, Eds.), pp. 183–190, Technical University Eindhoven, 1984.
9. A. RUDVALIS, (v, k, λ)-graphs and polarities of (v, k, λ)-designs, *Math. Z.* **120** (1971), 224–230.