On the Dynamic Analysis of Piecewise Linear Networks

W.P.M.H. Heemels, M.K. Çamlıbel, J.M. Schumacher

Abstract—**Piecewise linear (PL) modelling is often used to approximate the behavior of nonlinear circuits. One of the possible PL modelling methodologies is based on the linear complementarity problem, and this approach has already been used extensively in the circuits and systems community for *static* networks. In this paper the object of study will be dynamic electrical circuits that can be recast as linear complementarity systems, i.e., as interconnections of linear time-invariant differential equations and complementarity conditions (ideal diode characteristics). A mathematically precise framework is developed that formalizes the mixed discrete and continuous behavior of these switched networks. Within this framework the fundamental question of well-posedness (existence and uniqueness of solution trajectories given an initial condition) is studied and additional properties of the behavior are derived. For instance, a full characterization is presented of the inconsistent states.**

Keywords— **circuit analysis, piecewise linear networks, switched circuits, linear complementarity problem, passivity.**

I. Introduction

Many electrical networks consist of *dynamic* components like capacitors and inductors and *static* nonlinear elements such as resistors and transistors. To analyze the behaviour of such networks, the nonlinear elements are often approximated by piecewise linear (PL) descriptions. In the literature many explicit canonical representations of PL functions can be found that store the parameters in a minimal way [1–4]. Van Bokhoven [5] developed an implicit model based on the linear complementarity problem of mathematical programming [6]. Basically, the complementarity relations correspond to ideal diode characteristics. In [7,8] it has been shown that the complementarity framework includes the explicit canonical representations given in [1–3]. Consequently, static PL elements can be replaced by networks consisting of ideal diodes and linear resistors (see Section III for an example). For instance, in [5, Ch. 9] complementarity models have been presented for voltage controlled switches, MOS transistors and digital gates. Actually, Eaves and Lemke [9] showed that any static (continuous) PL mapping can be rewritten in the complementarity format. This explains the extensive use of the *linear complementarity problem* [6] (together with a number of variants) in the study of PL electrical networks [5,7,8,10–14].

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In many electrical networks switching elements like thyristors and diodes are already present for a great variety of applications in both power engineering and signal processing. To reduce the simulation time of the transient behavior of such networks [15–19] and for analysis purposes (of e.g. stability or chaos) [20,21] these switches are often modelled ideally.

As a consequence, two different motivations can be given for the use of ideal diode (or complementarity) models in the study of nonlinear and switched electrical circuits: as a modelling methodology for PL networks and as idealized descriptions of physical devices. In this paper we will consider PL networks that can be modelled (or realized) by using ideal diode characteristics (complementarity conditions) and linear resistors for the *static* (PL) part and inductors and capacitors for capturing the *dynamic* part of the network. This results in models that are combinations of linear electrical networks (described by linear time-invariant differential equations) and ideal diodes (complementarity conditions). As such, the systems at hand form a subclass of *linear complementarity systems* [11,22–25], which can be seen as dynamic extensions of the linear complementarity problem.

It is well-known that ideal network models may well be of a mixed discrete and continuous nature. In particular, the circuit evolves through multiple topologies (modes) depending on the (discrete) states of the diodes characteristics ("on" or "off") or equivalently, the complementarity conditions. For each combination of the discrete states of the diodes (blocking or conducting) other equations govern the evolution of the system's variables. The mode transitions are triggered by inequalities and may result in discontinuities and Dirac impulses in the network's variables, see e.g. [15,16,18,19,26–28].

In this paper we provide a mathematical framework that allows the precise formulation of a solution concept for the complementarity class of continuous/discrete networks. The introduction of a solution concept is coupled to the question of well-posedness, i.e., existence and uniqueness of solutions of the network model for all initial conditions. Much effort has been invested in considering existence and uniqueness of solutions to *static* (DC) models of electrical networks [29–35]. For the dynamic equivalent, the classical theory of ordinary differential equations guarantees existence and uniqueness of solutions under a Lipschitz continuity condition (see e.g. [36]). Here however we will be considering networks containing ideal diodes, for which such conditions are not fulfilled. The only papers known to the authors dealing with well-posedness for dynamic circuits containing non-Lipschitz elements are [37,38]. However, the obtained results in [37, 38] do not cover the networks considered here, since an ideal diode cannot be reformulated as a current or voltage-controlled resistor. To show that the well-posedness issue is nontrivial, we will present a network example containing a negative resistor that has multiple solutions for certain initial conditions and no solutions for others. Hence, not all PL circuits are well-posed and additional assumptions are required to guarantee the existence and uniqueness of trajectories.

The main purposes of the paper are the following.
(i) Define a mathematically precise solution concept for dynamic PL circuits that can be modelled by linear complementarity systems.
(ii) Prove (global) existence and uniqueness of solutions under a condition that all elements are passive (excluding negative resistors as in the example mentioned above).
(iii) Establish regularity properties of the solutions. In particular, it will be proven that derivatives of Dirac impulses do not occur (even for inconsistent initial states) and Dirac impulses may occur only at the initial time. The consistent states (also called 'regular states') will be characterized fully in terms of set inclusions and linear complementarity problems. Moreover, it will turn out that the set of switching times is a right-isolated set, meaning that following all time instants there exists a positive length time interval in which the diodes do not change their discrete state.

These results will be used to provide a rigorous basis for so-called "time-stepping" methods (see e.g. [5, 11, 39]) that are used for simulation of dynamic PL circuits. Although several numerical simulation methods have already been proposed to deal with phenomena that arise in non-smooth circuits [5, 8, 11, 12, 16, 17, 39], little attention has been paid to the question if and in what sense the computed time functions converge to the true solution of the network model. On the basis of the framework presented in the current paper, a companion paper [40] gives a formal statement and proof of the consistency – convergence of the approximated time functions to the exact solution of the network model – of time-stepping routines for the simulation of a class of internally switched electrical circuits. Another way of approximating dynamic circuits with ideal diodes can be obtained by replacing the ideal characteristic by smooth functions between diode current and voltage. The interested reader is referred to [41] for more details on the consistency of such 'regularization' or 'smoothing' methods.

The outline of the paper is as follows. After the notational conventions in the next section, complementarity modelling of PL dynamic circuits is discussed in Section III. In Section IV, we describe the evolution of the network model within a given mode, i.e., with the diodes replaced by either an open (blocking) or short (conducting) circuit. Next, an extension of the linear complementarity problem will be introduced, which will play an important role in the proof of well-posedness. In section VI the regular (or consistent) states are introduced and characterized explicitly. In Section VII the solution concept is introduced and the proof of global well-posedness is presented. Finally, we state the conclusions.

II. Notation

The following notational conventions will be in force. $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \ldots\}$, $\mathbb{R}$ the real numbers, $\mathbb{R}_+$ the nonnegative real numbers (including zero) and $\mathbb{C}$ the complex numbers. If a is a (column) vector, we denote its i-th component by a_i. $M^\top$ is the transpose of the matrix $M \in \mathbb{C}^{m\times n}$ and M^* denotes the complex conjugate transpose. A (not necessarily symmetric) matrix $M \in \mathbb{C}^{m\times m}$ is called nonnegative definite and we write $M \geqslant 0$ if Re $x^*Mx = \frac{1}{2}x^*(M + M^*)x \geqslant 0$ for all $x \in \mathbb{C}^m$. In case strict inequality holds for all nonzero vectors x, we call the matrix positive definite and write $M > 0$. By $\mathcal{I}$ we denote the identity matrix of any dimension. Given $M \in \mathbb{R}^{k\times l}$ and two subsets $I \subseteq \{1, \ldots, k\}$ and $J \subseteq \{1, \ldots, l\}$, the (I, J)-submatrix of M is defined as $M_{IJ} := (M_{ij})_{i\in I, j\in J}$. In case $J = \{1, \ldots, l\}$, we also write $M_{I\bullet}$. If $I = \{1, \ldots, k\}$, the notation $M_{\bullet J}$ is sometimes used.

A triple of matrices (A, B, C) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ is a called *minimal*, if the matrices $[B\ AB\ \ldots\ A^{n-1}B]$ and $[C^\top\ A^\top C^\top \ldots (A^\top)^{n-1}C^\top]$ have full rank.

By $\mathbb{R}(s)$ we denote the field of real rational functions in one variable. $M(s) \in \mathbb{R}^{k\times l}(s)$ means that $M(s)$ is a $k \times l$ matrix with entries in $\mathbb{R}(s)$. A rational vector or matrix is called (strictly) proper, if for all entries the degree of the numerator is smaller than or equal to (strictly smaller than) the degree of the denominator.

A vector $u \in \mathbb{R}^k$ is called nonnegative (positive), and we write $u \geqslant 0$ ($u > 0$), if $u_i \geqslant 0$ ($u_i > 0$) for all $i \in \{1, \ldots, k\}$. If two vectors u, $y \in \mathbb{R}^k$ are orthogonal, i.e., $u^\top y = 0$, we write $u\perp y$. Similarly, we write $u(s)\perp y(s)$ for two rational vectors $u(s)$, $y(s) \in \mathbb{R}^k(s)$, if $u^\top(s)y(s) = 0$ for all $s \in \mathbb{C}$.

The set of arbitrarily often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^m$ is denoted by $C^\infty(\mathbb{R}; \mathbb{R}^m)$. $\mathcal{L}_2^k(t_0, t_1)$ denotes the set of all measurable functions v from (t_0, t_1) to $\mathbb{R}^k$ for which the integral $\int_{t_0}^{t_1} \|v(\tau)\|^2 d\tau$ is finite.

III. Complementarity modelling

As already mentioned in the introduction, many dynamic piecewise-linear (PL) electrical networks can be modelled (or realized) by using linear resistors, capacitors, inductors, gyrators, transformers and ideal diodes. Kevenaar and Leenaerts [7] (see also [8]) show that all the explicit PL representations proposed by Chua and Kang [1, 2], Güzelis and Göknar [3], and Kahlert and Chua [4] are all covered by one *implicit* model based on the linear complementarity problem (see Definition V.10 below) of mathematical programming [6]. This implicit model was develop by Van Bokhoven [5] and can represent all static (continuous) PL functions as proven by Eaves and Lemke [9]. Van Bokhoven's model is of the form

$$z = Ax + Bu + g \tag{1a}$$

$$y = Cx + Du + h \tag{1b}$$

$$0 \leqslant y\perp u \geqslant 0, \tag{1c}$$

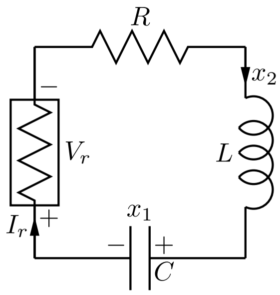

Fig. 1. An example of a piecewise linear circuit.

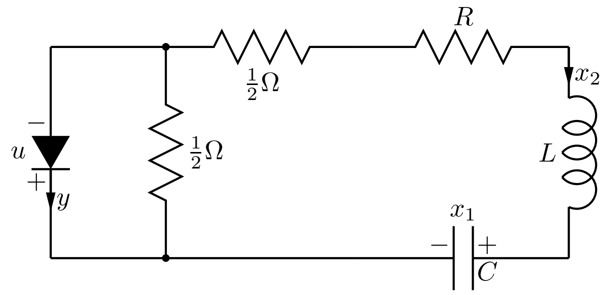

Fig. 2. An equivalent "complementarity" circuit of the network in Figure 1.

which describes a PL mapping from x to z. In (1) A, B, C, D are matrices and g, h are vectors of appropriate dimensions. Given $x \in \mathbb{R}^n$ one has to solve the linear complementarity problem (1b)–(1c) for the auxiliary variables y and u, after which u can be substituted in (1a) to obtain z.

To illustrate this modelling methodology, we consider the example of the nonlinear resistor in [11] given by the characteristic

$$V_r = \max(\frac{1}{2}I_r, I_r) = \begin{cases} I_r, & I_r \geqslant 0 \\ \frac{1}{2}I_r, & I_r < 0 \end{cases} \tag{2}$$

The voltage over the resistor is given by V_r, while I_r denotes the current through the resistor. This PL characteristic can be rewritten as

$$V_r = \frac{1}{2}I_r + u \tag{3a}$$
$$y = -I_r + 2u \tag{3b}$$
$$0 \leqslant y \perp u \geqslant 0. \tag{3c}$$

Indeed, $u = \max(\frac{1}{2}I_r, 0)$ and thus $V_r = \max(\frac{1}{2}I_r, I_r)$, which is equal to the PL function (2).

The nonlinear resistor given by (2) is now embedded in the dynamic network from [11], which is depicted in Figure 1. Taking $C = 1\ F$, $L = 1\ H$ and $R = 1\ \Omega$ we obtain the system description

$$\begin{array}{rcl} \dot{x}(t) &=& \begin{pmatrix} 0 & 1 \\ -1 & -\frac{3}{2} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} u(t) \\ y(t) &=& (0\ -1)\ x(t) + 2\ u(t) \\ 0 &\leqslant& y(t) \perp u(t) \geqslant 0, \end{array}$$

where x_1 is the voltage over the capacitor, x_2 is the current through the inductor and V_r is eliminated by using (3). From this reformulation we can now obtain the equivalent network as depicted in Figure 2 that consists of linear (positive) resistors, capacitors, inductors and ideal diodes only. In other words, we derived a "dynamic complementarity model" of the nonlinear network depicted in Figure 1.

In fact, [5, Section 2.3] presents a structured method that replaces any static PL two-pole element by an equivalent circuit consisting of ideal diodes, linear resistors and constant (current or voltage) sources. As we aim at providing sufficient conditions for the existence and uniqueness of solutions (so-called well-posedness) we will not consider

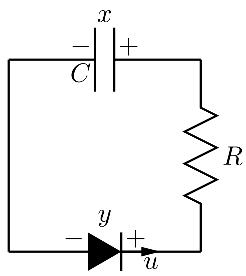

Fig. 3. A circuit containing a negative resistor.

networks including *negative* resistors as are used in [5, Section 2.3]. Indeed, negative resistors can result in ill-posed circuits as is illustrated by the simple example given in Figure 3. The circuit consists of a capacitor ($C = 1\ F$), a negative resistor ($R = -1\ \Omega$) and an ideal diode. The corresponding complementarity model is given by

$$\dot{x}(t) = u(t) \tag{5a}$$
$$y(t) = x(t) - u(t) \tag{5b}$$
$$0 \leqslant y(t) \perp u(t) \geqslant 0 \tag{5c}$$

with x the voltage across the capacitor, and u and y the current through and (minus) the voltage across the diode, respectively. In Figure 4 the linear relation between y and u given by (5b) and the complementarity conditions (5c) are drawn. It is obvious that in case the initial state satisfies $x(0) > 0$ *multiple* solutions exist, while for $x(0) < 0$ *no* solution trajectory can be found. Indeed, in case $x(0) = 1$ the diode can be both blocking ($u = 0$) and conducting ($y = 0$), which results in the solution trajectories $u(t) = 0$, $x(t) = y(t) = 1$ and $u(t) = x(t) = e^t$, $y(t) = 0$, respectively. This simple example shows that well-posedness does not hold for all PL systems and additional assumptions (like allowing only positive resistors) are required to guarantee the existence and uniqueness of trajectories.

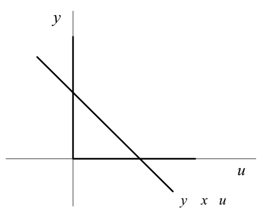

Fig. 4. A linear relation and complementarity conditions.

A second restriction that will be applied in this paper is that we assume absence of current and voltage sources. Unlike the positivity assumption on resistors, this restriction is imposed just to keep the presentation as uncluttered as possible. In this paper we therefore consider the basic case of networks realized by linear electrical networks consisting of (linear) positive resistors, inductors, capacitors, gyrators, transformers (RLCGT) and ideal diodes (like the one in Figure 1). An extension to the case including sources that generate even piecewise Bohl signals (e.g. constants, exponentials and (co)sines and combinations of these) can be given on the basis of the current paper as is outlined in [42].

The networks considered here lead directly to a complementarity model as mentioned in e.g. [5, 8]. Indeed, the linear (RLCGT)-part of the network can be described by the state space model

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (6a)$$
$$y(t) = Cx(t) + Du(t) \quad (6b)$$

under suitable conditions (the network does not contain all-capacitor loops or nodes with the only elements incident being inductors, see chapter 4 in [43] for more details.) In (6) $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ denote real matrices of appropriate dimensions, and x denotes the state variable of the network (typically consisting of linear combinations of the currents through the inductors and voltages across the capacitors). Moreover, the pair (u_i, y_i) denotes the voltage-current variables at the connections to the i-th diode, i.e.,

$$(u_i = -V_i \ \wedge \ y_i = I_i) \ \vee \ (u_i = I_i \ \wedge \ y_i = -V_i),$$

where V_i and I_i are the voltage across and current through the i-th diode, respectively, and $\vee$ denotes the Boolean (non-exclusive) "or" and $\wedge$ the Boolean "and"-operator. The ideal diode characteristics are described by the relations

$$V_i \leqslant 0 \ \wedge \ I_i \geqslant 0 \ \wedge \ (V_i = 0 \ \vee \ I_i = 0) \quad (7)$$

as shown in Figure 5.

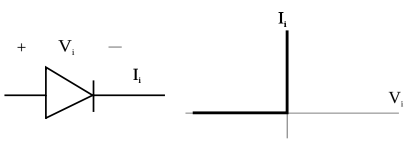

Fig. 5. The ideal diode characteristic.

By suitable substitutions the following system description is obtained:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (8a)$$
$$y(t) = Cx(t) + Du(t) \quad (8b)$$
$$0 \leqslant y(t) \perp u(t) \geqslant 0. \quad (8c)$$

In this formulation $t \in \mathbb{R}_+$ denotes the time variable, $x(t)$ the state, and $u(t)$ and $y(t)$ the complementarity variables at time t. The system (8) is called a *linear complementarity system*. System descriptions of this form were introduced in [23] and were further studied in [22–25, 41]. We use the notation LCS(A, B, C, D) to indicate the system given by (8). Note that (8c) means that for all $i \in \{1, \ldots, k\}$ $y_i(t) \geqslant 0 \wedge u_i(t) \geqslant 0 \wedge (y_i(t) = 0 \vee u_i(t) = 0)$. Rather than using this explicit expression, we shall below usually employ the more compact notation (9c). Observe that the description (4) for the nonlinear circuit in Figure 1 is exactly of the form (8).

Since (8a)-(8b) is a model for the RLCGT-multiport network consisting of positive resistors, capacitors, inductors, gyrators and transformers, the matrix quadruple (A, B, C, D) is not arbitrary, but satisfies a passivity condition. To be precise, (A, B, C, D) is passive (or in the terms of [44], *dissipative* with respect to the supply rate $u^\top y$) in the following sense.

Definition III.1 [44] A system (A, B, C, D) given by (6) is called *passive*, or *dissipative* with respect to the supply rate $u^\top y$, if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$, (a *storage function*), such that for all $t_0 \leqslant t_1$ and all time functions $(u, x, y) \in \mathcal{L}_2^{k+n+k}(t_0, t_1)$ satisfying (6) the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t)y(t)dt \geqslant V(x(t_1)).$$

The above inequality is called the *dissipation inequality*. The storage function represents a notion of "stored energy" in the network. The following proposition gives several equivalent characterizations of passivity.

Proposition III.2 [44] Consider a system (A, B, C, D) in which (A, B, C) is a minimal[1] representation. The following statements are equivalent.

- (A, B, C, D) is passive.
- The transfer matrix $G(s) := C(s\mathcal{I} - A)^{-1}B + D$ is *positive real*, i.e., $x^*[G(\lambda) + G^*(\lambda)]x \geqslant 0$ for all complex vectors x and all $\lambda \in \mathbb{C}$ such that Re $\lambda > 0$ and λ is not an eigenvalue of A.
- The matrix inequalities

$$\begin{pmatrix} -A^\top K - KA & -KB + C^\top \\ -B^\top K + C & D + D^\top \end{pmatrix} \geqslant 0 \quad (9a)$$

and

$$K = K^\top \geqslant 0 \quad (9b)$$

have a solution K.

Moreover, in case (A, B, C, D) is passive, all solutions to the linear matrix inequalities (9) are positive definite (i.e., (9b) holds with strict inequality) and a symmetric K is a solution to (9) if and only if $V(x) = \frac{1}{2}x^\top Kx$ defines a storage function of the system (A, B, C, D).

This proposition enables us to verify that the network in Figure 1 yields an LCS(A, B, C, D)-model with (A, B, C, D) passive for which sometimes the nomenclature *linear passive complementarity systems* is used. Indeed, it

[1] See Section II for a definition of "minimality."

is easily verified that the matrix inequalities (9) are satisfied for (4) with $K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Moreover, $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ is a storage function, which is physically clear as it represents the total electrical energy in the capacitor and the inductor in both Figure 1 and Figure 2.

A technical assumption that we will often use is the following.

Assumption III.3 B has full column rank and (A, B, C) is a minimal representation.

These assumptions imply that (specific kinds of) redundancy have been removed from the circuit. The minimality requirement of (A, B, C) indicates the fact that the number of states (i.e. the total number of capacitors and inductors) is the minimal number needed to realize the transfer function $C(sI - A)^{-1}B + D$ from u to y (see also [43, Ch. 8]). Minimality is a standard assumption in the literature on dissipative dynamic systems [44]. The full column rank condition is included to prevent redundancy in the collection of diodes. See [45] for two simple network examples that illustrate the implications and relevance of Assumption III.3.

We note the following consequence of passivity, which will be used frequently in the sequel.

Lemma III.4 Consider a system (A, B, C, D) in which (A, B, C) is a minimal representation and (A, B, C, D) is passive. If $v \in \mathbb{R}^k$ satisfies $(D + D^\top)v = 0$ (or equivalently, $v^\top Dv = 0$), then $C^\top v = KBv$ for any K satisfying (9).

Proof: According to Proposition III.2, passivity of the system implies that K is symmetric, $K > 0$ and satisfies

$$\begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{bmatrix} \leqslant 0. \tag{10}$$

Premultiplication of (10) by $(\gamma z^\top \;\; v^\top)$ and postmultiplication by $(\gamma z^\top \;\; v^\top)^\top$ for arbitrary $z \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ yields $\gamma^2 z^\top(A^\top K + KA)z + 2\gamma z^\top(KB - C^\top)v \leqslant 0$ due to $(D + D^\top)v = 0$. Considering this expression as an inequality for a quadratic form in γ, we find that $z^\top(KB - C^\top)v = 0$. Since z is arbitrary, we obtain $(KB - C^\top)v = 0$. □

IV. Dynamics in a given mode

Equation (8c) implies that, for all t, and for every $i = 1, \ldots, k$ $u_i(t) = 0$ or $y_i(t) = 0$ must be satisfied (the diode is conducting or blocking and can be replaced by a short or an open circuit, respectively). This results in a multimodal system with 2^k modes, where each mode is characterized by a subset I of $\{1, \ldots, k\}$, indicating that $y_i(t) = 0$ if $i \in I$ and $u_i(t) = 0$ if $i \in I^c$ with $I^c := \{i \in \{1, \ldots, k\} \mid i \notin I\}$. For each such mode (also called "topology," "configuration," or "discrete state") the laws of motion are given by differential and algebraic equations (DAEs). Specifically, in mode I they are given by (we omit the time arguments for brevity)

$$\dot{x} = Ax + Bu \tag{11a}$$
$$y = Cx + Du \tag{11b}$$
$$y_i = 0, \; i \in I \tag{11c}$$
$$u_i = 0, \; i \in I^c. \tag{11d}$$

Example IV.1 For an illustration of the ideas of this paper in the simplest possible context, consider the linear RLC circuit (with $R = 1\ \Omega$, $L = 1\ H$ and $C = 1\ F$) coupled to two ideal diodes as shown in Figure 6. The network is described by

$$\dot{x}_1 = x_2 - u_1 + u_2 \tag{12a}$$
$$\dot{x}_2 = -x_1 - x_2 - u_2 \tag{12b}$$
$$y_1 = -x_1 \tag{12c}$$
$$y_2 = x_1 + x_2 + u_2 \tag{12d}$$
$$0 \leqslant u \perp y \geqslant 0, \tag{12e}$$

where x_1 is the voltage across the capacitor C, x_2 is the current through the inductor L, u_1 and u_2 are the current through and y_1 and y_2 are (minus) the voltage across diode 1 and 2, respectively.

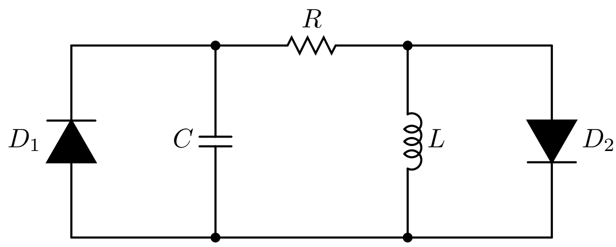

Fig. 6. RLC circuit with ideal diodes

Depending on whether the diodes are blocking or conducting, the system has $2^2 = 4$ modes or circuit topologies.
- *Mode $I = \varnothing$:* Both diodes are blocking in this mode, i.e., $u_1 = u_2 = 0$.
- *Mode $I = \{2\}$:* The first diode is blocking while the second one is conducting, i. e., $u_1 = y_2 = 0$ in this mode.
- *Mode $I = \{1\}$:* The first diode is conducting and the second one is blocking, i. e., $y_1 = u_2 = 0$ in this mode.
- *Mode $I = \{1, 2\}$:* In this mode both diodes are conducting, i. e., $y_1 = y_2 = 0$.

The mode will vary during the time evolution of the system (diodes go from conducting to blocking or vice versa). The system evolves in a certain mode as long as the inequality conditions in (8c) are satisfied. At the event of a mode transition, the system may in principle display jumps of the state variable x. Jumping phenomena are well-known in the theory of unilaterally constrained mechanical systems [46], where at impacts the change of velocity of the colliding bodies is often modelled as being instantaneous. These discontinuous and impulsive motions are also observed in electrical networks (see e.g. [15, 16, 18, 19, 26–28]) and consequently, a distributional framework will be needed to obtain a mathematically precise solution concept. We restrict ourselves to the Dirac distribution (supported at $t = 0$) denoted by δ and its derivatives, where $\delta^{(i)}$ denotes the i-th (distributional) derivative of δ.

Definition IV.2 [47] An *impulsive-smooth distribution* is a distribution $\mathtt{u}$ of the form $\mathtt{u} = \mathtt{u}_{imp} + \mathtt{u}_{reg}$, where

- $\mathtt{u}_{imp}$ is a linear combination of δ and its derivatives, i.e.,

$$\mathtt{u}_{imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)}$$

for vectors $u^{-i} \in \mathbb{R}^k$, $i = 0, \ldots, l$ and

- $\mathtt{u}_{reg}$ is an arbitrarily often differentiable function from $(0, \infty)$ to $\mathbb{R}^k$ such that $\mathtt{u}_{reg}^{(m)}(0+) := \lim_{t\downarrow 0} \frac{d^m \mathtt{u}_{reg}}{dt^m}(t)$ exists and is finite for all $m = 0, 1, 2, \ldots$.

The class of impulsive-smooth distributions is denoted by C^k_{imp}. For a distribution $\mathtt{u} \in C^k_{imp}$, $\mathtt{u}_{imp}$ is called the impulsive part and $\mathtt{u}_{reg}$ is called the smooth part. In case $\mathtt{u}_{imp} = 0$ we call $\mathtt{u}$ a *regular* or *smooth* distribution. If the Laplace transform of an impulsive-smooth distribution is rational, we call the distribution of *Bohl type* or a *Bohl distribution.* For a smooth Bohl distribution, we will use the term *Bohl function.*

We also would like to introduce the notion of the derivative of an impulsive-smooth distribution.

Definition IV.3 Let $\mathtt{u}$ be an impulsive-smooth distribution that can be written as $\mathtt{u} = \mathtt{u}_{imp} + \mathtt{u}_{reg}$, where

$$\mathtt{u}_{imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)}$$

for vectors $u^{-i} \in \mathbb{R}^k$, $i = 0, \ldots, l$ and $\mathtt{u}_{reg}$ is the smooth part. The derivative of $\mathtt{u}$ is denoted by $\dot{\mathtt{u}}$ and defined by

$$\dot{\mathtt{u}} = \sum_{i=0}^{l} u^{-i}\delta^{(i+1)} + \mathtt{u}_{reg}(0+)\delta + \dot{\mathtt{u}}_{reg}, \tag{13}$$

where $\dot{\mathtt{u}}_{reg}$ denotes the usual derivative of a function on $(0, \infty)$.

Lemma IV.4 Consider the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption III.3 is satisfied and (A, B, C, D) represents a passive system. Then the following holds.

1. For all $I \subseteq \{1, \ldots, k\}$ and for all initial states x_0, there exists a unique solution $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in C^{k+n+k}_{imp}$ satisfying the dynamics for mode I given by

$$\begin{aligned} \dot{\mathtt{x}} &= A\mathtt{x} + B\mathtt{u} + x_0\delta && \text{(14a)}\\ \mathtt{y} &= C\mathtt{x} + D\mathtt{u} && \text{(14b)}\\ \mathtt{y}_i &= 0,\ i \in I && \text{(14c)}\\ \mathtt{u}_i &= 0,\ i \in I^c && \text{(14d)} \end{aligned}$$

as equalities of distributions. We denote this solution by $(\mathtt{u}^{x_0,I}, \mathtt{x}^{x_0,I}, \mathtt{y}^{x_0,I})$.

2. For all modes I there exist matrices F^I and K^I such that for all initial states x_0 the smooth parts $(u, x, y) := (\mathtt{u}^{x_0,I}_{reg}, \mathtt{x}^{x_0,I}_{reg}, \mathtt{y}^{x_0,I}_{reg})$ of $(\mathtt{u}^{x_0,I}, \mathtt{x}^{x_0,I}, \mathtt{y}^{x_0,I})$ are Bohl functions and satisfy

$$\begin{aligned} \dot{x} &= F^I x && (15)\\ u &= K^I x && (16)\\ y &= Cx + Du. && (17) \end{aligned}$$

The matrices F^I and K^I only depend on the mode I and not on the particular x_0 at hand.

Proof:

1. The existence and uniqueness of a solution for (14) for all initial states x_0 is equivalent to the transfer matrix $G_{II} := C_{I\bullet}(s\mathcal{I} - A)^{-1}B_{\bullet I} + D_{II}$ being invertible as a rational matrix [47, Prop. 3.23, Thm. 3.24, Thm. 3.26]. This can also be seen from (22)-(23) below. Suppose on the contrary that $\det G_{II}(s) \equiv 0$. Then there exists a rational vector $v(s) \not\equiv 0$ such that $G_{II}(s)v(s) \equiv 0$. Take $\sigma > 0$ such that $v(\sigma) \neq 0$ and $\sigma\mathcal{I} - A$ is invertible. Define $\bar{u}$ as

$$\bar{u}_i := \begin{cases} 0 & \text{if } i \notin I \\ v_i(\sigma) & \text{if } i \in I \end{cases}$$

The triple

$$\begin{aligned} u(t) &= \bar{u}e^{\sigma t} && (18)\\ x(t) &= (\sigma\mathcal{I} - A)^{-1}B\bar{u}e^{\sigma t} && (19)\\ y(t) &= G(\sigma)\bar{u}e^{\sigma t} && (20) \end{aligned}$$

satisfies the system equations (6), where $G(s) = C(s\mathcal{I} - A)^{-1}B + D$. Since (A, B, C, D) is passive, there exists a $K > 0$ such that the dissipation inequality

$$x^\top(t_0)Kx(t_0) + \int_{t_0}^{t_1} u^\top(t)y(t)dt \geqslant x^\top(t_1)Kx(t_1) \tag{21}$$

holds for all t_0 and t_1 with $t_1 \geqslant t_0$. It can be verified that $u^\top(t)y(t) = e^{2\sigma t}\bar{u}^\top G(\sigma)\bar{u} = e^{2\sigma t}v(\sigma)^\top G_{II}(\sigma)v(\sigma) = 0$ for all t. By letting t_0 tend to $-\infty$, (21) results in

$$0 \geqslant x^\top(t_1)Kx(t_1)$$

for all t_1. Because $K > 0$, this implies that $x(t_1) = 0$ for all t_1. From (19) it follows that $B\bar{u} = 0$. Since B is of full column rank, $\bar{u} = 0$ and hence also $v(\sigma) = 0$. We reached a contradiction and consequently proved the first statement.

2. This statement follows from [47, Thm. 3.10]. □

Remark IV.5 In terms of Definition 3.2 in [24] the first property of Theorem IV.4 states that all modes are *autonomous.* To be specific, mode I is called autonomous (see also [24, Lemma 3.3]) if for all initial states x_0 there exists a unique impulsive-smooth solution to (14).

Remark IV.6 The positive realness of $G(s)$ implies that $G(\sigma)$ is nonnegative definite for all $\sigma > 0$. Since a nonnegative definite matrix has only nonnegative principal minors

[6, p. 153] and $\det G_{II}(s) \not\equiv 0$ (as shown in the proof of Lemma IV.4), it follows that there exists a $\sigma_0 \in \mathbb{R}$ such that for all $\sigma \geqslant \sigma_0$ the principal minors of $G(\sigma)$ are positive, i.e., $\det G_{II}(\sigma) > 0$ for all $I \subseteq \{1, \dots, k\}$. In terms of [6, Def. 3.3.1] this means that $G(\sigma)$ is a P-matrix for all sufficiently large σ.

Example IV.7 To demonstrate Lemma IV.4 we continue the running example IV.1. In particular, we will consider mode $I = \{2\}$ in which $u_1 = y_2 = 0$. Using (12d) and $y_2 = 0$ yields that $u_2 = -x_1 - x_2$. Since $u_1 = 0$, it holds that $u = K^{\{1\}}x$ with $K^{\{1\}} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$. Substituting $u_1 = 0$ and $u_2 = -x_1 - x_2$ in (12a)-(12b) leads to $\dot{x}_1 = -x_1$ and $\dot{x}_2 = 0$. Hence, $F^{\{1\}} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$.

The solutions $(\mathrm{u}^{x_0,I}, \mathrm{x}^{x_0,I}, \mathrm{y}^{x_0,I})$ have *rational* Laplace transforms $(\hat{\mathrm{u}}^{x_0,I}(s), \hat{\mathrm{x}}^{x_0,I}(s), \hat{\mathrm{y}}^{x_0,I}(s))$, which satisfy

$$\begin{aligned} s\hat{\mathrm{x}}^{x_0,I}(s) &= A\hat{\mathrm{x}}^{x_0,I}(s) + B\hat{\mathrm{u}}^{x_0,I}(s) + x_0 & \text{(22a)} \\ \hat{\mathrm{y}}^{x_0,I}(s) &= C\hat{\mathrm{x}}^{x_0,I}(s) + D\hat{\mathrm{u}}^{x_0,I}(s) & \text{(22b)} \\ \hat{\mathrm{y}}_I^{x_0,I}(s) &= 0 & \text{(22c)} \\ \hat{\mathrm{u}}_{I^c}^{x_0,I}(s) &= 0. & \text{(22d)} \end{aligned}$$

We introduce $G(s) = C(s\mathcal{I} - A)^{-1}B + D$ and $R(s) = C(s\mathcal{I} - A)^{-1}$. Since $G_{II}(s)$ is invertible as a rational matrix (see the proof of Lemma IV.4), the equations (22) can be solved explicitly. This yields that the Laplace transforms $(\hat{\mathrm{u}}^{x_0,I}(s), \hat{\mathrm{x}}^{x_0,I}(s), \hat{\mathrm{y}}^{x_0,I}(s))$ are given by

$$\hat{\mathrm{u}}_I^{x_0,I}(s) = -G_{II}^{-1}(s)R_{I\bullet}(s)x_0 \tag{23a}$$

$$\hat{\mathrm{u}}_{I^c}^{x_0,I}(s) = 0 \tag{23b}$$

$$\hat{\mathrm{x}}^{x_0,I}(s) = (s\mathcal{I} - A)^{-1}Bx_0 + (s\mathcal{I} - A)^{-1}B\hat{\mathrm{u}}^{x_0,I}(s) \tag{23c}$$

$$\hat{\mathrm{y}}_{I^c}^{x_0,I}(s) = [R_{I^c\bullet}(s) - G_{I^cI}(s)G_{II}^{-1}(s)R_{I\bullet}(s)]x_0 \tag{23d}$$

$$\hat{\mathrm{y}}_I^{x_0,I}(s) = 0. \tag{23e}$$

Hence, the solutions of the mode dynamics (14) are one-to-one related (by the Laplace transform and its inverse) to solutions satisfying (22). On the basis of this relation, we can prove that only Dirac impulses (and not its derivatives) show up in passive electrical networks with diodes. Note that this statement is implied by the fact that the Laplace transforms $(\hat{\mathrm{u}}^{x_0,I}(s), \hat{\mathrm{x}}^{x_0,I}(s), \hat{\mathrm{y}}^{x_0,I}(s))$ are proper for any $x_0 \in \mathbb{R}^n$ and $I \subseteq \{1, \dots, k\}$.

Theorem IV.8 Consider matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$ such that Assumption III.3 is satisfied and (A, B, C, D) represents a passive system. Then for each $x_0 \in \mathbb{R}^n$ and $I \subseteq \{1, \dots, k\}$ the Laplace transform $\hat{\mathrm{u}}^{x_0,I}(s)$ is proper.

Proof: Denote $\hat{\mathrm{u}}^{x_0,I}(s)$ by $u(s)$ for brevity. The triple

$$\bar{u}(t) = u(\sigma)e^{\sigma t} \tag{24}$$

$$\bar{x}(t) = (\sigma\mathcal{I} - A)^{-1}Bu(\sigma)e^{\sigma t} \tag{25}$$

$$\bar{y}(t) = G(\sigma)u(\sigma)e^{\sigma t} \tag{26}$$

satisfies (6) for all $\sigma \in \mathbb{R}$ such that $\sigma\mathcal{I} - A$ is nonsingular. It follows from passivity that there exists a $K > 0$ such that for all t_1 and t_0 with $t_1 \geqslant t_0$

$$\bar{x}^\top(t_1)K\bar{x}(t_1) - \bar{x}^\top(t_0)K\bar{x}(t_0) \leqslant \int_{t_0}^{t_1} \bar{u}^\top(t)\bar{y}(t)dt. \tag{27}$$

By substituting (24)-(26) into the dissipation inequality (27), one obtains

$$\begin{aligned} & u^\top(\sigma)B^\top(\sigma\mathcal{I} - A)^{-\top}K(\sigma\mathcal{I} - A)^{-1}Bu(\sigma) \leqslant \\ & \qquad \leqslant \frac{1}{2\sigma} \quad u^\top(\sigma)G(\sigma)u(\sigma). \end{aligned} \tag{28}$$

Since $K > 0$, B has full column rank, and $(\sigma\mathcal{I} - A)^{-1} = \frac{1}{\sigma}\mathcal{I} + \frac{1}{\sigma^2}A + \frac{1}{\sigma^3}A^2 + \dots$ is strictly proper, there exists an $\alpha > 0$ such that

$$\frac{\alpha}{\sigma^2}\|u(\sigma)\|^2 \leqslant u^\top(\sigma)B^\top(\sigma\mathcal{I} - A)^{-\top}K(\sigma\mathcal{I} - A)^{-1}Bu(\sigma) \tag{29}$$

for all sufficiently large σ. We know from (22) that $u^\top(s)y(s) = 0$, where $y(s) := \hat{\mathrm{y}}^{x_0,I} = C(s\mathcal{I} - A)^{-1}x_0 + G(s)u(s)$. Hence, the right-hand side of (28) satisfies

$$\begin{aligned} \frac{1}{2\sigma}u^\top(\sigma)G(\sigma)u(\sigma) &= -\frac{1}{2\sigma}u^\top(\sigma)C(\sigma\mathcal{I} - A)^{-1}x_0 \\ &\leqslant \frac{1}{2\sigma}\|C(\sigma\mathcal{I} - A)^{-1}x_0\|\|u(\sigma)\| \\ &\leqslant \frac{\beta}{2\sigma^2}\|u(\sigma)\|\|x_0\|. \end{aligned} \tag{30}$$

The last inequality follows from the existence of a $\beta > 0$ such that $\|C(\sigma\mathcal{I} - A)^{-1}\| \leqslant \frac{\beta}{\sigma}$ for all sufficiently large σ. Thus, (28), (29) and (30) yield $\|u(\sigma)\| \leqslant \frac{\beta}{2\alpha}\|x_0\|$ for all sufficiently large σ. Hence, $u(s)$ must be proper. □

The fact that solutions of linear passive networks with ideal diodes do not contain derivatives of Dirac impulses is widely believed true on "intuitive" grounds, but the authors are not aware of any previous rigorous proof. The framework proposed here makes it possible to prove the intuition.

To summarize the discussion so far, it has been shown that instead of considering impulsive-smooth distributions as the solution space within a mode, we can restrict ourselves to Bohl distributions with impulsive part containing only Dirac impulses and not its derivatives (i.e., Bohl distributions with *proper* rational Laplace transforms).

Consider a solution to (14) for mode I and initial state x_0. An important observation is that a nontrivial impulsive part of $\mathrm{u}^{x_0,I}$ will result in a re-initialization (jump) of the state. If $\mathrm{u}_{imp} = u^0\delta$ (i.e., $u^0 = \lim_{s\to\infty} \hat{\mathrm{u}}^{x_0,I}(s)$), then a jump will take place according to

$$\mathrm{x}_{reg}(0+) := \lim_{t\downarrow 0} \mathrm{x}_{reg}(t) = x_0 + Bu^0. \tag{31}$$

The proof can be found in [47].

The following properties can be proven for the impulsive part of an impulsive-smooth distribution satisfying the mode dynamics.

Lemma IV.9 Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption III.3 is satisfied and (A, B, C, D) represents a passive system. Consider the impulsive-smooth solution $(\mathtt{u}^{x_0,I}, \mathtt{x}^{x_0,I}, \mathtt{y}^{x_0,I})$ to (14) for mode I and initial state x_0. The impulsive part $\mathtt{u}^{x_0,I}_{imp}$ is given by $u^0\delta$ for some vector $u^0 \in \mathbb{R}^k$ that satisfies $u^{0\top} Du^0 = 0$ and $u^{0\top} C(x_0 + Bu^0) = 0$.

Proof: As stated before, the properness of $\hat{\mathtt{u}}^{x_0,I}(s)$ implies that $\mathtt{u}^{x_0,I}_{imp} = u^0\delta$ with $u^0 = \lim_{s\to\infty} \hat{\mathtt{u}}^{x_0,I}(s)$. For brevity we will denote $\hat{\mathtt{u}}^{x_0,I}(s)$ by $u(s)$ and $\hat{\mathtt{y}}^{x_0,I}(s)$ by $y(s)$ in this proof. Take the power series expansion of $u(s)$ around infinity as

$$u(s) = u^0 + u^1 s^{-1} + u^2 s^{-2} + \cdots \tag{32}$$

Because for all i either $u_i(s) \equiv 0$ or $y_i(s) \equiv 0$, we have that

$$u^\top(s)y(s) = u^\top(s)[C(s\mathcal{I} - A)^{-1}x_0 + G(s)u(s)] = 0.$$

Substituting (32) into this equality and considering the coefficients corresponding to s^0 and s^{-1} yield

$$u^{0\top} Du^0 = 0 \tag{34}$$

$$u^{0\top} Cx_0 + u^{0\top} Du^1 + u^{1\top} Du^0 + u^{0\top} CBu^0 = 0. \tag{35}$$

The relation (34) implies that

$$(D + D^\top)u^0 = 0. \tag{36}$$

Now, (35) and (36) give

$$u^{0\top} Cx_0 + u^{0\top} CBu^0 = 0, \tag{37}$$

which establishes together with (34) the desired identities. □

V. The rational complementarity problem

In the previous section the dynamics within a mode (i.e., with a fixed state of the diodes) has been considered, while the inequality conditions have been ignored. However, a solution $(\mathtt{u}^{x_0,I}, \mathtt{x}^{x_0,I}, \mathtt{y}^{x_0,I})$ within a mode (14) will in general only be valid for a limited amount of time, since a change of mode (diode going from conducting to blocking or vice versa) may be triggered by the inequality constraints. Therefore, we would like to express some kind of "local nonnegativity." We call a (smooth) Bohl function v *initially nonnegative* if there exists an $\varepsilon > 0$ such that $v(t) \geqslant 0$ for all $t \in [0, \varepsilon)$. Note that a Bohl function v is initially nonnegative if and only if there exists a $\sigma_0 \in \mathbb{R}$ such that its Laplace transform $\hat{v}(\sigma) \geqslant 0$ for all $\sigma \geqslant \sigma_0$. Hence, there is a connection between small time values for time functions and large values for the indeterminate s in the Laplace transform. This fact is closely related to the well-known initial value theorem (see e.g. [48]). The definition of initial nonnegativity for Bohl distributions will be based on this observation (see also [24, 25]).

Definition V.1 We call a Bohl distribution $\mathtt{v}$ *initially nonnegative*, if its Laplace transform $\hat{\mathtt{v}}(s)$ satisfies $\hat{\mathtt{v}}(\sigma) \geqslant 0$ for all sufficiently large real σ.

Remark V.2 To relate the definition to the time domain, note that a scalar-valued Bohl distribution $\mathtt{v}$ without derivatives of the Dirac impulse (i.e., $\mathtt{v}_{imp} = v^0\delta$ for some $v^0 \in \mathbb{R}$) is initially nonnegative if and only if
1. $v^0 > 0$, or
2. $v^0 = 0$ and there exists an $\varepsilon > 0$ such that $v_{reg}(t) \geqslant 0$ for all $t \in [0, \varepsilon)$.

Definition V.3 We call a Bohl distribution $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in C^{k+n+k}_{imp}$ an *initial solution* to (8) with initial state x_0, if there exists an $I \subseteq \{1, \ldots, k\}$ such that
1. $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ satisfies (14) for mode I and initial state x_0 in the distributional sense and
2. $\mathtt{u}, \mathtt{y}$ are initially nonnegative.

According to Lemma IV.4 condition 1 means that $(\mathtt{u}, \mathtt{x}, \mathtt{y}) = (\mathtt{u}^{x_0,I}, \mathtt{x}^{x_0,I}, \mathtt{y}^{x_0,I})$ for an LCS with (A, B, C, D) passive and satisfying Assumption III.3.

Example V.4 Consider the system $\dot{x}(t) = u(t)$, $y(t) = x(t)$ together with (8c). This represents a system consisting of a capacitor connected to a diode. The current in the network is equal to u and the voltage across the capacitor is equal to $y = x$. For initial state $x(0) = x_0 = 1$, $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ with $\mathtt{u} = 0$ (no current) and $\mathtt{y}(t) = \mathtt{x}(t) = 1$ for all $t \in \mathbb{R}$ is an initial solution. This corresponds to the case that the diode is always blocking and there is no (nonzero) current in the network. To demonstrate that the distributional framework is needed, consider the initial state $x_0 = -1$, for which $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ with $\mathtt{u} = \delta$, $\mathtt{x}(t) = \mathtt{y}(t) = 0$, $t > 0$ is the unique initial solution. This corresponds to an instantaneous discharge of the capacitor at time instant 0. Note that a state jump occurs at time 0 from -1 to 0.

We emphasize that an initial solution only satisfies the equations (8) in the following local sense. In case an initial solution has a nontrivial impulsive part, only the re-initialization as given in (31) forms a piece of the global solution. If the initial solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ is smooth, the largest interval on which $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ satisfies the equations (8) is equal to $[0, \varepsilon)$, where ε is equal to

$$\varepsilon = \inf\{t > 0 \mid \mathtt{u}_{reg,i}(t) < 0 \text{ or } \mathtt{y}_{reg,i}(t) < 0 \text{ for some } i\}. \tag{38}$$

Example V.5 Consider again the network in Example IV.1. We will compute the initial solutions for two initial states, to wit $(x_1(0), x_2(0))^\top = (-e, 1)^\top$ and $(x_1(0), x_2(0))^\top = (1, 1)^\top$.

If the response of mode $I = \{2\}$ is computed for initial state $(x_1(0), x_2(0))^\top = (-e, 1)^\top$ (see also Example IV.7), it can be seen that $\mathtt{x}_1(t) = -e^{(1-t)}$, $\mathtt{x}_2(t) = 1$, $\mathtt{y}_1(t) = e^{(1-t)}$, $\mathtt{u}_2(t) = e^{(1-t)} - 1$, $\mathtt{u}_1 = \mathtt{y}_2 = 0$. Hence, this is indeed an initial solution for initial state $(-e, 1)^\top$ as $\mathtt{u}$ and $\mathtt{y}$ are initially nonnegative. Note that the initial solution is smooth and satisfies the equations (8) on the interval $[0, 1)$ (i.e., $\varepsilon = 1$ in (38)).

For initial state $(x_1(0), x_2(0))^\top = (1, 1)^\top$ it can easily be verified that $\mathtt{x}_1 = 0$, $\mathtt{x}_2(t) = e^{-t}$, $\mathtt{y}_1 = 0$, $\mathtt{y}_2(t) = e^{-t}$,

$\mathtt{u}_1 = \delta + e^{-t}$, $\mathtt{u}_2 = 0$, which complies with mode $I = \{1\}$. As $\mathtt{u}$ and $\mathtt{y}$ are initially nonnegative, we have indeed derived an initial solution starting in $(1,1)^\top$. Note that there is a jump in the state component x_1 from 1 to zero caused by the presence of the δ. The physical interpretation is that there is an instantaneous discharge of the capacitor C.

In this manner the complete behaviour of the network can be derived, which results in the phase diagram as given in Figure 7.

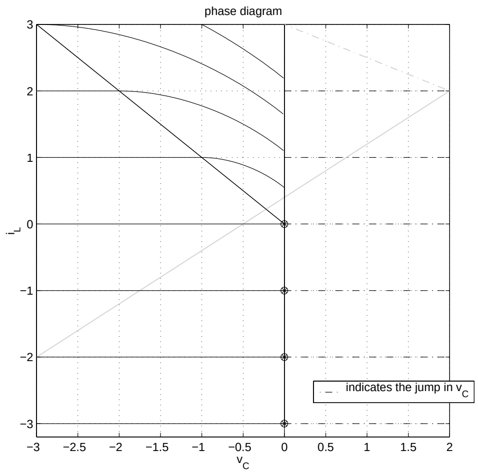

Fig. 7. Phase diagram of the circuit given in Example IV.1.

Even when a solution within some mode exists and is unique given an initial state, it still might be possible that different modes give rise to different initial solutions (see for instance, the example of the circuit in Figure 3 containing a negative resistor). It is also possible that there are no initial solutions at all, i.e., no solution within a mode satisfies the initial nonnegativity conditions. We will start our investigation of well-posedness for linear passive complementarity systems by studying existence and uniqueness of initial solutions. An important tool in existence and uniqueness of initial solutions is the *rational complementarity problem* (RCP) [22, 25].

Definition V.6 (The rational complementarity problem) Let the vector $x_0 \in \mathbb{R}^n$ and matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ be given. The *rational complementarity problem* $\mathrm{RCP}(x_0, A, B, C, D)$ is the problem of finding rational k-vectors $u(s) \in \mathbb{R}^k(s)$ and $y(s) \in \mathbb{R}^k(s)$ such that
1. for all $s \in \mathbb{C}$

$$y(s) = C(s\mathcal{I} - A)^{-1}x_0 + [C(s\mathcal{I} - A)^{-1}B + D]u(s) \quad (39a)$$

$$u(s) \perp y(s), \quad (39b)$$

and
2. there exists a $\sigma_0 \in \mathbb{R}$ satisfying for all $\sigma > \sigma_0$

$$y(\sigma) \geqslant 0 \quad \text{and } u(\sigma) \geqslant 0. \quad (40)$$

Any pair of rational vectors $(u(s), y(s))$ satisfying the above conditions is said to be a *solution* to $\mathrm{RCP}(x_0, A, B, C, D)$. If A, B, C and D are clear from the context, we also write $\mathrm{RCP}(x_0)$ for brevity.

From the definition of initial nonnegativity and (22), the following important relation is clear from [24].

Theorem V.7 Consider the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ and assume that all modes of $\mathrm{LCS}(A, B, C, D)$ are autonomous (see Remark IV.5). Then the following statements hold.
- All initial solutions are of Bohl type.
- There is a one-to-one correspondence between initial solutions to (8) and solutions to $\mathrm{RCP}(x_0)$. More specifically, $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ is an initial solution to (8) if and only if its Laplace transform $(\hat{\mathtt{u}}(s), \hat{\mathtt{x}}(s), \hat{\mathtt{y}}(s))$ is such that $(\hat{\mathtt{u}}(s), \hat{\mathtt{y}}(s))$ is a solution to $\mathrm{RCP}(x_0)$ and

$$\hat{\mathtt{x}}(s) = (s\mathcal{I} - A)^{-1}x_0 + (s\mathcal{I} - A)^{-1}B\hat{\mathtt{u}}(s). \quad (41)$$

- The following statements are equivalent.
  1. There exists a unique initial solution for initial state x_0 to $\mathrm{LCS}(A, B, C, D)$.
  2. $\mathrm{RCP}(x_0)$ has a unique solution.
- The initial solution is smooth if and only if the corresponding solution to $\mathrm{RCP}(x_0)$ is strictly proper. Similarly, the initial solution has an impulsive part containing only Dirac distributions (and not its derivatives) if and only if the corresponding solution to $\mathrm{RCP}(x_0)$ is proper.

As a consequence, studying existence and uniqueness of initial solutions is equivalent to studying existence and uniqueness of solutions to RCPs. In [25] necessary and sufficient conditions for existence and uniqueness of solutions to RCPs have been presented in terms of families of *linear complementarity problems* (cf. Definition V.10 below). Based on this relation and the literature on linear complementarity problems the following result has been proven in [25].

Theorem V.8 Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption III.3 is satisfied and (A, B, C, D) represents a passive system. Then $\mathrm{RCP}(x_0)$ has a unique solution for all x_0.

Theorem V.7 now yields the following.

Theorem V.9 Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption III.3 is satisfied and (A, B, C, D) represents a passive system. From each initial state x_0 there exists exactly one initial solution to $\mathrm{LCS}(A, B, C, D)$.

According to Theorem V.7 there exists a one-to-one relation between initial solutions and solutions to RCP. Since strictly proper Laplace transforms correspond to smooth Bohl distributions (without Dirac impulses and jumps of the state variable), it is interesting to characterize the set of initial states for which the corresponding solution to the RCP is strictly proper. In the following theorem such an explicit characterization will be given. To formulate the theorem, we need the following concepts.

Definition V.10 Let a real vector $q \in \mathbb{R}^k$ and a real matrix $M \in \mathbb{R}^{k\times k}$ be given. The linear complementarity problem with data q and M (LCP(q, M)) is the problem of finding a real vector $z \in \mathbb{R}^k$ such that $0 \leqslant z \perp (q + Mz) \geqslant 0$. Any such vector z is called a solution to LCP(q, M).

For an extensive survey on LCPs, we refer to [6]. The set of all solutions z to LCP(q, M) will be denoted by SOL(q, M).

Remark V.11 If $(u(s), y(s))$ is a solution to the problem RCP(x_0, A, B, C, D), then $u(\sigma)$ is a solution to LCP$(C(\sigma\mathcal{I} - A)^{-1}x_0, G(\sigma))$ for all sufficiently large (real) σ, where $G(s) = C(s\mathcal{I} - A)^{-1}B + D$.

Remark V.12 Several times we shall employ the following standard observation on solutions of LCP. If $z_i \in$ SOL(q_i, M_i) with $i \in \{1, 2\}$ then

$$(z_1 - z_2)^\top((q_1 + M_1z_1) - (q_2 + M_2z_2)) \\ = -z_1^\top(q_2 + M_2z_2) - z_2^\top(q_1 + M_1z_1) \leqslant 0.$$

Finally, a *dual cone* is defined as follows [6].

Definition V.13 Let $\mathcal{Q}$ be a nonempty set in $\mathbb{R}^k$. The *dual cone* of $\mathcal{Q}$, denoted by $\mathcal{Q}^*$, is defined as the set

$$\mathcal{Q}^* = \{w \in \mathbb{R}^k \mid w^\top v \geqslant 0 \text{ for all } v \in \mathcal{Q}\}.$$

Theorem V.14 Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption III.3 is satisfied and (A, B, C, D) represents a passive system. Denote the solution set of LCP$(0, D)$ by $\mathcal{Q} :=$ SOL$(0, D)$. Furthermore, let $(u_{x_0}(s), y_{x_0}(s))$ be the (unique) solution to $RCP(x_0)$. The following assertions hold:
1. For all $x_0 \in \mathbb{R}^n$, $C(x_0 + Bu^0) \in \mathcal{Q}^*$ where $u^0 = \lim_{s\to\infty} u_{x_0}(s)$.
2. $u_{x_0}(s)$ is strictly proper if and only if $Cx_0 \in \mathcal{Q}^*$.
3. $\lim_{s\to\infty} u_{x_0}(s) \in \mathcal{Q}$.

Proof:
1: In view of Remark V.11 and Remark V.12, we have for each $v \in \mathcal{Q} :=$ SOL$(0, D)$ that

$$(u_{x_0}(\sigma) - v)^\top(C(\sigma\mathcal{I} - A)^{-1}x_0 + G(\sigma)u_{x_0}(\sigma) - Dv) \leqslant 0$$

for all sufficiently large σ. Since $D \geqslant 0$ ((9a) yields $D + D^\top \geqslant 0$) and $G(\sigma) = C(\sigma\mathcal{I} - A)^{-1}B + D$, we obtain

$$(u_{x_0}(\sigma) - v)^\top[C(\sigma\mathcal{I} - A)^{-1}x_0 + \\ + C(\sigma\mathcal{I} - A)^{-1}Bu_{x_0}(\sigma)] \leqslant 0 \quad (42)$$

for all sufficiently large σ. Multiplying this relation by σ and letting σ tend to infinity yields, since $u_{x_0}(s)$ is proper,

$$(u^0 - v)^\top(Cx_0 + CBu^0) \leqslant 0$$

It follows from Lemma IV.9 that $v^\top(Cx_0 + CBu^0) \geqslant 0$ for all $v \in \mathcal{Q}$ and thus $C(x_0 + Bu^0) \in \mathcal{Q}^*$.

2: "only if": Suppose $u_{x_0}(s)$ is strictly proper. Statement *1* and $u^0 = 0$ yield $Cx_0 \in \mathcal{Q}^*$.

"if": Suppose that $Cx_0 \in \mathcal{Q}^*$. From Lemma III.4 and Lemma IV.9 we obtain that

$$\begin{aligned} u^{0\top}Du^0 &= 0 && (43)\\ u^{0\top}Cx_0 + u^{0\top}CBu^0 &= 0 && (44)\\ (KB - C^\top)u^0 &= 0 && (45)\end{aligned}$$

Since $(u_{x_0}(s), y_{x_0}(s))$ is the solution to RCP(x_0), $u^0 \geqslant 0$ and $Du^0 \geqslant 0$. Together with (43), this gives $u^0 = \lim_{s\to\infty} u_{x_0}(s) \in \mathcal{Q}$ (this proves statement *3*).

From (45), we obtain $u^{0\top}CBu^0 = u^{0\top}B^\top KBu^0$. Since $u^0 \in \mathcal{Q}$ and $Cx_0 \in \mathcal{Q}^*$, (44) gives

$$0 \geqslant -u^{0\top}Cx_0 = u^{0\top}CBu^0 = u^{0\top}B^\top KBu^0 \geqslant 0.$$

Finally, positive definiteness of K and the full column rank of B imply $u^0 = 0$, i.e., $u_{x_0}(s)$ is strictly proper.

3: This has already been shown in the proof of statement *2*. □

A direct implication of the statements *1* and *2* in Theorem V.14 is that, if smooth continuation is not possible for x_0, it is possible after one re-initialization. Indeed, by (31) the state after the re-initialization is equal to $x_0 + Bu^0$, if the impulsive part of the (unique) initial solution is equal to $u^0\delta$. According to the fact that the Laplace transform of an initial solution is a solution to the corresponding RCP (which is automatically proper), it follows that $\lim_{s\to\infty} u_{x_0}(s) = u^0$ is indeed the coefficient determining the impulsive part. Since $C(x_0+Bu^0) \in \mathcal{Q}^*$, it follows from statement *2* that from $x_0 + Bu^0$ there exists a smooth initial solution. To summarize this discussion, we formulate a local existence result.

Theorem V.15 Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption III.3 is satisfied and (A, B, C, D) represents a passive system. For all initial states x_0, there exists a unique Bohl distribution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ defined on $[0, \varepsilon)$ for some $\varepsilon > 0$ satisfying the following.
1. there exists an initial solution $(\bar{\mathtt{u}}, \bar{\mathtt{x}}, \bar{\mathtt{y}})$ such that

$$(\mathtt{u}_{imp}, \mathtt{x}_{imp}, \mathtt{y}_{imp}) = (\bar{\mathtt{u}}_{imp}, \bar{\mathtt{x}}_{imp}, \bar{\mathtt{y}}_{imp}).$$

with $\bar{\mathtt{u}}_{imp} = u^0\delta$ for some $u^0 \in \mathbb{R}^k$,
2. $\mathtt{x}_{reg}(0+) = x_0 + Bu^0$, and
3. for all $t \in (0, \varepsilon)$

$$\begin{aligned} \mathtt{x}_{reg}(t) &= \mathtt{x}_{reg}(0+) + \int_0^t [A\mathtt{x}_{reg}(\tau) + B\mathtt{u}_{reg}(\tau)]d\tau\\ \mathtt{y}_{reg}(t) &= C\mathtt{x}_{reg}(t) + D\mathtt{u}_{reg}(t)\\ 0 &\leqslant \mathtt{u}_{reg}(t) \perp \mathtt{y}_{reg}(t) \geqslant 0.\end{aligned}$$

VI. Regular states

Another consequence of Theorem V.14 is the characterization of so-called *regular states* (sometimes also called consistent states) as introduced in the following definition.

Definition VI.1 A state x_0 is called *regular* for LCS(A,B,C,D), if the corresponding initial solution is smooth. The collection of regular states for a given quadruple (A,B,C,D) is denoted by $\mathcal{R}$.

We have the following equivalent characterizations of regular states.

Theorem VI.2 Consider LCS(A,B,C,D) given by (8) such that (A,B,C,D) is passive and Assumption III.3 is satisfied. Define $\mathcal{Q} := \mathrm{SOL}(\imath,\mathcal{D})$ and let $\mathcal{Q}^*$ be the dual cone of $\mathcal{Q}$. The following statements are equivalent.
1. x_0 is a regular state for (8).
2. $Cx_0 \in \mathcal{Q}^*$.
3. LCP(Cx_0,D) has a solution.
4. $Cx_0 \in \mathrm{pos}(I,-D)$, which means that Cx_0 can be written as a positive combination of the columns of the identity matrix I and the matrix $-D$. In other words, $Cx_0 = v_1 - Dv_2$ for two nonnegative vectors $v_1 \geqslant 0$ and $v_2 \geqslant 0$.

Proof: Since strictly proper Laplace transforms correspond to smooth Bohl distributions, statement *2* in Theorem V.14 gives a characterization of the regular states: $x_0 \in \mathcal{R}$ if and only if $Cx_0 \in \mathcal{Q}^*$ with $\mathcal{Q} = \mathrm{SOL}(0,D)$. Hence, statement 1 and 2 are equivalent. Since $D \geqslant 0$, [6, Cor. 3.8.10] completes the proof. □

Hence, several tests are available for deciding the regularity of an initial state x_0. In [17] it is stated that a well-designed circuit does not contain Dirac impulses. As a consequence, the characterization of $\mathcal{R}$ forms a verification of the synthesis of the network.

Example VI.3 The circuit in Example IV.1 is of the form (8) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}; \qquad B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix};$$
$$C = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}; \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The cone $\mathcal{Q} = \mathrm{SOL}(0,D)$ is given by $\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mid u_1 \geqslant 0 \text{ and } u_2 = 0\}$ and thus $\mathcal{Q}^* = \{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid y_1 \geqslant 0\}$. As a consequence of Theorem VI.2, the set of regular states is given by

$$\mathcal{R} = \{x_0 \in \mathbb{R}^n \mid C_{1\bullet}x_0 \geqslant 0\} = \{x_0 \in \mathbb{R}^n \mid x_{0,1} \leqslant 0\}.$$

Note that this is in agreement with the phase diagram in Figure 7. Moreover, in Example V.5 the initial solution for the state $(1,1)^\top$ turned out to contain a non-trivial impulsive part and hence, $(1,1)^\top$ is not regular. This is in accordance with $(1,1)^\top \notin \mathcal{R}$. Similar statements hold for the initial state $(-e,1)^\top$.

For further illustration of the structure of the cones $\mathcal{Q}^*$ and $\mathcal{R}$, some additional examples are in order.

Example VI.4 Consider the following situations. In each case we assume that the quadruple (A,B,C,D) is passive and satisfies Assumption III.3.
(a) If $D = 0$, then $\mathcal{Q} = \mathbb{R}^k_+$ and $\mathcal{Q}^* = \mathbb{R}^k_+$. Hence, $\mathcal{R} = \{x_0 \in \mathbb{R}^n \mid Cx_0 \geqslant 0\}$.
(b) If $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\mathcal{Q} = \{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mid u_1 \geqslant 0 \text{ and } u_2 = 0\}$. Consequently, $\mathcal{Q}^* = \{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid y_1 \geqslant 0\}$ and thus $\mathcal{R} = \{x_0 \in \mathbb{R}^n \mid C_{1\bullet}x_0 \geqslant 0\}$.
(c) If D is positive definite, it follows that $\mathcal{Q} = \{0\}$, which implies that $\mathcal{Q}^* = \mathbb{R}^k$ and thus $\mathcal{R} = \mathbb{R}^n$.

In the next section, it will be shown that the characterization of the regular states plays a key role in the proof of global existence of solutions as the set of such initial states will be proven to be invariant under the dynamics.

VII. Solution concept and global well-posedness

In [24, 25] a (global) solution concept has been introduced that is based on concatenation of initial solutions. In principle, this allows impulses at any mode transition time (necessary for e.g. unilaterally constrained mechanical systems). In the context of linear passive electrical networks with diodes, such a general notion of solution will not be needed. In fact, the solution concept as formulated in Theorem V.15 will be extended such that mode changes are possible. This will be achieved by dropping the Bohl requirement and allowing $\mathcal{L}_2$ functions as regular parts. The function space $\mathcal{L}_\delta(0,T)$ consists of the distributions of the form $\mathtt{u} = \mathtt{u}_{imp} + \mathtt{u}_{reg}$, where $\mathtt{u}_{imp} = u^0\delta$ with $u_0 \in \mathbb{R}$ and $\mathtt{u}_{reg} \in \mathcal{L}_2(0,T)$.

Definition VII.1 Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption III.3 is satisfied and (A,B,C,D) represents a passive system. Let a time horizon $T > 0$ and initial state x_0 be given. $(\mathtt{u},\mathtt{x},\mathtt{y}) \in \mathcal{L}_\delta^{k+n+k}(0,T)$ is called a solution to LCS(A,B,C,D) on $[0,T]$, if
1. there exists an initial solution $(\bar{\mathtt{u}},\bar{\mathtt{x}},\bar{\mathtt{y}})$ such that

$$(\mathtt{u}_{imp},\mathtt{x}_{imp},\mathtt{y}_{imp}) = (\bar{\mathtt{u}}_{imp},\bar{\mathtt{x}}_{imp},\bar{\mathtt{y}}_{imp}),$$

2. $\mathtt{x}_{reg}(0+) = x_0 + Bu^0$ with $u^0 \in \mathbb{R}^k$ given by $\bar{\mathtt{u}}_{imp} = u^0\delta$, and
3. for almost all $t \in (0,T)$

$$\begin{aligned} \mathtt{x}_{reg}(t) &= \mathtt{x}_{reg}(0+) + \int_0^t [A\mathtt{x}_{reg}(\tau) + B\mathtt{u}_{reg}(\tau)]d\tau \\ \mathtt{y}_{reg}(t) &= C\mathtt{x}_{reg}(t) + D\mathtt{u}_{reg}(t) \\ 0 &\leqslant \mathtt{u}_{reg}(t) \perp \mathtt{y}_{reg}(t) \geqslant 0. \end{aligned}$$

We have already proven local well-posedness (Theorem V.15). The question arises whether global well-posedness is also guaranteed.

A. Global existence

We now come to the main existence result of this paper.

Theorem VII.2 Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption III.3 is satisfied and (A, B, C, D) represents a passive system. Then, for all initial states x_0 and all $T > 0$ the system LCS(A, B, C, D) has a solution on $[0, T]$ in the sense of Definition VII.1.

Proof: The construction of a solution will be based on concatenation of initial solutions. Theorem V.15 implies that a solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ exists on $[0, \tau_1)$ (take τ_1 as large as possible, i.e., equal to ε as in (38)) from initial state x_0. Note that $\mathtt{x}(0+) \in \mathcal{R}$ and that $(\mathtt{u}_{reg}, \mathtt{x}_{reg}, \mathtt{y}_{reg})$ is part of a smooth initial solution with initial state $\mathtt{x}_{reg}(0+)$. Since $t \mapsto (\mathtt{u}_{reg}, \mathtt{x}_{reg}, \mathtt{y}_{reg})(t+\rho)$ forms a smooth initial solution for any $\rho \in (0, \tau_1)$, we have that $\mathtt{x}_{reg}(\rho) \in \mathcal{R}$ for all $\rho \in (0, \tau_1)$. Since $(\mathtt{u}_{reg}, \mathtt{x}_{reg}, \mathtt{y}_{reg})$ is a Bohl function, the limit $\lim_{t\uparrow\tau_1} \mathtt{x}_{reg}(t) = \mathtt{x}_{reg}(\tau_1)$ exists. The closedness of $\mathcal{R}$ (follows from statement *2* in Theorem V.14) implies that $\mathtt{x}(\tau_1) \in \mathcal{R}$. Due to local existence of solutions and $\mathtt{x}(\tau_1) \in \mathcal{R}$, there exists a smooth continuation (a smooth initial solution) from $\mathtt{x}(\tau_1)$ that defines a solution on $[0, \tau_2)$ with $\tau_2 > \tau_1$. This construction can be repeated as long as the limit $\lim_{t\uparrow\tau} \mathtt{x}(t)$ exists, where $[0, \tau)$ is the time-interval on which a solution has been generated so far. An obstruction to the existence of a global solution (on $[0, T]$) might be that the intervals of continuation $[\tau_i, \tau_{i+1})$ are getting smaller and smaller such that $\lim_{i\to\infty} \tau_i = \tau^* < T$ and $\lim_{t\uparrow\tau^*} \mathtt{x}(t)$ does not exist. To complete the proof we will show the existence of the latter limit under any circumstances.

Suppose the maximal interval on which a solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ can be defined is $[0, \tau^*)$, $\tau^* < T$. According to Lemma IV.4 there is at most exponential growth ($\dot{\mathtt{x}} = F^I \mathtt{x}$) between mode changes. For shortness we drop the subscript *reg* in the remainder of the proof. Since $\mathtt{x}$ is continuous on $(0, \tau^*)$ and governed by at most a finite number of linear dynamics ($\dot{\mathtt{x}} = F^I \mathtt{x}$), $\mathtt{x}$ is bounded (say $\|\mathtt{x}(t)\| \leqslant M$ for all $t \in [0, \tau^*)$). On an interval $(s, t) \subseteq [0, \tau^*)$ where $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ is governed by the dynamics $\dot{\mathtt{x}} = F^I \mathtt{x}$ of mode I, the following estimate holds

$$\begin{aligned} \|\mathtt{x}(t) - \mathtt{x}(s)\| = \|e^{F^I(t-s)}\mathtt{x}(s) - \mathtt{x}(s)\| \leqslant \\ \leqslant c_I \mid t - s \mid \|\mathtt{x}(s)\| \leqslant c_I M \mid t - s \mid . \quad (46) \end{aligned}$$

Indeed, note that the matrix function $t \mapsto \frac{e^{tF^I} - \mathcal{I}}{t}$ is bounded (by c_I) on $[0, \tau^*)$. Hence, for $(s, t) \subseteq [0, \tau^*)$ with $\mathtt{x}$ possibly evolving through several modes we get from (46) that

$$\|\mathtt{x}(t) - \mathtt{x}(s)\| \leqslant M \max_{I \subseteq \{1,\dots,k\}} c_I \mid t - s \mid .$$

This implies that $\mathtt{x}$ is Lipschitz continuous on $[0, \tau^*)$ and thus also uniformly continuous. It follows from a standard result in mathematical analysis [49, ex. 4.13] that $\mathtt{x}^* := \lim_{t\uparrow\tau^*} \mathtt{x}(t)$ exists. From the construction above it can be derived that $\mathtt{x}(t) \in \mathcal{R}$ for all $t \in [0, \tau^*)$ and hence, $\mathtt{x}^* \in \mathcal{R}$, which implies that smooth continuation is possible (local existence) from $\mathtt{x}^*$ beyond τ^*. This contradicts the definition of τ^*. Hence, existence of a solution on $[0, T]$ is guaranteed. □

B. Uniqueness

It can easily be seen that the solutions obtained by the construction in Theorem VII.2 must be unique, because the initial solutions are unique (see e.g. [25]). However, it might be possible that a different construction yields other solutions. The following theorem states that this is not the case.

Theorem VII.3 Consider matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ such that Assumption III.3 is satisfied and (A, B, C, D) represents a passive system. Then for all initial states x_0 and all final times $T > 0$ there exists at most one solution $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in \mathcal{L}_\delta^{k+n+k}(0, T)$ to LCS(A, B, C, D) in the sense of Definition VII.1.

Proof: Suppose that two solutions $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ and $(\mathtt{u}', \mathtt{x}', \mathtt{y}')$ exist in the sense of Definition VII.1. According to Corollary V.9 there exists exactly one initial solution from the initial state x_0. This implies that the impulsive parts of $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ and $(\mathtt{u}', \mathtt{x}', \mathtt{y}')$ must be the same and moreover, that the re-initialization from x_0 must be unique so that $\mathtt{x}(0+) = \mathtt{x}'(0+)$. Clearly, $(\mathtt{u} - \mathtt{u}', \mathtt{x} - \mathtt{x}', \mathtt{y} - \mathtt{y}')$ satisfies (6) from initial state 0 and is smooth. The dissipation inequality yields

$$\begin{aligned} \int_0^t [\mathtt{u}(\tau) - \mathtt{u}'(\tau)]^\top [\mathtt{y}(\tau) - \mathtt{y}'(\tau)] d\tau \geqslant \\ [\mathtt{x}(t) - \mathtt{x}'(t)]^\top K [\mathtt{x}(t) - \mathtt{x}'(t)] \end{aligned}$$

for all $t \in (0, \infty)$. From the fact that $\mathtt{u}, \mathtt{u}', \mathtt{y}$ and $\mathtt{y}'$ are nonnegative almost everywhere and the complementarity of $(\mathtt{u}, \mathtt{y})$ and $(\mathtt{u}', \mathtt{y}')$, we obtain

$$\int_0^t [\mathtt{u}(\tau) - \mathtt{u}'(\tau)]^\top [\mathtt{y}(\tau) - \mathtt{y}'(\tau)] d\tau \leqslant 0.$$

Hence,

$$[\mathtt{x}(t) - \mathtt{x}'(t)]^\top K [\mathtt{x}(t) - \mathtt{x}'(t)] \leqslant 0$$

for all $t \in (0, \infty)$. Since $K > 0$, we obtain $\mathtt{x}(t) = \mathtt{x}'(t)$ for all t. Since B is of full column rank, it follows that $\mathtt{u} = \mathtt{u}'$ and $\mathtt{y} = \mathtt{y}'$ almost everywhere. □

Since the global solution is unique, the solution must be equal to the one constructed in the proof of Theorem VII.2. This characterizes the nature of solutions to linear passive complementarity systems. Between mode changes the trajectories are of Bohl type and thus real-analytic. Moreover, the set $\mathcal{E}$ of mode transition times is right-isolated, i.e., for all $\tau \in \mathcal{E}$ there exists an $\alpha > 0$ such that $(\tau, \tau + \alpha) \cap \mathcal{E}$ is empty.

Remark VII.4 The fact that the set of mode transition times $\mathcal{E}$ is right-isolated can also be formulated as follows: there do no exist left-accumulation points[2] of mode transition times in the solutions defined by Definition VII.1. However, we cannot exclude the existence of right-accumulation points in general on the basis of this paper. Using a result in [50] it can be proven that for a linear passive network with one diode satisfying Assumption III.3 and $D = 0$ also right-accumulations do not occur.

VIII. Conclusions

In this paper we studied all dynamic piecewise linear (PL) networks that can be realized by linear passive electrical circuits with ideal diodes. As a result, the systems under study fall within the realm of linear complementarity systems for which a mathematical framework has been established in this paper. This framework has led to a precise definition of a transient true solution and formal proofs were given for the existence and uniqueness of solutions (well-posedness). Moreover, several regularity properties of the solutions have been proven. In particular, it has been shown that derivatives of Dirac impulses do not occur and that Dirac impulses happen only at the initial time instant; also the set of regular states has been exactly characterized.

Such a rigorous basis is needed for many analysis issues of switched electrical circuits. For instance, the paper [40] deals with the question whether the approximated time functions obtained by a time-stepping method [5, 11, 39] converge to the true transient solution of the network model. The theory developed in this paper is indispensable for answering the consistency question for this numerical simulation technique.

Networks with internally triggered switches have discrete as well as continuous characteristics. From this point of view, the paper proposes a systematic modelling framework and a precise notion of solutions for a class of networks of such a mixed nature. Systems consisting of continuous dynamics (differential equations) and switching logic are sometimes called "hybrid systems" and receive currently much attention from both control theorists [51, 52] and computer scientists [53]. Hybrid systems are encountered in various research programs ranging from switching controllers, unilaterally constrained mechanical systems, piecewise linear systems, and switched electrical networks to hydraulic systems with valves. Since the underlying problems for these systems are essentially the same, all these research programs may benefit from a general theory as is currently being developed for complementarity systems.

[2] A point τ is called a left-accumulation point of $\mathcal{E} \subseteq \mathbb{R}$, if there exists a sequence $\{\tau_i\}_{i \in \mathbb{N}}$ with $\tau_i \in \mathcal{E}$ such that $\tau_i > \tau$ and $\lim_{i \to \infty} \tau_i = \tau$. A right-accumulation point is defined by changing ">" into "<".

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