On the p-ranks of the adjacency matrices of distance-regular graphs

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Abstract

Let Γ be a distance regular graph with adjacency matrix A. Let I be the identity matrix and J the all-1 matrix. Let p be a prime. We study the p-rank of the matrices $A + bJ - cI$ for integral b, c.

1 Introduction

In this paper we will deal with the following problem: Given the spectrum of a (regular) graph, what can we say about the p-ranks of $A - cI$ or, more general, $A + bJ - cI$, where A is the adjacency matrix and b and c integers. Which of these p-ranks are completely determined by the spectrum and which are not? Some of these remaining p-ranks can be determined if we furthermore assume that the graph is distance-regular. Our main tool to determine the p-ranks is the minimal polynomial of A considered as a matrix over $\mathbb{F}_p$. We will first mention some properties of the minimal polynomial and then show how this determines most of the p-ranks of some regular graph, given its spectrum. In the third section we consider the Hamming and Doob graphs as our main example. Some more examples are considered in the last section. Most results of the first two sections can also be found in [3].

1.1 The minimal polynomial

Let F be any field and A a $v \times v$-matrix over F. A polynomial $f(x) \in F[x]$ is called an *annihilating polynomial* of A if $f(A) = O$. The *minimal polynomial* of A is the unique monic annihilating polynomial of A that has minimal degree. Let $\varphi_0(x)$ be the minimal polynomial of A and let $c \in F$, then equivalent are:

1. $r(A - cI) < v$,
2. c is a root of $\det(xI - A)$,
3. c is a root of $\varphi_0(x)$.

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Let $\varphi_0(x) = (x-\lambda_1)^{h_1}\cdots(x-\lambda_n)^{h_n}\varphi_1(x)$, where $\varphi_1(x)$ has no roots in F and let $\det(xI-A) = (x-\lambda_1)^{m_1}\cdots(x-\lambda_n)^{m_n}f(x)$ where $f(x)$ has no roots in F, then $\varphi_1(x)$ divides $f(x)$ and there exists a regular $v\times v$-matrix S such that

$$S^{-1}AS = \mathrm{diag}(A_1,\ldots,A_n,B)$$

where A_i is a $m_i\times m_i$-matrix such that $(A_i-\lambda_i I)$ is nilpotent with index h_i and B is a $(v-\sum m_i)\times(v-\sum m_i)$-matrix that has no eigenvalues in F. Now $r((A-\lambda_i I)^{h_i}) = v-m_i$ and

$$v-m_i+h_i-1 \le r(A-\lambda_i I) \le v-\frac{m_i}{h_i}$$

so λ_i is a simple root of $\varphi_0(x)$ if and only if $r(A-\lambda_i I) = v-m_i$.

Furthermore, $\ker((A-\lambda_i I)^{h_i})\oplus\langle(A-\lambda_i I)^{h_i}\rangle = F^v$ and $\ker((A-\lambda_i I)^{h_i})\cap\langle(A-\lambda_i I)^{h_i}\rangle = \{0\}$.

In this paper we will only consider the minimal polynomial of the adjacency matrix of a graph considered as a matrix over $\mathbb{R}$ or $\mathbb{F}_p$. Dealing with this situation, we have the following lemma:

Lemma 1 *(cf. [5]) Let A be an integral $v\times v$-matrix, then the minimal polynomial of A over $\mathbb{R}$, $\varphi_0(x)$ say, has integral coefficients.* □

So if we consider A as a matrix over $\mathbb{F}_p$, $\varphi_0(x) \pmod p$ is an annihilating polynomial of A and the minimal polynomial of A modulo p divides $\varphi_0(x) \pmod p$.

A connected graph Γ is called *distance-regular* if it is regular of valency k, and if for any two points $\gamma,\delta\in\Gamma$ at (graph)distance i, there are precisely c_i neighbours of δ at distance $i-1$ from γ and b_i neighbours of δ at distance $i+1$ from γ. The sequence

$$\iota(\Gamma) = \{b_0,b_1,\ldots,b_{d-1};c_1,c_2,\ldots,c_d\},$$

where d is the diameter of Γ, is called the *intersection array* of Γ; the numbers c_i, b_i and a_i, where

$$a_i = k-b_i-c_i$$

is the number of neighbours of δ at distance i from γ, are called the *intersection numbers* of Γ. The *intersection matrix* of Γ is the tridiagonal matrix

$$\begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ c_1 & a_1 & b_1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & \cdots & 0 & c_d & a_d \end{pmatrix}$$

and has the same eigenvalues as Γ. More about distance-regular graphs can be found in [2].

For determining the minimal polynomial of the adjacency matrix of a distance-regular graph over $\mathbb{F}_p$, the following lemma is very useful.

Lemma 2 *Let A be the adjacency matrix of a distance-regular graph with intersection matrix B and let p be a prime, then, when calculating modulo p, A and B have the same minimal polynomial. Let $e := \min\{d+1, i | c_i \equiv 0 \pmod p),\ i \geq 0\}$ and B_e the $e \times e$-submatrix of B consisting of the first e rows and columns, then this minimal polynomial is equal to* $\det(xI - B_e)$.

Proof: Let $f(x)$ be a polynomial with integral coefficients, then $f(A) \equiv O \pmod p$ if and only if $f(B) \equiv O \pmod p$. In fact $f(A) \equiv O \pmod p$ if and only if the first column of $f(B)$ is all-zero. Let

$$B = \begin{pmatrix} B_e & C \\ O & B_{d-e} \end{pmatrix}, \text{ then } f(B) = \begin{pmatrix} f(B_e) & C' \\ O & f(B_{d-e}) \end{pmatrix}$$

for each polynomial $f(x)$. Since the matrices $B_e^0 = I, B_e^1, B_e^2, \ldots, B_e^{e-1}$ are clearly linearly independent, the minimal polynomial of B_e has degree at least e, so it must be equal to $\det(xI - B_e)$. □

2 Integral matrices with given spectrum

In this section we consider three special cases of the following problem: Given the spectrum of a symmetric integral matrix with constant row sum, M say, (think of the adjacency matrix of some regular graph with prescribed spectrum). Which p-ranks of $M - cI$ (for integral c) are determined by the spectrum of M and which are not (in general)? The three special cases we will consider here cover all possibilities of regular graphs with at most four eigenvalues, see [5]. Further cases can be treated similarly.

Since the spectrum of M is known, also its minimal polynomial over $\mathbb{R}$, $\varphi_0(x)$ say, is known as well as its characteristic polynomial. Most p-ranks of the matrices $M - cI$ follow from the properties of the minimal polynomial of M over $\mathbb{F}_p$.

2.1 CASE 1

Let M be a symmetric, integral $v \times v$-matrix with constant row sum θ_0 and n distinct integral eigenvalues $\theta_0^{1=m_0}, \theta_1^{m_1}, \theta_2^{m_2}, \ldots, \theta_{n-1}^{m_{n-1}}$, where the exponents denote the multiplicities and $\sum_{i=0}^{n-1} m_i = v$. So the minimal polynomial of M over $\mathbb{R}$ is $\prod_{i=0}^{n-1}(x - \theta_i)$. Let p be a prime. We are interested in the p-ranks of the matrices $M + cI$ for integral c.

If $\theta_i \not\equiv 0 \pmod p$ for $i = 0, 1, 2, \ldots, n-1$ then $r_p(M) = v$.

If $\theta_i \equiv 0 \pmod p$ for some i and $\theta_j \not\equiv 0 \pmod p$ for all $j \neq i$ then $r_p(M) = v - m_i$.

If $\theta_0 \equiv \theta_i \equiv 0 \pmod p$ for some $i \neq 0$ and $\theta_j \not\equiv 0 \pmod p$ for all $j \neq 0, i$ then

$$\begin{array}{rcll} r_p(M) & = & v - m_i - 1 & \text{if } \frac{1}{v}\prod_{j=1}^{n-1}(\theta_0 - \theta_j) \equiv 0 \pmod p \\ & & v - m_i & \text{otherwise} \end{array}$$

Indeed, first of all $r_p(M) \geq \sum\{m_i | \theta_i \not\equiv 0 \pmod p\}$ and $r_p(M) \leq v - m_i$ for each eigenvalue θ_i that is divisible by p. These remarks prove the first two statements. Now suppose we are in the third case. Then $r_p(M)$ is either $v - m_i$ or $v - m_i - 1$. It follows from the eigenvalues that $\prod_{j=1}^{n-1}(M - \theta_j I) = \frac{1}{v}\prod_{j=1}^{n-1}(\theta_0 - \theta_j)J$. So if $\frac{1}{v}\prod_{j=1}^{n-1}(\theta_0 - \theta_j) \equiv 0 \pmod p$ then the minimal polynomial of M over $\mathbb{F}_p$ contains one factor x and hence $r_p(M) = v - m_i - 1$ and it contains two factors x (and hence $r_p(M) = v - m_i$) otherwise.

The p-ranks that are in general not determined by the spectrum are:

$$\boxed{r_p(M - \theta_i I) \quad \text{for } p|(\theta_j - \theta_i) \quad \text{with } i \neq j \in \{1, 2, \ldots, n-1\}}$$

2.2 CASE 2

Let M be a symmetric integral $v \times v$-matrix with constant row sum θ_0 and n eigenvalues $\theta_0^{1=m_0}, \theta_1^{m_1}, \ldots, \theta_{n-3}^{m_{n-3}}$, $\theta_{n-2} = \frac{1}{2}(a + \sqrt{b})^{m_{n-2}}, \theta_{n-1} = \frac{1}{2}(a - \sqrt{b})^{m_{n-2}}$ with integral $\theta_1, \ldots, \theta_{n-3}, a, b, \frac{1}{4}(a^2 - b)$. Let p be a prime.

If $p \not| \prod_{i=1}^{n-1}\theta_i$ then $r_p(M) = v$.

If for some $i \in \{0, 1, 2, \ldots, n-3\}$ $\theta_i \equiv 0 \pmod p$ and $p \not| \prod_{j \neq i}\theta_j$ then $r_p(M) = v - m_i$.

If for some $i \in \{1, 2, \ldots, n-3\}$ both $\theta_0 \equiv \theta_i \equiv 0 \pmod p$ and $p \not| \prod_{j \neq 0, i}\theta_j$ then

$$\begin{array}{rcll} r_p(M) & = & v - m_i - 1 & \text{if } \frac{1}{v}\prod_{j=1}^{n-1}(\theta_0 - \theta_j) \equiv 0 \pmod p \\ & & v - m_i & \text{otherwise} \end{array}$$

For integral c the matrix $M - cI$ has eigenvalues $\theta_i - c$ ($i = 0, 1, 2, \ldots, n-1$). Now $p|(\theta_{n-2} - c)(\theta_{n-1} - c)$ if c is a solution of $x^2 - ax + \frac{a^2-b}{4} \equiv 0 \pmod p$. This equation has either no solution (irreducible polynomial), or one solution with multiplicity 2 or two different solutions. If $p = 2$ we have no solutions if both a and $\frac{a^2-b}{4}$ are odd, we have one solution ($c = \frac{b}{4}$) if a is even and we have two solutions if a is odd and $\frac{a^2-b}{4}$ is even. If p is odd then $x^2 - ax + \frac{a^2-b}{4} \equiv 0 \pmod p$ is equivalent with $(2x - a)^2 \equiv b \pmod p$, so we have 0,1 or 2 solutions depending on whether b is a nonsquare, zero or a square $\pmod p$.

Suppose $x^2 - ax + \frac{a^2-b}{4} \equiv 0 \pmod p$ has two different solutions c_1 and c_2 and that $\theta_1, \theta_2, \ldots, \theta_{n-3}$ are not solutions of this equation. Let c be either c_1 or c_2. Over $\mathbb{F}_p$, the polynomial $\det(xI - M)$ contains m_{n-2} (if $\theta_0 \not\equiv 0 \pmod p$) or $m_{n-2} + 1$ factors $(x - c)$, so $r_p(M - cI)$ is either $v - m_{n-2}$ or $v - m_{n-2} - 1$. Clearly $r_p(M - cI) = v - m_{n-2}$ if $\theta_0 \not\equiv c \pmod p$. If $c \equiv \theta_0 \pmod p$ then

$$\begin{array}{rcll} r_p(M - cI) & = & v - m_{n-2} - 1 & \text{if } \frac{1}{v}\prod_{j=1}^{n-1}(\theta_0 - \theta_j) \equiv 0 \pmod p \\ & & v - m_{n-2} & \text{otherwise.} \end{array}$$

Suppose p is an odd prime that divides $(\theta_{n-2} - \theta_{n-1})^2$ precisely once. Suppose furthermore that none of $\theta_1, \theta_2, \ldots, \theta_{n-3}$ is a solution of $x^2 - ax + \frac{a^2-b}{4} \equiv 0 \pmod p$. Let $A := M - \frac{1}{2}(\theta_{n-2} + \theta_{n-1})I$.

If $\theta_0 \not\equiv \frac{1}{2}(\theta_{n-2}+\theta_{n-1}) \pmod p$ then $p^{m_{n-2}} \| \det(A)$ and $r_p((A)^2) = v - 2m_{n-2}$, so $r_p(A) = v - m_{n-2}$.

If $\theta_0 \equiv \frac{1}{2}(\theta_{n-2}+\theta_{n-1})$ then $r_p(A)$ is either $v - m_{n-2}$ or $v - m_{n-2} - 1$. If furthermore $v \not\equiv 0 \pmod p$ then $r_p(A) = v - m_{n-2} - 1$. (Indeed $\mathbf{1} \notin \langle A \rangle_p$ and $\mathbf{1} \in \langle A + J \rangle_p$).

Now suppose that $v \equiv 0 \pmod p$ and $\theta_0 \equiv \frac{1}{2}(\theta_{n-2}+\theta_{n-1})$, then

$$\begin{array}{rcll} r_p(A) & = & v - m_{n-2} - 1 & \text{if } \frac{1}{v}\prod_{i=1}^{n-1}(\theta_0 - \theta_i) \equiv 0 \pmod p \\ & & v - m_{n-2} & \text{otherwise.} \end{array}$$

Indeed, $\prod_{i=1}^{n-3}(M - \theta_i I)$ has p-rank $1 + 2m_{n-2}$. If A has p-rank $v - m_{n-2}$ then $r_p(A^2) = v - 2m_{n-2}$, so $\prod_{i=1}^{n-1}(M - \theta_i I) = \frac{1}{v}\prod_{i=1}^{n-1}(\theta_0 - \theta_i) J \not\equiv O \pmod p$. If $\frac{1}{v}\prod_{i=1}^{n-1}(\theta_0 - \theta_i) \not\equiv 0 \pmod p$ then $\mathbf{1} \in \langle A^2 \rangle_p$ and if $(\theta_0 - \theta_{n-2})(\theta_0 - \theta_{n-1})$ contains precisely e factors p, also v contains precisely e factors p. Let $\underline{x}$ and $\underline{y}$ be two vectors such that $A^2\underline{y}^T = \mathbf{1}^T$ and $\underline{x}A = \mathbf{1}$. Then $\underline{x}\mathbf{1}^T = \underline{x}A^2\underline{y}^T = 0$, so there exists a vector $\underline{x}$ such that $\underline{x}A = \mathbf{1}$ and $\underline{x}\mathbf{1}^T = 0$. It follows that A has the same p-rank as the $(v+1)\times(v+1)$-matrix B defined by

$$B := \left(\begin{array}{c|c} \theta_0 - \frac{1}{2}(\theta_{n-2}+\theta_{n-1}) & \mathbf{1} \\ \hline \mathbf{1}^T & A \end{array}\right)$$

We have that p divides v precisely once since $(\theta_0 - \theta_{n-2})(\theta_0 - \theta_{n-1}) = (\theta_0 - \frac{1}{2}(\theta_{n-2} + \theta_{n-1}))^2 - \frac{(\theta_{n-2}-\theta_{n-1})^2}{4}$. If $\eta_i = \theta_i - \frac{1}{2}(\theta_{n-2}+\theta_{n-1})$ for $i = 0, 1, \ldots, n-1$ are the eigenvalues of A, then B has spectrum:

$$(\eta_0 + \sqrt{v})^1, (\eta_0 - \sqrt{v})^1, \eta_1^{m_1}, \ldots, \eta_{n-1}^{m_{n-1}}$$

It follows that $\det(B)$ contains $m_{n-2}+1$ factors p (1 factor from $(\eta_0 + \sqrt{v})(\eta_0 - \sqrt{v}) = \eta_0^2 - v$ and m_{n-2} factors from $(\eta_{n-2}\eta_{n-1})^{m_{n-2}}$ since $\eta_{n-2}\eta_{n-1} = -\frac{(\theta_{n-2}-\theta_{n-1})^2}{4}$) so $r_p(A) = r_p(B) \le v - m_{n-2}$. It follows that $r_p(A) = v - m_{n-2}$.

The p-ranks that are in general not determined by the spectrum are:

$r_p(M - \theta_i I)$	with $p \mid (\theta_i - \theta_j)$	for $i \neq j \in \{1, 2, \ldots, n-3\}$
$r_p(M - \theta_i I)$	with $p \mid (\theta_i - \theta_{n-2})(\theta_i - \theta_{n-1})$	for $i \in \{1, 2, \ldots, n-3\}$
$r_2(M - \theta_{n-2}\theta_{n-1} I)$	if $\theta_{n-2} + \theta_{n-1}$ is even	
$r_p(M - \frac{1}{2}(\theta_{n-2}+\theta_{n-1}) I)$	for odd $p^2 \mid (\theta_{n-2} - \theta_{n-1})^2$	

Example

According to Haemers and Spence [7] there are ten graphs on 24 vertices with spectrum $7^1, \sqrt{7}^8, -1^7, -\sqrt{7}^8$. One of these, the so-called Klein graph, is distance-regular with intersection array $\{7, 4, 1; 1, 2, 7\}$. The p-ranks that are still open in this case are $r_2(A + I)$ and $r_3(A + I)$. Figure 1 denotes the ranks we find for the ten graphs with the mentioned spectrum (The number after the value for the rank is the unique b_0 such that $r_p(A - cI + b_0 J) = r_p(A - cI + bJ) - 1$ for all $b \neq b_0$, or '−' in case $r_p(A - cI + bJ)$ is independent of b).

	$\lvert\mathrm{Aut}(\Gamma)\rvert$	$r_2(A+I)$		$r_3(A+I)$		
1	2	14	–	16	–	
2	12	14	–	16	–	
3	3	15	1	16	–	
4	2	13	1	16	–	
5	16	12	–	16	–	
6	8	11	1	16	–	
7	6	13	1	16	–	
8	6	15	1	16	–	
9	42	15	1	16	–	
10	336	9	1	16	–	Klein graph

Figure 1: The relevant p-ranks of all graphs cospectral with the Klein graph

2.3 CASE 3

Let A be the $v \times v$-adjacency matrix of a k-regular graph with spectrum k^1, θ^k, θ_2^k, θ_3^k, so $v = 3k+1$, $\theta_1 + \theta_2 + \theta_3 = -1$, $\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1 = -k$, $q = \theta_1\theta_2\theta_3 = k + \frac{1}{3}(q_3+1)$, with $q_3 = \theta_1^3 + \theta_2^3 + \theta_3^3 = 6\Delta - k^2 + \frac{2\Delta}{k}$, where Δ is the number of triangles per point (cf. [5]). We study the p-ranks of the matrices $M - cI$ with $M := A + bJ$ for integral b and c. So $M - cI$ has spectrum $(k+bv-c)^1, (\theta_1 - c)^k, (\theta_2 - c)^k, (\theta_3 - c)^k$. Let p be a prime.

If $p \not| (k+bv-c)(\theta_1 - c)(\theta_2 - c)(\theta_3 - c)$ then $r_p(M) = v$.

If $p | (k+bv-c)$ and $p \not| (\theta_1 - c)(\theta_2 - c)(\theta_3 - c)$ then $r_p(M) = v - 1$.

If $p | (\theta_1 - c)(\theta_2 - c)(\theta_3 - c)$ then c is a solution of

$$x^3 + x^2 - kx - q \equiv 0 \pmod{p}. \tag{1}$$

Now there are five possibilities:

1. This equation has no solutions, so it is an irreducible polynomial over $\mathbb{F}_p$.
2. It has one solution, c say, with multiplicity 1.
3. It has three different solutions, c_1, c_2, c_3 say. So $x^3 + x^2 - kx - q \equiv (x - c_1)(x - c_2)(x - c_3) \pmod{p}$.
4. It has one solution with multiplicity 3.
5. It has two different solutions. One, c_1 say, with multiplicity 1 and one (c_2) with multiplicity 2.

Suppose that the equation (1) has one solution (c say) with multiplicity 1. So $x^3 + x^2 - kx - q \equiv (x - c)(\text{monic irr. pol. of deg. 2}) \pmod{p}$. Now $r_p(M - cI)$ is either $v - k$ or $v - k - 1$. Clearly if $p \not| (k + bv - c)$ then $r_p(M - cI) = v - k$. In general

$r_p(M-cI) = v-k-1$ if $\frac{1}{v}\prod_{i=1}^{3}(k+bv-\theta_i) \equiv 0 \pmod{p}$ and $r_p(M-cI) = v-k$ otherwise.

Suppose that (1) has three different solutions c_1, c_2 and c_3. Let c be one of these three solutions. If $c \not\equiv (k+bv) \pmod{p}$ then $r_p(M-cI) = v-k$. If $c \equiv (k+bv) \pmod{p}$ then

$$r_p(M-cI) \;=\; \begin{array}{ll} v-k-1 & \text{if } \frac{1}{v}\prod_{i=1}^{3}(k+bv-\theta_i) \equiv 0 \pmod{p} \\ v-k & \text{otherwise} \end{array}$$

Suppose that (1) has one solution (c say) with multiplicity 3, so $x^3+x^2-kx-q \equiv (x-c)^3 \equiv x^3-3cx^2+3c^2x-c^3 \pmod{p}$. Since $(x-c)^3 \equiv x^3-c \pmod{3}$ we may assume that $p \neq 3$. It follows that $c = -\frac{1}{3}$ and the equivalence holds if $p|v$.

Suppose that (1) has two solutions c_1 and c_2, so $x^3+x^2-kx-q \equiv (x-c_1)(x-c_2)^2 \pmod{p}$. If $c_1 \not\equiv (k+bv) \pmod{p}$ then $r_p(M-c_1I) = v-k$. If $c_1 \equiv (k+bv) \pmod{p}$ then $r_p(M-c_1I) = v-k-1$ if $\frac{1}{v}\prod_{i=1}^{3}(k+bv-\theta_i) \equiv 0 \pmod{p}$ and $r_p(M-c_1) = v-k$ otherwise. If $p=3$ then $x^3+x^2-kx-q \equiv (x+k)^2(x+k+1) \pmod{3}$ if and only if $3|(k^2+k+q)$. If $p=2$ then $x^3+x^2-kx-q \equiv x^2(x+1) \pmod{2}$ if and only if both k and q are even. If $p \neq 2,3$ then it follows that $c_1 = -k+\frac{6\Delta}{k}+2$ and $c_2 = -\frac{1}{2}(c_1+1)$ with the condition that p divides $(k-3-3\frac{2\Delta}{k})^2 - 4(1+\frac{2\Delta}{k})$.

The remaining p-ranks are:

$r_p(M+\frac{1}{3}I)$	for $3 \neq p\|v$
$r_2(M)$	if $2\|k$ and $2\|q$
$r_3(M+kI)$	if $3\|(k^2+k+q)$
$r_p(M-\frac{1}{2}(k-\frac{6\Delta}{k}-3)I)$	for $2,3 \neq p\|((k-3-3\frac{2\Delta}{k})^2-4(1+\frac{2\Delta}{k}))$

3 Hamming and Doob graphs

3.1 Definitions

The *Kronecker product* of two matrices A and B is the matrix with blocks $a_{ij}B$ and is denoted by $A \otimes B$. By definition $(A \otimes P)(B \otimes Q) = AB \otimes PQ$ where A and B resp. P and Q have fitting sizes.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with adjacency matrices A_1 and A_2 respectively. The *direct product* of G_1 and G_2 is the graph with vertex set $V_1 \times V_2$ and two vertices (v_1, w_1) and (v_2, w_2) are adjacent iff $((v_1 = v_2) \wedge (w_1, w_2) \in E_2)$ or $((w_1 = w_2) \wedge (v_1, v_2) \in E_1)$. So by definition the direct product of G_1 and G_2 has adjacency matrix $I \otimes A_1 + A_2 \otimes I$. Usually this is called the *Kronecker sum* of A_1 and A_2 (notation: $A_1 \oplus A_2$). If $\{\lambda_i\}$ and $\{\mu_j\}$ are the eigenvalues of A_1 and A_2 respectively, then $\{\lambda_i + \mu_j\}$ are the eigenvalues of $A_1 \oplus A_2$.

Let X be a finite set of cardinality $q \geq 2$. The *Hamming graph* Γ (with diameter d on X) has vertex set X^d, the cartesian product of d copies of X; two vertices of Γ are adjacent whenever they differ in precisely one coordinate. An equivalent definition is that a Hamming graph is the direct product of d copies of a complete graph on q

vertices. Because only d and q will be relevant here, we denote Γ as well as its adjacency matrix by $H(d,q)$.

Hamming graphs are distance-regular with diameter d and parameters $b_j = (d-j)(q-1)$, $c_j = j$ and $a_j = j(q-2)$ for $0 \leq j \leq d$. As distance-regular graphs they are uniquely determined by their parameters, except when $q = 4$. In this case the *Doob graphs*, which we will denote here by $D(n,m)$ and which are defined as the direct product of a Hamming graph $H(n,4)$ with m copies of a Shrikhande graph, are distance-regular with the same parameters as $H(n+2m,4)$. There are no other exceptions, see [2] Section 9.2. Hamming graphs have eigenvalues $\theta_j = q(d-j) - d$ with multiplicities $f_j = \binom{d}{j}(q-1)^j$ $(j = 0,1,\ldots,d)$. Because the eigenvalues of a distance-regular graph are determined by its parameters the Doob graph $D(n,m)$ has the same eigenvalues as $H(2n+m,4)$.

3.2 The p-ranks

In this section we will determine the p-ranks of matrices $A - cI$ for integral c where A is the adjacency matrix of a Pseudo Hamming graph, that is a distance-regular graph with the same parameters as some Hamming graph $H(d,q)$. It turns out that almost all of these p-ranks follow from the minimal polynomial of A, considered as matrix over $\mathbb{F}_p$. In order to determine the remaining p-ranks we have to use the structure of the considered Hamming or Doob graph.

Theorem 3 *Let A be the adjacency matrix of a distance-regular graph with the same parameters as the Hamming graph $H(d,q)$, then*

$$\begin{array}{lll} A^p - A \equiv O & \pmod{p} & \text{if } p \not| q \\ A^p + dI \equiv O & \pmod{p} & \text{if } p | q \end{array}$$

Proof: The theorem is trivial if $d < p$ since then the characteristic polynomial of B divides $x^p - x$ if $p \not| q$ and $x^p + d$ if $p|q$. If $d \geq p$ then $c_p \equiv 0 \pmod{p}$, so A has characteristic polynomial $\varphi_0(x)$ of degree p. If $p|q$ then all eigenvalues are equal to $-d$ modulo p, so $\varphi_0(x) = (x+d)^p = x^p + d$. If $p \not| q$ then modulo p the first p eigenvalues of B are all different, so $\varphi_0(x)$ contains a factor $x - c$ for each $c \in \mathbb{F}_p$ which implies that $\varphi_0(x) = x^p - x$. $\square$

More precisely, the minimal polynomial $\varphi_0(x)$ of A over $\mathbb{F}_p$ is

$$\varphi_0(x) = \begin{cases} \prod_{i=0}^{d}(x-\theta_i) & \text{if } p \not| q \text{ and } d < p \\ x^p - x & \text{if } p \not| q \text{ and } d \geq p \\ (x+d)^d & \text{if } p|q \text{ and } d < p \\ x^p + d & \text{if } p|q \text{ and } d \geq p \end{cases}$$

Since each element of $\mathbb{F}_p$ is a simple root of $x^p - x$, almost all p-ranks of Pseudo Hamming graphs follow from the eigenvalues and the intersection numbers. In particular we have:

Corollary 4 *Let A be the adjacency matrix of a distance regular graph with the same parameters as a Hamming graph $H(d,q)$. Let c be an integer and p a prime, then*

$$\begin{array}{lcll} r_p(A-cI) & = & q^d - \sum_{\theta_i - c \equiv 0 \pmod p} f_i & \text{if } p \not| q \\ r_p(A-cI) & = & q^d & \text{if } p|q \text{ and } c+d \not\equiv 0 \pmod p \\ r_p((A+dI)^e) & \le & \frac{p-e}{p} q^d & \text{for } 0 \le e \le p \text{ if } p|q \end{array}$$

So using the minimal polynomial of A over $\mathbb{F}_p$ we can determine all p-ranks except

$$r_p(A+dI) \text{ for } p|q.$$

We will now determine the p-rank of the matrix $A_H(d,q)+dI$ for every prime number p dividing q and the corresponding 2-rank for the Doob graphs. Hamming and Doob graphs are the only distance-regular Pseudo Hamming graphs (cf. [2]). Using the fact that these graphs are direct products of complete graphs and/or Shrikhande graphs, we derive a recurrence relation for the considered p-rank. We will denote the matrix $A_H(d,q)+dI$ by $B(d,q)$. For two matrices M and N, the expression $M \sim_p N$ will mean that, considered as matrices over $\mathbb{F}_p$, M and N have the same rank.

Lemma 5 *Let p be a prime dividing q, let M be an integral matrix and $l \not\equiv 0 \pmod p$ an integer, then*

$$(I_q \otimes M^k) + l(J_q \otimes M^{k-1}) \sim_p \operatorname{diag}(M^{k-1}, (M^k)^{q-2}, M^{k+1}).$$

Proof:

$$\begin{pmatrix} M^k + lM^{k-1} & lM^{k-1} & \cdots & \cdots & lM^{k-1} \\ lM^{k-1} & M^k + lM^{k-1} & lM^{k-1} & \cdots & lM^{k-1} \\ \vdots & lM^{k-1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & lM^{k-1} \\ lM^{k-1} & lM^{k-1} & \cdots & lM^{k-1} & M^k + lM^{k-1} \end{pmatrix} \sim_p$$

$$\sim_p \begin{pmatrix} M^k + lM^{k-1} & lM^{k-1} & \cdots & \cdots & lM^{k-1} \\ -M^k & M^k & 0 & \cdots & 0 \\ \vdots & 0 & M^k & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -M^k & 0 & \cdots & 0 & M^k \end{pmatrix} \sim_p$$

$$\sim_p \begin{pmatrix} M^k & lM^{k-1} & \cdots & \cdots & lM^{k-1} \\ 0 & M^k & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & M^k \end{pmatrix} \sim_p \begin{pmatrix} M^k & lM^{k-1} & 0 & \cdots & 0 \\ 0 & M^k & 0 & \cdots & 0 \\ \vdots & \ddots & M^k & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & M^k \end{pmatrix} \sim_p$$

$$\sim_p \begin{pmatrix} 0 & lM^{k-1} & 0 & \cdots & 0 \\ -\frac{1}{l}M^{k+1} & M^k & 0 & \cdots & 0 \\ 0 & 0 & M^k & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & M^k \end{pmatrix} \sim_p \mathrm{diag}(M^{k-1}, M^{k+1}, (M^k)^{q-2})$$

□

Lemma 6 *Let Sh be the adjacency matrix of the Shrikhande graph, then*

$$(I_{16} \otimes M) + (Sh \otimes I) \sim_2 diag((I)^6, (M)^4, (M^2)^6)$$

Proof: The proof is as straightforward as the proof of the previous lemma, but contains too many steps to write down here. □

3.2.1 Hamming graphs

Because $B(d,q) = H(d,q) + dI$, we have the following recurrence relation for $B(d,q)$:

$$B(d,q) = I \otimes B(d-1,q) + J \otimes I$$

from which we derive that

$$B(d,q)^k = I_q \otimes B(d-1,q)^k + kJ_q \otimes B(d-1,q)^{k-1} \tag{2}$$

$B(0,q) = (0)$, so by induction $B(d,q)^p \equiv 0 \pmod p$. From Lemma 5 and (2) we get the following recurrence relation for the p-rank of powers of the matrix $B(d,q)$:

Theorem 7

$$r_p B(d,q)^k = r_p B(d-1,q)^{k-1} + (q-2) r_p B(d-1,q)^k + r_p B(d-1,q)^{k+1}$$

for $k = 1, \ldots, p-1$ and with $p|q$.

Together with the obvious relations

$$\begin{aligned} r_p B(0,q)^k &= 0 \quad \text{for } k > 0 \\ r_p B(d,q)^p &= 0 \quad \text{for } d \geq 0 \\ r_p B(d,q)^0 &= q^d \quad \text{for } d \geq 0 \end{aligned}$$

these determine the p-ranks of the powers of $B(d,q)$ completely. If we define $\underline{r}_d$ to be the vector of length p with the k-th coefficient equal to $r_p B(d,q)^{k-1}$, then the above relations can be rewritten as

$$\underline{r}_d = Q \underline{r}_{d-1}$$

with

$$\underline{r}_0 = (1,0,0,\ldots,0)^T \text{ and } Q = \begin{pmatrix} q & 0 & 0 & \cdots & \cdots & 0 \\ 1 & q-2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & q-2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & q-2 \end{pmatrix}$$

So

$$\underline{r}_d = Q^d \underline{r}_0$$

For $p \neq 2,3$ this will be the most useful expression we get for the p-rank of $H(d,q)+dI$ and its powers. For $p = 2,3$ we can derive explicit formulas for the p-ranks we are looking for by diagonalizing the matrix Q. Namely if $p = 2$ then

$$Q = \begin{pmatrix} q & 0 \\ 1 & q-2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & q-2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^{-1}$$

and if $p = 3$ then

$$Q = \begin{pmatrix} q & 0 & 0 \\ 1 & q-2 & 1 \\ 0 & 1 & q-2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} q & 0 & 0 \\ 0 & q-1 & 0 \\ 0 & 0 & q-3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}^{-1}$$

which yields:

Corollary 8 *If $p = 2$, then*

$$\underline{r}_d = \begin{pmatrix} q^d \\ \frac{1}{2}(q^d - (q-2)^d) \end{pmatrix} \tag{3}$$

If $p = 3$, then

$$\underline{r}_d = \begin{pmatrix} q^d \\ \frac{2}{3}q^d - \frac{1}{2}(q-1)^d - \frac{1}{6}(q-3)^d \\ \frac{1}{3}q^d - \frac{1}{2}(q-1)^d + \frac{1}{6}(q-3)^d \end{pmatrix} \tag{4}$$

In fact we can find for every p and $k \in \{1,2,\ldots,p-1\}$ coefficients $\alpha_1, \alpha_2, \ldots, \alpha_p$ depending on p and k only, such that

$$r_p B(d,q)^k = \alpha_1 q^d + \alpha_2 (q-2+x_1)^d + \cdots + \alpha_p (q-2+x_{p-1})^d$$

with $x_1, x_2, \ldots, x_{p-1}$ zero's of the polynomial $S_{p-1}(x)$ defined by $S_n(x) = \det(S_n + xI_n)$, where S_n is the $n \times n$ matrix with one's on the codiagonals and zero's elsewhere. One can prove that the roots x_i lie symmetric with respect to 0 and $|x_i| < 2$ for $i = 1,2,\ldots,n$.

	$\lvert\mathrm{Aut}(\Gamma)\rvert$	$r_3(A)$		$r_2(A+I)$		
1	288	13	–	19	1	
2	72	14	–	17	1	
3	1296	12	–	19	1	$H'(3,3)$
4	1296	14	–	13	1	$H(3,3)$

Figure 2: The relevant p-ranks of all graphs cospectral with the Hamming graph $H(3,3)$

3.2.2 Doob graphs

Let $Sh(m)$ denote the adjacency matrix of the direct product of m copies of a Shrikhande graph, then it satisfies the following recurrence relation:

$$Sh(m) = I_{16} \otimes Sh(m-1) + Sh \otimes I_{16^{m-1}}$$

So by induction $Sh(m)^2 \equiv 0 \pmod 2$. Using Lemma 6 we get that

$$r_2 Sh(m) = 6 \cdot 16^{m-1} + 4 \cdot r_2 Sh(m-1)$$

which yields:

$$r_2 Sh(m) = \frac{1}{2}(16^m - 4^m) = r_2 B(2m,4) \tag{5}$$

If we denote the adjacency matrix of the Doob graph $D(d,m)$ also by $D(d,m)$, we find for its 2-rank:

Corollary 9

$$r_2(D(d,m)+dI) = \frac{1}{2}(4^{d+2m} - 2^{d+2m}) = r_2(H(d+2m,4)+dI)$$

Proof: By induction on d (or m) using Lemma 5 and (5) (or Lemma 6 and (3)). □

From Corollary 9 we get that each p-rank of a Doob graph is the same as the one for the Hamming graph with the same parameters as this Doob graph. Finally, it follows from the recurrence relations for $B(d,q)$ and $D(d,m)$ that for p dividing q we have $\mathbf{1} \in \langle H(d,q)+dI\rangle_p$ and $\mathbf{1} \in \langle D(d,m)+dI\rangle_2$, so $r_p(H(d,q)+dI+bJ)$ and $r_2(D(d,m)+dI+bJ)$ (for integral b) are independent of b.

Example

The Hamming graph $H(3,3)$ on 27 vertices has spectrum $6^1, 3^6, 0^{12}, -3^8$. Given a graph with this spectrum, Γ say, with adjacency matrix A, all p-ranks of $A - cI$ (c integral) are determined by the spectrum, except possibly for $r_3(A)$ and $r_2(A+I)$. According to Haemers and Spence [7] there are 4 graphs with this spectrum. They have relevant p-ranks as shown in Figure 2. Note that $H(3,3)$ is uniquely determined by its spectrum and the minimality of the 2-rank of $A+I$. In general we have:

Theorem 10 *Let for some odd q, Γ be a graph with the same spectrum as $H(3,q)$ and let A be its adjacency matix. If $r_2(A+I) = 1+3(q-1)^2$, then Γ is isomorphic to $H(3,q)$.*

Proof: It follows from the spectrum of Γ that $r_2(A) = 3(q-1)+(q-1)^3$, so modulo 2 Γ has minimal polynomial x^2+x. So $A^2 \equiv A \pmod 2$ and two vertices at distance 2 have at least 2 common neighbours, implying that each vertex has at most $3(q-1)^2$ vertices at distance 2. It is proved in Haemers [6] that each vertex has at least this many vertices at distance 2 and that if equality holds for each vertex, the graph is distance-regular. Since $H(3,q)$ is the unique distance-regular graph with this spectrum, the result follows. $\square$

4 Some other distance-regular graphs

4.1 Johnson graphs

The *Johnson graph* $J(n,k)$ is the graph with vertex set the k-subsets of a set with n elements, two vertices being adjacent if they have an intersection of size $k-1$. The Johnson graph $J(n,k)$ has $\binom{n}{k}$ vertices, diameter $d = \min\{k, n-k\}$ and is distance-regular with intersection numbers $a_i = (n-2i)i$, $b_i = (k-i)(n-k-i)$, $c_i = i^2$. It has eigenvalues $\theta_i = k(n-k) - i(n+1-i) = k(n-k) - (\frac{n+1}{2})^2 + (i - \frac{n+1}{2})^2$ with multiplicities $f_i = \binom{n}{i} - \binom{n}{i-1}$ $(0 \le i \le d)$.

Let p be a prime and let A be the adjacency matrix of a distance-regular graph with the same parameters as $J(n,k)$, then A modulo p has minimal polynomial

$$\begin{array}{ll}
(x+k)^2 & \text{if } p=2 \text{ and } n \text{ even},\\
x(x+1) & \text{if } p=2 \text{ and } n \text{ odd},\\
\prod_{i=0}^{d}(x-\theta_i) \pmod p & \text{if } p \text{ is odd and } p>d,\\
\prod_{i=0}^{p-1}(x-k(n-k)+(\frac{n+1}{2})^2-i^2) & \text{if } p \text{ is odd and } p \le d.
\end{array}$$

So we have

$$r_p((A-cI)^2) = \sum_{\theta_i \not\equiv c \pmod p} f_i$$

for all prime numbers p and integral c, and even

$$r_p(A-cI) = \sum_{\theta_i \not\equiv c \pmod p} f_i$$

for all prime numbers p and integers c, except possibly for

$$\begin{array}{ll}
r_2(A+kI) & \text{if } n \text{ is even},\\
r_p(A-(k(n-k)-(\frac{n+1}{2})^2+c)I) & \text{with } p \text{ odd and } c \text{ is a non-zero square} \pmod p.
\end{array}$$

Concerning these cases we could determine only a few more p-ranks. From now on we will denote the adjacency matrix of $J(n,k)$ by $J(n,k)$ as well. If n is even then $r_2(J(n,k)+kI) = r_2(J(n-1,k)+kI)$ since the sum modulo 2 of all rows of $J(n,k)+kI$

corresponding to the k-subsets that contain some fixed $k-1$-set is equal to the zero vector. From this we get the following values for the 2-ranks of $J(n,k)$ which were also found by R. Riebeek (personal communication):

$$\begin{aligned} r_2(J(n,k)+kI) &= \tbinom{n-2}{k-2} \quad \text{if } n \text{ is even,} \\ & \tbinom{n-1}{k-1} \quad \text{if } n \text{ is odd,} \\ r_2(J(n,k)+(k+1)I) &= \tbinom{n}{k} \quad \text{if } n \text{ is even,} \\ & \tbinom{n-1}{k} \quad \text{if } n \text{ is odd.} \end{aligned}$$

Some other results follow from the work of Wilson (see [9]) who, for given integers t, k and n, determined the p-ranks of the $\binom{n}{t}$ by $\binom{n}{k}$ matrix $N_{t,k}(n)$ (or simply $N_{t,k}$) of 0's and 1's, the rows of which are indexed by the t-subsets T of an n-set X, whose columns are indexed by the k-subsets K of the same set X, and where the entry $N_{t,k}(T,K)$ in row T and column K is 1 if $T \subseteq K$ and is 0 otherwise. He proved the following theorem:

Theorem 11 (Wilson) *(cf. [9]) For $t \leq \min\{k, n-k\}$, the rank of $N_{t,k}$ modulo a prime p is*

$$\sum \binom{n}{i} - \binom{n}{i-1}$$

where the sum is extended over those indices i such that p does not divide the binomial coefficient

$$\binom{k-i}{t-i}.$$

□

Now by definition $J(n,k)+kI = N_{k-1,k}^T N_{k-1,k}$. By Wilson's theorem

$$r_p(N_{k-1,k}) = \sum_{\substack{i=0 \\ i \not\equiv k \pmod p}}^{k-1} \binom{n}{i} - \binom{n}{i-1},$$

so for instance $r_p(N_{k-1,k}) = \binom{n}{k-1}$ if $p > k$, $r_p(N_{p-1,p}) = \binom{n}{p-1} - 1$ and $r_3(N_{3,4}) = \binom{n}{3} - (n-1)$ from which, after considering the kernel of $N_{k-1,k}$, the following results follow:

$$\begin{aligned} r_p(J(n,k)+kI) &= \tbinom{n}{k-1} \quad \text{if } p > k, \\ r_p(J(n,p)+pI) &= \tbinom{n}{p-1} - 2 \quad \text{if } p \text{ divides } \tbinom{n}{p-1}, \\ & \tbinom{n}{p-1} - 1 \quad \text{if } p \text{ does not divide } \tbinom{n}{p-1}, \end{aligned}$$

$$
\begin{aligned}
r_3(J(n,4)+4I) \quad = \quad & \binom{n}{3} - 2(n-1) + 1 && \text{if } n \equiv 0 \pmod 3, \\
& \binom{n}{3} - (n-1) && \text{if } n \equiv 1 \pmod 3, \\
& \binom{n}{3} - 2(n-1) && \text{if } n \equiv 2 \pmod 3.
\end{aligned}
$$

4.2 GQ minus a spread

Let $GQ(s,t)$ be a generalized quadrangle with point set $\mathcal{P}$ and line set $\mathcal{L}$. A *spread* is a collection of lines partitioning the point set. Let $\mathcal{S}$ be a spread of $GQ(s,t)$, then (cf. [2] section 12.5 or [1]) the collinearity graph Γ of $(\mathcal{P}, \mathcal{L} \setminus \mathcal{S})$ is distance-regular of diameter 3, with $v = (s+1)(st+1)$ vertices, spectrum $st^1, -1^{st=m_2}, s^{st(st+1)/(s+t)=m_1}, -t^{s^2(st+1)/(s+t)=m_3}$ and intersection array $\{st, s(t-1), 1; 1, t-1, st\}$, an antipodal $(s+1)$-cover of the complete graph K_{st+1}. More generally, given a strongly regular graph Δ with parameters $(v,k,\lambda,\mu) = ((s+1)(st+1), s(t+1), s-1, t+1)$ such that there is a partition $\mathcal{S}$ of its point set into $(s+1)$-cliques, the partial graph Γ obtained by deleting the edges contained in the members of $\mathcal{S}$ is distance-regular of diameter 3 with intersection array as given above. Conversely, any graph Γ with these parameters arises in this way.

Let G be a graph with the same spectrum as Γ and let A be its adjacency matrix, then the only p-ranks that are not necessarily determined by the spectrum are

$$
\begin{array}{ll}
r_p(A+I) & \text{for } p|(s+1) \text{ or } p|(t-1) \\
r_p(A+tI) & \text{for } p|(s+t)
\end{array}
$$

If furthermore G is distance-regular we can say more.

Theorem 12 *Let Γ be a distance-regular graph with adjacency matrix A, intersection array $\{st, s(t-1), 1; 1, t-1 st\}$ and spectrum $st^1, s^{m_1}, -1^{m_2}, -t^{m_3}$. Let $\mathcal{S}$ be the partition of the point set into the $st+1$ antipodal $(s+1)$-tuples and let Δ be the strongly regular graph with adjacency matrix B, parameters $(v,k,\lambda,\mu) = ((s+1)(st+1), s(t+1), s-1, t+1)$ and spectrum $s(t+1)^1, (s-1)^{m_1+m_2}, -(t+1)^{m_3}$ obtained from Γ by adding the edges between antipodal pairs of vertices.*

If p divides $t-1$, but not $s+1$ or $s+t$, then

$$r_p(A+I) = 1 + m_1.$$

If p divides $s+1$, but not $t-1$ or $s+t$, then

$$r_p(A+I) = 1 + m_2 + m_3.$$

If p divides $s+t$, but not $s+1$ or $t-1$, then

$$r_p(A+tI) = r_p(B+(t+1)I) + st - \epsilon,$$

where $\epsilon = 1$ if p divides t and $\epsilon = 0$ otherwise.

Proof: Let p be a prime dividing $t-1$, then since Γ is distance-regular we have that $(x+1)(x-s)$ is the minimal polynomial of A over $\mathbb{F}_p$, so if p does not divide $s+1$, then $r_p(A+I) = 1 + m_1$ and $r_p(A - sI) = m_2 + m_3$.

Let p be a prime dividing $s+1$, but not $t-1$ or $s+t$. Denote the $st+1$ antipodal $(s+1)$-tuples of Γ as well as their characteristic vectors by $l_1, l_2, \ldots, l_{st+1}$ and let S ($=J_{s+1} \otimes I_{st+1}$) be the matrix for which $S_{ij} = 1$ if $i = j$ or if i and j are antipodal in Γ and $S_{ij} = 0$ otherwise, then $B + I = A + S$ and

$$\langle A+I\rangle_p \oplus \langle l_1, \ldots, l_{st+1}\rangle_p = \langle B+2I\rangle_p \oplus \langle l_1, \ldots, l_{st+1}\rangle_p.$$

Claim 1:

$$l_i \in \langle A+I\rangle_p \quad \text{for } i = 1, 2, \ldots, st+1$$

Summing all rows of $A+I$ gives $(st+1)\mathbf{1} \equiv (1-t)\mathbf{1} \pmod p$, so the all-one vector is in $\langle A+I\rangle_p$. Since Δ is strongly regular with the given spectrum, we have $(B + 2I)(B+(t+1)I) \equiv (t+1)J \pmod p$. Furthermore we have that $S(B+2I) \equiv J \pmod p$ and $S^2 \equiv O \pmod p$, so $(A+tI)(A+I) =(B+(t+1)I-S)(B+2I-S) = (B+(t+1)I)(B+2I) - S(2B+(t+3)I) + S^2 \equiv (t+1)J - 2J - (t-1)S = (t-1)(J-S) \pmod p$, so $\mathbf{1} - l_i$ and hence l_i is in $\langle A+I\rangle_p$.

Claim 2:

$$\langle B+2I\rangle_p \cap \langle l_1, \ldots, l_{st+1}\rangle_p = \langle \mathbf{1}\rangle_p$$

Note that $\langle l_1, \ldots, l_{st+1}\rangle_p = \langle \mathbf{1}, l_1 - l_2, \ldots, l_1 - l_{st+1}\rangle_p$ since $\mathbf{1} + \sum_{i=1}^{st+1}(l_1 - l_i) = \mathbf{1} - \sum l_i + (st+1)l_1 \equiv (1-t)l_1 \pmod p$. Now $l_i(B+2I) = \mathbf{1}$, so $l_i - l_j \in \ker_p(B+2I)$ for all $i, j = 1, 2, \ldots, st+1$. Since B (over $\mathbb{F}_p$) has minimal polynomial $(x+t+1)^2(x+2)$ we have $\ker_p(B+2I) \cap \langle B+2I\rangle_p = \{0\}$ and the claim follows.

Now

$$r_p(A+I) = \dim(\langle A+I\rangle_p \oplus \langle l_1, \ldots, l_{st+1}\rangle_p) = \dim(\langle B+2I\rangle_p \oplus \langle l_1, \ldots, l_{st+1}\rangle_p) = r_p(B+2I) + st = 1 + m_2 + m_3.$$

Let p be a prime dividing $s+t$ but not $s+1$ or $t-1$, then again

$$\langle A+tI\rangle_p \oplus \langle l_1, \ldots, l_{st+1}\rangle_p = \langle B+(t+1)I\rangle_p \oplus \langle l_1, \ldots, l_{st+1}\rangle_p \tag{6}$$

Since $l_i(A+tI) = \mathbf{1} + (t-1)l_i$ the left hand side of (6) has dimension $r_p(A+tI)$ or $r_p(A+tI)+1$ depending on whether $\mathbf{1} \in \langle A+tI\rangle_p$ or not. Summing the rows of $A+tI$ yields $(s+1)t\mathbf{1}$, so $\mathbf{1} \in \langle A+tI\rangle_p$ if $p \not| t$. If on the other hand $p|t$, then $\langle A+tI\rangle_p \subset \mathbf{1}^\perp$, but $\mathbf{1}\mathbf{1}^T = (s+1)(st+1) \neq 0$, so $\mathbf{1} \notin \langle A+tI\rangle_p$.

For each l_i we have that $l_i(B+(t+1)I) = \mathbf{1}$, so $l_i - l_j \in \ker_p(B+(t+1)I)$ for any two $(s+1)$-tuples of $\mathcal{S}$. Let $\chi := \sum_{i=1}^{st+1} \alpha_i l_i$ be a vector from $\langle l_1, \ldots l_{st+1}\rangle_p$ and suppose that $\chi \in \langle B+(t+1)I\rangle_p$. Then, since $l_i - l_j \in \ker_p(B+(t+1)I)$ and B is symmetric, we must have that $l_i\chi^T$ is constant for all l_i. Now $l_i\chi^T = \alpha_i(s+1)$ and p does not divide $s+1$, so $\chi \in \langle \mathbf{1}\rangle_p$. Since $\mathbf{1} \in \langle B+(t+1)I\rangle_p$ the dimension of the right hand side of (6) is equal to $r_p(B+(t+1)I) + st$. $\square$

	$\vert\mathrm{Aut}(\Gamma)\vert$	$r_2(A)$		$r_3(A+I)$		
1	4	12	0	14	–	
2	36	12	0	14	–	
3	2	14	0	14	–	
4	12	14	0	14	–	
5	12	14	0	12	–	
6	16	14	0	14	–	
7	4	12	0	14	–	
8	12	12	0	14	–	
9	16	10	0	14	–	
10	24	12	0	14	–	
11	1296	8	0	14	–	Γ_3 of $H(3,3)$
12	1296	14	0	10	–	$GQ(2,4)$ minus 'plane-ovoid'-spread
13	324	14	0	12	–	$GQ(2,4)$ minus 'tripod'-spread

Figure 3: The relevant p-ranks of all graphs cospectral with $GQ(2,4)$ minus a spread

So the only p-rank(s) that can depend on the particular spread that is deleted is

$$r_p(A+tI) \quad \text{for } p \text{ dividing } s+t \text{ as well as } t-1.$$

Example
Up to isomorphism there are two distance-regular graphs on 27 vertices with intersection array $\{8,6,1;1,3,8\}$ (cf. [1]). Both are the collinearity graph of the unique $GQ(2,4)$ minus a spread. This GQ possesses exactly two non-isomorphic spreads (see [4]). In the dual $GQ(4,2)$ one spread corresponds to a 'plane-ovoid' and the other to a 'tripod'. The considered graphs have spectrum $8^1, 2^{12}, -1^8, -4^6$, so if A is the adjacency matrix of a graph Γ with this spectrum, $r_2(A)$ and $r_3(A+I)$ are in general not determined by this spectrum. If Γ is $GQ(2,4)$ minus a spread, then $r_2(A) = r_2(B+I) + st - 1 = 14$, where B is the adjacency matrix of $GQ(2,4)$ for which $r_2(B+I) = 7$. In [7] all graphs with spectrum $8^1, 2^{12}, -1^8, -4^6$ are determined. There are 13 of these. Their ranks are denoted in Figure 3.

4.3 Square 2-designs

Any connected bipartite graph Γ is the incidence graph of a design $(X, \mathcal{B})$. It can be found in [2] (section 1.6) that Γ is a (bipartite) distance-regular graph of diameter 3 if and only if $(X, \mathcal{B})$ is a square 2-design. If the square 2-design has parameters 2-(w,k,μ), then Γ has $2w$ vertices, intersection array $\{k, k-1, k-\mu; 1, \mu, k\}$ and spectrum $\pm k^1, \pm\sqrt{k-\mu}^{\,w-1}$. It is proved in [6] that any graph with this spectrum is distance-regular and hence the incidence graph of a square 2-design.

Let Γ be a graph with spectrum $\pm k^1, \pm\sqrt{k-\mu}^{\,w-1}$. Let A be its adjacency matrix and N the incidence matrix of the corresponding square 2-design with parameters 2-(w,k,μ)

(so $k(k-1)=\mu(w-1)$), then the following p-ranks are still open:

$$\begin{array}{ll} r_p(A+kI) & \text{for } p|k^2-k+\mu=\mu w \\ r_2(A+(k-\mu)I) & \\ r_p(A) & \text{for odd } p \text{ for which } p^2|(k-\mu) \end{array}$$

Notice that

$$A=\begin{pmatrix} O & N \\ N^T & O \end{pmatrix}.$$

Suppose that $p|k^2-k+\mu$ and $p \not| k$ then $r_p(A+kI) = r_p \mathrm{diag}(I_w, N^TN-k^2I) = w+r_p(\mu J+(k^2-k+\mu)I) = w+r_p(\mu J)$. Similarly, if $k-\mu$ is odd then $r_2(A+I)=w+r_2(\mu J)$. If $2\|(k-\mu)$ then $r_2(N)=\frac{w+1}{2}$ and hence $r_2(A)=w+1$ if μ is odd and $r_2(N)=\frac{w-1}{2}$ and hence $r_2(A)=w-1$ if μ is odd. The p-ranks that remain are:

$$r_p(A)=2r_p(N) \quad \text{for } p^2|(k-\mu).$$

The p-ranks $r_p(N)$ with $p^2|(k-\mu)$ are precisely those that are not determined by the parameters of the design, see [8].

4.4 Taylor graphs

A distance-regular graph with intersection array $\{k,\mu,1;1,\mu,k\}$ is called a Taylor graph. For a Taylor graph the number of vertices at distance 2 and 3 from a point is k and 1 respectively, so Γ is an antipodal double cover of K_{k+1} and has spectrum $k^1, -1^k, \lambda_2^{m_2}, \lambda_3^{m_3}$, with $\lambda_2+\lambda_3=k-1-2\mu$, $\lambda_2\lambda_3=-k$, $m_2=\frac{-\lambda_3}{\lambda_2-\lambda_3}(k+1)$, and $m_3=\frac{\lambda_2}{\lambda_2-\lambda_3}(k+1)$.

Let A be the adjacency matrix of a graph Γ with this spectrum, then the following p-ranks are not necessarily determined by the spectrum:

$$\begin{array}{ll} r_p(A+I) & \text{for } p|2\mu \\ r_p(A-\frac{1}{2}(k-1-2\mu)I) & \text{for odd } p \text{ with } p^2|(k-1-2\mu)^2+4k. \end{array}$$

If Γ is distance-regular and $p|\mu$, then (x+1)(x-k) is the minimal polynomial of A modulo p, so if $p|\mu$ but $p\not| k+1$, the geometric multiplicities of the eigenvalues -1 and k are equal to their algebraic multiplicities.

Suppose that Γ is a Taylor graph, let x be a vertex of Γ and let B_x be the adjacency matrix of $\overline{\Gamma(x)}$, the complement of the subgraph of Γ induced by the neighbours of x. The graph $\overline{\Gamma(x)}$ is strongly regular with parameters $(k,\mu,\frac{1}{2}(-k-1+3\mu),\frac{1}{2}\mu)$ and has eigenvalues μ and $-\frac{1}{4}(k+1-2\mu)\pm\frac{1}{4}\sqrt{(k-1-2\mu)^2+4k}$. Let

$$B'_x := \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0}^T & B_x \end{array}\right),$$

then

$$A=\left(\begin{array}{c|c} J-I-B'_x & B'_x \\ \hline B'_x & J-I-B'_x \end{array}\right)$$

and it follows that the p-ranks of the Taylor graphs that are not yet determined by the spectrum, can be expressed in terms of the p-ranks of $\overline{\Gamma(x)}$:

Theorem 13

$$r_p(A+I) = 2 + r_p(B_x)$$

If p is an odd prime for which $p^2|(k-1-2\mu)^2+4k$ and p does not divide μ, then

$$r_p(A - \frac{1}{2}(k-1-2\mu)I) = k + \epsilon + r_p(B_x + \frac{1}{4}(k+1-2\mu)I)$$

where $\epsilon = 0$ if $p|k$ and $\epsilon = 1$ otherwise.

Note that $r_p(B_x)$ is not necessarily determined by its spectrum if p divides both $\frac{1}{2}\mu$ and $\frac{1}{2}(k+1)$. For an odd prime p with $p^2|(k-1-2\mu)^2+4k$ the rank $r_p(B_x + \frac{1}{4}(k+1-2\mu)I))$ is in general not determined by the spectrum of B_x.

Proof: The first identity follows straight forward from the fact that $\mathbf{1} \notin \langle B'_x \rangle_p$. Now suppose we are in the second case. Then

$$A - \frac{1}{2}(k-1-2\mu)I \sim_p \left(\begin{array}{c|c} 2J - 2B'_x - (k+1-2\mu)I & 2B'_x \\ \hline 2B'_x & 2J - 2B'_x - (k+1-2\mu)I \end{array}\right) \sim_p$$

$$\sim_p \left(\begin{array}{c|c} 2J - (k+1-2\mu)I & 2B'_x \\ \hline 2J - (k+1-2\mu)I & 2J - 2B'_x - (k+1-2\mu)I \end{array}\right) \sim_p$$

$$\sim_p \left(\begin{array}{c|c} 2J - (k+1-2\mu)I & 2B'_x \\ \hline O & 2J - 4B'_x - (k+1-2\mu)I \end{array}\right) \sim_p$$

$$\sim_p \left(\begin{array}{c|c} 2J - (k+1-2\mu)I & O \\ \hline O & 2J - 4B'_x - (k+1-2\mu)I \end{array}\right).$$

Now $(-(k+1-2\mu)|2\mathbf{1}) \in \langle (2\mathbf{1}^T|2J-4B_x-(k+1-2\mu)I) \rangle_p$ and $\mathbf{1} \in \langle 4B_x + (k+1-2\mu)I \rangle_p$ and the result follows. □

Example

The Johnson graph $J(6,3)$ is the unique distance-regular graph on 20 vertices with intersection array $\{9,4,1;1,4,9\}$. So $J(6,3)$ is a Taylor graph and has spectrum $9^1, 3^5, -1^9, -3^5$. The relevant p-ranks are $r_2(A+I)$ and $r_3(A)$. The neighbour graph of any vertex of $J(6,3)$ is the Paley graph $P(9)$ which is self-complementary. So for $J(6,3)$ we have $r_2(A+I) = 2+4 = 6$ and $r_3(A) = 9+0+4 = 13$. According to Haemers and Spence [7] there are six graphs with the same spectrum as $J(6,3)$ with the following ranks:

	$\lvert\mathrm{Aut}(\Gamma)\rvert$	$r_2(A+I)$		$r_3(A)$		
1	1440	6	–	13	0	$J(6,3)$
2	96	8	–	13	0	
3	32	8	–	13	0	
4	16	10	–	13	0	
5	48	10	–	13	0	
6	12	10	–	13	0	

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