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A dynamic reinsurance theory

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Abstract: In this paper we present a technique that can be used by the insurer, who reinsured part of his risk by means of a proportional stop-loss contract, to evaluate his residual risk position. Part of this technique consists of the calculation of the optimal reinsurance strategy. We also show how this same technique can be used by the reinsurer to evaluate his risk position.

Keywords: Proportional stop-loss reinsurance, Residual risk, Optimal reinsurance strategy, Carré du Champ operator.

1. Introduction

An insurer who wants to reduce his risk can do this by underwriting a reinsurance policy. He hereby has the choice between several different sorts of contracts. We suppose that the insurer wants to reduce his risk by contracting a *proportional stop-loss reinsurance*. This is a contract between the insurer and the reinsurer, where the reinsurer promises to cover a certain fraction j (the reinsurance fraction) of that part of the losses that exceeds a certain bound X (the stop-loss bound) during a given time period $[0, T]$. In return, the insurer promises to hand over a certain fraction p of his received premiums to the reinsurer. Clearly, the insurer will have to choose the fraction j and the stop-loss bound X, taking into account the residual risk and the fraction p that logically follow on this choice. Therefore he needs criteria to evaluate his residual risk, given a certain choice of j and X. In this article we give three such criteria, namely

- the conditional expectation R_t of the residual loss (this is the total loss over the reinsured period reduced with the part that is covered by the reinsurer) at a certain time t, $0 \le t \le T$, and conditional to the information that the insurer knows about the risk process at time t.
- the conditional variance V_t of the residual loss at a certain time t, $0 \le t \le T$, and conditional to the information that the insurer knows about the risk process at time t.
- the *optimal reinsurance strategy* $(j_s)_{0 \le s \le T}$. This is a continuous time stochastic process where for each time t, $0 \le t \le T$, the reinsurance fraction j_t is chosen in such a way that $V[R_T]$ becomes minimal.

These three criteria permit the insurer to evaluate the residual risk at a certain time t for all his proportional stop loss reinsurance contracts at that time, whatever the remaining term of the contract may be.

When we replace the *residual loss for the insurer* with the *part of the loss that is covered by the reinsurer* these same three criteria can be used by the reinsurer to evaluate his risk position at a certain time t.

Furthermore, we consider the situation where the insurer wants to reduce his risk by contracting a proportional stop loss reinsurance on the claim height process *in combination with* a proportional stop

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loss reinsurance on the claim number process. We will show how an optimal reinsurance strategy can be constructed in this case.

2. The model

Let $\{N_t : t \in \mathbb{R}^+\}$ be the random process that counts the claims of an insurance portfolio and let $\{X_n : n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables representing the sizes of the succesive claims. In what follows, we will denote P_{X_i} for the distribution of X_i and f_{X_i} for the density function of this distribution. We suppose that the claim number process is a homogeneous Poisson process with parameter λ, i.e.,

$$P(N_t = n) = \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}.$$

Furthermore, $\{S_t : t \in \mathbb{R}^+\}$ is the risk process with $\forall t \in \mathbb{R}^+$,

$$S_t = \sum_{i=1}^{N_t} X_i$$

and $\mathscr{F}_t = \sigma\{S_s : 0 \le s \le t\}$ i.e., $\mathscr{F}_t$ is the sigma-algebra containing the history of the process $\{S_s : s \in \mathbb{R}^+\}$ up to time t.

All these processes are defined on some probability space $(\Omega, \mathscr{F}, P)$. We now define the Markov semi-group $(P_t)_{t \in \mathbb{R}^+}$ by

$$P_t = P_{S_t - S_0} = P_{S_t}.$$

It is easy to see that this is a stationary convolution semi-group and that the process $\{S_t : t \in \mathbb{R}^+\}$ is a realisation of this semi-group. Therefore it follows that the process $\{S_t : t \in \mathbb{R}^+\}$ is a right Markov process [see Getoor (1975)].

Since this will cause no problems of interpretation, we will also denote $(P_t)_{t \in \mathbb{R}^+}$ for the family of linear operators linked with this Markov semi-group, i.e.,

$$P_t f(x) = \int f(x+y) P_{S_t}(\mathrm{d}y) \quad \text{for all } f \in L(\Omega, \mathscr{F}, P_{S_t}).$$

For more information, see Dellacherie and Meyer (1987).

3. The scheme to evaluate residual risk

We will now introduce a scheme that can be used by the insurer to evaluate his residual risk at a certain time t for all the *proportional stop-loss* reinsurance contracts that he has running at that time. This scheme consists of three steps which we will now deal with in detail.

Step 1. It is clear that some essential information about the residual risk of the insurer lies in the conditional expectation of the residual loss of the insurer, conditional to the information that he knows about the risk process at that time. The residual loss of the insurer is given by

$$L = S_T - j(S_T - X)^+.$$

Hence the conditional expectation R_t is given by

$$\begin{aligned} R_t &= \mathrm{E}\left[S_T - j(S_T - X)^+ \mid \mathscr{F}_t\right] \\ &= H(t, S_t), \end{aligned}$$

where

$$H(t, x) = P_{T-t}h(x), \quad h(x) = x - j(x - X)^{+}.$$

Step 2. Still, this conditional expectation does not give the insurer sufficient information to get a thorough view of his residual risk. Consider for example the case where at a certain time $t < T$ the risk process exceeds the stop-loss bound X, and $j = 1$. In this case, the conditional expectation of the residual loss will be maximal ($R_t = X$). Yet, one can hardly claim that there is a great risk, since the insurer knows *for sure* that his residual loss will be X at time T. Therefore we introduce a second criterion to evaluate the residual risk of the insurer, namely the conditional variance V_t of the residual loss of the insurer. V_t is given by

$$\begin{aligned} V_t &= V\left[S_T - j(S_T - X)^{+} \mid \mathscr{F}_t\right] \\ &= E\left[L^2 \mid \mathscr{F}_t\right] - R_t^2 \\ &= K(t, S_t) - R_t^2, \end{aligned}$$

where

$$K(t, x) = P_{T-t}k(x), \quad k(x) = \left(x - j(x - X)^{+}\right)^2.$$

Step 3. The third criterion to evaluate the residual risk of the insurer consists of the calculation of the *optimal reinsurance strategy*. This optimal reinsurance strategy consists of a continuous time stochastic process $\{j_s : 0 \leq s \leq T\}$ which is constructed in such a way that the variance $V[R_T]$ of the conditional expectation of the residual loss at time T becomes minimal if the insurer, at each time $t \leq T$, chooses the insurance fraction to be equal to j_t. This implies that, in order to be able to minimize $V[R_T]$, the reinsurance portfolio should be continuously modified. It is clear that this is not possible in reality. Still the calculation of this optimal reinsurance strategy can be very worthwile, since it provides the insurer with some essential information about the optimality of his own risk position. To derive this optimal reinsurance strategy, we use a technique which was developed by Bouleau and Lamberton (1989) to calculate the optimal hedge position between two financial instruments. In fact, we calculate the optimal hedge position between two different risks, namely the original risk and the reinsured risk.

We will therefore first sketch the result of Bouleau and Lamberton. For more details and for the proofs, see Bouleau and Lamberton (1989).

Consider a portfolio with *stock price* $\{S_t : t \in \mathbb{R}^+\}$, and a *contingent claim* portfolio $\{M_t : t \in \mathbb{R}^+\}$ with expiration date T. The aim is to hedge the risk of the contingent claim portfolio by means of a self-financing hedge portfolio in $\{S_t : t \in \mathbb{R}^+\}$. This hedge position has to be optimal in the sense that the variance of the residu at time T has to become minimal. If one writes j_s for the hedge position at time s, and R_t for the residu at time t, we have

$$R_t = M_t - M_0 - \int_{]0, t]} j_s \, dS_s.$$

Consider a right Markov process $\{X_t : t \in \mathbb{R}^+\}$ with state space $(E, \mathscr{E})$, canonical filtration $(\mathscr{F}_t)$ and transition semi-group (P_t).

Definition (by Bouleau and Lamberton). Let f be a universally measurable function on E.

(i) We shall say that f belongs to $\mathscr{D}_1(A)$ if there exists a universally measurable function g satisfying

$$\int_0^t |g|(X_s) \, ds < +\infty \quad \forall t \geq 0, \ P^x\text{-a.s.}, \ \forall x \in E,$$

such that

$$C_t^f = f(X_t) - f(X_0) - \int_0^t g(X_s) \, ds$$

is a right continuous local martingale under every P^x, $\forall x \in E$.

(ii) We shall say that $f \in \mathscr{D}_2(A)$ if $f \in \mathscr{D}_1(A)$ and if C_t^f is locally square integrable under every P^x, $\forall x \in E$.

The function g in (i) is called the *extended infinitesimal generator* of f, and is denoted Af. The *Carré du Champ* operator is now defined by

$$\Gamma(f, g) = Afg - fAg - gAf.$$

Remark that it is not necessary for $f \in \mathscr{D}_2(A)$ to be bounded! This is very important for our results since we will have to cope with unbounded functions.

Theorem (by Bouleau and Lamberton). *If*

(*i*) (X_t) *permits a* Carré du Champ operator.
(*ii*) *For all* $x \in E$, *the stock price* S_t *is a* $(\mathscr{F}_t, P^x)$*-martingale of the form* $S_t = G(t, X_t)$ *for some* $G \in \mathscr{D}_2(\mathscr{A})$.
(*iii*) *For all* $x \in E$, *the contingent claim* M_t *is a* $(\mathscr{F}_t, P^x)$*-martingale of the form* $M_t = E^x[H(S_T) \mid \mathscr{F}_t] = F(t, X_t)$, *where* $f = H(G(T, \cdot))$ *satisfies* $P_T f^2(x) < \infty$ *for all* $x \in E$.

Then the process $\{j_s : 0 \le s \le T\}$ *of optimal hedging is given, under every* P^x, *by*

$$j_t = \frac{\Gamma(F, G)}{\Gamma(G, G)}(t, X_{t-}), \quad 0 \le t \le T,$$

where Γ *is the* Carré du Champ operator *of the process* $\{(t, X_t) : t \in \mathbb{R}^+\}$.

We will now transform this result into a technique to calculate an insurance strategy $(j_s)_{0 \le s \le T}$ that minimizes the variance of the residual loss of the insurer. The conditional expectation at time t of the part of the loss covered by the reinsurer at time T is given by

$$V_t = \int_{]0, t]} j_s \, \mathrm{d}C_s$$

where

$$C_s = \mathrm{E}\left[(S_T - X)^+ \mid \mathscr{F}_s\right].$$

The conditional expectation at time t of the part of the loss covered by the insurer at time T is given by

$$\bar{S}_t = \mathrm{E}[S_T \mid \mathscr{F}_t].$$

So the conditional expectation of the residual loss of the insurer is given by

$$R_t = \bar{S}_t - \bar{S}_0 - \int_{]0, t]} j_s \, \mathrm{d}C_s.$$

It is our aim to find the strategy $(j_s)_{0 \le s \le T}$ that minimizes $V[R_T]$. Since $(R_t)_{t \in \mathbb{R}^+}$ is a martingale and $\mathrm{E}[R_0] = 0$ this is equivalent to minimizing $\mathrm{E}[R_T^2]$.

Theorem.

$$V[R_T] \textit{ is minimal} \Leftrightarrow j_s = \frac{\Gamma(F, G)}{\Gamma(G, G)}(s, S_{s-}), \quad \forall s \in \mathbb{R}^+,$$

where Γ *is the* Carré du Champ operator *of the process* $\{(t, S_t) : t \in \mathbb{R}^+\}$ *and*

$$G(t, x) = \int (x + y - X)^+ P_{S_{T-t}}(\mathrm{d}y),$$

$$F(t, x) = x + \mathrm{E}[S_{T-t}].$$

Proof. It is clear that we only have to verify wether the three sufficient conditions in the theorem by Bouleau and Lamberton are satisfied.

(i) *The first condition.* We denote A for the infinitesimal generator of the Markov semi-group $(P_t)_{t\in\mathbb{R}^+}$ defined by $P_t = P_{S_t}$ and $\mathscr{D}(A)$ for its domain.

The semi-group $(P_t)_{t\in\mathbb{R}^+}$ is strongly continuous on $\mathscr{C}_0(\mathbb{R})$ and

$$\mathscr{D}(A) = \left\{ f\in C_0(\mathbb{R}) : \lim_{t\to\infty} \frac{P_t f - f}{t} \text{exists} \right\}$$
$$= C_0(\mathbb{R}).$$

Furthermore, $\forall f \in C_0(\mathbb{R})$

$$Af(x) = \lambda \int (f(x+y) - f(x)) P_{X_i}(\mathrm{d}y).$$

We denote $(\bar{P}_t)_{t\in\mathbb{R}}$ for the semi-group of the stochastic process $Y_t = (t, S_t)$, $\mathscr{A}$ for its infinitesimal generator, and $\mathscr{D}(\mathscr{A})$ for the domain of this generator. The semi-group $(\bar{P}_t)_{t\in\mathbb{R}^+}$ is strongly continuous on the Banach space $\mathscr{C}_0(\mathbb{R}^+\times\mathbb{R})$ and

$$\mathscr{D}(\mathscr{A}) = \left\{ f\in \mathscr{C}_0(\mathbb{R}^+\times\mathbb{R}) : \frac{\partial f}{\partial t} \in \mathscr{C}_0(\mathbb{R}^+\times\mathbb{R}) \right\}.$$

Furthermore,

$$\mathscr{A}f(t, x) = Af(t, \cdot)(x) + \frac{\partial f}{\partial t}(t, x) \quad \forall f \in \mathscr{D}(\mathscr{A}).$$

Since $\mathscr{D}(\mathscr{A})$ is an algebra, we can conclude that the process $\{(t, S_t): t\in\mathbb{R}^+\}$ permits a *Carré du Champ operator* Γ [see Dellacherie and Meyer (1987)].

(ii) *The second condition.* Consider the stochastic process $(C_t)_{t\in\mathbb{R}^+}$ defined by

$$C_t = \mathrm{E}\left[(S_T - X)^+ \mid \mathscr{F}_t \right] \quad \forall t\in\mathbb{R}^+.$$

It is clear that $C_t = G(t, S_t)$ is a martingale with

$$G(t, x) = \int (x + y - X)^+ P_{S_{T-t}}(\mathrm{d}y)$$
$$= P_{T-t} g(x)$$

with

$$g(x) = (x - X)^+.$$

Furthermore,

$$P_T g^2(x) < \infty.$$

So we see that $G \in \mathscr{D}_2(\mathscr{A})$.

(iii) *The third condition.* Consider the stochastic process $(\bar{S}_t)_{t\in\mathbb{R}^+}$ defined by

$$\bar{S}_t = \mathrm{E}[S_T \mid \mathscr{F}_t] \quad \forall t\in\mathbb{R}^+.$$

It is clear that $\bar{S}_t = F(t, S_t)$ is a martingale with

$$F(t, x) = \int (x + y) P_{S_{T-t}}(dy)$$
$$= P_{T-t} f(x)$$

with

$$f(x) = x.$$

Furthermore,

$$P_T f^2(x) < \infty.$$

So we can conclude that $F \in \mathscr{D}_2(\mathscr{A})$.

Since all conditions are fulfilled, it now follows from the results proven in Bouleau and Lamberton (1989) that

$$\mathrm{E}\left[R_T^2\right] \text{ is minimal} \Leftrightarrow j_s = \frac{\Gamma(F, G)}{\Gamma(G, G)}(s, S_{s-}) \quad \forall s \in \mathbb{R}^+$$

where

$$\Gamma(K, H)(t, x) = \mathscr{A}(KH)(t, x) - K(t, x)\mathscr{A}H(t, x) - H(t, x)\mathscr{A}K(t, x)$$

for all $K, H \in \mathscr{D}_2(\mathscr{A})$.

4. Numerical calculations

In the previous section we derived formulas for the three criteria to evaluate the risk position of the insurer who wants to reduce his risk by underwriting some proportional stop loss reinsurance contracts. In this section it is our aim to show that these formulas can also be calculated numerically.

We will perform the calculations in the case where the process $\{X_i : i \in \mathbb{N}\}$ is i.i.d. exponentially distributed with $\mathrm{E}[X_i] = \mu > 0$.

Step 1. To know the conditional expectation of the residual loss of the insurer at a certain time t, we have to calculate $H(t, x)$, where x is the value of S_t (which is well known at that time). Clearly,

$$\begin{aligned} H(t, x) &= \int \left(x + y - j(x + y - X)^+\right) P_{S_{T-t}}(\mathrm{d}y) \\ &= x + \mathrm{E}[S_{T-t}] - j\int_{X-x}^{\infty} (y - X + x) P_{S_{T-t}}(\mathrm{d}y) \\ &= x + \mathrm{E}[S_{T-t}] - j(\mathrm{E}[S_{T-t}] - X + x) + j\int_0^{X-x} (y - X + x) P_{S_{T-t}}(\mathrm{d}y) \qquad x \le X, \\ &= x + \mathrm{E}[S_{T-t}] - j(\mathrm{E}[S_{T-t}] - X + x) \qquad x \ge X. \end{aligned}$$

The problem now is to calculate

$$\int_0^{X-x} (y + x - X) P_{S_{T-t}}(\mathrm{d}y),$$

where in general the density function of the stochastic variable S_{T-t} is not known analytically. The solution to this problem was given by Panjer (1981).

With the notation

$$p_n = P(N_{T-t} = n),$$

$$g_{T-t}(x) = \sum_{n \ge 1} p_n f_{X_i}^{*n}(x),$$

it is clear that

$$\int_0^{X-x} (y + x - X) P_{S_{T-t}}(\mathrm{d}y) = p_0(x - X) + \int_0^{X-x} (y + x - X) g_{T-t}(y)\, \mathrm{d}y.$$

Panjer proved that the function g_{T-t} is the solution of the following integral equation:

$$g(0) = \lambda(T-t)\ \mathrm{e}^{-\lambda(T-t)} f_{X_i}(0),$$

$$g(y) = \lambda(T-t)\ \mathrm{e}^{-\lambda(T-t)} f_{X_i}(y) + \lambda(T-t) \int_0^y \frac{y-x}{y} f_{X_i}(y-x) g(x)\ \mathrm{d}x \quad \forall y > 0.$$

We solved this equation in a finite amount of discretization points ih, $i = 0, 1, 2, \ldots, n$, by using a quadrature rule. This provided us with approximations

$$g_i \approx g_{T-t}(ih), \quad i = 0, 1, 2, \ldots, n.$$

Subsequently, we used these approximations to calculate an approximation to the integral

$$\int_0^{X-x} (y + x - X) g_{T-t}(y)\ \mathrm{d}y$$

again by using a quadrature rule.

For a detailed exposition of the numerical methods used, see Baker (1977).

Step 2. To know the conditional variance of the residual loss of the insurer at a certain time t, we have to calculate $K(t, x)$, where x is the value of S_t. It is clear that the calculations are basically the same as those for $H(t, x)$.

Step 3. To know the optimal reinsurance strategy for the insurer at a certain time t, we have to calculate

$$j_t = \frac{\Gamma(F, G)}{\Gamma(G, G)}(t, x)$$

where x is the value of S_{t-}. Clearly,

$$F(t, x) = x + \mathrm{E}[S_{T-t}],$$

$$\begin{aligned} G(t, x) &= x - X + \mathrm{E}[S_{T-t}] - \int_0^{X-x} (x + y - X) P_{S_{T-t}}(\mathrm{d}y), \quad && x \le X, \\ &= x - X + \mathrm{E}[S_{T-t}], && x \ge X. \end{aligned}$$

Since $\{F(t, S_t): t \in \mathbb{R}^+\}$ and $\{G(t, S_t): t \in \mathbb{R}^+\}$ are martingales, we have

$$\mathscr{A}F \equiv 0, \quad \mathscr{A}G \equiv 0.$$

Consequently,

$$\begin{aligned} \Gamma(F, G)(t, x) &= \mathscr{A}(FG)(t, x) \\ &= \frac{\partial FG}{\partial t}(t, x) + \lambda \int (FG(t, x+y) - FG(t, x)) P_{X_i}(\mathrm{d}y) \\ &= F(t, x) \frac{\partial G}{\partial t}(t, x) + F(t, x) \lambda \int (G(t, x+y) - G(t, x)) P_{X_i}(\mathrm{d}y) \\ &\quad - \lambda\mu G(t, x) + \lambda \int y G(t, x+y) P_{X_i}(\mathrm{d}y) \\ &= F(t, x) \mathscr{A}G(t, x) + \lambda \int y (G(t, x+y) - G(t, x)) P_{X_i}(\mathrm{d}y) \\ &= \lambda \int y (G(t, x+y) - G(t, x)) P_{X_i}(\mathrm{d}y) \\ &= I_1(x) + I_2(x) - \lambda\mu G(t, x) \quad \text{if} \quad x < X \\ &= I_3(x) - \lambda\mu G(t, x) \quad \text{if} \quad x \ge X, \end{aligned}$$

where

$$I_1(x) = \lambda \int_0^{X-x} y G(t, x+y) f_{X_i}(y) \, \mathrm{d}y$$

(can be calculated numerically),

$$I_2(x) = \lambda \int_{X-x}^{\infty} y(x+y-X+\mathrm{E}[S_{T-t}]) f_{X_i}(y) \, \mathrm{d}y$$
$$= \lambda \, \mathrm{e}^{-(X-x)/\mu}\left((X-x)^2 + 2\mu(X-x) + 2\mu^2 + (x-X+\mathrm{E}[S_{T-t}])(X-x+\mu)\right),$$

$$I_3(x) = \lambda \int_0^{\infty} y(x+y-X+\mathrm{E}[S_{T-t}]) f_{X_i}(y) \, \mathrm{d}y$$
$$= 2\lambda\mu^2 + \lambda\mu(x-X+E[S_{T-t}]).$$

Analogously,

$$\begin{aligned} \Gamma(G, G)(t, x) &= \mathscr{A}(G^2)(t, x) \\ &= \lambda \int (G(t, x+y) - G(t, x))^2 P_{X_i}(\mathrm{d}y) \\ &= I_4(x) + I_5(x) \quad \text{if} \quad x < X \\ &= I_6(x) \quad \text{if} \quad x \geq X, \end{aligned}$$

where

$$I_4(x) = \lambda \int_0^{X-x} (G(t, x+y) - G(t, x))^2 f_{X_i}(y) \, \mathrm{d}y$$

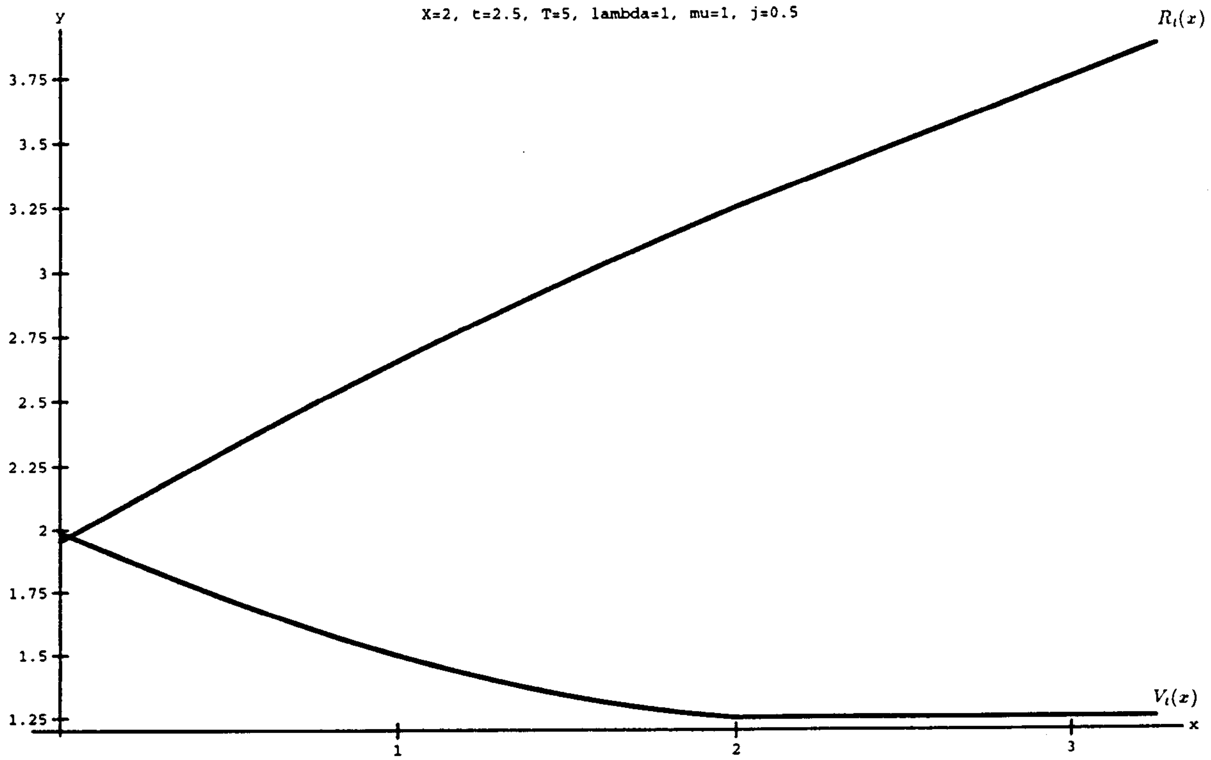

Fig. 1.

(can be calculated numerically),

$$I_5(x) = \lambda \int_{X-x}^{\infty} y^2 f_{X_i}(y)\, \mathrm{d}y$$
$$= \lambda e^{-(X-x)/\mu}\left((X-x)^2 + 2\mu(X-x) + 2\mu^2\right),$$
$$I_6(x) = \lambda \int_0^{\infty} y^2 f_{X_i}(y)\, \mathrm{d}y$$
$$= 2\lambda\mu^2.$$

These formulas can again be calculated numerically by using quadrature rules [Baker (1977)].

5. Numerical results

In all our results we used the following values:

(i) the expected value of the claim height $\mu = 1$,
(ii) the expected value of the inter arrival time $\lambda = 1$,
(iii) the stop loss bound $X = 2$,
(vi) the term of the contract $T = 5$.

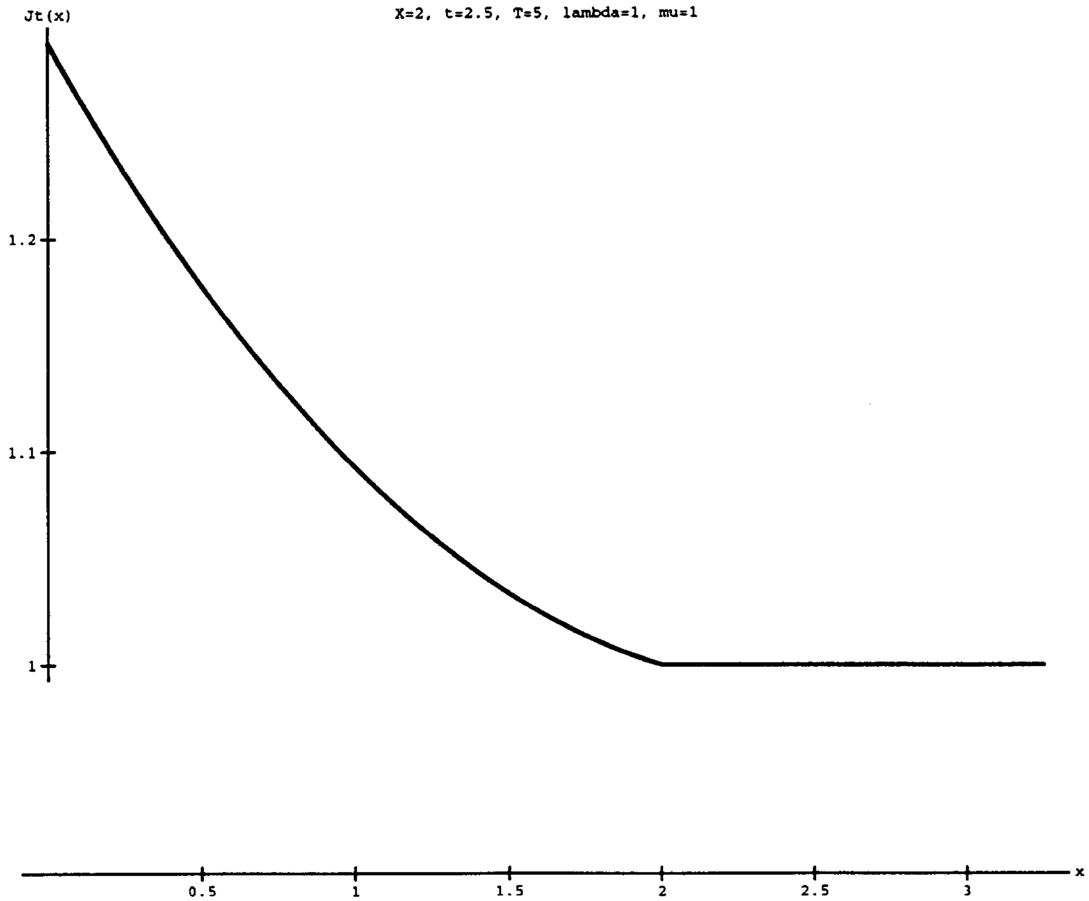

Fig. 2.

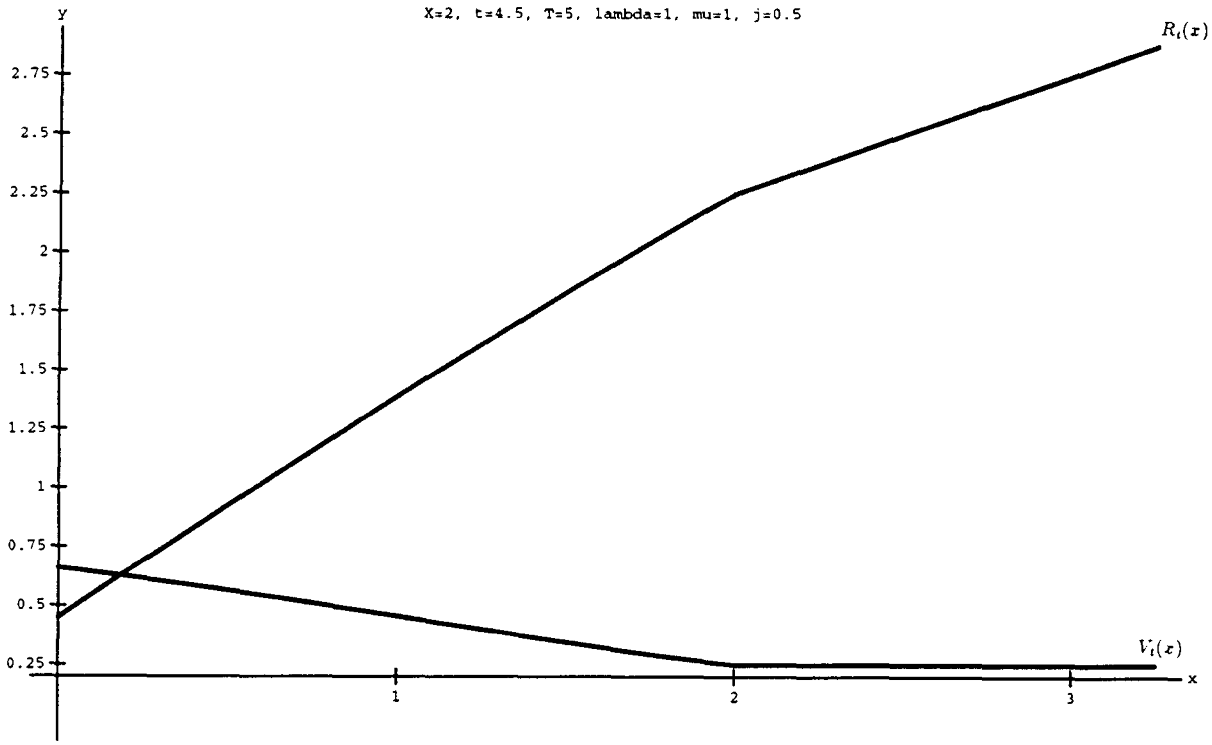
X=2, t=4.5, T=5, lambda=1, mu=1, j=0.5
y
2.75
2.5
2.25
2
1.75
1.5
1.25
1
0.75
0.5
0.25
1
2
3
x
$R_t(x)$
$V_t(x)$

Fig. 3.

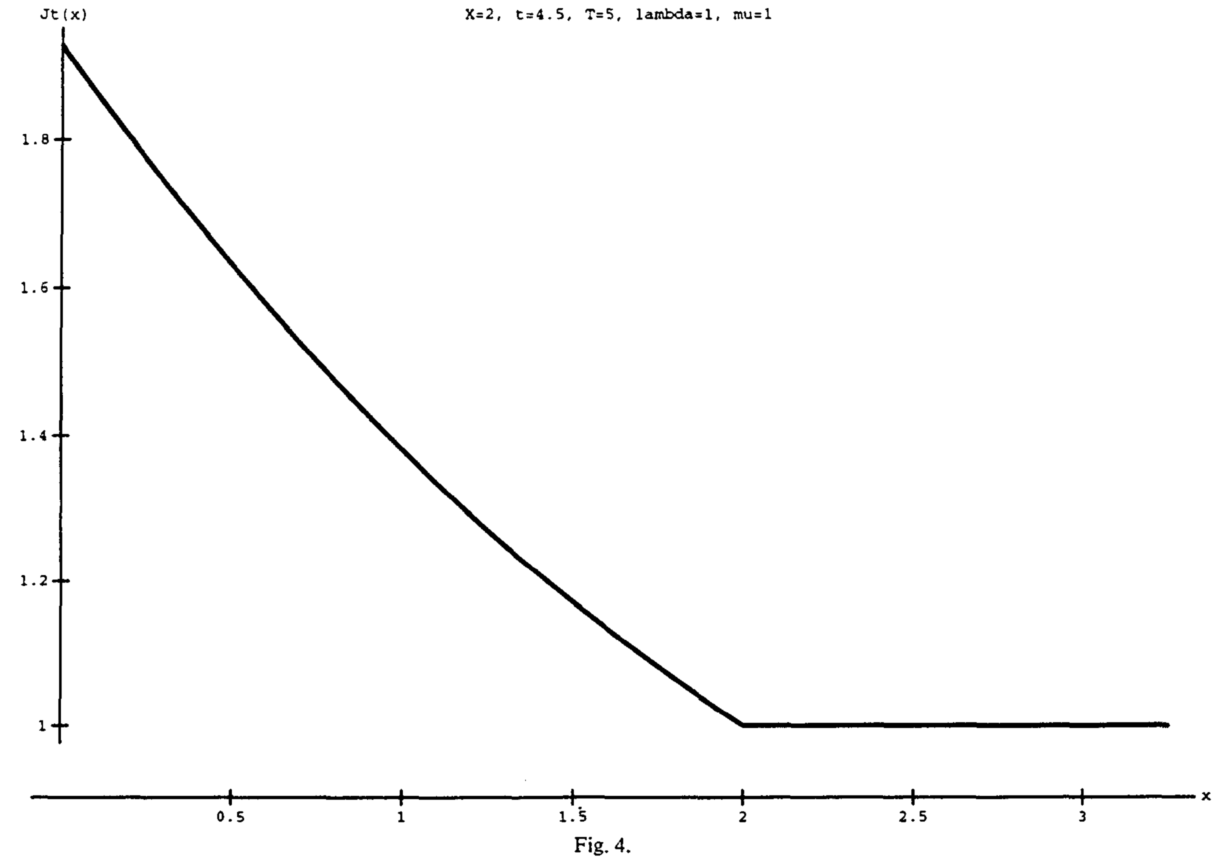
Jt(x)
X=2, t=4.5, T=5, lambda=1, mu=1
1.8
1.6
1.4
1.2
1
0.5
1
1.5
2
2.5
3
x

Fig. 4.

In Figure 1 we plotted the conditional expectation and the conditional variance of the residual loss of the insurer at time 2.5, as a function of the total claim height $x = S_{t-}$ at that moment. We assume that the insurance fraction was chosen to be 50%.

In Figure 2 we plotted the third criterion to evaluate residual risk, namely the *optimal reinsurance strategy* j, at time 2.5 and as a function of the total claim height at that moment.

In Figures 3 and 4 we did the same for $t = 4.5$.

6. The viewpoint of the reinsurer

We now consider the case where insurer and reinsurer have underwritten a reinsurance contract with the following specifications:

- when the total claim height at time T is less than or equal to a certain bound X, the total loss will be covered by the insurer. So there will be no intervention of the reinsurer.
- as soon as the total claim height S_T at time T exceeds the bound X but stays bellow a second bound $Y(>X)$, the insurer will cover X and the reinsurer will cover the part $S_T - X$ that exceeds this bound, without any franchise for the insurer.
- when the total claim height exceeds the second bound Y, the insurer will cover X, the reinsurer will cover the part $Y - X$ of the loss between X and Y without any franchise, plus $100(1-j)\%$ of the part of the loss $S_T - Y$ that exceeds the bound Y. This leaves a franchise of $j(S_T - Y)$ for the insurer to cover.

We will now introduce a scheme that can be used by the reinsurer in the above situation, to evaluate his risk position at every time $t \leq T$, and for every contract of this type that he has running at that time. This scheme consists of the same three steps as those described in Section 3 for evaluating the risk position of the insurer, namely

Step 1. At first we will calculate the conditional expectation at time t of the part of the loss that will have to be covered by the reinsurer at time T. The part of the loss to be covered by the reinsurer at time T is given by

$$LRI = (S_T - X)^+ - j(S_T - Y)^+.$$

Conditional expectations can be calculated using the techniques explained in the previous section.

Step 2. The second step consists of calculating the conditional variance of the part of the loss to be covered by the reinsurer at time T. Again, this can be done by using the techniques explained in the previous section.

Step 3. The third step consists of the calculation of the *optimal reinsurance strategy*. This optimal reinsurance strategy consists of a continuous time stochastic process $\{j_s : 0 \leq s \leq T\}$ which is constructed in such a way that the variance $V[R_T]$ of the conditional expectation of the part of the loss to be covered by the reinsurer at time T becomes minimal if the reinsurer, at each time $t \leq T$, chooses the franchise for the insurer to be equal to j_t.

To derive the optimal reinsurance strategy $(j_s)_{0 \leq s \leq T}$ from the viewpoint of the reinsurer, we now consider the processes

$$\mathrm{E}\left[(S_T - X)^+ \mid \mathcal{F}_t\right] = G(t, S_t),$$

$$\mathrm{E}\left[(S_T - Y)^+ \mid \mathcal{F}_t\right] = Q(t, S_t).$$

We write R_t for the conditional expectation at time t of the part of the loss to be covered by the reinsurer at time T, so

$$R_t = G(t, S_t) - G(0, S_0) - \int_{]0,t]} j_s \, \mathrm{d}Q(s, S_s).$$

It is clear that these functions G and Q satisfy the sufficient conditions of the theorem by Bouleau and Lamberton. So we can conclude that

$$\mathrm{E}\left[R_T^2\right] \text{ is minimal} \Leftrightarrow j_s = \frac{\Gamma(G, Q)}{\Gamma(Q, Q)}(s, S_{s-}) \quad \forall s \in \mathbb{R}^+.$$

7. Numerical results

In all our results we used the following values

(i) the expected value of the claim height $\mu = 1$,
(ii) the expected value of the inter claim time $\lambda = 1$,
(iii) the first stop loss bound $X = 2$,
(iv) the second stop loss bound $Y = 3$,
(v) the term of the contract $T = 5$.

In Figure 5 we plotted the conditional expectation and the conditional variance at time 2.5 of the part of the loss to be covered by the reinsurer at time T, as a function of the total claim height $x = S_{t-}$ at that moment. We assumed that the insurance fraction j was chosen to be 50%.

In Figure 6 we plotted the third criterion to evaluate residual risk for the reinsurer, namely the *optimal reinsurance strategy*, at time 2.5 and as a function of the total claim height at that moment.

In figures 7 and 8 we did the same for $t = 4.5$.

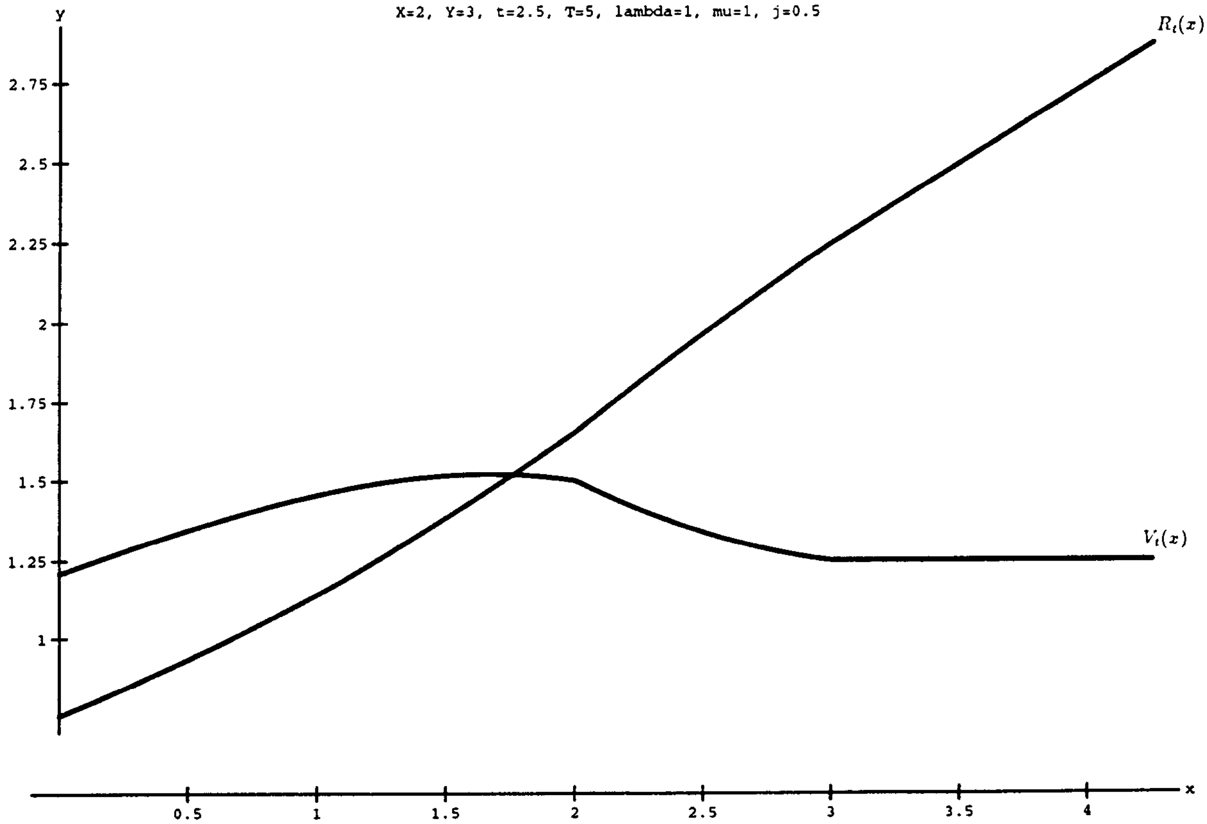

Fig. 5.

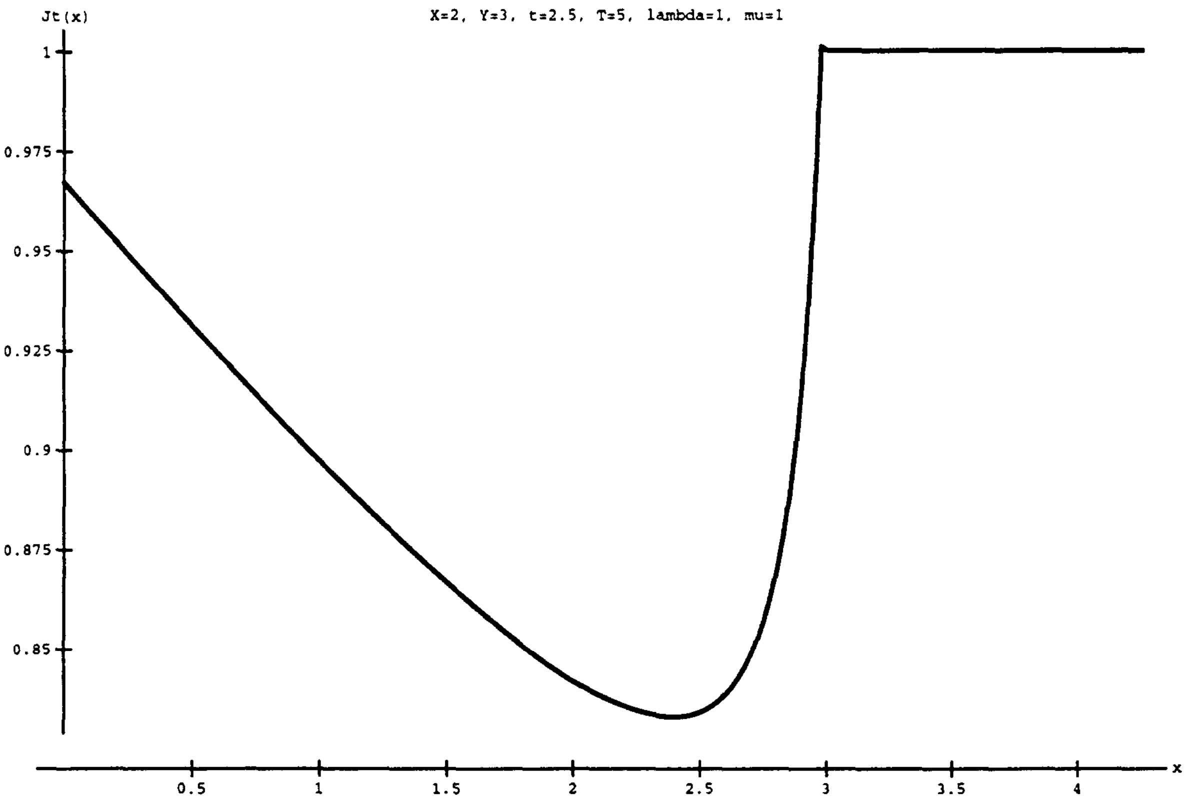
Jt(x)
X=2, Y=3, t=2.5, T=5, lambda=1, mu=1
1
0.975
0.95
0.925
0.9
0.875
0.85
0.5
1
1.5
2
2.5
3
3.5
4
x

Fig. 6.

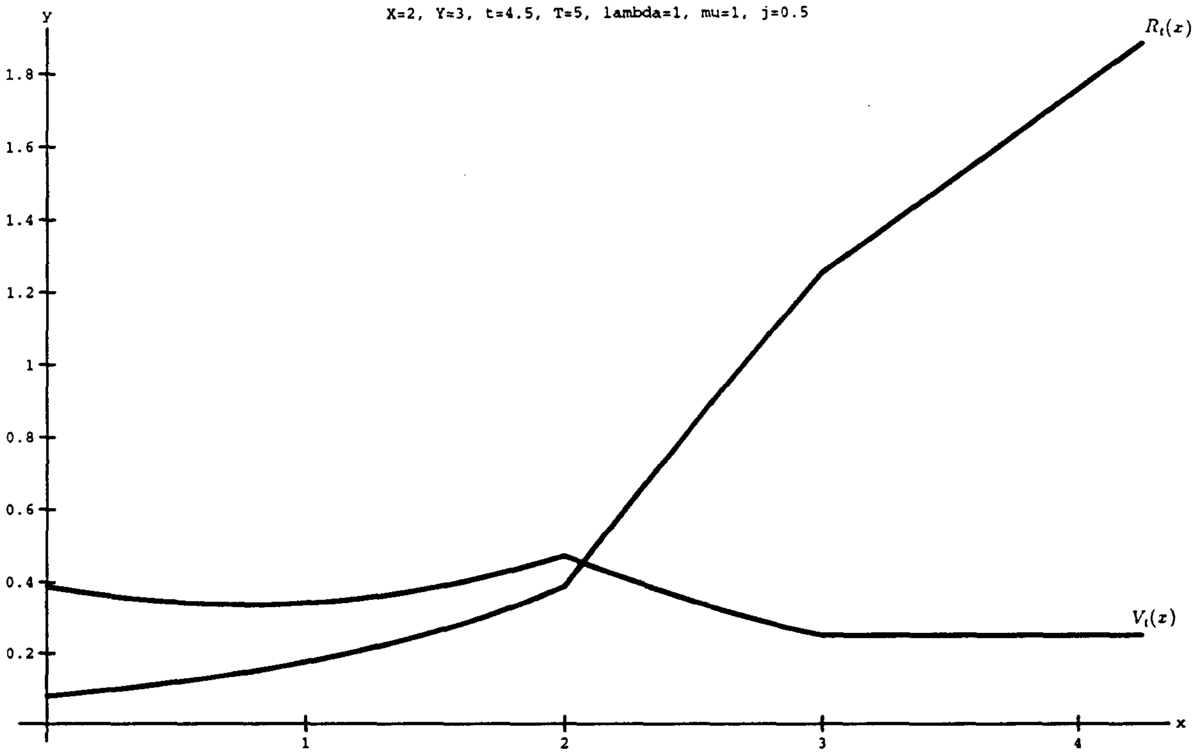
y
X=2, Y=3, t=4.5, T=5, lambda=1, mu=1, j=0.5
$R_t(x)$
$V_t(x)$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
1
2
3
4
x

Fig. 7.

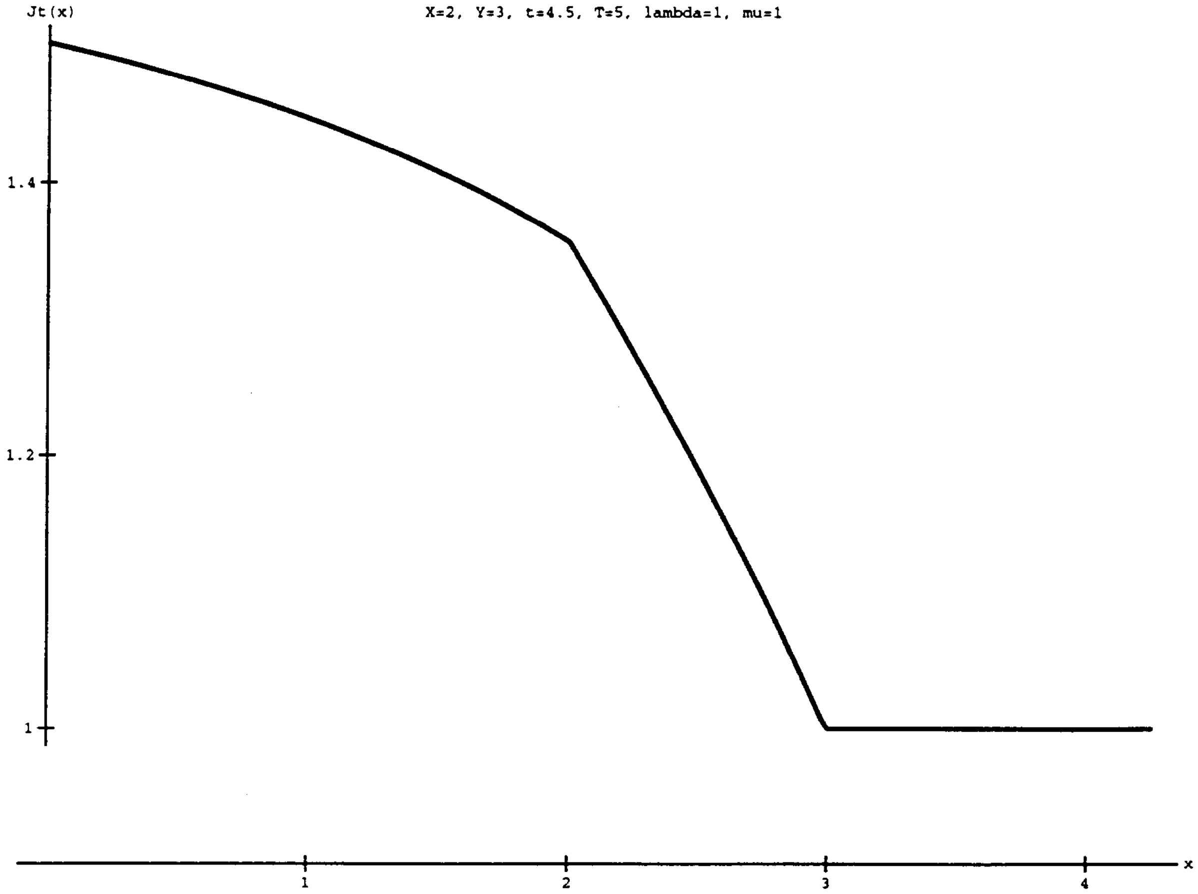

Fig. 8.

8. Multidimensional reinsurance

We suppose that the insurer wants to reduce his risk by contracting a proportional stop loss reinsurance on the claim height process with reinsurance fraction j^1 and stop loss bound X in combination with a proportional stop loss reinsurance on the claim number process with reinsurance fraction j^2 and stop loss bound M. In this section we will construct an optimal reinsurance strategy $(j_s^1, j_s^2)_{0 \le s \le T}$. This is a continuous time s.p. where j^1 and j^2 are chosen in such a way that $V[R_T]$ is minimized. Again R_t stands for the conditional expectation of the residual loss of the insurer at time t, and conditional to the information that the insurer has about the claim height process and the claim number process at that time.

To establish this result we define some new processes, namely

$$Y_t = (N_t, S_t),$$

$$K_t = \mathrm{E}\left[\mu(N_T - M)^+ \mid \mathcal{F}_t\right],$$

where $\mu = \mathrm{E}[X_i]$ and

$$\mathcal{F}_t = \sigma(\{Y_u : 0 \le u \le t\}).$$

Furthermore, we define the semi-group $(P_t)_{t \in \mathbb{R}^+}$ as follows:

$$P_t = P_{(N_t, S_t)-(N_0, S_0)}$$
$$= P_{(N_t, S_t)}.$$

Since the process (Y_t) has stationary and independent increments, we know that (P_t) is a *stationary convolution semi-group*. It is also clear that (Y_t) is a realization of this semi-group. So we can conclude that (Y_t) is a right Markov process [see Getoor (1975)]. Let A be the infinitesimal generator of this process.

Since this will cause no problems of interpretation, we will also denote $(P_t)_{t\in\mathbb{R}^+}$ for the family of linear operators linked with this Markov semi-group, i.e.

$$P_t f(n, x) = \int f(n+m, x+y) P_{(N_t, S_t)}(\mathrm{d}m, \mathrm{d}y) \quad \text{for all } f \in L(\Omega, \mathscr{F}, P_{(N_t,S_t)}).$$

Furthermore, we define the s.p.

$$Z_t = (t, Y_t).$$

We denote $(\bar{P}_t)_{t\in\mathbb{R}}$ for the semi-group of the stochastic process Z_t, $\mathscr{A}$ for its infinitesimal generator, and $\mathscr{D}(\mathscr{A})$ for the domain of this generator.

Lemma. *The semi-group $(\bar{P}_t)_{t\in\mathbb{R}^+}$ is strongly continuous w.r.t. the Banach space $\mathscr{C}_0(\mathbb{R}^+\times\mathbb{N}\times\mathbb{R})$ and*

$$\mathscr{D}(\mathscr{A}) = \left\{ f \in \mathscr{C}_0(\mathbb{R}^+\times\mathbb{N}\times\mathbb{R}) : \frac{\partial f}{\partial t} \in \mathscr{C}_0(\mathbb{R}^+\times\mathbb{N}\times\mathbb{R}) \right\}.$$

Furthermore,

$$\begin{aligned}\mathscr{A}f(t, n, x) &= Af(t, \cdot, \cdot)(n, x) + \frac{\partial f}{\partial t}(t, n, x) \\ &= \frac{\partial f}{\partial t}(t, n, x) + \lambda \int (f(t, n+1, x+y) - f(t, n, x)) P_{X_i}(\mathrm{d}y) \quad \forall f \in \mathscr{D}(\mathscr{A}).\end{aligned}$$

Proof. Straightforward.

We will use the following notations:

$$\begin{aligned}C_t &= E^x\left[(S_T - X)^+ \mid \mathscr{F}_t\right] \\ &= G_1(t, N_t, S_t), \\ K_t &= E^x\left[\mu(N_T - M)^+ \mid \mathscr{F}_t\right] \\ &= G_2(t, N_t, S_t), \\ \bar{S}_t &= E^x[S_T \mid \mathscr{F}_t] \\ &= F(t, N_t, S_t).\end{aligned}$$

Furthermore, we denote G for the column array with components G_1, G_2, $\Gamma(G, G')$ for the matrix with components $\Gamma(G_i, G_j)$, $1 \le i, j \le 2$, and $\Gamma(G, F)$ [resp. $\Gamma(F, G')$] for the column (resp. row) array with components $\Gamma(G_i, F)$ [resp. $\Gamma(F, G_i)$], $1 \le i \le 2$.

The conditional expectation of the residual loss of the insurer is then given by

$$R_t = F(t, S_t) - F(0, S_0) - \int_{]0,t]} j_s \, \mathrm{d}(C_s, K_s), \quad \text{where} \quad j_s = (j_s^1, j_s^2).$$

Theorem.

$$\mathrm{E}\left[R_T^2\right] \text{ is minimal} \Leftrightarrow j_s = \Gamma(F, G') B(s, N_{s-}, S_{s-})$$

where B is the Moore–Penrose inverse of the matrix $\Gamma(G, G')$.

Proof. We see that

(i) $(Z_t)_{t\in\mathbb{R}^+}$ is a right Markov process and $\mathscr{D}(\mathscr{A})$ is an algebra. So the process $(Z_t)_{t\in\mathbb{R}^+}$ admits a *Carré du Champ operator* [see Dellacherie and Meyer (1987), Getoor (1975)].

(ii) $G_1(t, N_t, S_t)$ is a martingale in $L^2(\Omega, \mathscr{F}, P)$, with

$$G_1(t, n, x) = P_{T-t}g_1(n, x),$$
$$g_1(n, x) = (x - X)^+.$$

So we can conclude that $G_1 \in L^2(\Omega, \mathscr{F}, P) \cap \mathscr{D}(\mathscr{A})$.
Analogously,

$$G_2(t, n, x) = P_{T-t}g_2(n, x),$$
$$g_2(n, x) = \mu(n - M)^+,$$

is a martingale in $L^2(\Omega, \mathscr{F}, P) \cap \mathscr{D}(\mathscr{A})$.

(iii)

$$F(t, n, x) = P_{T-t}f(n, x),$$
$$f(n, x) = x.$$

It is clear that $F \in L^2(\Omega, \mathscr{F}, P) \cap \mathscr{D}(\mathscr{A})$.

It now follows from the results proven in Bouleau and Lamberton (1989) that

$$\mathrm{E}\left[R_T^2\right] \text{ is minimal} \Leftrightarrow j_s = \Gamma(F, G')B(s, N_{s-}, S_{s-})$$

where B is the *Moore–Penrose* inverse of the matrix $\Gamma(G, G')$. If $\Gamma(G, G')$ is a regular matrix, this becomes

$$j_s = \Gamma(F, G')\Gamma(G, G')^{-1}(s, N_{s-}, S_{s-})$$

or

$$j^1 = \frac{\Gamma(G_1, G_2)\Gamma(F, G_2) - \Gamma(G_2, G_2)\Gamma(F, G_1)}{\Gamma(G_1, G_2)^2 - \Gamma(G_1, G_1)\Gamma(G_2, G_2)},$$

$$j^2 = \frac{\Gamma(G_1, G_2)\Gamma(F, G_1) - \Gamma(G_1, G_1)\Gamma(F, G_2)}{\Gamma(G_1, G_2)^2 - \Gamma(G_1, G_1)\Gamma(G_2, G_2)}.$$

With the following notations:

$$I_1(x) = \int y(G_1(t, x + y) - G_1(t, x))P_{X_i}(\mathrm{d}y),$$

$$I_2(x) = \int (G_1(t, x + y) - G_1(t, x))^2 P_{X_i}(\mathrm{d}y),$$

$$I_3(x) = \int (G_1(t, x + y) - G_1(t, x))P_{X_i}(\mathrm{d}y),$$

$$I_4(n) = G_2(t, n + 1) - G_2(t, n),$$

and after some simple but tedious computations, we obtain

$$j^1 = \frac{\mu I_3(x) - I_1(x)}{I_3(x)^2 - I_2(x)},$$

$$j^2 = \frac{I_3(x)I_1(x) - \mu I_2(x)}{I_4(n)I_3(x)^2 - I_2(x)I_4(n)}, \quad \text{where} \quad \mu = \mathrm{E}[X_i].$$

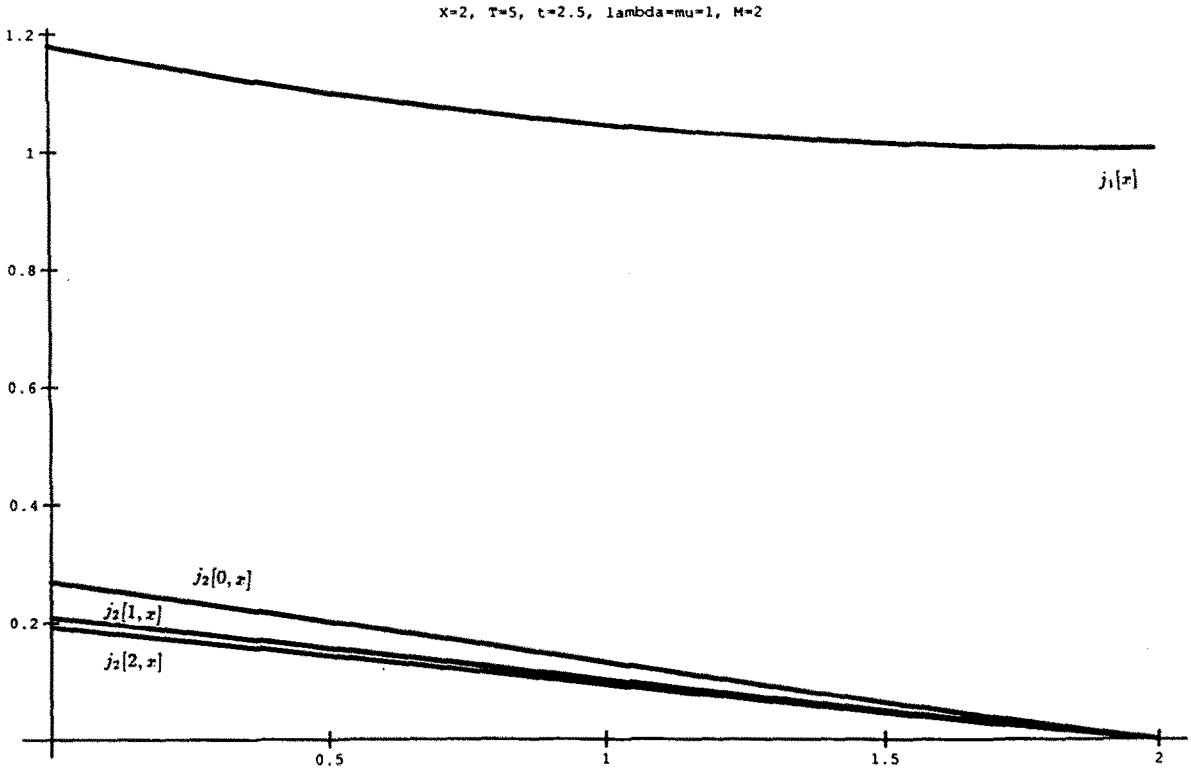

Fig. 9.

Numerical computations can be done with the techniques explained in Section 4. For some values of the stop loss bound X on the claim height and the stop-loss bound M on the claim number, numerical instabilities can cause some problems in calculating the above formulas for j^1 and j^2. We will therefore present an example where no such instabilities occurred.

9. Numerical results

In Figure 9 we plotted the optimal reinsurance strategy (j^1, j^2) at time $t = 2.5$, as a function of the total claim height at that moment and the number of claims at that moment. We used the following values:

(i) the expected value of the claim height $\mu = 1$,
(ii) the expected value of the inter claim time $\lambda = 1$,
(iii) the stop loss bound on the claim height $X = 2$,
(vi) the stop loss bound on the claim number $M = 2$,
(v) the term of the contract $T = 5$.

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