ELSEVIER
Insurance: Mathematics and Economics 14 (1994) 205–218
INSURANCE MATHEMATICS & ECONOMICS

Equilibria in a mixed financial–reinsurance market with constrained trading possibilities

Anja De Waegenaere

Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

Received January 1994; revised April 1994

Abstract

In this paper we consider a model for redistribution of risk by means of reinsurance contracts as well as financial assets. There is an important difference between the trade on financial markets and the trade on reinsurance markets. The trade of reinsurance contracts is constrained in the sense that agents can only buy reinsurance contracts for those risks that they insured initially. Such a constraint does not apply for financial markets. Therefore, the existing equilibrium models for redistribution of risk are adapted to the situation where financial markets are included in the model, where the trade of reinsurance is constrained and where markets are potentially incomplete. We use 'General Equilibrium Theory for Incomplete financial markets' to prove that equilibria exist on such a mixed financial–reinsurance market. We show that the existence of constraints on the reinsurance portfolios that can be traded can have an important influence on the structure of the equilibrium prices. More precisely, we show that limited arbitrage possibilities can exist at equilibrium. As a consequence, there does not necessarily exist a risk neutral probability distribution. Furthermore, we study the constrained Pareto optimality of the equilibria.

Key words: Optimal reinsurance; Incomplete markets; Trading constraints; Arbitrage possibilities; Constrained Pareto optimality; Risk neutral probability distribution

1. Introduction

It is well known that in general insurance agents will redistribute the insured risks amongst each other by means of reinsurance contracts. In doing so, they try to *optimize their risk position*, given the prices of reinsurance. Vaguely stated (we will be more precise in the sequel), this means that each of the agents has a certain rule to decide whether he prefers a risk position to another risk position. According to this rule, he will choose his 'optimal' risk position.

Rules which are used quite often to 'choose' between risk positions are based on actuarial calculations. This means that several stochastic characteristics of a risk, such as its mean and variance, are calculated. Prices are then calculated using well-known *actuarial pricing principles*. Given the prices for (re)insurance and the stochastic characteristics of the insured risks, the agent has to decide upon reinsurance.

In De Waegenaere and Delbaen (1992), the authors show how, for stop loss reinsurance contracts, these rules can be refined using conditional expectation and conditional variance of the residual risk.

SSDI 0167-6687(94)00014-6

Expected value and variance are calculated conditional to the information that the agent has about the claim height process at that time. Furthermore, the optimal hedge between residual risk and reinsured risk is calculated. These three criteria allow for a continuous adjustment of the reinsurance portfolio.

A criticism on these kind of rules however, is that they are only based on the stochastic characteristics of the insured risks, and therefore don't keep track of the surrounding market conditions such as the possibilities on the financial markets. It is clear however, that financial markets are very important for insurers and reinsurers, because they provide a means to invest premiums. Therefore, it would be interesting to have a model for optimal reinsurance where these market conditions, as well as the stochastic characteristics of the risks, can play a role in determining redistributions of risks and prices for reinsurance of risks. In such a model, agents will construct a financial portfolio and a risk portfolio ((re)insurances) according to their own preferences. Of course, these 'optimal' choices will depend on the prices of the financial assets and the (re)insurance contracts that can be traded. Now the question is whether prices for financial assets and (re)insurance contracts can be found such that this behaviour of the agents leads to an equilibrium, i.e. the net trade of contracts equals zero.

If we formulate it like this, it becomes clear that the problem fits into the framework of *General Equilibrium Theory*, or shortly G.E. theory. Indeed, G.E. theory is especially concerned with the existence of equilibrium prices and equilibrium allocations of goods, under the assumption that the agents each use a certain rule to determine their optimal position, given the prices of the goods. The word 'good' should be interpreted in a broad sense: it could be an apple as well as a random payment conditional to the occurrence of a certain event, for example a payment if a house burns down.

The idea of applying G.E. theory to (re)insurance markets has already been raised by several other authors. Some very interesting papers on this subject are Borch (1962), Bühlmann (1980, 1984), Gerber (1984), Lienhard (1986), and Pressacco (1979). In Borch (1990), one finds a very clear explanation of how G.E. theory can be applied to the (re)insurance markets. This leads to very nice results about the structure of equilibrium prices for (re)insurance and the Pareto optimality of equilibrium allocations of risk.

It is our aim in this paper to study a general equilibrium model where the trade on reinsurance markets is combined with the trade on financial markets. So we consider a *mixed financial–reinsurance market* with two types of agents, (re)insurers and financial agents. This market is considered to be potentially incomplete (see for instance Magill and Shafer (1991) for a definition of complete and incomplete markets). This means that we allow for the possibility that the trade of reinsurance is constrained to a finite number of standard contracts such as proportional, excess of loss or stop loss contracts. The complete markets case (as in Bühlmann (1980)) where there is a reinsurance contract for every possible risk, and therefore even for risks which are not in any of the agents' portfolios, is a special case of our model. Furthermore, as opposed to the existing equilibrium models for reinsurance, we don't allow (re)insurers to reinsure risk they didn't insure in the first place. So, in our model, we take into account that, as opposed to financial assets, reinsurance contracts for a risk can only be bought by those (re)insurers who insured the risk initially. This yields a model where the trade of reinsurance is constrained by asymmetric constraints. The constraints on portfolio holdings are asymmetric because they depend upon the initial risk portfolio of the agent. The reasons we are interested in this case are the following:

- We believe that the interaction between financial markets and (re)insurance markets is very important. One obvious reason is that, in the absence of financial markets, (re)insurance agents would not be able to invest their premium incomes.
- Since (re)insurance contracts are often standardized to be proportional, excess of loss, stop loss, or combinations of these, it seems reasonable to assume that (re)insurance markets might be incomplete, i.e. not every possible combination of risks can be insured. In this case, as in Gerber (1984), the

reinsurance contracts that can be traded belong to the class of linear combinations of a fixed number of *standard contracts*.
- Recent developments in the theory of incomplete markets have made clear that if the trade on a certain market is constrained, equilibrium prices need not necessarily be discounted expected values with respect to a risk neutral probability distribution (see for instance De Waegenaere (1993)). An important consequence is that equilibrium prices of reinsurance are not necessarily C.A.P.M. prices. So the results in Müller's paper (Müller, 1986) are no longer true in the case of constrained trade. Therefore, it becomes clear that the presence of trading constraints can have drastic effects on the structure of the equilibrium prices.

The model we are going to describe in this paper can be seen as an extension to an incomplete markets framework with trading constraints of models previously presented by Bühlmann (1980, 1984) and Gerber (1984). In Bühlmann (1980, 1984), the author used G.E. theory to determine an optimal redistribution of risks and corresponding prices for reinsurance. Since this idea is fundamental to our model, we will briefly explain it in the next section.

2. Bühlmann's economic premium principles

Insurance clearly takes place in a world of uncertainty. Indeed, premiums are deterministic, but the payoff of claim heights is stochastic. In a two period setting, this uncertainty could be described by the fact that there are a certain number of different states of the world that can occur at a later date, called date one in the sequel. We will denote s for a state of the world and Ω for the set of all possible states of the world. A risk is therefore described by a stochastic variable $X\colon \Omega \to \mathbb{R}$.

Each agent $i \in \{1, 2, \dots, I\}$ has a preference relation $\succcurlyeq^i$ on risks. So $X \succcurlyeq^i Y$ means that agent i prefers risk X to risk Y, or that he is indifferent between the two risks. Following G.E. theory, a criterion to decide upon reinsurance would be such that, given the prices for reinsurance, each agent would reinsure in order to obtain the risk position that maximizes his utility, according to his own preference relation. A question which arises then naturally is whether prices for reinsurance can be found such that these optimization processes lead to market clearing in risks. Such prices are called *equilibrium prices*. The pricing principle leading to these prices is called an *economic premium principle* (as opposed to an actuarial premium principle).

Bühlmann (1980, 1984) used G.E. theory, more specifically the Walrasian equilibrium concept of a pure exchange economy, to obtain an economic premium principle. More precisely, he proves that equilibrium prices for reinsurance exist for arbitrary risk averse *von Neumann–Morgenstern* utility functions. [1] The original risk (before reinsurance) of an agent i is denoted by a stochastic variable $X^i\colon \Omega \to \mathbb{R}$. So $X^i(s)$ denotes the claim height to be paid by the agent if state s occurs at date one. Redistribution of risks goes by means of the trading of reinsurance contracts $Z\colon \Omega \to \mathbb{R}$. By reinsuring $Z\colon \Omega \to \mathbb{R}$, agent i can transform his original risk X^i into a new risk Y (after reinsurance) given by

$$Y = X^i - Z.$$

In Bühlmann (1980, 1984) the price of reinsuring $Z\colon \Omega \to \mathbb{R}$ is considered to be a linear functional of the form

$$\mathscr{P}_\phi[Z] = \int_\Omega Z(s)\phi(s)\,\mathrm{d}P(s).$$

[1] See for instance Debreu (1972).

Here P is a given probability measure on Ω, and ϕ is called the *price density*. Let w^i: $\Omega \to \mathbb{R}$ denote the initial wealth of agent i. Then before reinsurance, the date one wealth of an agent would be the stochastic variable w^i-X^i: $\Omega \to \mathbb{R}$. If the agent buys reinsurance $Z: \Omega \to \mathbb{R}$, then his date one wealth would be the stochastic variable

$$V = w^i - X^i + Z - \mathscr{P}_{\phi}[Z]: \Omega \to \mathbb{R}.$$

Each agent has a von Neumann–Morgenstern utility on date one wealth variables, i.e. there exist utility functions u^i: $\mathbb{R} \to \mathbb{R}_+$ such that for variables V, $W: \Omega \to \mathbb{R}$, one has

$$V \succcurlyeq^i W \Leftrightarrow \mathrm{E}_P\left[u^i(V)\right] \geq \mathrm{E}_P\left[u^i(W)\right].$$

Now the idea of G.E. theory is that each agent will choose reinsurance Z in order to maximize the utility of his date one wealth. An equilibrium price density $\tilde{\phi}$ is a price density such that there exist risks ($\tilde{Y}^1$, $\tilde{Y}^2, \ldots, \tilde{Y}^I$) satisfying

$$\tilde{Y}^i = \underset{Y:\,\Omega \to \mathbb{R}}{\operatorname{argmax}}\left(\mathrm{E}_P\left[u^i\left(w^i - Y - \mathscr{P}_{\tilde{\phi}}[X^i - Y]\right)\right]\right) \quad \text{for} \quad \text{all } i \in \{1, 2, \ldots, I\}, \tag{1a}$$

$$\sum_{i=1}^{I} \tilde{Y}^i = \sum_{i=1}^{I} X^i \quad \text{a.s.} \tag{1b}$$

Eq. (1a) expresses that each agent chooses reinsurance $Z = X^i - \tilde{Y}^i$ in order to obtain a date one wealth which maximizes his utility. Eq. (1b) expresses that these optimal choices must lead to market clearing. The proof of existence of a solution of (1), i.e. of the existence of an equilibrium price density, is established in Bühlmann (1984) for arbitrary risk-averse utility functions u^i, $i \in \{1, 2, \ldots, I\}$. Furthermore in this same paper, a link is made between the equilibrium price density $\tilde{\phi}$ and an exponential premium calculation principle.

Some remarks can be made about condition (1a):

- Markets are complete. Indeed, all possible reinsurance contracts, i.e. all possible random variables Z: $\Omega \to \mathbb{R}$, can be traded. In general however, we see that reinsurance contracts are often standardized to be either proportional, excess of loss, stop loss, or combinations of these.
- There is no constraint on the reinsurance contracts that can be bought by an agent. So, regardless of his initial risk portfolio, the agent can buy reinsurance contracts. In particular, agents are allowed to buy reinsurance contracts for risks they didn't insure in the first place.
- Prices are considered to be expected values with respect to some price density.
- There is essentially only one time period, i.e. prices are paid when risks occur. This implies that one can't take into account that between payment of premiums and occurrence of claims, premiums can be invested, for instance at a fixed interest rate.

For the first remark, an extension of Bühlmann's model can be found in Gerber (1984). In this paper, the author considers redistribution of risk through a finite number of fixed reinsurance contracts Y_1, $Y_2, \ldots, Y_n$. So agents can 'only' reinsure linear combinations of these contracts, i.e. reinsurance of the form

$$Z = \sum_{j=1}^{n} c_j Y_j, \quad \text{with} \qquad c_j \in \mathbb{R}, \quad j \in \{1, 2, \ldots, n\}.$$

This implies that the reinsurance market can be incomplete in this case. Furthermore, Bühlmann's model for a finite state space $\Omega = \{1, 2, \dots, S\}$ is a special case of Gerber's model. Indeed, let there be S contracts, one for each possible state $s \in \{1, 2, \dots, S\}$ at date one, such that

$$Y_s(s) = 1,$$
$$Y_s(t) = 0 \quad \forall t \neq s,$$

then it is clear that one gets Bühlmann's model.

The other remarks hold for both models (the fact that equilibrium prices in Gerber (1984) in the case of a finite state space are discounted expected values with respect to a risk-neutral probability distribution cannot be seen directly from the model, but it is a well-known result in G.E. theory for incomplete markets). Therefore, the aim in this paper is to extend these models to a general equilibrium model for the reinsurance market where

- (re)insurance markets are treated as being (potentially) incomplete markets,
- the trade on (re)insurance markets is constrained by institutional rules such as the fact that reinsurance contracts for a certain risk can only be bought by those agents who insured (part of) the risk initially,
- prices are not necessarily expected values with respect to some price density,
- financial markets are included in the model,
- there are two time periods, at date zero prices for (re)insurance and financial assets are paid, at date one risks occur and assets pay off.

We will proceed in the following way: in Section 3, we will motivate why a general equilibrium model for a mixed financial–reinsurance market should be different from a general equilibrium model for a purely financial market. We will come to the conclusion that the right framework for these mixed markets is the one for incomplete markets with trading constraints. In Section 4, we treat the mixed financial–reinsurance market in detail. So we consider a market where reinsurance contracts as well as financial assets can be traded. We prove that equilibria exist, and we study the structure of the equilibrium prices. In Section 5, we show that the introduction of trading constraints can have drastic effects on the structure of the equilibrium prices. An example will make clear that the no-arbitrage principle can be violated in equilibrium. Therefore there does not necessarily exist a risk-neutral probability distribution. In Section 6, we show that the equilibrium allocations of risk are in some sense *constrained Pareto optimal.*

3. Insurance markets versus financial markets

From the mathematical point of view, there is no difference between a (re)insurance contract and any other financial asset such as for instance equity of a firm. Indeed, both have a deterministic price, and a stochastic payoff at a later date. So they both are fully described by

- a random variable A: $\Omega \to \mathbb{R}$, where Ω is the state space. For each state $s \in \Omega$, $A(s)$ denotes the payoff of the asset or (re)insurance contract at date one if the world is in state s.
- A price $q \in \mathbb{R}$ to be paid at date zero.

The difference lies in the way they are traded. Suppose for example that a certain agent (called agent 1) insures a certain house against fire. Then every insurance agent is allowed to write a reinsurance contract on that house, but agent 1 is the only agent who is allowed to buy such a contract. For financial assets, every agent is allowed to buy every asset written by the other agents. This example indicates that there is a basic difference between the trade of (re)insurance contracts on the one hand and financial assets on the other hand. It makes clear that the trade on (re)insurance markets is constrained by very specific rules which do not apply for arbitrary financial markets.

4. The mixed financial–reinsurance market

In this section, we will give an appropriate general equilibrium model for mixed financial–reinsurance markets. It is a model for incomplete markets with trading constraints. We will prove that equilibrium prices and allocations of risk exist. We show that the structure of the equilibrium prices for the reinsurance contracts is different from the structure of the equilibrium prices in Bühlmann's (1980, 1984) models and Gerber's (1984) model.

We consider a market where $I(\geq 2)$ agents are present. Some of the agents are (re)insurers, indexed by $i \in \mathcal{I} \neq \emptyset$, the others are financial agents, indexed by $i \in \mathcal{F}$. By convention, any agent who is both financial agent and (re)insurer, will be denoted as a (re)insurer $i \in \mathcal{I}$. Therefore we can assume that $\mathcal{I} \cap \mathcal{F} = \emptyset$.

Before any reinsurance took place, each of the (re)insurers $i \in \mathcal{I}$ has a portfolio of risks X_j^i: $\Omega \to \mathbb{R}_+$, $j = 1, 2, \ldots,$ where Ω denotes the state space and is considered to be finite. In the sequel, we will denote $\Omega = \{1, 2, \ldots, S\}$. It is clear that, in deciding how these risks should be redistributed, financial markets can play a very important role. Indeed, they provide a means to invest premiums. Therefore, we consider consider a model where redistribution of risks is combined with the possibility of asset trading.

We denote K for the number of assets that can be traded, and J for the number of risks to be redistributed. The case of redistribution of risk without asset trading, i.e. $K = 0$ (as in Bühlmann (1980, 1984) and Gerber (1984)) is a special case of our model.

Since the state space Ω consists of S states, any stochastic variable (and therefore the payoff of any (re)insurance contract or financial asset) is fully defined by a vector $A = (A_1, A_2, \ldots, A_S)^t \in \mathbb{R}^S$, where for each $s \in \Omega$, A_s denotes the value of the stochastic variable in state s.

Notations

- Let R denote the $S \times J$ matrix of risks to be redistributed. So, for each $s \in \{1, 2, \ldots, S\}$ and for each $j \in \{1, 2, \ldots, J\}$, $R_{sj} \in \mathbb{R}_+$ denotes the claim height to be paid for risk j in state s. We denote γ_j for the price of reinsuring risk j. As in Gerber (1984), the price of reinsuring a fraction $p \in [0, 1]$ of risk j equals $p\gamma_j$. We denote γ for the vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_J)$.
- For each insurer $i \in \mathcal{I}$ and each risk $j \in \{1, 2, \ldots, J\}$, we denote $c_j^i \in [0, 1]$ for the fraction of that risk carried by that insurer (before redistribution takes place). We allow for the possibility that before reinsurance, some of the risks R_j are covered by several agents on a proportional basis, if not, c_j^i would be equal to zero or one for all agents $i \in \{1, 2, \ldots, I\}$ and all contracts $j \in \{1, 2, \ldots, J\}$. Furthermore,
$$\sum_{i \in \mathcal{I}} c_j^i = 1 \quad \text{for} \quad \text{all } j \in \{1, 2, \ldots, J\}.$$
- Let C denote the $S \times K$ matrix of financial assets. So, for each $s \in \{1, 2, \ldots, S\}$ and for each $k \in \{1, 2, \ldots, K\}$, asset k promises the delivery of C_{sk} units of account in state s. We denote q_k for the price of asset k. We denote q for the vector $q = (q_1, q_2, \ldots, q_K)$.
- We denote $A = (C \mid R)$ for the $S \times (K + J)$ matrix where the first K columns are financial assets, and the last J columns are risks.
- A portfolio consists of a column vector $z = (z_1, z_2, \ldots, z_K)^t \in \mathbb{R}^K$ of numbers of assets, and a column vector $v = (v_1, v_2, \ldots, v_J)^t \in \mathbb{R}^J$ of numbers of reinsurance contracts. We will denote $(q, \gamma) = (q_1, q_2, \ldots, q_K, \gamma_1, \gamma_2, \ldots, \gamma_J)$, $(z, v) = (z_1, z_2, \ldots, z_K, v_1, v_2, \ldots, v_J)$, and $\binom{z}{v} = (z_1, z_2, \ldots, z_K, v_1, v_2, \ldots, v_J)^t$.

Definition 4.1. A risk position is a vector $x \in \mathbb{R}^{S+1}$, where

- x_0 is the wealth at date zero,
- for each $s \in \{1, 2, \ldots, S\}$, x_s is the wealth at date one if the world is in state s.

For each vector $x = (x_0, x_1, \dots, x_S)^t \in \mathbb{R}^{S+1}$, we denote $\mathbf{x_1}$ for the date one components, i.e. $\mathbf{x_1} = (x_1, x_2, \dots, x_S)^t$.

The initial risk position of agent $i \in \{1, 2, \dots, I\}$ (before trading of reinsurance and assets) will be denoted w^i.

Through the trade of reinsurance contracts and financial assets, the agents can obtain new risk positions. The problem now is to search for a redistribution of risks and a trade of assets such that each agent $i \in \{1, 2, \dots, I\}$ obtains a risk position that maximizes his utility (according to his personal preference relation $\succcurlyeq^i$ on risk positions).

The trade of (re)insurance is restricted by a very important condition, namely the fact that reinsurance contracts for a risk can only be bought by those agents who insured the risk initially. Furthermore, risks are redistributed amongst insurance agents only. Financial agents are not allowed to trade reinsurance contracts. Therefore, we define in the following definition the set of portfolios $Z^i \subset \mathbb{R}^{K+J}$ that can be traded by agent i, for all $i \in \{1, 2, \dots, I\}$.

Definition 4.2. We define the trade set of an insurance agent $i \in \mathscr{I}$ as follows:

$$Z^i = \mathbb{R}^K \times \prod_{j=1}^{J} \left] -\infty, c_j^i \right].$$

We define the trade set of an financial agent $i \in \mathscr{F}$ as follows:

$$Z^i = \mathbb{R}^K \times \{0\}^J.$$

The interpretation is as follows:
- $z \in \mathbb{R}^K$ implies that agent i is allowed to sell and buy assets without restrictions.
- For insurers, $v_j \le c_j^i$ for $j \in \{1, 2, \dots, J\}$ implies that agent i is allowed to reinsure part of (and maximum all of) the risk that he initially insured. For financial agents, $v_j = 0$ implies that they are not allowed to trade reinsurance.

As stated before, agents will trade in order to maximize their utility over the set of risk positions that they can obtain by means of an allowed trade of assets and reinsurance. With the previous notations, we see that this set equals

$$B^i(q, \gamma, A, Z^i) = \left\{ x \in X^i \mid \exists (z, v) \in Z^i : \begin{array}{l} x_0 = w_0^i - qz - \gamma v \\ x_s = w_s^i + \sum_{k=1}^{K} C_{sk} z_k + \sum_{j=1}^{J} R_{sj} v_j, \quad s \in \{1, 2, \dots, S\} \end{array} \right\}$$

$$= \left\{ x \in X^i \mid \exists (z, v) \in Z^i : x = w^i + \begin{pmatrix} -(q, \gamma) \\ A \end{pmatrix} \begin{pmatrix} z \\ v \end{pmatrix} \right\},$$

where X^i denotes the set of risk positions from which agent i wants to choose an optimal one.

This finally allows us to define equilibrium prices and allocations for mixed financial-reinsurance markets:

Definition 4.3. A system of reinsurance prices $(\gamma_1, \gamma_2, \dots, \gamma_J) \in \mathbb{R}^J$ and asset prices $(q_1, q_2, \dots, q_K) \in \mathbb{R}^K$ is an equilibrium price system if and only if there exist risk positions $\bar{x}^i \in X^i$, $i \in \{1, 2, \dots, I\}$, and portfolios $(\bar{z}^i, \bar{v}^i) \in Z^i$, $i \in \{1, 2, \dots, I\}$, satisfying the following conditions:

$$\bar{x}^i = w^i + \begin{pmatrix} -(q, \gamma) \\ A \end{pmatrix} \begin{pmatrix} \bar{z}^i \\ \bar{v}^i \end{pmatrix} \in B^i(q, \gamma, A, Z^i), \tag{2a}$$

$$\bar{x}^i \succcurlyeq^i y \quad \text{for} \quad \text{all } y \in B^i(q, \gamma, A, Z^i), \tag{2b}$$

$$\sum_{i=1}^{I} \bar{z}^i = 0, \quad \sum_{i=1}^{I} \bar{v}^i = 0. \tag{2c}$$

The corresponding allocation $\{\bar{x}^i \colon i \in \{1, 2, \ldots, I\}\}$ will be called an equilibrium allocation.

Assumptions $\tilde{A}$

$\tilde{A}_1$. The preference relations of the agents are continuous, strictly monotone and convex (see for instance Debreu (1972) or Hildenbrand and Kirman (1988)).

$\tilde{A}_2$. There is no redundancy in the financial assets C_k, $k \in \{1, 2, \ldots, K\}$, i.e. $\operatorname{rank}(C) = K$.

$\tilde{A}_3$. $\langle C \rangle \cap \langle R \rangle = \{0\}$.

$\tilde{A}_4$. For all $j \in \{1, 2, \ldots, J\}$, $R_j \in \mathbb{R}_+^S \setminus \{0\}$.

$\tilde{A}_5$. For all $i \in \{1, 2, \ldots, I\}$, X^i is bounded from below, closed and convex, and $w^i \in \operatorname{int}(X^i)$.

Remarks. (i) It is clear that assumption $\tilde{A}_2$ can be made without loss of generality. Indeed, since there are no constraints on the trade of financial assets, the problem can always be written such that $\tilde{A}_2$ is satisfied.
(ii) Assumption $\tilde{A}_3$ says that there is no financial portfolio that exactly duplicates a reinsurance portfolio. This is very reasonable because the set of states s which influence the payoff of reinsurance (fire, accidents, ...) is different from the set of states that influence the payoff of financial assets (politics, economics, ...).

The aim now is to prove that, under these assumptions, equilibria exist, i.e. problem (2) has a solution. In the sequel, we will denote $AS(Z)$ for the asymptotic cone of a set $Z \subset \mathbb{R}^{K+J}$, and $\operatorname{Ker}(A)$ for the null space of the matrix A, so for each $i \in \{1, 2, \ldots, I\}$, we have

$$AS(Z^i) = \left\{ \begin{pmatrix} z \\ v \end{pmatrix}^t \in Z^i \,|\, \text{for all } t \in \mathbb{R}_+ \colon t \begin{pmatrix} z \\ v \end{pmatrix}^t \in Z^i \right\},$$

and

$$\operatorname{Ker}(A) = \left\{ \begin{pmatrix} z \\ v \end{pmatrix} \in \mathbb{R}^{K+J} \,|\, A \begin{pmatrix} z \\ v \end{pmatrix} = 0 \right\}.$$

Lemma 4.1. *By Definition 4.2, we have for all $i \in \{1, 2, \ldots, I\}$:*
(1) Z^i is a closed and convex subset of $\mathbb{R}^{K+J}$,
(2) $0 \in Z^i$.
Furthermore,
(3) $0 \in \operatorname{int}(\sum_{i=1}^{I} Z^i)$.
If assumptions $\tilde{A}_2$, $\tilde{A}_3$, $\tilde{A}_4$ are satisfied, we have
(4) $\operatorname{Ker}(A) \cap AS(Z^i) = \{0\}$ for all $i \in \{1, 2, \ldots, I\}$.

Proof. (1) is clear.
(2) and (3) follow from the fact that for all $j \in \{1, 2, \ldots, J\}$, we have

$$\sum_{i \in \mathscr{I}} c_j^i = 1,$$

$$c_j^i \in [0,1] \quad \text{for all } i \in \mathscr{I}.$$

We will now show that (4) is satisfied. For each insurer $i \in \mathscr{I}$ we have

$$AS(Z^i) = \mathbb{R}^K \times \mathbb{R}^J_- .$$

Now suppose that $\binom{z}{v}^t \in \operatorname{Ker}(A) \cap AS(Z^i)$. By assumption $\tilde{A}_3$, this implies that $Cz = 0$ and $Rv = 0$. From assumption $\tilde{A}_2$ it then follows that $z = 0$. Now since $v_j \leq 0$ for all $j \in \{1, 2, \dots, J\}$ it follows from $\tilde{A}_4$ that $R_{sj} v_j = 0$ for all states $s \in \{1, 2, \dots, S\}$ and for all contracts $j \in \{1, 2, \dots, J\}$. Now $\tilde{A}_4$ implies that for each contract $j \in \{1, 2, \dots, J\}$, there is at least one state $s \in \{1, 2, \dots, S\}$ such that $R_{sj} > 0$. Therefore, it follows that $v = 0$.

For financial agents $i \in \mathscr{F}$, the idea is analogous.

Theorem 4.1. *Under assumptions $\tilde{A}$, the mixed financial–reinsurance market can reach equilibrium prices and equilibrium allocations.*

Proof. Trivial consequence of Lemma 4.1 and Theorem 2.4.1 in De Waegenaere (1993).

5. Properties of equilibrium prices

In Section 4 we proved that, under certain (rather weak) conditions on the structure of the contracts, equilibria exist. A very interesting question is whether a relation can be found between the equilibrium prices and C.A.P.M. pricing, as in the case of Müller's paper (Müller (1986)).

It is well known that for incomplete markets without trading constraints ($Z^i = \mathbb{R}^L$, for all $i \in \{1, 2, \dots, I\}$, where L denotes the number of contracts), equilibrium prices are *arbitrage free*. This means that it is impossible that for a matrix of asset returns $A \in \mathbb{R}^{S \times L}$, there exist equilibrium asset prices $(q_1, q_2, \dots, q_L) \in \mathbb{R}^L$, such that there exists a portfolio $z \in \mathbb{R}^L$ satisfying

$$\begin{pmatrix} -q \\ A \end{pmatrix} z > 0,$$

i.e. by buying this portfolio, the agent can only gain, because the price of the portfolio is negative ($qz \leq 0$), and the payoff of the portfolio is positive in each state ($(Az)_s \geq 0$, $s \in \{1, 2, \dots, S\}$), with at least one strict inequality.

If prices are arbitrage free, we know that there exists a risk-neutral probability distribution such that equilibrium asset prices are the (discounted) expected value of the asset payoff with respect to this probability distribution. Therefore, in unconstrained markets, equilibrium prices are C.A.P.M. prices. In the next example however, we show that the mixed financial–reinsurance markets we studied in Section 4 may allow for limited arbitrage possibilities at equilibrium. As a consequence, there does not necessarily exist a risk-neutral probability distribution!

Example. We consider an economy with two insurance agents, three assets (a riskless bond and two reinsurance contracts, so $J = 2$ and $K = 1$), and four possible states at date one. The matrix of returns $A = (C \mid R) \in \mathbb{R}^{4 \times (1+2)}$ is given by

$$A = (C \mid R) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

We consider the case of proportional redistribution of insured risks. Before redistribution, there are two risks (column 2 and 3) which are each insured for 50% by each of the agents. So they can each buy reinsurance for a maximum of 50% of the risk. There is no constraint on the trade of the riskless bond. Therefore, the trade sets of the agents are given by

$$Z^i = \mathbb{R} \times \left] -\infty, \tfrac{1}{2} \right]^2, \quad i = 1, 2.$$

The initial risk position of agent 1 is equal to

$$\left(w_0^1, w_1^1, w_2^1, w_3^1, w_4^1\right) = (5.5, 5, 4.5, 4, 4).$$

The initial risk position of agent 2 is equal to

$$\left(w_0^2, w_1^2, w_2^2, w_3^2, w_4^2\right) = (11.5, 12, 12.5, 13, 13).$$

The utility functions of the agents are of the form:

$$u^1(x) = a\sqrt{x_0} + \sqrt{x_1} + \sqrt{x_2} + b\sqrt{\frac{x_3 + x_4}{2}},$$

$$u^2(x) = \sqrt{x_0} + c\sqrt{x_1} + d\left(\sqrt{x_2} + \sqrt{\frac{x_3 + x_4}{2}}\right).$$

So the problem we have to solve is whether there exist prices q for the bond, and γ_1, γ_2 for the reinsurance contracts such that there exist risk positions $\bar{x}^i \in B^i(q, \gamma, A, Z^i)$ satisfying

$$\bar{x}^i \in \underset{x \in B^i(q, \gamma, A, Z^i)}{\operatorname{argmax}} \; u^i(x), \tag{3a}$$

$$\bar{x}^1 + \bar{x}^2 = w^1 + w^2, \tag{3b}$$

where

$$B^i(q, \gamma, A, Z^i) = \left\{ x \in \mathbb{R}_+^5 \mid \exists (z, v) \in Z^i : \begin{array}{l} x_0 = w_0^i - qz - \gamma_1 v_1 - \gamma_2 v_2 \\ x_s = w_s^i + \left(A \begin{pmatrix} z \\ v \end{pmatrix} \right)_s, \quad s = 1, 2, 3, 4 \end{array} \right\}.$$

Since $\operatorname{rank}(A) = 3$, we know that for each (q, γ_1, γ_2), there exists a vector $\pi_1 = (\pi_1, \pi_2, \pi_3, \pi_4) \in \mathbb{R}^4$, such that $(q, \gamma_1, \gamma_2) = \pi_1 A$. Therefore problem (3) is equivalent to the following problem: do there exist state prices $\pi_s \in \mathbb{R}$, $s = 1, 2, 3, 4$, such that there exist $\bar{x}^i \in \mathscr{B}^i(\pi, A, Z^i)$ satisfying

$$\bar{x}^i \in \underset{x \in \mathscr{B}^i(\pi, A, Z^i)}{\operatorname{argmax}} \; u^i(x)$$

and

$$\bar{x}^1 + \bar{x}^2 = w^1 + w^2,$$

where $\pi_0 = 1$, $\pi = (\pi_0, \pi_1) \in \mathbb{R}^5$ and for $i = 1, 2$,

$$\mathscr{B}^i(\pi, A, Z^i) = \left\{ x \in \mathbb{R}_+^5 \,\middle|\, \begin{array}{c} \pi x = \pi w^i \\ x_2 - x_1 \le w_2^i - w_1^i + \frac{1}{2} \\ x_2 - x_3 \le w_2^i - w_3^i + \frac{1}{2} \\ x_3 - x_4 = w_3^i - w_4^i \end{array} \right\}.$$

Using the technique of Lagrange multipliers (see for instance Luenberger (1973)), we see quite easily that for $a = 8$, $b = 4$, $c = 8$, $d = 2$, $\pi = (1, 2, -1, 1, 1)$ is an equilibrium price system with corresponding equilibrium allocations $\bar{x}^1 = (16, 1, 1, 1, 1)$ and $\bar{x}^2 = (1, 16, 16, 16, 16)$.

Furthermore, it is clear that $\binom{z}{v}=(-\frac{1}{4},\frac{1}{4},\frac{1}{4})^t(\in Z^1\cap Z^2)$ is an arbitrage possibility. Indeed, $q=3$, $\gamma_1=1$, $\gamma_2=1$ and

$$\begin{pmatrix} -(q,\gamma) \\ A \end{pmatrix}\begin{pmatrix} z \\ v \end{pmatrix}=\begin{pmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix}>0.$$

So in this example, we clearly have an equilibrium price system which violates the no arbitrage principle and therefore, there does not exist a risk-neutral probability. So equilibrium reinsurance prices are not necessarily C.A.P.M. prices.

In the next theorem, we show that for the equilibrium prices of the financial assets in the mixed financial–reinsurance market, there still exists a risk-neutral probability such that equilibrium prices are discounted expected values with respect to this probability distribution. The equilibrium prices for reinsurance contracts however, are bounded above by the discounted expected value of their payoff with respect to the same probability distribution. First we need a lemma.

Lemma 5.1. *A system of inequalities*

$$\begin{cases} f_k(z)\geq 0 & k\in\{1,2,\dots,K\} \\ g_l(z)\geq 0 & l\in\{1,2,\dots,L\}, \end{cases}$$

has a solution z that satisfies $f_k(z)\neq 0$ for some $k\in\{1,2,\dots,K\}$, if and only if there does not exist $(\pi,\lambda)\in\mathbb{R}^K_{++}\times\mathbb{R}^L_+$ such that

$$\sum_{k=1}^K \pi_k f_k+\sum_{l=1}^L \lambda_l g_l=0.$$

Proof. Is a slight modification of a proof by Fan (1956).

Theorem 5.1. *For all equilibrium prices $(q,\gamma)\in\mathbb{R}^{K+J}$, there exists a vector $\pi\in\mathbb{R}^S_{++}$ such that*:

$$q_k=(\pi C)_k \quad \forall k\in\{1,2,\dots,K\},$$
$$\gamma_j\leq(\pi R)_j \quad \forall j\in\{1,2,\dots,J\}.$$

Proof. Clearly for each insurer $i\in\mathcal{I}$ we have

$$AS(Z^i)=\mathbb{R}^K\times\mathbb{R}^J_-,$$

and for each financial agent $i\in\mathcal{F}$, we have

$$AS(Z^i)=\mathbb{R}^K\times\{0\}^J.$$

Now we know that (see for instance De Waegenaere (1993)) the set of equilibrium prices is a subset of the set

$$Q=\left\{(q,\gamma)\in\mathbb{R}^{K+J}\,|\,\forall\binom{z}{v}\in\bigcup_{i=1}^I AS(Z^i):\begin{pmatrix} -(q,\gamma) \\ A \end{pmatrix}\begin{pmatrix} z \\ v \end{pmatrix}\notin\mathbb{R}^{S+1}_+\setminus\{0\}\right\}.$$

Now we define two systems of inequalities

$$\begin{cases} f_0(z, v) = -(q, \gamma)\binom{z}{v} \geq 0 \\ f_s(z, v) = \left(A\binom{z}{v}\right)_s \geq 0 \quad \forall s \in \{1, 2, \dots, S\} \\ g_j(z, v) = -v_j \geq 0 \quad \forall j \in \{1, 2, \dots, J\} \end{cases} \tag{S$_1$}$$

and

$$\begin{cases} f_0(z, v) = -(q, \gamma)\binom{z}{v} \geq 0 \\ f_s(z, v) = \left(A\binom{z}{v}\right)_s \geq 0 \quad \forall s \in \{1, 2, \dots, S\} \\ g_j(z, v) = v_j = 0 \quad \forall j \in \{1, 2, \dots, J\}. \end{cases} \tag{S$_2$}$$

Then $(q, \gamma) \notin Q$ if and only if one of the systems of inequalities S_1 or S_2 has a solution satisfying

$$\exists s \in \{0, 1, \dots, S\}: f_s(z, v) \neq 0. \tag{4}$$

But clearly, this is equivalent to the statement that S_1 has a solution satisfying (4). By Lemma 5.1, this is equivalent to the statement that there does not exist a vector $(\pi, \lambda) = (\pi_0, \pi_1, \dots, \pi_S, \lambda_1, \lambda_2, \dots, \lambda_J) \in \mathbb{R}^{S+1}_{++} \times \mathbb{R}^J_+$ with

$$\begin{aligned} &\pi_0 q_k = (\pi_1 C)_k \quad \forall k \in \{1, 2, \dots, K\}, \\ &\pi_0 \gamma_j = (\pi_1 R)_j - \lambda_j \quad \forall j \in \{1, 2, \dots, J\}. \end{aligned} \tag{5}$$

So, $(q, \gamma) \in Q$ if and only if there exists a vector $\pi \in \mathbb{R}^S_{++}$ satisfying

$$\begin{aligned} &q_k = (\pi C)_k \quad \forall k \in \{1, 2, \dots, K\}, \\ &\gamma_j \leq (\pi R)_j \quad \forall j \in \{1, 2, \dots, J\}. \end{aligned}$$

Since equilibrium prices are in Q, this concludes the proof.

So if there is a riskless bond available on the financial market, i.e. one of the columns of C equals $(1, 1, \dots, 1)^t$, then the equilibrium prices of assets and risks satisfy

$$\begin{aligned} &q_k = \tilde{q} E_P[C_k] \quad \forall k \in \{1, 2, \dots, K\}, \\ &\gamma_j \leq \tilde{q} E_P[R_j] \quad \forall j \in \{1, 2, \dots, J\}, \end{aligned} \tag{6}$$

where $\tilde{q} = \pi_1(1, 1, \dots, 1)^t = \sum_{s=1}^{S} \pi_s$ denotes the price of the riskless bond, and E_P denotes the expected value of the corresponding stochastic variable with respect to the probability measure P on the state space Ω, given by

$$P(\{s\}) = \frac{\pi_s}{\sum_{t=1}^{S} \pi_t}, \quad s \in \{1, 2, \dots, S\}.$$

The example at the beginning of this section makes clear that it is possible that for some equilibrium prices γ of reinsurance, there does not exist a probability distribution that gives an equality in (6), instead of an inequality.

6. Constrained Pareto optimality

It is easy to see that the equilibrium allocations for the mixed financial–reinsurance markets defined in Section 4 are *constrained Pareto optimal* in the sense that there does not exist another redistribution of risk and assets satisfying the constraints defined by the sets Z^i such that every agent is better off, and at least one agent is strictly better of. Formally, this means that we define the set of *feasible allocations* as follows:

$$\mathscr{F} := \left\{ \left(x^i, \begin{pmatrix} z^i \\ v^i \end{pmatrix} \right)^t_{i=1,\ldots,I} \in \prod_{i=1}^{I} (X^i \times Z^i) \;\middle|\; \begin{array}{l} x_1^i = w_1^i + A\begin{pmatrix} z^i \\ v^i \end{pmatrix} \quad \forall i \in \{1, 2, \ldots, I\} \\ \sum_{i=1}^{I} z^i = 0, \quad \sum_{i=1}^{I} v^i = 0, \quad \sum_{i=1}^{I} x_0^i = \sum_{i=1}^{I} w_0^i \end{array} \right\}.$$

Then for equilibrium allocations $\{(\bar{x}^i, (\begin{smallmatrix}\bar{z}^i \\ \bar{v}^i\end{smallmatrix})^t) : i \in \{1, 2, \ldots, I\}\} \in \Pi_{i=1}^{I}(X^i \times Z^i)$ there do not exist new allocations $\{(x^i, (\begin{smallmatrix}z^i \\ v^i\end{smallmatrix})) : i \in \{1, 2, \ldots, I\}\}$ such that

$$\begin{cases} \left(\left(x^i, \begin{pmatrix} z^i \\ v^i \end{pmatrix} \right)^t, \quad i \in \{1, 2, \ldots, I\} \right) \in \mathscr{F} \\ \forall i \in \{1, 2, \ldots, I\} : u^i(x^i) \geq u^i(\bar{x}^i) \\ \exists i \in \{1, 2, \ldots, I\} : u^i(x^i) > u^i(\bar{x}^i). \end{cases}$$

The proof can be found in De Waegenaere (1993).

Furthermore, we would like to remark that, even if we would consider the case of *von Neumann–Morgenstern* utilities, we would not get the same characterization of Pareto optimal exchanges as in the models of Bühlmann (1980, 1984) or Gerber (1984). Clearly, the reason for this difference is that in these models the only constraint that matters in finding Pareto optimal allocations is the market clearing constraint. In our model however, each of the agents faces his own trading constraints, defined in the sets Z^i, $i \in \{1, 2, \ldots, I\}$. Therefore, it is clear that one cannot expect to get the same result.

7. Concluding remarks

The main issue in this paper was to show that the mixed financial–reinsurance markets can reach an equilibrium, but that the structure of the equilibrium prices can be drastically influenced by the existence of trading constraints. Indeed, if the trade of reinsurance contracts is constrained by the fact that reinsurance for a risk can only be bought by those agents who insured the risk initially, then limited arbitrage possibilities may exist at equilibrium. As a consequence, these equilibrium prices cannot be considered as being the discounted expected value of the payoffs with respect to some probability measure on the state space Ω. Equilibrium prices therefore have a different structure than in the models of Bühlmann (1980, 1984) and Gerber (1984).

Finally, I would like to remark that it is not essential that we restrict ourselves in Section 4 to the trade of reinsurance. The results remain valid if one includes the insurance market. It was only for notational convenience that we restricted to reinsurance. The restriction to two time periods is also not essential, and only made for notational convenience. We could as well consider assets and reinsurance contracts paying off at different times in the future. We would only have to redefine the state space.

References

Bowers, N., H. Gerber, J. Hickman, D. Jones and C. Nesbitt (1986). *Actuarial Mathematics*. The Society of Actuaries, Ikasca, IL.

Borch, K. (1962). Equilibrium in a reinsurance market. *Econometrica* 30, 424–444.

Borch, K. (1986). Risk theory and serendipity, *Insurance: Mathematics and Economics* 5, 103–112.

Borch,K. (1990). *Economics of Insurance*. North-Holland, Amsterdam.

Bühlmann, H. (1980). An economic premium principle. *Astin Bulletin* 11, 52–60.

Bühlmann, H. (1984). The general economic premium principle. *Astin Bulletin* 14, no 1, 13–21.

Debreu, G. (1972). *Theory of value*, Cowles Foundation Monograph 17. Yale University Press, New Haven, CT.

De Waegenaere, A. (1993). Incomplete markets with trading constraints. Ph.D. thesis, University of Antwerp (U.I.A.), Antwerp.

De Waegenaere, A. and Delbaen F. (1992). A dynamic reinsurance theory. *Insurance: Mathematics and Economics* 11, no 1, 31–48.

Fan, K. (1956). On systems of linear inequalities. In: H. Kuhn and A. Tucker, eds., *Linear inequalities and related systems*. Annals of mathematical studies. Princeton University Press, Princeton, NJ.

Gerber, H.U. (1984). Equilibria in a proportional reinsurance market. *Insurance: Mathematics and Economics* 3, 97–100.

Goovaerts, M.J., F. De Vylder and J. Haezendonck (1984). Insurance premiums. North-Holland, Amsterdam.

Gollier, Ch. and H. Schlesinger (1992). Second-best insurance contract design in an incomplete market. Discussion paper 9212. CORE, Louvain-la-Neuve.

Hildenbrand, W. and A. Kirman (1988). *Equilibrium Analysis*, 2nd Edition. North-Holland, Amsterdam.

Lienhard, (1986). Calculation of price equilibria for utility functions of the HARA class. *Astin Bulletin* 17, 591–597.

Luenberger, D. (1973). *Introduction to Linear and Non-linear Programming*. Addison-Wesley, Reading, MA.

Magill, M. and W. Shafer (1991). Incomplete Markets., In: W. Hildenbrand and H. Sonnenschein, eds., *Handbook of Mathematical Economics*, Vol. IV. North-Holland, Amsterdam.

Müller, H.M. (1986). Economic premium principles in insurance and the Capital Asset Pricing Model. *Astin Bulletin* 17, no. 2, 141–150.

Pressacco, F. (1979). Value and prices in a reinsurance market. *Astin Bulletin* 10, 263–273.