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Realization and Partial Fractions

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ABSTRACT

We discuss the relation between two intrinsically different proposals that have been made in the literature concerning the representation by constant matrices of rational matrices given in fractional form. It turns out that the relation is most naturally studied in the framework of partial-fraction decompositions. We develop the realization theory for decompositions with respect to arbitrary complementary parts of the extended complex plane which may, for instance, correspond to stability and instability. An isomorphism is obtained which connects the spaces used in the two methods, and several identities relating to the McMillan degree are derived in a direct way. Finally, a new computational procedure is given to obtain the partial-fraction decomposition of a rational matrix given in fractional form.

1. INTRODUCTION

The representation of rational matrices through a tuple of constant matrices has proved to be a powerful tool in a large variety of problems that

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involve rational matrices; see for instance [2, 9]. Certainly the most popular form has been the *standard state-space realization*, which represents the rational matrix $G(s)$ by a four-tuple of constant matrices (A, B, C, D) through the formula $G(s) = C(sI - A)^{-1}B + D$. However, only rational matrices without poles at infinity can be represented in this way. Although the restriction to proper rational matrices is often justified in applications, it is of interest to look for alternative representations that cover the whole class of rational matrices. An example of such a representation is the *descriptor form*, which specifies a general rational matrix $G(s)$ by a five-tuple (E, A, B, C, D) through the formula $G(s) = C(sE - A)^{-1}B + D$.

It was shown by Fuhrmann (see for instance [11, §I.10]) how to obtain a natural standard state-space realization for a rational matrix that is given in *fractional form*, that is, as a quotient of polynomial matrices $[G(s) = D^{-1}(s)N(s)$, where $N(s)$ and $D(s)$ are polynomial matrices, and $D(s)$ is nonsingular]. The Fuhrmann realization serves as an intermediary between state-space techniques on the one hand and techniques based on the fractional form on the other hand. The theory as given by Fuhrmann only applies to proper rational matrices, and there is a natural question whether his methods can be adapted to cover nonproper rational matrices as well. Solutions to this problem were provided by Conte and Perdon [5] and by Wimmer [27]. In both these papers, a decomposition is made of the given rational matrix into a strictly proper rational part and a polynomial part. The standard method is used for the strictly proper part, and a suitably adapted version of the same method is employed for the polynomial part. Finally, the two parts are added to obtain a representation in descriptor form of the given matrix.

A different approach was taken by the present authors in [17]. Instead of going to the descriptor form directly, we used an alternative representation of rational matrices by means of constant matrices, which we termed the *pencil form*. (In fact, in the cited paper we were concerned with the representation of "behaviors," which involves a bit more; however, in this paper we shall only consider the representation of rational matrices.) The constant matrices appearing in the pencil form are constructed from mappings between various spaces that are defined in a way that is inspired by the Fuhrmann realization. However, the assumption of properness is no longer needed. Once the pencil form is obtained, it is possible to obtain a representation in descriptor form by a simple procedure, as was also shown in [17].

It is natural to ask what the relation is between the two realization procedures, of which one is based on a decomposition into a proper part and a polynomial part, and the other on the pencil form, which is in a certain sense a "homogeneous" representation. To answer this question, it turns out to be useful to take a slightly generalized viewpoint. Note that the decomposition of a rational matrix into a proper part and a polynomial part is a special case of a (two-term) partial-fraction expansion, that is, an additive decomposition into two parts that have poles in two prescribed complementary regions of the extended complex plane. It turns out that the Fuhrmann realization theory can be reformulated easily as a method for providing a state-space description of one term in this decomposition when the rational matrix is given in a suitably adapted fractional form. Special cases are both the standard realization method and the adaptation of it that was used by Conte and Perdon and by Wimmer to obtain a representation of polynomial matrices. The realization via the pencil form can also be performed at this level of generality, and the connection with the realization via the partial-fraction expansion can be made. The virtue of looking at the problem from this point of view is that no point in the extended complex plane is assigned any special role *a priori*, and that also situations are covered in which both prescribed regions in the plane contain more than one point.

The plan of this paper is as follows. In the next section, we shall introduce some notation. Then, we shall present the generalized version of Fuhrmann's realization that was alluded to above. In Section 4, we shall generalize the pencil-form realization in the same way. After that, we construct the isomorphism between the spaces that are used in the two realization methods. The relation with the McMillan degree is worked out in Section 6, and some computational issues are discussed in Section 7. The final Section 8 contains some additional remarks and conclusions.

2. NOTATION AND PRELIMINARIES

The extended complex plane $\mathbb{C}^e$ is the set $\mathbb{C}$ of complex numbers together with the point at infinity. It will be assumed throughout this paper that two nonempty subsets Γ_+ and Γ_- of $\mathbb{C}^e$ have been given which satisfy

$$\Gamma_+ \cap \Gamma_- = \varnothing, \qquad \Gamma_+ \cup \Gamma_- = \mathbb{C}^e. \tag{2.1}$$

We shall take the complex numbers as the basic field in this paper, but everything done below can also be done over the real numbers if it is additionally assumed that Γ_+ and Γ_- are symmetric with respect to the real axis, and that their intersections with the extended real axis are both nonempty.

The ring of rational functions that have no poles in Γ_+ $[\Gamma_-]$ will be denoted by $\mathbb{C}_{0+}(s)$ $[\mathbb{C}_{0-}(s)]$. Elements of $\mathbb{C}_{0+}(s)$ $[\mathbb{C}_{0-}(s)]$ will sometimes also be called *plus functions* [*minus functions*]. If $\Gamma_+ = \mathbb{C}$ and $\Gamma_- = \{\infty\}$, then the plus functions are the polynomials and the minus functions are the proper rational functions. This is the special case which corresponds to the "standard" realization theory.

Every rational function can be written as the sum of a plus function and a minus function, but not uniquely so, because the intersection of $\mathbb{C}_{0+}(s)$ and $\mathbb{C}_{0-}(s)$ consists of all constant functions. To get a unique decomposition, we introduce the following device. Fix a point α_+ in Γ_+ and a point α_- in Γ_-. (If one wants to work over the reals, let both α_+ and α_- be real.) Define

$$\mathbb{C}_+(s) = \{f \in \mathbb{C}_{0+}(s) \,|\, f(\alpha_+) = 0\}, \tag{2.2}$$

$$\mathbb{C}_-(s) = \{f \in \mathbb{C}_{0-}(s) \,|\, f(\alpha_-) = 0\}. \tag{2.3}$$

We now have:

LEMMA 2.1. *Every rational function $f(s)$ can be decomposed in a unique way as*

$$f(s) = f_-(s) + f_0 + f_+(s) \tag{2.4}$$

where

$$f_-(s) \in \mathbb{C}_-(s), \qquad f_+(s) \in \mathbb{C}_+(s), \tag{2.5}$$

and f_0 is a constant.

The associated projections (in obvious notation) will be denoted by π_-, π_0, and π_+, respectively. It follows from the uniqueness of the decomposition that these are indeed linear mappings. We shall also use

$$\pi_{0+} = \pi_0 + \pi_+, \qquad \pi_{0-} = \pi_0 + \pi_-. \tag{2.6}$$

We shall assume that a rational function has been fixed which has one zero, at α_+, and one pole, at α_-. Since multiplication by such a function can change a minus function to a plus function, we shall denote the chosen function by $\chi(s)$ after the mythological ferryman Χάρων. Of course, $\chi(s)$ can also be seen as a Möbius transformation. Some examples of situations we have in mind are:

(1) $\Gamma_+ = \mathbb{C}$, $\Gamma_- = \{\infty\}$, $\alpha_+ = 0$, $\alpha_- = \infty$, $\chi(s) = s$.
(2) $\Gamma_+ = \{\infty\}$, $\Gamma_- = \mathbb{C}$, $\alpha_+ = \infty$, $\alpha_- = 0$, $\chi(s) = s^{-1}$.
(3) $\Gamma_+ = \{s \in \mathbb{C} \,|\, |s| < 1\}$, $\Gamma_- = \{s \in \mathbb{C} \,|\, |s| \geqslant 1\}$, $\alpha_+ = 0$, $\alpha_- = \infty$, $\chi(s) = s$.
(4) $\Gamma_+ = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s < 0\}$, $\Gamma_- = \{s \in \mathbb{C} \,|\, \mathrm{Re}\, s \geqslant 0\}$, $\alpha_+ = -1$, $\alpha_- = 1$, $\chi(s) = (s+1)/(s-1)$.

We note the following simple rules, in which we use χ to denote the operation of multiplication by $\chi(s)$:

$$\chi\pi_{0+} = \pi_{+}\chi, \tag{2.7}$$

$$\pi_{0-}\chi = \chi\pi_{-}. \tag{2.8}$$

More generally, we shall be in the habit of writing R for the operation of multiplication by the rational matrix $R(s)$, leaving it to the context to determine between which spaces the mapping R acts. With any rational matrix $R(s)$ of size $p \times q$ we associate the following vector spaces over $\mathbb{C}$ (cf. for instance [7]):

$$X_{-}(R) = \{f(s) \in \mathbb{C}^{p}_{0+}(s) | \exists g(s) \in \mathbb{C}^{q}_{-}(s) : f(s) = R(s)g(s)\} \tag{2.9}$$

and its twin

$$X_{+}(R) = \{f(s) \in \mathbb{C}^{p}_{0-}(s) | \exists g(s) \in \mathbb{C}^{q}_{+}(s) : f(s) = R(s)g(s)\}. \tag{2.10}$$

Spaces of this type will serve as state spaces for the various realizations we shall consider.

Any rational matrix that doesn't have a pole at α_{-} can be represented, in what might be called "modified state-space form," by the formula

$$G(s) = C[\chi(s)I - A]^{-1}B + D. \tag{2.11}$$

Realizations in this modified form are being used in system theory and operator theory for a variety of purposes. Use of $\chi(s) = s + \alpha$ [or $\chi(s) = \alpha s$] makes it possible to find a representation in which the matrix A has all its eigenvalues in the left half plane (or in the unit circle); such a modification was in fact used in the very first appearance of what is now known as the Fuhrmann realization [10]. If the rational matrix $G(s)$ is square and nonsingular, it can be ensured by a suitable choice of $\chi(s)$ that the matrix D in the above representation is nonsingular, which is sometimes useful (cf. [2]). The use of the Möbius transformation $\chi(s) = (s-1)/(s+1)$ to relate discrete-time systems to continuous-time systems is standard. *Delta-operator realizations* for discrete-time systems, which are modified forms with $\chi(s) = s - 1$, are strongly advocated in the recent book [22]; one of the advantages of the use of this modified form is the reduction of roundoff noise in the finite-wordlength implementation of digital controllers (see also [18]).

3. REALIZATION BY PARTIAL FRACTIONS

In this section we shall show how the Fuhrmann realization procedure can be extended to our present setting. Let a rational matrix $G(s)$ of size $p \times m$ be given in the form

$$G(s) = D_{+}^{-1}(s) N_{+}(s), \tag{3.1}$$

where $D_{+}(s)$ and $N_{+}(s)$ are both matrices over $\mathbb{C}_{0+}(s)$, and $D_{+}(s)$ is nonsingular. Since the quotient field of $\mathbb{C}_{0+}(s)$ is $\mathbb{C}(s)$, every rational matrix can be represented in this way. Our purpose will be to construct a state-space realization for $\pi_{-} G(s)$ in modified form, assuming that $G(s)$ is given in the form (3.1).

REMARK 3.1. The idea of finding a realization from a representation of a transfer matrix as a quotient of two matrices which are analytic over a given part of the complex plane can be traced back to the infinite-dimensional realization theory that was developed in the seventies (see [11, Chapter III] and the references given therein). It should be noted, though, that the factorizations were usually not written as a quotient in those days; a typical form is $G(s) = Q(s)H^{*}(s)$ where $Q(s)$ is inner (with respect to the unit circle) and $H^{*}(s) = H^{\mathrm{T}}(s^{-1})$ with $H(s) \in H^{\infty}$. Our matrices $D_{+}(s)$ and $N_{+}(s)$ correspond to $Q^{-1}(s)$ and $H^{*}(s)$, respectively; these are indeed both matrices that are analytic outside the unit disk. Because the realization problem was considered for H^{∞}-functions, the aspect of a partial-fraction decomposition did not come into play. The fact that a small modification of the Fuhrmann realization can be applied to find a state-space representation for the stable part of an L^{∞}-function given in fractional form was apparently noted only much later in [8]. In the theory of finite-dimensional linear systems, the use of fractional representations over other rings than the ring of polynomials emerged as an important tool by the end of the seventies; see for instance [6]. (It is interesting to note that this development followed, rather than preceded, the infinite-dimensional theory.) The representation of rational matrices as a quotient of matrices over the ring of stable proper rational functions has since become a standard tool in H^{∞}-optimization, as evidenced for instance in [9, 26]. Perhaps as a result of the fact that the availability of a state-space representation is usually taken for granted in H^{∞}-optimization, no realization theory associated with the representation over the ring of stable proper rational functions seems to have been developed within this theory. The emphasis put in this paper on the relation between realization theory and partial-fraction decomposition may be of interest in this connection, since partial-fraction decomposition is a standard tool in H^∞-optimization.

In the realization to be presented below, we shall use the following vector space as a state space:

$$X_-(D_+) = \{f(s) \in \mathbb{C}^p_{0+}(s) \mid \pi_{0+} D_+^{-1} f = 0\}. \tag{3.2}$$

Introduce the following mappings:

$$A_- : f \mapsto D_+ \pi_- D_+^{-1} \chi f \tag{3.3}$$

from $X_-(D_+)$ into itself,

$$B_- : u \mapsto D_+ \pi_- D_+^{-1} N_+ u \tag{3.4}$$

from $\mathbb{C}^m$ to $X_-(D_+)$, and

$$C_- : f \mapsto \pi_0 D_+^{-1} \chi f \tag{3.5}$$

from $X_-(D_+)$ into $\mathbb{C}^p$. We note that $D_+\pi_- D_+^{-1} f = f - D_+ \pi_{0+} D_+^{-1} f \in \mathbb{C}^p_{0+}(s)$ for any $f \in \mathbb{C}^p_{0+}(s)$; this justifies the definitions of the mappings A_- and B_-. The following result is a direct generalization of Theorem I.10-1 in [11]; cf. also [8]. We shall present a proof for completeness, although the argument is just a direct extension of the one in [11]. Formulation and proof of minimality properties as in [11] will be left to the reader.

THEOREM 3.2. *If $G(s)$ is given by* (3.1) *where both $D_+(s)$ and $N_+(s)$ are matrices over $\mathbb{C}_{0+}(s)$, then the triple (A_-, B_-, C_-) defined by* (3.3)–(3.5) *gives a realization in modified state-space form of $\pi_- G(s)$.*

Proof. The terms in the expansion

$$C_-[\chi(s)I - A_-]^{-1}B_- = \sum_{k=1}^{\infty} C_- A_-^{k-1} B_- \chi^{-k}(s) \tag{3.6}$$

can be computed as follows:

$$C_- A_-^{k-1} B_- = \pi_0 D_+^{-1} \chi D_+ \pi_- D_+^{-1} \chi^{k-1} D_+ \pi_- D_+^{-1} N_+$$
$$= \pi_0 \chi \pi_- \chi^{k-1} \pi_- D_+^{-1} N_+ = \pi_0 \chi^k \pi_- D_+^{-1} N_+. \quad (3.7)$$

The result therefore follows from the formula

$$f(s) = \sum_{k=1}^{\infty} (\pi_0 \chi^k f) \chi^{-k}(s), \quad (3.8)$$

which is valid for every $f \in \mathbb{C}_-^p(s)$. ■

By interchanging "+" and "−" in the above [which includes replacing $\chi(s)$ by $\chi^{-1}(s)$] we can construct a state-space realization for $\pi_+ G(s)$ from a factorization $G(s) = D_-^{-1}(s) N_-(s)$ over $\mathbb{C}_{0-}(s)$. It is then possible to find $\pi_0 G$ by the formula

$$\pi_0 G = G(\alpha) - [(\pi_+ G)(\alpha) + (\pi_- G)(\alpha)], \quad (3.9)$$

which holds for any complex α that is not a pole of $G(s)$. Alternatively, we may use the formula

$$\pi_{0+} = \chi^{-1} \pi_+ \chi \quad (3.10)$$

to construct a realization for $\pi_{0+} G(s)$ from a realization for $\pi_+ \chi G(s)$. This leads to the following:

$$X_{0+}(D_-) = \{ f(s) \in \mathbb{C}_{0-}^p(s) \mid \pi_- D_-^{-1} f = 0 \}, \quad (3.11)$$

$$A_{0+} : f \mapsto D_- \pi_{0+} D_-^{-1} \chi^{-1} f, \quad (3.12)$$

$$B_{0+} : u \mapsto D_- \pi_{0+} D_-^{-1} N_- u, \quad (3.13)$$

$$C_{0+} : f \mapsto \pi_0 D_-^{-1} f. \quad (3.14)$$

One easily verifies that the triple (A_{0+}, B_{0+}, C_{0+}) represents $\pi_{0+} G(s)$ through the formula

$$\pi_{0+} G(s) = C_{0+} [I - \chi(s) A_{0+}]^{-1} B_{0+}. \quad (3.15)$$

This representation in what might be called a *modified descriptor form* can be merged with the realization we found for $\pi_- G(s)$. This results in a representation in modified descriptor form for the complete matrix $G(s)$, as follows:

$$G(s) = C[\chi(s)E - A]^{-1}B, \tag{3.16}$$

where

$$E = \begin{pmatrix} I & 0 \\ 0 & A_{0+} \end{pmatrix}, \qquad A = \begin{pmatrix} A_- & 0 \\ 0 & I \end{pmatrix}, \qquad B = \begin{pmatrix} B_- \\ B_{0+} \end{pmatrix},$$

$$C = \begin{bmatrix} C_- & -C_{0+} \end{bmatrix}. \tag{3.17}$$

This is essentially the construction in [5, 27]. If the factorizations that one starts with are coprime, then the construction leads to a realization that is minimal (in the sense that the size of E is minimal) among all realizations of the form (3.16). A reduction in size is possible, however, if one allows the presence of a constant D-term (cf. for instance [17]).

REMARK 3.3. If one is interested only in finding a state-space representation for $\pi_- G(s)$, then the assumption that $\chi(s)$ has its zero in Γ_+ is not really needed. The most interesting special case occurs when $\infty \in \Gamma_-$. We can then take $\chi(s) = s - \alpha$ with $\alpha \in \Gamma_+$ and apply the theorem to obtain a realization in modified form

$$\pi_- G(s) = C_- \left[(s-\alpha)I - A_-\right]^{-1} B_- \tag{3.18}$$

with parameters defined by (3.3)–(3.5). Because

$$D_+ \pi_- D_+^{-1}(s-\alpha)f = D_+ \pi_- D_+^{-1} sf - \alpha f \tag{3.19}$$

and

$$\pi_0 D_+^{-1}(s-\alpha)f = \pi_0 D_+^{-1} sf \tag{3.20}$$

for $f \in X_-(D_+)$, the function $\pi_- G(s)$ may also be represented in standard state-space form

$$\pi_- G(s) = C(sI - A)^{-1} B \tag{3.21}$$

with state space $X_-(D_+)$ and parameters defined by

$$A : f \mapsto D_+ \pi_- D_+^{-1} sf, \tag{3.22}$$

$$B : u \mapsto D_+ \pi_- D_+^{-1} N_+ u, \tag{3.23}$$

$$C : f \mapsto \pi_0 D_+^{-1} sf. \tag{3.24}$$

This representation is valid whether or not the origin of the complex plane belongs to Γ_+.

4. REALIZATION IN PENCIL FORM

We have seen above that it is possible to obtain a state-space representation of the plus part of a rational matrix from a factorization of that matrix over the ring of minus functions, and vice versa. In this section, we shall show that one can also obtain a representation by means of constant matrices for the *complete* rational matrix from a fractional representation over $\mathbb{C}_{0+}(s)$ [or $\mathbb{C}_{0-}(s)$]. The representation involved is the so-called *pencil representation* [17], which can be introduced as follows. Note that there is a one-one relation between the set of rational matrices of size $p \times m$ and the set of m-dimensional subspaces of $\mathbb{C}^{p+m}(s)$ that are complementary to span$[I_p \;\; 0]^{\mathrm{T}}$. The relation is given by

$$G(s) \mapsto V(s) = \ker\begin{bmatrix} I_p & -G(s) \end{bmatrix}. \tag{4.1}$$

Now, it is a fact (as shown below) that every m-dimensional subspace $V(s)$ of $\mathbb{C}^{p+m}(s)$ can be represented in the form

$$V(s) = H \ker(sG - F) \tag{4.2}$$

where F, G, and H are constant matrices; the dimensions of F and G are $n \times (n+m)$, where n depends on $V(s)$, and the dimension of H is $(p+m) \times (n+m)$. The representation (4.2) has been called the pencil representation in [17]. Any rational matrix, be it proper or nonproper, can be represented in pencil form through the associations (4.1) and (4.2).

If $(D(s), N(s))$ is a pair of rational matrices of size $p \times p$ and $p \times m$ respectively, and the matrix $[D(s) \quad N(s)]$ has full row rank, then this pair determines an m-dimensional subspace of $\mathbb{C}^{p+m}(s)$ by

$$V(s) = \ker\left[D(s) \quad -N(s) \right]. \tag{4.3}$$

The subspace defined in this way is complementary to $\operatorname{span}[I_p \;\; 0]^{\mathrm{T}}$ if and only if $D(s)$ is invertible, and in this case the rational matrix associated with $V(s)$ through (4.1) is $G(s) = D^{-1}(s)N(s)$. In view of this, it is natural to generalize our representation problem slightly. So, given a $p \times (p+m)$ matrix $R(s)$ of full row rank over the ring $\mathbb{C}_{0+}(s)$, we shall be looking for a representation of the vector space $\ker R(s)$ in (modified) pencil form:

$$\ker R(s) = H \ker\{\chi(s)G - F\}. \tag{4.4}$$

Remark 4.1. Here, as well as below, we make no distinction in notation between a mapping $H: X \to Y$ between vector spaces over $\mathbb{C}$ and the induced mapping between the corresponding vector spaces $X(s)$ and $Y(s)$ over $\mathbb{C}(s)$.

Remark 4.2. The correspondence between different representations is defined in the above by associating an m-dimensional subspace of $\mathbb{C}^{p+m}(s)$ to each representation, as in (4.1), (4.2), and (4.3). This type of equivalence was introduced in [1] and was termed "input/output equivalence" in [16]. The terminology was suggested by the use of the phrase "input/output relation" in [12] for m-dimensional subspaces of $\mathbb{C}^{p+m}(s)$. It may be noted that a subspace of $\mathbb{C}^{p+m}(s)$ of $\mathbb{C}(s)$-dimension m induces a continuous mapping from the Riemann sphere $\mathbb{C}^e$ into the Grassmannian manifold $G^m(\mathbb{C}^{p+m})$ of m-dimensional subspaces of $\mathbb{C}^{p+m}$ in the following way: in a neighborhood of any given $s_0 \in \mathbb{C}^e$, define the mapping by

$$s \mapsto \operatorname{span} V(s), \tag{4.5}$$

where $V(s)$ is any basis matrix for the given subspace that has no pole at s_0. It has been shown essentially in [20] that this prescription does indeed define a unique mapping from $\mathbb{C}^e$ to $G^m(\mathbb{C}^{p+m})$. The pencil representation (4.2) [or (4.4)] can then be seen as a first-order representation of this mapping.

Our main result on representation in the form (4.4) is given below. The proof technique can of course no longer be based on expansion around a point in the complex plane as in (3.6). The result has been gleaned from [17], but the proof is different because we are now working under input/output equivalence rather than under external equivalence. For the definition of the space $X_{-}(R)$, see [7].

THEOREM 4.3. *Let $R(s) \in \mathbb{C}_{0+}^{p\times q}(s)$ be of full row rank. Introduce the following complex vector spaces:*

$$X^{-}(R) = \{w(\lambda) \in \mathbb{C}_{-}^{q}(\lambda) \,|\, \pi_{-} Rw = 0\} \tag{4.6}$$

$$N^{-}(R) = \{w(\lambda) \in \mathbb{C}_{-}^{q}(\lambda) \,|\, Rw = 0\} \tag{4.7}$$

$$X_{-}(R) = \{p(\lambda) \in \mathbb{C}_{0+}^{p}(\lambda) \,|\, \exists w(\lambda) \in \mathbb{C}_{-}^{q}(\lambda) \ s.t.\ p = Rw\}. \tag{4.8}$$

The elements of the quotient space $X^{-}(R)/\chi^{-1}N^{-}(R)$ will be denoted by $[w(\lambda)]$ or $[w]$, where $w(\lambda) \in X^{-}(R)$. Define mappings F and G from $X^{-}(R)/\chi^{-1}N^{-}(R)$ to $X_{-}(R)$ by

$$F : [w] \mapsto R\pi_{-}\chi w \tag{4.9}$$

and

$$G : [w] \mapsto Rw, \tag{4.10}$$

and define a mapping H from $X^{-}(R)/\chi^{-1}N^{-}(R)$ to $\mathbb{C}^{q}$ by

$$H : [w] \mapsto \pi_{0}\chi w. \tag{4.11}$$

These mappings are well defined, and the relation (4.4) holds. That is to say, the triple (F, G, H) provides a modified pencil-form realization under input / output equivalence for the given matrix $R(s)$ over $\mathbb{C}_{0+}(s)$.

Proof. It is straightforward to verify that the mappings F, G, and H are indeed well defined. Now, let $(w(\lambda))(s)$ be an element of $\ker\{\chi(s)G - F\}$. We then have

$$\chi(s)R(\lambda)(w(\lambda))(s) - R(\lambda)\pi_{-}\chi(\lambda)(w(\lambda))(s) = 0. \tag{4.12}$$

Because $(w(\lambda))(s) \in (\mathbb{C}^q_-(\lambda))(s)$, we obviously have

$$\pi_-\chi(\lambda)(w(\lambda))(s) = \chi(\lambda)(w(\lambda))(s) - \pi_0\chi(\lambda)(w(\lambda))(s), \quad (4.13)$$

so that (4.12) may be rewritten as

$$\chi(s)R(\lambda)(w(\lambda))(s) = R(\lambda)\chi(\lambda)(w(\lambda))(s) - R(\lambda)\pi_0\chi(\lambda)(w(\lambda))(s). \quad (4.14)$$

By the definition of H, this gives

$$R(\lambda)(H(w(\lambda)))(s) = \{\chi(s) - \chi(\lambda)\}R(\lambda)(w(\lambda))(s). \quad (4.15)$$

Taking $\lambda = s$, we get in particular

$$R(s)(H(w(\lambda)))(s) = 0. \quad (4.16)$$

We have proved that $\ker R(s) \supset H \ker\{\chi(s)G - F\}$.

It remains to show that the dimensions of the two spaces are equal. Because the mapping G is surjective, as is clear from the definition, the rational matrix $\chi(s)G - F$ has full row rank, so that we have

$$\dim_{\mathbb{C}(s)} \ker\{\chi(s)G - F\} = \dim_{\mathbb{C}} \ker G = \dim_{\mathbb{C}} N^-(R)/\chi^{-1}N^-(R) = \dim_{\mathbb{C}(s)} \ker R(s). \quad (4.17)$$

Finally, we note that the mapping $[G^{\mathsf{T}} \ H^{\mathsf{T}}]^{\mathsf{T}}$ is injective; for, if for some $w \in X^-(R)$ we have both $Rw = 0$ and $\pi_0\chi w = 0$, then $\chi w \in \mathbb{C}_-(\lambda)$, so $w \in \chi^{-1}N^-(R)$ and hence $[w] = 0$. Therefore, the rational matrix $\{\chi(s)G^{\mathsf{T}} - F^{\mathsf{T}} \ H^{\mathsf{T}}\}^{\mathsf{T}}$ has full column rank, and we may conclude

$$\dim_{\mathbb{C}(s)} H \ker\{\chi(s)G - F\} = \dim_{\mathbb{C}(s)} \ker\{\chi(s)G - F\} = \dim_{\mathbb{C}(s)} \ker R(s). \quad (4.18)$$

■

It can be shown that the realization above is minimal (in the sense that no pencil representation with a lesser value of n exists for the same input/output relation) if $R(s)$ has full row rank as a matrix over $\mathbb{C}$ for every $s \in \Gamma_+$.

5. CONSTRUCTION OF THE ISOMORPHISM

In the previous two sections, we have seen two ways of representing a general rational matrix $G(s)$ by means of constant matrices, when factorizations of $G(s)$ over $\mathbb{C}_{0+}(s)$ and $\mathbb{C}_{0-}(s)$ are given:

(1) by two modified state-space representations for the partial fractions of $G(s)$ corresponding to Γ_+ and Γ_- respectively;
(2) by a modified pencil representation.

When using the first method, we obtain two state spaces: $X_-(D_+)$ and $X_+(D_-)$. Application of the second method leads to a space $X_-([D_+ - N_+])$, which might be considered as a "state space" for the pencil representation. It is a natural question to ask in what way these spaces are connected. We shall now establish an isomorphism which ties the three spaces together.

THEOREM 5.1. *Let $G(s)$ be a rational matrix of size $p \times m$, and suppose that factorizations of $G(s)$ both over $\mathbb{C}_{0+}(s)$ and over $\mathbb{C}_{0-}(s)$ have been given:*

$$G(s) = D_+^{-1}(s)N_+(s) = D_-^{-1}(s)N_-(s). \tag{5.1}$$

Also suppose that the pair $(D_-(s), N_-(s))$ is left coprime. Under these conditions, the mapping

$$\Psi : [f] \mapsto D_- \chi \pi_{0+} D_+^{-1} f \; \left(= D_- \pi_+ \chi D_+^{-1} f \right) \tag{5.2}$$

from $X_-([D_+ - N_+])/X_-(D_+)$ to $X_+(D_-)$ is well defined, and is an isomorphism.

Proof. Define Φ from $X_-([D_+ - N_+])$ to $X_+(D_-)$ by

$$\Phi : f \mapsto D_- \chi \pi_{0+} D_+^{-1} f \; \left(= D_- \pi_+ \chi D_+^{-1} f \right). \tag{5.3}$$

The statement in the theorem will follow if we can show that this mapping is well defined, that it is surjective, and that its kernel is exactly $X_-(D_+)$.

We first prove that Φ maps $X_-([D_+ \; -N_+])$ into $\mathbb{C}^p_-(s)$. Take $f \in X_-([D_+ \; -N_+])$, and let $g \in \mathbb{C}^p_-(s)$ and $h \in \mathbb{C}^m_-(s)$ be such that $f = D_+ g - N_+ h$. Then

$$D_+^{-1} f = g - D_+^{-1} N_+ h = g - D_-^{-1} N_- h. \tag{5.4}$$

Note that $\pi_{0+} g = 0$, so

$$\Phi f = -D_- \pi_+ \chi D_-^{-1} N_- h = -N_- \chi h + D_- \pi_{0-} \chi D_-^{-1} N_- h, \tag{5.5}$$

which indeed belongs to $\mathbb{C}^p_-(s)$. Next we have to show that $\pi_{0-} D_-^{-1} \Phi = 0$; this is obvious from the definition. It is also obvious that $\ker \Phi$ coincides with $X_-(D_+)$.

Finally, we prove that Φ is surjective. Let $p \in X_+(D_-)$, and define $f = \chi^{-1} D_+ D_-^{-1} p$. It is easily verified that $f \in \mathbb{C}^p_{0+}(s)$ and that $\Phi f = p$. In order to show that $f \in X_-([D_+ \; -N_+])$, we have to find $g \in \mathbb{C}^p_-(s)$ and $h \in \mathbb{C}^m_-(s)$ such that $f = D_+ g - N_+ h$. Since the pair $(D_-(s), N_-(s))$ is left coprime, the matrix $[D_-(s) - N_-(s)]$ has a right inverse, say $[K^{\mathrm{T}}(s) \;\; L^{\mathrm{T}}(s)]^{\mathrm{T}}$, in $\mathbb{C}_{0-}^{(p+m)\times p}(s)$. Define $g = \chi^{-1} K p$ and $h = \chi^{-1} L p$; then $g \in \mathbb{C}^p_-(s)$, $h \in \mathbb{C}^m_-(s)$, and

$$D_- g - N_- h = \chi^{-1} p. \tag{5.6}$$

This gives

$$D_+ g - N_+ h = D_+ D_-^{-1}(D_- g - N_- h) = D_+ D_-^{-1} \chi^{-1} p = f,$$

and so the proof is complete. ∎

We immediately have the following corollary.

COROLLARY 5.2. *Let $G(s)$ be a rational matrix of size $p \times m$, and suppose that factorizations of $G(s)$ both over $\mathbb{C}_{0+}(s)$ and over $\mathbb{C}_{0-}(s)$ have been given as in* (5.1). *Also suppose that the pair $(D_-(s), N_-(s))$ is left coprime. Under these conditions, we have*

$$\dim X_-([D_+ \; -N_+]) = \dim X_-(D_+) + \dim X_+(D_-). \tag{5.7}$$

We shall elaborate on this dimensional equality in the next section.

6. THE MCMILLAN DEGREE

In this section we consider some alternative expressions for the dimensions of the vector spaces that have appeared in the development above. The following version of a Wiener-Hopf factorization theorem for rational matrix functions will be needed. We still use the notation and the assumptions of Section 2.

THEOREM 6.1. *Every rational matrix $R(s)$ of full row rank p can be written in the form*

$$R(s) = U_+(s)\begin{bmatrix}\Delta(s) & 0\end{bmatrix}U_-(s), \tag{6.1}$$

where $U_+(s)$ is $\mathbb{C}_{0+}(s)$-unimodular, $U_-(s)$ is $\mathbb{C}_{0-}(s)$-unimodular, and

$$\Delta(s) = \operatorname{diag}(\chi^{k_1}(s),\dots,\chi^{k_p}(s)), \qquad k_1,\dots,k_p \in \mathbb{Z}. \tag{6.2}$$

Moreover, the indices $k_1,\dots,k_p$ are the same (up to order) in any factorization of the form (6.1).

Proof. Let $g(s)$ be a rational function such that $g(s)R(s)$ is a matrix over $\mathbb{C}_{0-}(s)$. For instance by reduction to Hermite form [19], one can find a $\mathbb{C}_{0-}(s)$-unimodular matrix $V(s)$ such that

$$g(s)R(s)V(s) = \begin{bmatrix}\hat{R}(s) & 0\end{bmatrix}, \tag{6.3}$$

where $\hat{R}(s)$ is square and nonsingular. It follows that for the construction of the factorization (6.1) it is sufficient to consider the case in which $R(s)$ is invertible. For this case, a construction method is given in [4, Chapter 1] under some extra conditions, which are however inessential in the present context.

It remains to prove the uniqueness of the indices $k_1,\dots,k_p$. Define, for $k \in \mathbb{Z}$,

$$n_k(R) = \dim_{\mathbb{C}} X_-(\chi^{-k}R). \tag{6.4}$$

It is easily seen that these integers are invariant under left multiplication of $R(s)$ by $\mathbb{C}_{0+}(s)$-unimodular matrices and right multiplication of $R(s)$ by

$\mathbb{C}_{0-}(s)$-unimodular matrices. Consequently, we can use the factorization (6.1) to compute the indices $n_k(R)$ in terms of the indices k_i:

$$n_k(R) = \sum_{i \in I_+(k)} k_i - k, \qquad I_+(k) = \{i \in \{1,\dots,p\} \mid k_i - k \geqslant 0\}. \quad (6.5)$$

Since we have

$$n_{k-1}(R) = \sum_{i \in I_+(k-1)} k_i - k + 1 = \sum_{i \in I_+(k)} k_i - k + 1 = n_k(R) + \operatorname{card} I_+(k), \quad (6.6)$$

the number of i's for which $k_i \geqslant k$ is equal to $n_{k-1}(R) - n_k(R)$, so that

$$\operatorname{card}\{i \in \{1,\dots,p\} \mid k_i = k\} = n_{k+1}(R) - 2n_k(R) + n_{k-1}(R). \quad (6.7)$$

This shows that, conversely, the indices k_i are uniquely determined (up to order) by the integers $n_k(R)$. Since the latter are directly determined by $R(s)$ through (6.4), the proof is complete. ■

DEFINITION 6.2. The indices $k_1,\dots,k_p$ that are defined by the factorization (6.1) are called the *factorization indices* of $R(s)$ with respect to (Γ_+,Γ_-).

COROLLARY 6.3. *For any rational matrix $R(s)$ of full row rank, the integer* $\dim_{\mathbb{C}} X_-(R)$ *is equal to the sum of the nonnegative factorization indices of $R(s)$ with respect to (Γ_+,Γ_-).*

Proof. Use (6.4) and (6.5) with $k = 0$. ■

COROLLARY 6.4. *For any nonsingular rational matrix $D(s)$, the number of zeros of $D(s)$ in Γ_+ minus the number of poles of $D(s)$ in Γ_+ is equal to the sum of the factorization indices of $D(s)$ with respect to (Γ_+,Γ_-).*

Proof. Both sides of the equality are invariant under left multiplication of $D(s)$ by $\mathbb{C}_{0+}(s)$-unimodular matrices and right multiplication of $D(s)$ by $\mathbb{C}_{0-}(s)$-unimodular matrices. (To see the invariance under multiplication by $\mathbb{C}_{0-}(s)$-unimodular matrices, note that the number of zeros of $D(s)$ in Γ_+

minus the number of poles of $D(s)$ in Γ_+ is equal to the number of poles of $D(s)$ in Γ_- minus the number of zeros of $D(s)$ in Γ_-, by the fact that the total number of zeros of $D(s)$ in $\mathbb{C}^e$ is equal to the total number of poles in $\mathbb{C}^e$ [14, Exercise 6.5.12b].) It is therefore sufficient to prove the theorem for the case in which $D(s)$ has the form (6.2), which is straightforward. ■

PROPOSITION 6.5. *If $R(s)$ is a $\mathbb{C}_{0+}(s)$-matrix of full row rank, then the factorization indices of $R(s)$ with respect to (Γ_+, Γ_-) are nonnegative.*

Proof. From (6.1), we have

$$U_+^{-1}(s)R(s) = \begin{bmatrix}\Delta(s) & 0\end{bmatrix}U_-(s). \tag{6.8}$$

On the left-hand side we have a matrix over $\mathbb{C}_{0+}(s)$, whereas $U_-(s)$ on the right-hand side is unimodular over $\mathbb{C}_{0-}(s)$. So each $\chi^{k_i}(s)$ on the diagonal of $\Delta(s)$ multiplies at least one nonzero minus function into a plus function, which can only happen if k_i is nonnegative. ■

COROLLARY 6.6. *If $D(s)$ is a nonsingular rational matrix having no poles in Γ_+, then* $\dim X_-(D)$ *is equal to the number of zeros of $D(s)$ in Γ_+.*

Proof. The statement follows from Corollary 6.3, Corollary 6.4, and Proposition 6.5. ■

REMARK 6.7. It can be verified that this corollary allows a sharpening of Theorem 5.1 in the following sense. If the pair $(D_-(s), N_-(s))$ is not assumed to be coprime, then the mapping Ψ of the theorem is still well defined and injective. The mapping is surjective if *and only if* the pair $(D_-(s), N_-(s))$ is left coprime.

The *degree* of a rational matrix $G(s)$ is defined in [21] as the total number of poles of $G(s)$ in $\mathbb{C}^e$. We shall write $\deg G$ for the degree of $G(s)$, and likewise we shall write $\deg_+ G$ for the number of poles of $G(s)$ in Γ_+ and $\deg_- G$ for the number of poles of $G(s)$ in Γ_-. Obviously,

$$\deg G = \deg_+ G + \deg_- G. \tag{6.9}$$

We can now easily identify the terms on the right-hand side, if left coprime factorizations over $\mathbb{C}_{0-}(s)$ and $\mathbb{C}_{0+}(s)$ are given.

COROLLARY 6.8. *If* $G(s) = D_+^{-1}(s)N_+(s)$ *is a left coprime factorization over* $\mathbb{C}_{0+}(s)$, *then*

$$\deg_+ G = \dim X_-(D_+). \tag{6.10}$$

Proof. Under the coprimeness condition, the number of poles of $G(s)$ in Γ_+ is equal to the number of zeros of $D_+(s)$ in Γ_+. Therefore, the statement follows from Corollary 6.6. ■

Together with the main result of the previous section, this leads immediately to the following characterization of the McMillan degree of a rational matrix in terms of the factors in a coprime factorization over the ring $\mathbb{C}_{0+}(s)$.

COROLLARY 6.9. *If* $G(s) = D_+^{-1}(s)N_+(s)$ *is a left coprime factorization over* $\mathbb{C}_{0+}(s)$, *then*

$$\deg G = \dim X_-([D_+ - N_+]). \tag{6.11}$$

Proof. The statement is immediate from (6.9), Corollary 6.8 and its twin version, and Corollary 5.2. ■

In view of Corollary 6.3 and Proposition 6.5, the above can be reformulated as follows:

COROLLARY 6.10. *If* $G(s) = D_+^{-1}(s)N_+(s)$ *is a left coprime factorization over* $\mathbb{C}_{0+}(s)$, *then* $\deg G$ *is equal to the sum of the factorization indices of* $[D_+(s) - N_+(s)]$ *with respect to* (Γ_+, Γ_-).

The factorization indices of a polynomial matrix with respect to $(\mathbb{C}, \{\infty\})$ are also known as the minimal row degrees (i.e., the degrees of the rows of a unimodularly related row reduced matrix—see [14, §6.3.2]). So a particular case of the above corollary is:

COROLLARY 6.11. *If* $G(s) = D^{-1}(s)N(s)$ *is a left coprime factorization over* $\mathbb{C}[s]$, *then* $\deg G$ *is equal to the sum of the minimal row degrees of* $[D(s) \;\; N(s)]$.

This result is immediate from the Fuhrmann realization theory in case $G(s)$ is proper, because then the sum of the minimal row degrees of $[D(s) \;\; N(s)]$ is equal to the sum of the minimal row degrees of $D(s)$, and one can use the well-known characterization of the degree of a proper rational matrix as the dimension of the state space in a minimal realization. The fact that the statement is also true in the nonproper case can be inferred by combining results from the literature which connect the degree to the rank of the matrix E in a minimal descriptor-form representation with results which connect this rank to the sum of the row degrees (cf. [3, 17, 24, 25, 28]). The present development, however, provides a direct proof based on the isomorphism of Theorem 5.1.

REMARK 6.12. The dimensional equality (5.7) can also be used to express $\deg_- G$ in terms of a factorization over $\mathbb{C}_{0+}(s)$, which doesn't even have to be coprime.

COROLLARY 6.13. *If $G(s) = D_+^{-1}(s)N_+(s)$ is a factorization over $\mathbb{C}_{0+}(s)$, then*

$$\deg_- G = \dim \frac{X_-([D_+ \; -N_+])}{X_-(D_+)}. \tag{6.12}$$

Proof. The statement is immediate from Corollary 5.2 and Corollary 6.8 (with "+" and "−" interchanged).

7. COMPUTATIONAL ISSUES

In Sections 3 and 4, we have described two different methods for obtaining a representation in terms of constant matrices for a general rational matrix which is given in fractional form. Any discussion of the merits of these two methods from the computational point of view should take into account the question what kind of representation one is looking for. If the form one wants to obtain is the descriptor form, then the method of Section 4 (the pencil realization) would seem to be preferable for the following reasons:

(1) in the realization by partial fractions, one has to compute basis matrices for two state spaces, rather than for one as in the pencil realization;

(2) in the realization by partial fractions, there are six parameter matrices to compute, rather than three as in the pencil realization; moreover, the definitions of the parameter matrices in (3.3)–(3.5) and (3.12)–(3.14) are more complicated than those in (4.9)–(4.11).

REMARK 7.1. The fact that the pencil representation is formed on the basis of a factorization over either $\mathbb{C}_{0+}(s)$ or $\mathbb{C}_{0-}(s)$ may seem to present an additional advantage over the realization by partial fractions, which needs factorizations over both rings. However, if the matrix parameters defined in Theorem 4.1 are computed via the Wiener-Hopf factorization, then there is no real gain, since the upper rows of the matrix $U_-(s)$ in the factorization (6.1) already provide a factorization over $\mathbb{C}_{0-}(s)$.

Neither the pencil realization nor the realization by partial fractions leads immediately to a representation in descriptor form, but in both cases such a representation can be obtained by a simple rearrangement of the data which doesn't involve any computation (see for instance [17] and (3.17) above). Both methods require the computation of basis matrices for spaces of the form $X_-(R)$. The most obvious tool to use for this is the Wiener-Hopf factorization (6.1). It should be noted that the computation of this factorization is simple when Γ_- is a singleton, in which case the procedure is essentially the same as the well-known algorithm for reducing a polynomial matrix to row reduced form [14, p. 386].

A different perspective appears when the ultimate goal of the computation is not a representation in descriptor form but rather a representation in (modified) state-space form of the components in the partial-fraction expansion. Such representations are relevant in several applications. For instance, a well-known method to compute the norm of a Hankel operator with rational symbol is based on the state-space representation for the term in the partial-fraction decomposition of the symbol that has poles outside the unit circle [9, Chapter 5]. In econometrics, the principal part of a transfer matrix associated with a pole at the point 1 is of interest because of its role in the description of "co-integration" [13, 23]. For the first example, the methods discussed in this paper require a factorization of the given rational matrix over the ring of rational functions whose poles are inside the unit circle; representations of this form are indeed often used [9]. In the second example, we need a factorization over the ring of rational functions having no pole at 1. This is certainly provided by the standard ARMA representations of econometrics, which actually use the ring of polynomials.

State-space representations for the terms in a partial-fraction decomposition can be obtained from the pencil realization by first going to the descriptor form and then using an appropriate solution method for the generalized eigenvalue problem, such as the one in [15]. However, the method of realization by partial fractions would seem to be more naturally adapted to the problem, at least when the starting point is a fractional representation, and so it is of interest to see what this method can do. At first sight, it seems that the definition (3.4) already requires the computation of the partial-fraction expansion, so that the only gain obtained from the method would be that the terms in this expansion are displayed in state-space form. However, it is possible to avoid the computation of the partial-fraction expansion, and even to avoid computing the quotient $D_+^{-1}(s)N_+(s)$ altogether, by using the following trick. We shall be looking for representations in the form

$$G(s) = C_-\{\chi(s)I - A_-\}^{-1}B_- + \pi_0 G + C_+\{\chi^{-1}(s)I - A_+\}^{-1}B_+, \quad (7.1)$$

where the left-hand term on the right represents $\pi_- G(s)$ and the right-hand term represents $\pi_+ G(s)$. We start from factorizations $G(s) = D_+^{-1}N_+(s) = D_-^{-1}(s)N_-(s)$, which are both assumed to be coprime. Suppose that we already have computed a basis matrix $M_+(s)$ for $X_-(D_+)$ and a basis matrix $M_-(s)$ for $X_+(D_-)$. The constant matrix B_- is defined, according to (3.4), by

$$D_+^{-1}(s)M_+(s)B_- = \pi_- G(s). \quad (7.2)$$

Likewise, we have

$$D_-^{-1}(s)M_-(s)B_+ = \pi_+ G(s). \quad (7.3)$$

This means that the three matrices B_-, $\pi_0 G$, and B_+ must satisfy the equation

$$G(s) = \begin{bmatrix} D_+^{-1}M_+(s) & I_p & D_-^{-1}(s)M_-(s) \end{bmatrix} \begin{pmatrix} B_- \\ \pi_0 G \\ B_+ \end{pmatrix}. \quad (7.4)$$

The columns of the rational matrix on the right-hand side are independent over $\mathbb{C}$, so that the solution is unique. By multiplying through with $D_+(s)$, we get the equivalent equation

$$N_+(s) = \begin{bmatrix} M_+(s) & D_+(s) & D_+(s)D_-^{-1}(s)M_-(s) \end{bmatrix} \begin{pmatrix} B_- \\ \pi_0 G \\ B_+ \end{pmatrix}, \quad (7.5)$$

which is stated entirely in terms of plus functions. For instance, by using the bases in $X_-(D_+)$ and $X_+(D_-)$ we can write (7.5) as an equation in constant matrices of size $n+p$ times m, where $n = \dim X_-(D_+) + \dim X_+(D_-) =$

the McMillan degree of $G(s)$. Solving these equations will provide the parameters B_-, $\pi_0 G$, and B_+.

To illustrate the computational procedure, we shall work out an example. The example is the same as in [17], where the pencil realization method is applied to it; here we shall apply the method of realization by partial fractions. The setting is the "standard" one, with $\Gamma_+ = \mathbb{C}$, $\Gamma_- = \{\infty\}$, and $\chi(s) = s$. The rational matrix $G(s)$ is given in fractional form by $G(s) = D_+^{-1}(s)N_+(s)$, with

$$D_+(s) = \begin{pmatrix} s+1 & 0 \\ s+2 & 2s \end{pmatrix}, \qquad N_+(s) = \begin{pmatrix} -s^2 & -2 \\ -1 & -s+1 \end{pmatrix}. \tag{7.6}$$

The matrix $[D_+(s) - N_+(s)]$ is row reduced, and so it is easy to write down a WH factorization

$$[D_+(s) - N_+(s)] = \begin{pmatrix} s^2 & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} \dfrac{s+1}{s^2} & 0 & 1 & \dfrac{2}{s^2} \\ \dfrac{s+2}{s} & 2 & \dfrac{1}{s} & \dfrac{s-1}{s} \end{pmatrix}, \tag{7.7}$$

and to obtain from this a factorization over $\mathbb{C}_{0-}(s)$: $G(s) = D_-^{-1}(s)N_-(s)$ with

$$D_-(s) = \begin{pmatrix} \dfrac{s+1}{s^2} & 0 \\ \dfrac{s+2}{s} & 2 \end{pmatrix}, \qquad N_-(s) = \begin{pmatrix} -1 & -\dfrac{2}{s^2} \\ -\dfrac{1}{s} & -\dfrac{s+1}{s} \end{pmatrix}. \tag{7.8}$$

A basis matrix for $X_-(D_+)$ can be computed from the WH factorization

$$D_+(s) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} \dfrac{s+1}{s} & 0 \\ \dfrac{s+2}{s} & 2 \end{pmatrix} \tag{7.9}$$

of $D_+(s)$. This shows that we can take

$$M_+(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.10}$$

To compute a basis matrix for $X_+(D_-)$, we have to obtain a WH factorization of $D_-(s)$ with respect to (Γ_-, Γ_+). This can be done by transposing a WH factorization with respect to (Γ_+, Γ_-) of the polynomial matrix $s^2 D_-^{\mathrm{T}}(s)$:

$$s^2 D_-^{\mathrm{T}}(s) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} s^2 & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} \dfrac{s+1}{s^2} & \dfrac{s+2}{s} \\ -2\dfrac{s+1}{s} & -4 \end{pmatrix}, \tag{7.11}$$

so

$$D_-(s) = \begin{pmatrix} \dfrac{s+1}{s^2} & -2\dfrac{s+1}{s} \\ \dfrac{s+2}{s} & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{s} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{7.12}$$

As a basis matrix for $X_+(D_-)$, we can therefore take

$$M_-(s) = \begin{pmatrix} (s+1)/s \\ 2 \end{pmatrix}. \tag{7.13}$$

We can now start computing the parameters in the state-space representations of the terms in the partial-fraction decomposition. We first use (7.5), which in this case reads

$$\begin{pmatrix} -s^2 & -2 \\ -1 & -s+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & s+1 & 0 & s(s+1) \\ 0 & 1 & s+2 & 2s & 2s \end{pmatrix} \begin{pmatrix} B_- \\ \pi_0 G \\ B_+ \end{pmatrix}. \tag{7.14}$$

The solution is obtained by equating coefficients:

$$\begin{pmatrix} B_- \\ \pi_0 G \\ B_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -2 \\ 0 & 0 \\ -1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -3 & 1 \\ 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 \end{pmatrix}. \tag{7.15}$$

Next we compute the parameters C_- and C_+. According to (3.5), the first is

$$C_- = \pi_0\big(sD_+^{-1}(s)M_+(s)\big) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{7.16}$$

For the second, we have analogously

$$C_+ = \pi_0\big(s^{-1}D_-^{-1}(s)M_-(s)\big) = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}. \tag{7.17}$$

Finally, the matrix A_- is computed from the formula

$$M_+(s)A_- = \chi(s)M_+(s) - D_+(s)C_-, \tag{7.18}$$

which follows from (3.3) and (3.5), and which gives

$$A_- = \begin{pmatrix} -1 & 0 \\ -2 & 0 \end{pmatrix}. \tag{7.19}$$

The analogous formula

$$M_-(s)A_+ = \chi^{-1}(s)M_-(s) - D_-(s)C_+ \tag{7.20}$$

gives

$$A_+ = 0, \tag{7.21}$$

as of course it should be, since the matrix A_+ in the type of representation we consider is necessarily nilpotent.

Summarizing the computation, we have found the partial-fraction decomposition

$$\begin{aligned}\begin{pmatrix} s+1 & 0 \\ s+2 & 2s \end{pmatrix}^{-1}\begin{pmatrix} -s^2 & -2 \\ -1 & -s+1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\left[sI - \begin{pmatrix} -1 & 0 \\ -2 & 0 \end{pmatrix}\right]^{-1}\begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}(s^{-1}-0)^{-1}[-1 \quad 0].\end{aligned} \tag{7.22}$$

The left-hand side was the starting point of our computation, a fractional representation in terms of polynomial matrices; the right-hand side is the result, an additive representation which separates the finite and the infinite poles. The transformation to descriptor form is not difficult from here, but would perhaps rather hide than add information. A comparison with the computation in [17] will readily show that the pencil realization is much more convenient if all one wants is a minimal descriptor representation without separation between finite and infinite modes.

8. FINAL REMARKS AND CONCLUSIONS

We have discussed two realization methods for rational matrices given in fractional form: the pencil realization, and the realization by partial fractions. The two methods are alike in that they both apply to the whole class of rational matrices, and that they provide representations in terms of constant matrices. The pencil realization requires one fractional representation whereas the alternative method uses two, but we do not believe this difference to be very significant (see Remark 7.1). The realization by partial fractions leads to a more specific result than the pencil realization does, because it actually provides the partial-fraction decomposition of the given rational matrix. There is a computational price to be paid for this bonus, and so the pencil realization would seem to be preferable as a computational tool when all one wants is a representation of the given rational matrix in terms of constant matrices (say, descriptor form or pencil form). However, in case one is interested in obtaining state-space representations for the components in a partial-fraction expansion, realization by partial fractions may be an interesting option to compare with the alternative route via the pencil realization, the descriptor representation, and the generalized eigenvalue problem.

The isomorphism of Section 5 provides the connection between the two realization methods, inasmuch as it establishes a relation between the state spaces as vector spaces. Of course, the space $X_-(D_+)$ is a $\mathbb{C}_{0+}(s)$-module under the multiplication defined by $p \cdot f = D_+ \pi_- D_+^{-1} pf$ [$p \in \mathbb{C}_{0+}(s)$, $f \in X_-(D_+)$], and likewise $X_+(D_-)$ is a $\mathbb{C}_{0-}(s)$-module. These module structures are closely related with the (modified) state-space realizations on the two spaces [5, 27]. However, the space $X_-([D_+ - N_+])$ has no apparent module structure, and it seems hard to interpret the isomorphism between $X_-([D_+ - N_+])/X_-(D_+)$ and $X_+(D_-)$ in a module-theoretic sense. It remains an open question whether a natural connection exists between the parameters in a pencil realization and the parameters of the state-space realizations of the terms in a partial-fraction decomposition.

We have considered general partial-fraction decompositions in the sense that the decomposition is made with respect to arbitrary nonempty complementary parts of the complex plane. These parts may for instance correspond to stability and instability regions. The classical Fuhrmann realization [11, Chapter I.10] appears in our framework as the special case of realization by partial fractions (with respect to $\Gamma_+ = \mathbb{C}$ and $\Gamma_- = \{\infty\}$) in which it is assumed *a priori* that the rational matrix $G(s)$ has its elements in $\mathbb{C}_{0-}(s)$. In this case of course one term in the partial-fraction decomposition becomes trivial. In addition, there is a simplification in the definition of the B-parameter for the other term [see (3.4)] which makes it unnecessary to solve a system of linear equations as we did in Section 7.

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